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Rough sets models based on (L, M)-G-fuzzy remote neighborhood operators

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Abstract. In this paper, the concept of (L, M)-G-fuzzy remote neighborhood operators is introduced. Subsequently, approximation pairs grounded in (L, M)-G-fuzzy remote neighborhood operators are defined and explored. In particular, approximation pairs (operators) based on convex (L, M)-fuzzy remote neighborhood operators are investigated and characterized. Furthermore, a measure for approximation operators based on convex (L, M)-fuzzy remote neighborhood operators is defined. In a certain sense, this measure quantifies the quality of the derived approximation operators.

1. Introduction

Rough set theory was introduced by Pawlak [11, 12] to deal with the incompleteness and uncertain information. Many scholars applied this theory to various fields (see, for example, [8–10, 30, 39]). At present, rough set theory has been extended to binary relation-based rough sets (see, for example, [24, 41]), covering-based rough sets (see, for example, [3, 26, 29, 32, 40]) and generalized neighborhood system-based rough sets (see, for example, [17, 22, 28, 33]). It is emphasized here that the generalized neighborhood system-based rough sets (see [33, 34]) are more general than the neighborhood-based (binary relation-based) rough sets and covering-based rough sets. In a more general framework, for L and M to be different integral commutative complete lattice monoids, Šostak et al. [20] introduced the notion of M-valued L-fuzzy relations and used it to define a theory of L-fuzzy rough sets, called the many-level version of rough approximation for L-fuzzy sets. Subsequently, El-Saady et al. [4] developed a model of rough sets based on M-level L-fuzzy G-neighborhood systems. They proved that L-generalized neighborhood systems-based approximation operators (see [34]) and M-level L-fuzzy relation-based approximation operators (see [20])

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can be regarded as special cases of (L, M)-fuzzy generalized neighborhood systems-based approximation operators.

The notion of remote neighborhood systems is fundamental in the theory of Wang's topological molecular lattice (see [23]), and it is abstracted from the geometric notion of "remote". The notion of remote neighborhood systems can be considered as the dual notion of neighborhood systems in general topology. Therefore, Sun et al.[21] developed a theory of rough sets based on generalized remote neighborhood operators. It is worth mentioning that neighborhood systems also have certain advantages in dissymmetric lattices; they play an important role in topological molecular lattice and L-topology. Using the idea of remote neighborhood systems, Yue and Fang [31] proposed the concept of fuzzy remote neighborhood systems in fuzzy topological molecular lattice, and studied the connections between fuzzy co-topologies and topological fuzzy remote neighborhood systems. More generally, Yang and Li [27] introduced the concept of topological (L, M)-fuzzy remote neighborhood systems of fuzzy points, and gave the relationship between (L, M)-fuzzy co-topologies and topological (L, M)-fuzzy remote neighborhood systems. Recently, Zhao et al.[37] studied two kinds of convex (L, M)-fuzzy remote neighborhood operators in (L, M)-fuzzy setting, and proved that these two kinds of convex (L, M)-fuzzy remote neighborhood operators and (L, M)-fuzzy convex structures are categorically isomorphic; Gégény and Radeleczki [6] defined quasiorders by means of lower and upper fuzzy approximations, and they established conditions under which fuzzy rough sets form lattices. Inspired by the concepts of generalized remote neighborhood operators and M-level L-fuzzy G-neighborhood systems, we will consider how to construct rough set models based on (L, M)-G-fuzzy remote neighborhood operators, and will consider some issues regarding the lattice structure of the approximation pairs (operators) of these rough set models.

The contents of this paper are organized as follows. In Section 2, we recall some necessary concepts and results. In Section 3, we present the concept of (L, M)-G-fuzzy remote neighborhood operators as a generalization of generalized remote neighborhood operators and an extension of convex (L, M)-fuzzy remote neighborhood operators. Subsequently, we discuss in detail the approximation pairs (operators) based on (L, M)-G-fuzzy remote neighborhood operators. In Section 4, we not only discuss the approximation pairs (operators) inspired by convex (L, M)-fuzzy remote neighborhood operators, but also define the measure of approximation operators based on convex (L, M)-fuzzy remote neighborhood operators. In Section 5, we draw a conclusion.

2. Preliminaries

In this paper, let M be a complete lattice with the smallest element $\bot_M$ and the largest element $\top_M$. $M_{\bot_M} = M - \{\bot_M\}$. An element a in M is said to be coprime if $a \leq b \vee c$ implies that $a \leq b$ or $a \leq c$. The set of all coprime elements in $M_{\bot_M}$ is denoted by $J(M)$. $\forall u, v \in M$, we say that u is wedge below v in M (in symbols, $u \prec v$) if for all subsets $D \subseteq M$, $v \leq \bigvee D$ implies that $u \leq d$ for some $d \in D$. We denote $\beta(x) = \{y \in M \mid y \prec x\}$. A complete lattice M is said to be completely distributive iff for each $x \in M$, $x = \bigvee \beta(x) = \bigvee \beta^*(x)$, where $\beta^*(x) = \beta(x) \cap J(M)$ is called the standard greatest minimal family of x (see [23]).

Let L be a completely distributive lattice with order-reversing involution $'$. For a nonempty set X, L^X denotes the set of all L-subsets on X. The operators on L can be translated onto L^X in a pointwise way. The smallest element and the largest element in L^X are denoted by $\bot_L^X$ and $\top_L^X$, respectively. For each $a \in L$, let $\underline{a}$ denote the constant L-fuzzy subset of X with the value a. We say $\{A_i : i \in J\}$ is a directed (resp. co-directed) subset of L^X, in symbols $\{A_i : i \in J\} \overset{dir}{\subseteq} L^X$ (resp. $\{A_i : i \in J\} \overset{cdir}{\subseteq} L^X$) if for each $A_1, A_2 \in \{A_i : i \in J\}$, there exists $A_3 \in \{A_i : i \in J\}$ such that $A_1, A_2 \leq A_3$ (resp. $A_1, A_2 \geq A_3$). We usually use the symbol $\bigvee_{i\in I}^{d} A_i$ to represent the supremum of a directed subset $\{A_i\}_{i\in I} \subseteq L^X$. Let X, Y be two nonempty sets and let $f : X \longrightarrow Y$ be a mapping. Define $f^{\rightarrow} : L^X \longrightarrow L^Y$ and $f^{\leftarrow} : L^Y \longrightarrow L^X$ as follows: (1) $\forall A \in L^X$, $\forall y \in Y$, $f^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x)$; (2) $\forall A \in L^Y, \forall x \in X$, $f^{\leftarrow}(A)(x) = A(f(x))$. It can be verified that the pair $(f^{\rightarrow}, f^{\leftarrow})$ is a Galois connection on $(L^X, \leq)$ and $(L^Y, \leq)$ (see [13]). An L-fuzzy inclusion (see [18, 19]) on X is a mapping $\widetilde{\subset} : L^X \times L^X \longrightarrow L$ defined by the equality $\widetilde{\subset}(E, F) = \bigwedge_{x\in X}(E'(x) \vee F(x))$. The degree of intersection (see [2]) on X is a mapping $\widetilde{\sqcap} : L^X \times L^X \longrightarrow L$ defined by the equality $\widetilde{\sqcap}(E, F) = \bigvee_{x\in X}(E(x) \wedge F(x))$ $(\forall E, F \in L^X)$.

Throughout this paper, if not stated otherwise, we always assume that L denotes a completely distributive De Morgan algebra and M denotes a completely distributive lattice. In a completely distributive lattice L, there exists a binary operation $\to$, and this implication operation is defined by $a \to b = \bigvee\{x \in L \mid a \wedge x \leq b\}$. The following properties of the implication operator $\to$ are readily verified.

(1) $(a \to b) \geq c \iff a \wedge c \leq b$;
(2) $a \to b = \top \iff a \leq b$;
(3) $a \leq b \implies a \to c \geq b \to c$ and $c \to a \leq c \to b$;
(4) $(a \wedge b) \to c = a \to (b \to c)$;
(5) $a \to \bigwedge_{i\in I} b_i = \bigwedge_{i\in I}(a \to b_i)$ for any nonempty index set I;
(6) $\left(\bigvee_{i\in I} a_i\right) \to b = \bigwedge_{i\in I}(a_i \to b)$ for any nonempty index set I.

Some concepts related to category theory can be found in [1].

Definition 2.1. ([21]) A function $\mathbb{G} : X \longrightarrow 2^{2^X}$ is called a generalized remote neighborhood operator on X if for any $x \in X, \mathbb{G}(x)$ is nonempty. Usually, $\mathbb{G}(x)$ is called the generalized remote neighborhood system of x and each $A \in \mathbb{G}(x)$ is called a remote neighborhood of x.

The set of all generalized remote neighborhood operators on X is denoted by $\mathbf{GRN}(X)$. Define a relation $\leq$ on $\mathbf{GRN}(X)$ as follows: $\forall x \in X, \mathbb{G}_1 \leq \mathbb{G}_2 \iff \mathbb{G}_1(x) \subseteq \mathbb{G}_2(x)$. Then $(\mathbf{GRN}(X), \leq)$ is a poset.

Definition 2.2. ([21]) Let $\mathbb{G} : X \longrightarrow 2^{2^X}$ be a generalized remote neighborhood operator. Then for each subset A of X, the upper and lower rough approximation operators $\overline{\mathbb{G}}(A)$ and $\underline{\mathbb{G}}(A)$ are defined as follows:

$$\underline{\mathbb{G}}(A) = \{x \in X \mid \exists B \in \mathbb{G}(x), A' \subseteq B\}, \quad \overline{\mathbb{G}}(A) = \{x \in X \mid \forall B \in \mathbb{G}(x), A \nsubseteq B\}$$

A is called a definable set if $\overline{\mathbb{G}}(A) = \underline{\mathbb{G}}(A)$, and a rough set otherwise.

Definition 2.3. ([37]) A mapping $\mathscr{R} : L^X \times M_{\perp_M} \times X \longrightarrow L$ is called a convex (L, M)-fuzzy remote neighborhood operator if it satisfies the following seven axioms: for each $x \in X$, $A, B \in L^X$ and $a, b \in M_{\perp_M}$,

(CFR1) $\mathscr{R}(\perp_L^X, a, x) = \top_L$.

(CFR2) $\mathscr{R}(A, a, x) \leq A'(x)$.

(CFR3) If $A \leq B$, then $\mathscr{R}(B, a, x) \leq \mathscr{R}(A, a, x)$.

(CFR4) If $a \leq b$, then $\mathscr{R}(A, b, x) \leq \mathscr{R}(A, a, x)$.

(CFR5) $\mathscr{R}(A, a, x) = \bigvee\left\{\mathscr{R}(B, a, x) \mid B'(y) \leq \mathscr{R}(A, a, y), \forall y \in X\right\}$.

(CFR6) $\mathscr{R}(\bigvee^d_{j\in J} A_j, a, x) = \bigwedge_{j\in J} \mathscr{R}(A_j, a, x)$.

(CFR7) For any nonempty subset $\{a_j\}_{j\in J}$ of $M_{\perp_M}$ the implication holds:

$$A'(x) = \mathscr{R}(A, a_j, x) \quad \forall j \in J,\ \forall x \in X \implies A'(x) = \mathscr{R}\Big(A, \bigvee_{j\in J} a_j, x\Big).$$

If $\mathscr{R}$ is a convex (L, M)-fuzzy remote neighborhood operator on X, then the pair $(X, \mathscr{R})$ is called a convex (L, M)-fuzzy remote neighborhood space.

Let $(X, \mathscr{R}_X)$ and $(Y, \mathscr{R}_Y)$ be two convex (L, M)-fuzzy remote neighborhood spaces, then a function $g : X \longrightarrow Y$ is called a convex (L, M)-fuzzy remote neighborhood preserving mapping if $\mathscr{R}_Y(A, a, g(x)) \leq \mathscr{R}_X(g^{\leftarrow}(A), a, x)$ for each $x \in X$, $A \in L^Y$ and $a \in M_{\perp_M}$. The category of all convex (L, M)-fuzzy remote neighborhood spaces as objects and all their (L, M)-RNP mappings as morphisms is denoted by (L, M)-**FR**.

Definition 2.4. (Fang and Yue [5]) A mapping $\mathscr{C} : L^X \longrightarrow M$ is called an (L, M)-fuzzy closure system on X if it satisfies:

(LMC1) $\mathscr{C}(\bot_L^X) = \mathscr{C}(\top_L^X) = \top_M$;

(LMC2) $\bigwedge_{i\in I} \mathscr{C}(U_i) \leq \mathscr{C}(\bigwedge_{i\in I} U_i)$ $(\forall \{U_i\}_{i\in I} \subseteq L^X)$.

Definition 2.5. (Pang [14], Pang [15], Shi and Xiu [16]) A closure system $\mathscr{C}$ is called an (L, M)-fuzzy convex structure if one of the following conditions holds (the second then follows as a consequence):

(LMC3) If $\{U_k\}_{k\in K} \subseteq L^X$ is totally ordered, then $\bigwedge_{k\in K} \mathscr{C}(U_k) \leq \mathscr{C}(\bigvee_{k\in K} U_k)$.

(LMC3)* If $\{U_i\}_{i\in I} \subseteq L^X$ is directed, then $\bigwedge_{i\in I} \mathscr{C}(U_i) \leq \mathscr{C}(\bigvee^d_{i\in I} U_i)$.

If $\mathscr{C}$ is an (L, M)-fuzzy convex structure on X, then the pair $(X, \mathscr{C})$ is called an (L, M)-fuzzy convex space.

Let $(X, \mathscr{C}_X)$ and $(Y, \mathscr{C}_Y)$ be (L, M)-fuzzy convex spaces and $g : X \longrightarrow Y$ be a mapping. g is called (L, M)-convexity preserving ((L, M)-CP, in short) (see [14, 16]) if $\mathscr{C}_Y(S) \leq \mathscr{C}_X(g^{\leftarrow}(S))$ for all $S \in L^Y$. It is easy to check that all (L, M)-fuzzy convex spaces as objects and all corresponding (L, M)-CP mappings as morphisms form a category, denoted by (L, M)-**FC**.

Theorem 2.6. ([37]) $\mathbf{R^C}$ *and* $\mathbf{C^R}$ *are isomorphic functors, where two functors are given as follows:*

$$\mathbf{R^C} : \begin{cases} (L, M)\text{-}\mathbf{FC} \longrightarrow (L, M)\text{-}\mathbf{FR}, \\ (X, \mathscr{C}) \longmapsto (X, \mathscr{R}_{\mathscr{C}}), \\ g \longmapsto g, \end{cases} \qquad \mathbf{C^R} : \begin{cases} (L, M)\text{-}\mathbf{FR} \longrightarrow (L, M)\text{-}\mathbf{FC}, \\ (X, \mathscr{R}) \longmapsto (X, \mathscr{C}_{\mathscr{R}}), \\ g \longmapsto g, \end{cases}$$

and

$$\mathscr{R}_{\mathscr{C}}(A, a, x) = \bigvee \Big\{D'(x) \in L : D \geq A,\ \mathscr{C}(D) \geq a\Big\} (\forall x \in X, A \in L^X, a \in M_{\bot_M}),$$

$$\mathscr{C}_{\mathscr{R}}(A) = \bigvee \Big\{a \in M_{\bot_M} : A'(y) = \mathscr{R}(A, a, y), \forall y \in X\Big\}\ (\forall A \in L^X).$$

Definition 2.7. ([35]) A mapping $\mathscr{CO} : L^X \times M_{\bot_M} \longrightarrow L^X$ is called a convex (L, M)-fuzzy hull operator on X if it satisfies the following conditions: for any $A, B \in L^X$ and $r, s \in M_{\bot_M}$,

(CO1) $\mathscr{CO}(\bot_L^X, r) = \bot_L^X$.

(CO2) $B \leq \mathscr{CO}(B, r)$.

(CO3) If $r \leq s$, then $\mathscr{CO}(B, r) \leq \mathscr{CO}(B, s)$.

(CO4) If $A \leq B$, then $\mathscr{CO}(A, r) \leq \mathscr{CO}(B, r)$.

(CO5) $\mathscr{CO}(\mathscr{CO}(B, r), r) = \mathscr{CO}(B, r)$.

(CO6) If $\{B_i : i \in J\} \subseteq L^X$ is nonempty and totally ordered by inclusion, then

$$\mathscr{CO}\Big(\bigvee_{i\in J} B_i, r\Big) = \bigvee_{i\in J} \mathscr{CO}(B_i, r).$$

(CO7) If $r = \bigvee\{s \in M_{\bot_M} : A = \mathscr{CO}(A, s)\}$, then $\mathscr{CO}(A, r) = A$.

If $\mathscr{CO}$ is a convex (L, M)-fuzzy hull operator on X, then the pair $(X, \mathscr{CO})$ is called a convex (L, M)-fuzzy hull space.

Let $(X, \mathscr{CO}_X)$ and $(Y, \mathscr{CO}_Y)$ be two convex (L, M)-fuzzy hull spaces and let $g : X \longrightarrow Y$ be a mapping. g is called a convex (L, M)-fuzzy hull preserving function if $g^{\rightarrow}(\mathscr{CO}_X(A, r)) \leq \mathscr{CO}_Y(g^{\rightarrow}(A), r)$ for all $A \in L^X$ and $r \in M_{\bot_M}$. It is easy to check that all convex (L, M)-fuzzy hull spaces as objects and all corresponding convex (L, M)-fuzzy hull preserving functions as morphisms form a category, denoted by (L, M)-**FCH**.

Theorem 2.8. ([35]) $\mathbf{CO^C}$ *and* $\mathbf{C^{CO}}$ *are isomorphic functors, where two functors are given as follows:*

$$\mathbf{CO^C}:\begin{cases}(L,M)\text{-}\mathbf{FC}\longrightarrow(L,M)\text{-}\mathbf{FCH},\\ (X,\mathscr{C})\longmapsto(X,\mathscr{CO}_{\mathscr{C}}),\\ g\longmapsto g,\end{cases}\qquad \mathbf{C^{CO}}:\begin{cases}(L,M)\text{-}\mathbf{FCH}\longrightarrow(L,M)\text{-}\mathbf{FC},\\ (X,\mathscr{CO})\longmapsto(X,\mathscr{C}_{\mathscr{CO}}),\\ g\longmapsto g,\end{cases}$$

and

$$\mathscr{CO}_{\mathscr{C}}(A,a)=\bigwedge\left\{B\in L^X: A\leq B,\ \mathscr{C}(B)\geq a\right\},$$

$$\mathscr{C}_{\mathscr{CO}}(A)=\bigvee\left\{a\in M_{\perp_M}: A=\mathscr{CO}(A,a)\right\}.$$

Definition 2.9. ([36]) A mapping $\mathscr{I}: L^X\times M_{\perp_M}\longrightarrow L^X$ is called a concave (L,M)-fuzzy interior operator on X if it satisfies the following conditions: for any $A,B\in L^X$ and $r,s\in M_{\perp_M}$,

(I1) $\mathscr{I}(\top_L^X,r)=\top_L^X$.

(I2) $\mathscr{I}(A,r)\leq A$.

(I3) If $A\leq B$, then $\mathscr{I}(A,r)\leq\mathscr{I}(B,r)$.

(I4) If $r\leq s$, then $\mathscr{I}(A,s)\leq\mathscr{I}(A,r)$.

(I5) $\mathscr{I}(\mathscr{I}(A,r),r)=\mathscr{I}(A,r)$.

(I6) If $\{A_i:i\in J\}\overset{cdir}{\subseteq}L^X$, then $\mathscr{I}(\bigwedge_{i\in J}A_i,r)=\bigwedge_{i\in J}\mathscr{I}(A_i,r)$.

(I7) $\mathscr{I}(B,\bigvee\{r\in M_{\perp_M}: B=\mathscr{I}(B,r)\})=B$.

If $\mathscr{I}$ is a concave (L,M)-fuzzy interior operator on X, then the pair $(X,\mathscr{I})$ is called a concave (L,M)-fuzzy interior space.

Let $(X,\mathscr{I}_X)$ and $(Y,\mathscr{I}_Y)$ be two concave (L,M)-fuzzy interior spaces, then a function $g: X\longrightarrow Y$ is called a concave (L,M)-fuzzy interior preserving function if $g^{\leftarrow}(\mathscr{I}_Y(A,r))\leq\mathscr{I}_X(g^{\leftarrow}(A),r)$ for all $A\in L^Y$ and $r\in M_{\perp_M}$. It is easy to check that all concave (L,M)-fuzzy interior spaces as objects and all corresponding concave (L,M)-fuzzy interior preserving functions as morphisms form a category, denoted by (L,M)-**CFI**.

If **LMFA** is represented as the category of all (L,M)-fuzzy concave spaces as objects and all (L,M)-FCAPs as morphisms (see [25]), then the following conclusion holds.

Theorem 2.10. ([36]) (L,M)-***CFI*** *is isomorphic to* **LMFA**.

For the clarity of notation employed in all subsequent definitions, theorems and illustrative examples, we compile a comprehensive "Symbols and Definitions" reference table in Table 1.

3. (L,M)-G-fuzzy remote neighborhood operators-based rough sets

In this section, we will give the definition of (L,M)-G-fuzzy remote neighborhood operators. Then, we will develop a theory of (L,M)-G-fuzzy remote neighborhood operator-based rough sets from a constructive approach.

Definition 3.1. Let X be the universe of discourse. A mapping $\mathscr{G}: L^X\times M_{\perp_M}\times X\longrightarrow L$ is called an (L,M)-G-fuzzy remote neighborhood operator on X if it satisfies the following condition: for any $r\in M_{\perp_M}, x\in X$,

$$\bigvee_{B\in L^X}\mathscr{G}(B,r,x)=\top_L.$$

Table 1: Symbols and Definitions Used in the Text

Notation	Meaning/Definition
L	Completely distributive De Morgan algebra (with order-reversing involution $'$)
M	Completely distributive lattice
$\bot_L, \top_L$	Smallest and largest elements of L
$\bot_M, \top_M$	Smallest and largest elements of M
$M_{\bot_M}$	$M - \{\bot_M\}$ (non-$\bot_M$ elements of M)
$J(L), J(M)$	Set of coprime elements in $L_{\bot_L}$ (resp. $M_{\bot_M}$)
$\prec$	Wedge below relation: $u \prec v \iff \forall D \subseteq M, v \leq \bigvee D \Rightarrow \exists d \in D, u \leq d$
$\beta(x)$	$\{y \in M \mid y \prec x\}$ (wedge below set of x)
$\beta^*(x)$	$\beta(x) \cap J(M)$ (standard greatest minimal family of x)
L^X	Set of all L-subsets on nonempty set X
$\bot_L^X, \top_L^X$	Smallest and largest elements of L^X
$\underline{a}$	Constant L-fuzzy subset of X with value $a \in L$
$\bigvee_{i\in I}^{d} A_i$	Supremum of a directed subset $\{A_i\}_{i\in I} \subseteq L^X$
$\overset{dir}{\subseteq}$ / $\overset{cdir}{\subseteq}$	Directed/co-directed subset inclusion in L^X
$f^{\rightarrow} : L^X \to L^Y$	Forward image mapping of $f : X \to Y$
$f^{\leftarrow} : L^Y \to L^X$	Inverse image mapping of $f : X \to Y$
$\to$	Implication operator on L: $a \to b = \bigvee\{x \in L \mid a \wedge x \leq b\}$
$\widetilde{\subset} : L^X \times L^X \to L$	L-fuzzy inclusion on X
$\widetilde{\cap} : L^X \times L^X \to L$	Degree of intersection on X
$\mathbf{GRN}(X)$	Set of all generalized remote neighborhood operators on X
$\mathscr{R} : L^X \times M_{\bot_M} \times X \to L$	Convex (L, M)-fuzzy remote neighborhood operator
$\mathscr{C} : L^X \to M$	(L, M)-fuzzy convex structure (closure system)
$\mathscr{CO} : L^X \times M_{\bot_M} \to L^X$	Convex (L, M)-fuzzy hull operator
$\mathscr{I} : L^X \times M_{\bot_M} \to L^X$	Concave (L, M)-fuzzy interior operator
$(X, \mathscr{R})$	Convex (L, M)-fuzzy remote neighborhood space
$(X, \mathscr{C})$	(L, M)-fuzzy convex space
$(X, \mathscr{CO})$	Convex (L, M)-fuzzy hull space
$(X, \mathscr{I})$	Concave (L, M)-fuzzy interior space
(L, M)-**FR**	Category of convex (L, M)-fuzzy remote neighborhood spaces
(L, M)-**FC**	Category of (L, M)-fuzzy convex spaces
(L, M)-**FCH**	Category of convex (L, M)-fuzzy hull spaces
(L, M)-**CFI**	Category of concave (L, M)-fuzzy interior spaces
LMFA	Category of (L, M)-fuzzy concave spaces
$\mathbf{R}^{\mathbf{C}}, \mathbf{C}^{\mathbf{R}}$	Isomorphic functors between (L, M)-**FC** and (L, M)-**FR**
$\mathbf{CO}^{\mathbf{C}}, \mathbf{C}^{\mathbf{CO}}$	Isomorphic functors between (L, M)-**FC** and (L, M)-**FCH**

Let

$$\mathbf{GFR}(X,L,M)=\{\mathscr{G}:L^X\times M_{\perp_M}\times X\longrightarrow L\mid \forall r\in M_{\perp_M},x\in X,\bigvee_{B\in L^X}\mathscr{G}(B,r,x)=\top_L\}$$

denote the collection of all (L,M)-G-fuzzy remote neighborhood operators on X.

Define a relation $\leq$ on $\mathbf{GFR}(X,L,M)$ as follows: $\mathscr{G}_1\leq\mathscr{G}_2$ if and only if $\mathscr{G}_1(C,r,x)\leq\mathscr{G}_2(C,r,x)$ for each $(C,r,x)\in L^X\times M_{\perp_M}\times X$. Then $(\mathbf{GFR}(X,L,M),\leq)$ is a poset. Moreover, let $\emptyset\neq\{\mathscr{G}_i\}_{i\in\Lambda}\subseteq(\mathbf{GFR}(X,L,M),\leq)$, define the mapping $\bigvee_{i\in\Lambda}\mathscr{G}_i:L^X\times M_{\perp_M}\times X\longrightarrow L$ as follows: for each $(C,r,x)\in L^X\times M_{\perp_M}\times X$,

$$\Big(\bigvee\nolimits_{i\in\Lambda}\mathscr{G}_i\Big)(C,r,x)=\bigvee\nolimits_{i\in\Lambda}\mathscr{G}_i(C,r,x).$$

Then, we can verify that $\bigvee_{i\in\Lambda}\mathscr{G}_i\in\mathbf{GFR}(X,L,M)$. And, $\bigvee\limits_{i\in\Lambda}\mathscr{G}_i$ is the supremum of $\{\mathscr{G}_i\}_{i\in\Lambda}$. Therefore, $(\mathbf{GFR}(X,L,M),\leq)$ is a join-complete lattice, but not a complete lattice (see Example 3.2).

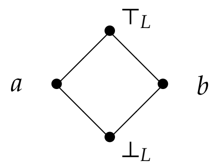

Fig. 1 The structure of L

Example 3.2. (1) Obviously, a convex (L,M)-fuzzy remote neighborhood operator must be an (L,M)-G-fuzzy remote neighborhood operator.

(2) Let $X=\{x\}$ be a singleton set, $M=\{\perp_M,\top_M\}$, and $L=\{\perp_L,a,b,\top_L\}$ be a diamond-type lattice (see Fig. 1), where $a'=b$, $\top_L'=\perp_L$. Then $L^X=\{\perp_L^X,\underline{a},\underline{b},\top_L^X\}$. Define two mappings $\mathscr{G}_1:L^X\times M_{\perp_M}\times X\longrightarrow L$ and $\mathscr{G}_2:L^X\times M_{\perp_M}\times X\longrightarrow L$ as follows:

$$\mathscr{G}_1(A,\top_M,x)=\begin{cases}a, & \text{if } A=\underline{a},\\ b, & \text{if } A=\underline{b},\\ \perp_L, & \text{otherwise},\end{cases}\qquad \mathscr{G}_2(A,\top_M,x)=\begin{cases}b, & \text{if } A=\underline{a},\\ a, & \text{if } A=\underline{b},\\ \perp_L, & \text{otherwise}.\end{cases}$$

Then $\mathscr{G}_i$ $(i\in\{1,2\}),\mathscr{G}_1\vee\mathscr{G}_2\in\mathbf{GFR}(X,L,M)$. However, there is no $\mathscr{G}\in\mathbf{GFR}(X,L,M)$ such that $\mathscr{G}\leq\mathscr{G}_1$ and $\mathscr{G}\leq\mathscr{G}_2$. If not, for each $A\in L^X$, we have

$$\perp_L\leq\mathscr{G}(A,\top_M,x)\leq\mathscr{G}_1(A,\top_M,x)\wedge\mathscr{G}_2(A,\top_M,x)=\perp_L.$$

This implies that $\bigvee_{B\in L^X}\mathscr{G}(B,\top_M,x)=\perp_L$, which contradicts $\mathscr{G}\in\mathbf{GFR}(X,L,M)$.

Theorem 3.3. *Ordered sets* $(\mathbf{GRN}(X),\leq)$ *and* $(\mathbf{GFR}(X,\mathbf{2},\mathbf{2}),\leq)$ *are order-isomorphic, where* $L=M=\mathbf{2}=\{\perp,\top\}$ *is the two-element complete lattice (completely distributive De Morgan algebra).*

Proof. Firstly, define a mapping $f:(\mathbf{GRN}(X),\leq)\longrightarrow(\mathbf{GFR}(X,\mathbf{2},\mathbf{2}),\leq)$ as follows:

$$\forall\mathbb{G}\in\mathbf{GRN}(X),x\in X,\text{ and }\chi_A\in\{\perp,\top\}^X,\ f(\mathbb{G})(\chi_A,\top,x)=\mathscr{G}_{\mathbb{G}}(\chi_A,\top,x)=\chi_{\mathbb{G}(x)}(A).$$

Then $f(\mathbb{G})\in\mathbf{GFR}(X,\mathbf{2},\mathbf{2})$. Define a mapping $g:(\mathbf{GFR}(X,\mathbf{2},\mathbf{2}),\leq)\longrightarrow(\mathbf{GRN}(X),\leq)$ as follows:

$$\forall\mathscr{G}\in\mathbf{GFR}(X,\mathbf{2},\mathbf{2}),\text{and } x\in X, g(\mathscr{G})(x)=\mathbb{G}_{\mathscr{G}}(x)=\{A\in 2^X\mid\mathscr{G}(\chi_A,\top,x)=\top\}.$$

Then $g(\mathscr{G})\in\mathbf{GRN}(X)$.

Secondly, $\forall \mathscr{G} \in \mathbf{GFR}(X, \mathbf{2}, \mathbf{2})$,

$$\begin{array}{rcl}(f \circ g)(\mathscr{G})(\chi_A, \top, x) & = & f(\mathbb{G}_{\mathscr{G}})(\chi_A, \top, x) \\ & = & \mathscr{G}_{\mathbb{G}_{\mathscr{G}}}(\chi_A, \top, x) \\ & = & \begin{cases} \top, & \text{if } A \in \mathbb{G}_{\mathscr{G}}(x), \\ \bot, & \text{if } A \notin \mathbb{G}_{\mathscr{G}}(x) \end{cases} \\ & = & \begin{cases} \top, & \text{if } \mathscr{G}(\chi_A, \top, x) = \top, \\ \bot, & \text{if } \mathscr{G}(\chi_A, \top, x) = \bot \end{cases} \\ & = & \mathscr{G}(\chi_A, \top, x),\end{array}$$

i.e., $f \circ g = id_{\mathbf{GFR}(X,\mathbf{2},\mathbf{2})}$ (here $id_{\mathbf{GFR}(X,\mathbf{2},\mathbf{2})}$ denotes the identity map on $\mathbf{GFR}(X, \mathbf{2}, \mathbf{2})$).

$\forall \mathbb{G} \in \mathbf{GRN}(X)$, $(g \circ f)(\mathbb{G})(x) = g(\mathscr{G}_{\mathbb{G}})(x) = \{A \in 2^X \mid \mathscr{G}_{\mathbb{G}}(\chi_A, \top, x) = \top\} = \{A \in 2^X \mid \chi_{\mathbb{G}(x)}(A) = \top\} = \mathbb{G}(x)$, i.e., $g \circ f = id_{\mathbf{GRN}(X)}$ (here $id_{\mathbf{GRN}(X)}$ denotes the identity map on $\mathbf{GRN}(X)$). Hence, f is a bijection. In addition, it is easily verified that f and g are order-preserving mappings. Therefore, f is an order-isomorphism. It follows that $(\mathbf{GRN}(X), \leq) \cong (\mathbf{GFR}(X, \mathbf{2}, \mathbf{2}), \leq)$. □

Remark 3.4. By Theorem 3.3, we immediately know that every generalized remote neighborhood operator can be regarded as a special (L, M)-G-fuzzy remote neighborhood operator.

Note that $A \nsubseteq B$ if and only if $B' \cap A \neq \emptyset$. And, $A' \subseteq B$ if and only if $B' \subseteq A$. So, $\overline{\mathbb{G}}(A)$ and $\underline{\mathbb{G}}(A)$ in Definition 2.2 can be rewritten as follows:

$$\underline{\mathbb{G}}(A) = \{x \in X \mid \exists B \in \mathbb{G}(x), B' \subseteq A\},$$

$$\overline{\mathbb{G}}(A) = \{x \in X \mid \forall B \in \mathbb{G}(x), B' \cap A \neq \emptyset\}.$$

Naturally, the approximation pair based on an (L, M)-G-fuzzy remote neighborhood operator can be defined as follows:

Definition 3.5. Let $\mathscr{G} : L^X \times M_{\perp_M} \times X \longrightarrow L$ be an (L, M)-G-fuzzy remote neighborhood operator on X. The upper rough approximation operator $\overline{\mathscr{G}} : L^X \times M_{\perp_M} \longrightarrow L^X$ and the lower rough approximation operator $\underline{\mathscr{G}} : L^X \times M_{\perp_M} \longrightarrow L^X$ are defined as follows: for any $A \in L^X$, $r \in M_{\perp_M}$ and $x \in X$,

$$\underline{\mathscr{G}}(A, r)(x) = \bigvee_{B \in L^X} \Big(\mathscr{G}(B, r, x) \wedge \widetilde{\subset}(B', A)\Big),$$

$$\overline{\mathscr{G}}(A, r)(x) = \bigwedge_{B \in L^X} \Big(\mathscr{G}(B, r, x) \to \widetilde{\cap}(B', A)\Big).$$

If $\mathscr{G} : L^X \times M_{\perp_M} \times X \longrightarrow L$ is an (L, M)-G-fuzzy remote neighborhood operator on X, then the pair $(\underline{\mathscr{G}}, \overline{\mathscr{G}})$ is called an approximation pair on X. And, for each $(A, r) \in L^X \times M_{\perp_M}$,

$$(\underline{\mathscr{G}}, \overline{\mathscr{G}})(A, r) = (\underline{\mathscr{G}}(A, r), \overline{\mathscr{G}}(A, r))$$

is called an r-rough set on X.

Theorem 3.6. *Let $\mathbb{G}$ be a generalized remote neighborhood operator defined as in Definition 2.1, and $\mathscr{G}_{\mathbb{G}}$ be an (L, M)-G-fuzzy remote neighborhood operator defined as in Theorem 3.3. Then, for each $A \in 2^X$,*

$$(\underline{\mathscr{G}_{\mathbb{G}}}, \overline{\mathscr{G}_{\mathbb{G}}})(\chi_A, \top) = (\chi_{\underline{\mathbb{G}}(A)}, \chi_{\overline{\mathbb{G}}(A)}).$$

Proof. $\forall x \in X$.

(1) Let $\underline{\mathscr{G}_{\mathbb{G}}}(\chi_A, \top)(x) = \top$, then

$$\bigvee_{\chi_B \in \{\bot, \top\}^X} \Big(\mathscr{G}_{\mathbb{G}}(\chi_B, \top, x) \wedge \widetilde{\subset}(\chi_{B'}, \chi_A)\Big) = \top.$$

There exists $\chi_{B_0} \in \{\bot, \top\}^X$ such that $\mathscr{G}_{\mathbb{G}}(\chi_{B_0}, \top, x) \wedge \widetilde{\subset}(\chi_{B'_0}, \chi_A) = \top$. So, $\mathscr{G}_{\mathbb{G}}(\chi_{B_0}, \top, x) = \widetilde{\subset}(\chi_{B'_0}, \chi_A) = \top$. Further, $B_0 \in \mathbb{G}_{\mathscr{G}_{\mathbb{G}}}(x) = \mathbb{G}(x)$, and $B'_0 \subseteq A$. Hence, $x \in \underline{\mathbb{G}}(A)$. It follows that $\chi_{\underline{\mathbb{G}}(A)}(x) = \top$.

On the other hand, if $\chi_{\underline{\mathbb{G}}(A)}(x) = \top$, then $x \in \underline{\mathbb{G}}(A)$. Thus, there exists $B_0 \in \mathbb{G}(x)$ such that $B'_0 \subseteq A$. So,

$$\begin{aligned}\underline{\mathscr{G}_{\mathbb{G}}}(\chi_A, \top)(x) &= \bigvee_{\chi_B \in \{\bot, \top\}^X} \Big(\mathscr{G}_{\mathbb{G}}(\chi_B, \top, x) \wedge \widetilde{\subset}(\chi_{B'}, \chi_A)\Big) \\ &\geq \mathscr{G}_{\mathbb{G}}(\chi_{B_0}, \top, x) \wedge \widetilde{\subset}(\chi_{B'_0}, \chi_A) \\ &= \top.\end{aligned}$$

It follows that $\underline{\mathscr{G}_{\mathbb{G}}}(\chi_A, \top)(x) = \top \Longleftrightarrow \chi_{\underline{\mathbb{G}}(A)}(x) = \top$. Therefore, $\chi_{\underline{\mathbb{G}}(A)} = \underline{\mathscr{G}_{\mathbb{G}}}(\chi_A, \top)$.

(2) Let $\overline{\mathscr{G}_{\mathbb{G}}}(\chi_A, \top)(x) = \top$, then

$$\bigwedge_{\chi_B \in \{\bot, \top\}^X} \Big(\mathscr{G}_{\mathbb{G}}(\chi_B, \top, x) \to \widetilde{\cap}(\chi_{B'}, \chi_A)\Big) = \top.$$

So, for each $\chi_B \in \{\bot, \top\}^X$, we have $\mathscr{G}_{\mathbb{G}}(\chi_B, \top, x) \to \widetilde{\cap}(\chi_{B'}, \chi_A) = \top$. Thus, $\chi_{\mathbb{G}(x)}(B) \leq \widetilde{\cap}(\chi_{B'}, \chi_A)$ $(\forall \chi_B \in \{\bot, \top\}^X)$. In particular, $\forall B \in \mathbb{G}(x)$, we have $\widetilde{\cap}(\chi_{B'}, \chi_A) = \top$. This implies that $\chi_{B'}(x_0) = \chi_A(x_0) = \top$ for some $x_0 \in X$. Hence, $\forall B \in \mathbb{G}(x)$, $x_0 \in B' \cap A \neq \emptyset$. It follows that $x \in \overline{\mathbb{G}}(A)$, i.e., $\chi_{\overline{\mathbb{G}}(A)}(x) = \top$.

On the other hand, let $\chi_{\overline{\mathbb{G}}(A)}(x) = \top$, then $x \in \overline{\mathbb{G}}(A)$. So, $\forall B \in \mathbb{G}(x)$, $B' \cap A \neq \emptyset$. There exists $x_0 \in B' \cap A$ such that $\widetilde{\cap}(\chi_{B'}, \chi_A) \geq \chi_{B'}(x_0) \wedge \chi_A(x_0) = \top$. Thus, $\widetilde{\cap}(\chi_{B'}, \chi_A) = \top$. If $C \in \mathbb{G}(x)$, then $\mathscr{G}_{\mathbb{G}}(\chi_C, \top, x) = \chi_{\mathbb{G}(x)}(C) = \top$. Hence, $\mathscr{G}_{\mathbb{G}}(\chi_C, \top, x) \to \widetilde{\cap}(\chi_{C'}, \chi_A) = \top \to \top = \top$; If $C \notin \mathbb{G}(x)$, then $\mathscr{G}_{\mathbb{G}}(\chi_C, \top, x) = \chi_{\mathbb{G}(x)}(C) = \bot$. Hence,

$$\mathscr{G}_{\mathbb{G}}(\chi_C, \top, x) \to \widetilde{\cap}(\chi_{C'}, \chi_A) = \bot \to \widetilde{\cap}(\chi_{C'}, \chi_A) = \top.$$

So,

$$\overline{\mathscr{G}_{\mathbb{G}}}(\chi_A, \top)(x) = \bigwedge_{\chi_C \in \{\bot, \top\}^X} \Big(\mathscr{G}_{\mathbb{G}}(\chi_C, \top, x) \vee \widetilde{\cap}(\chi_{C'}, \chi_A)\Big) = \top.$$

It follows that $\overline{\mathscr{G}_{\mathbb{G}}}(\chi_A, \top)(x) = \top \Longleftrightarrow \chi_{\overline{\mathbb{G}}(A)}(x) = \top$. Therefore, $\overline{\mathscr{G}_{\mathbb{G}}}(\chi_A, \top) = \chi_{\overline{\mathbb{G}}(A)}$.

By (1) and (2), we have $(\underline{\mathscr{G}_{\mathbb{G}}}, \overline{\mathscr{G}_{\mathbb{G}}})(\chi_A, \top) = (\chi_{\underline{\mathbb{G}}(A)}, \chi_{\overline{\mathbb{G}}(A)})$. $\square$

The set of all approximation pairs on X based on (L, M)-G-fuzzy remote neighborhood operators is denoted by $\mathbf{AP}(X, L, M)$, that is,

$$\mathbf{AP}(X, L, M) = \{(\underline{\mathscr{G}}, \overline{\mathscr{G}}) \mid \mathscr{G} \in \mathbf{GFR}(X, L, M)\}.$$

Define an order relation $\lesssim$ on the set $\mathbf{AP}(X, L, M)$ as follows: for each $(C, r) \in L^X \times M_{\bot_M}$, $x \in X$, $(\underline{\mathscr{G}}_1, \overline{\mathscr{G}}_1) \lesssim (\underline{\mathscr{G}}_2, \overline{\mathscr{G}}_2)$ iff $\underline{\mathscr{G}}_1(C, r)(x) \leq \underline{\mathscr{G}}_2(C, r)(x)$, and $\overline{\mathscr{G}}_2(C, r)(x) \leq \overline{\mathscr{G}}_1(C, r)(x)$. Then $(\mathbf{AP}(X, L, M), \lesssim)$ is a poset.

Theorem 3.7. *Let $\emptyset \neq \{\mathscr{G}_i\}_{i\in I} \subseteq (\boldsymbol{GFR}(X, L, M), \leq)$, define the approximation pair $\widetilde{\bigvee}_{i\in I}\big(\underline{\mathscr{G}_i}, \overline{\mathscr{G}}_i\big)$ as follows:*

$$\widetilde{\bigvee}_{i\in I}\big(\underline{\mathscr{G}_i}, \overline{\mathscr{G}}_i\big) = \Big(\underline{\bigvee_{i\in I} \mathscr{G}_i}, \overline{\bigvee_{i\in I} \mathscr{G}_i}\Big).$$

Then $\widetilde{\bigvee}_{i\in I}\big(\underline{\mathscr{G}_i}, \overline{\mathscr{G}}_i\big)$ is the supremum of $\{(\underline{\mathscr{G}_i}, \overline{\mathscr{G}}_i)\}_{i\in I}$. Further, $(\boldsymbol{AP}(X, L, M), \lesssim)$ is a join-complete lattice.

Proof. Firstly, since $\bigvee_{i\in\Lambda}\mathscr{G}_i \in \mathbf{GFR}(X, L, M)$, we have

$$\widetilde{\bigvee_{i\in I}}\left(\underline{\mathscr{G}}_i, \overline{\mathscr{G}}_i\right) = \left(\underline{\bigvee_{i\in I}\mathscr{G}_i}, \overline{\bigvee_{i\in I}\mathscr{G}_i}\right) \in \mathbf{AP}(X, L, M).$$

And, for any $(A, r) \in L^X \times M_{\perp_M}$ and $x \in X$,

$$\left(\underline{\bigvee_{i\in\Lambda}\mathscr{G}_i}, \overline{\bigvee_{i\in\Lambda}\mathscr{G}_i}\right)(A, r)(x) = \left(\bigvee_{i\in\Lambda}\underline{\mathscr{G}_i}(A, r)(x), \bigwedge_{i\in\Lambda}\overline{\mathscr{G}_i}(A, r)(x)\right).$$

Indeed, for any $(A, r) \in L^X \times M_{\perp_M}$ and $x \in X$,

$$\begin{aligned}
\underline{\left(\bigvee_{i\in\Lambda}\mathscr{G}_i\right)}(A, r)(x) &= \bigvee_{B\in L^X}\left(\left(\bigvee_{i\in\Lambda}\mathscr{G}_i\right)(B, r, x) \wedge \widetilde{\subset}(B', A)\right) \\
&= \bigvee_{B\in L^X}\left(\bigvee_{i\in\Lambda}\mathscr{G}_i(B, r, x) \wedge \widetilde{\subset}(B', A)\right) \\
&= \bigvee_{i\in\Lambda}\bigvee_{B\in L^X}\left(\mathscr{G}_i(B, r, x) \wedge \widetilde{\subset}(B', A)\right) \\
&= \bigvee_{i\in\Lambda}\underline{\mathscr{G}_i}(A, r)(x).
\end{aligned}$$

$$\begin{aligned}
\overline{\left(\bigvee_{i\in\Lambda}\mathscr{G}_i\right)}(A, r)(x) &= \bigwedge_{B\in L^X}\left(\left[\left(\bigvee_{i\in\Lambda}\mathscr{G}_i\right)(B, r, x)\right] \to \widetilde{\cap}(B', A)\right) \\
&= \bigwedge_{B\in L^X}\left(\left[\bigvee_{i\in\Lambda}\mathscr{G}_i(B, r, x)\right] \to \widetilde{\cap}(B', A)\right) \\
&= \bigwedge_{i\in\Lambda}\bigwedge_{B\in L^X}\left(\mathscr{G}_i(B, r, x) \to \widetilde{\cap}(B', A)\right) \\
&= \bigwedge_{i\in\Lambda}\overline{\mathscr{G}_i}(A, r)(x).
\end{aligned}$$

Therefore,

$$\left(\underline{\bigvee_{i\in\Lambda}\mathscr{G}_i}, \overline{\bigvee_{i\in\Lambda}\mathscr{G}_i}\right)(A, r)(x) = \left(\bigvee_{i\in\Lambda}\underline{\mathscr{G}_i}(A, r)(x), \bigwedge_{i\in\Lambda}\overline{\mathscr{G}_i}(A, r)(x)\right).$$

Secondly, $\left(\underline{\bigvee_{i\in\Lambda}\mathscr{G}_i}, \overline{\bigvee_{i\in\Lambda}\mathscr{G}_i}\right)$ is an upper bound of $\{(\underline{\mathscr{G}}_i, \overline{\mathscr{G}}_i)\}_{i\in\Lambda}$. Indeed, let $\mathscr{G}_1 \leq \mathscr{G}_2$, then $\mathscr{G}_1(A, r, x) \leq \mathscr{G}_2(A, r, x)$ for each $(A, r, x) \in L^X \times M_{\perp_M} \times X$. Thus $(\mathscr{G}_1 \vee \mathscr{G}_2)(A, r, x) = \mathscr{G}_1(A, r, x) \vee \mathscr{G}_2(A, r, x) = \mathscr{G}_2(A, r, x)$. It implies that $\mathscr{G}_1 \vee \mathscr{G}_2 = \mathscr{G}_2$. So, for each $(A, r, x) \in L^X \times M_{\perp_M} \times X$, we have

$$\underline{\mathscr{G}_2}(A, r)(x) = \underline{\mathscr{G}_1 \vee \mathscr{G}_2}(A, r)(x) = \underline{\mathscr{G}_1}(A, r)(x) \vee \underline{\mathscr{G}_2}(A, r)(x),$$

$$\overline{\mathscr{G}_2}(A, r)(x) = \overline{\mathscr{G}_1 \vee \mathscr{G}_2}(A, r)(x) = \overline{\mathscr{G}_1}(A, r)(x) \wedge \overline{\mathscr{G}_2}(A, r)(x).$$

So, $\underline{\mathscr{G}_1}(A, r)(x) \leq \underline{\mathscr{G}_2}(A, r)(x)$, and $\overline{\mathscr{G}_2}(A, r)(x) \leq \overline{\mathscr{G}_1}(A, r)(x)$. It follows that $(\underline{\mathscr{G}}_1, \overline{\mathscr{G}}_1) \lesssim (\underline{\mathscr{G}}_2, \overline{\mathscr{G}}_2)$. Therefore,

$$(\underline{\mathscr{G}}_i, \overline{\mathscr{G}}_i) \lesssim \left(\underline{\bigvee_{i\in\Lambda}\mathscr{G}_i}, \overline{\bigvee_{i\in\Lambda}\mathscr{G}_i}\right)$$

for each $i \in \Lambda$, that is, $\left(\underline{\bigvee_{i\in\Lambda}\mathscr{G}_i}, \overline{\bigvee_{i\in\Lambda}\mathscr{G}_i}\right)$ is an upper bound of $\{(\underline{\mathscr{G}}_i, \overline{\mathscr{G}}_i)\}_{i\in\Lambda}$.

Now, let $\mathscr{G} : L^X \times M_{\perp_M} \times X \longrightarrow L$ be another (L, M)-G-fuzzy remote neighborhood operator on X such that $(\underline{\mathscr{G}}_i, \overline{\mathscr{G}}_i) \lesssim (\underline{\mathscr{G}}, \overline{\mathscr{G}})$ for each $i \in \Lambda$. Then, for any $C \in L^X$, $r \in M_{\perp_M}$ and $x \in X$, $\underline{\mathscr{G}}_i(C, r)(x) \leq \underline{\mathscr{G}}(C, r)(x)$, and $\overline{\mathscr{G}}(C, r)(x) \leq \overline{\mathscr{G}}_i(C, r)(x)$. So,

$$\underline{\left(\bigvee_{i\in\Lambda}\mathscr{G}_i\right)}(C, r)(x) = \bigvee_{i\in\Lambda}\underline{\mathscr{G}}_i(C, r)(x) \leq \underline{\mathscr{G}}(C, r)(x),$$

and

$$\overline{\mathscr{G}}(C,r)(x) \leq \bigwedge_{i\in\Lambda} \overline{\mathscr{G}}_i(C,r)(x) = \overline{\Big(\bigvee_{i\in\Lambda} \mathscr{G}_i\Big)}(C,r)(x).$$

It implies that $\Big(\underline{\bigvee_{i\in\Lambda} \mathscr{G}_i}, \overline{\bigvee_{i\in\Lambda} \mathscr{G}_i}\Big) \lesssim (\underline{\mathscr{G}}, \overline{\mathscr{G}})$, i.e., $\widetilde{\bigvee_{i\in I}}\Big(\underline{\mathscr{G}_i}, \overline{\mathscr{G}}_i\Big) = \Big(\underline{\bigvee_{i\in\Lambda} \mathscr{G}_i}, \overline{\bigvee_{i\in\Lambda} \mathscr{G}_i}\Big)$ is the supremum of $\{(\underline{\mathscr{G}_i}, \overline{\mathscr{G}}_i)\}_{i\in I}$.

From the above proof, we can see that $(\mathbf{AP}(X,L,M), \lesssim)$ is a join-complete lattice. □

Remark 3.8. By Theorem 3.7, we have the following proposition: if $\mathscr{G}_2 \leq \mathscr{G}_1$, then $(\underline{\mathscr{G}}_2, \overline{\mathscr{G}}_2) \lesssim (\underline{\mathscr{G}}_1, \overline{\mathscr{G}}_1)$. However, the converse proposition does not hold. For example, let $\mathscr{G}_1$ and $\mathscr{G}_2$ be the mappings defined as in Example 3.2. Then, for any $A \in L^X$ and $x \in X$, $\underline{\mathscr{G}}_1(A, \top_M)(x) = \top_L$, $\overline{\mathscr{G}_1}(A, \top_M)(x) = \bot_L$,

$$\underline{\mathscr{G}}_2(A, \top_M)(x) = \overline{\mathscr{G}_2}(A, \top_M)(x) = \begin{cases} \bot_L, & \text{if } A = \bot_L^X, \\ a, & \text{if } A = \underline{a}, \\ b, & \text{if } A = \underline{b}, \\ \top_L, & \text{otherwise.} \end{cases}$$

It implies that $\underline{\mathscr{G}_2}(A, \top_M)(x) \leq \underline{\mathscr{G}_1}(A, \top_M)(x)$, and $\overline{\mathscr{G}_1}(A, \top_M)(x) \leq \overline{\mathscr{G}_2}(A, \top_M)(x)$, i.e., $(\underline{\mathscr{G}}_2, \overline{\mathscr{G}}_2) \lesssim (\underline{\mathscr{G}}_1, \overline{\mathscr{G}}_1)$, but there is no $\mathscr{G}_2 \leq \mathscr{G}_1$. Naturally, we have the following problem:

Problem 3.9. *Is $(\boldsymbol{AP}(X,L,M), \lesssim)$ a complete lattice?*

4. Approximation operators (pairs) based on convex (L, M)-fuzzy remote neighborhood operators

In this section, L denotes a regular[1)] completely distributive De Morgan algebra, we will focus on discussing approximation operators (pairs) based on convex (L, M)-fuzzy remote neighborhood operators and their properties.

The following lemma proves that $\overline{\mathscr{G}}$ and $\underline{\mathscr{G}}$ are dual.

Lemma 4.1. *Let $\mathscr{G} : L^X \times M_{\perp_M} \times X \longrightarrow L$ be an (L, M)-G-fuzzy remote neighborhood operator on X. Then*

$$(\underline{\mathscr{G}}, \overline{\mathscr{G}})(A, r) = (\underline{\mathscr{G}}(A, r), \overline{\mathscr{G}}(A, r)) = ((\overline{\mathscr{G}}(A', r))', (\underline{\mathscr{G}}(A', r))')$$

for each $A \in L^X$ and $r \in M_{\perp_M}$.

Proof. $\forall x \in X$,

$$\begin{aligned} (\overline{\mathscr{G}}(A', r))'(x) &= [\overline{\mathscr{G}}(A', r)(x)]' \\ &= \bigvee_{B\in L^X} \Big[\Big(\mathscr{G}(B, r, x) \to \widetilde{\cap}(B', A')\Big)\Big]' \\ &= \bigvee_{B\in L^X} \Big(\mathscr{G}(B, r, x) \wedge \Big(\widetilde{\cap}(B', A')\Big)'\Big) \\ &= \bigvee_{B\in L^X} \Big(\mathscr{G}(B, r, x) \wedge \widetilde{\subset}(B', A)\Big) \\ &= \underline{\mathscr{G}}(A, r)(x). \end{aligned}$$

Hence, $(\overline{\mathscr{G}}(A', r))' = \underline{\mathscr{G}}(A, r)$. Analogously, $(\underline{\mathscr{G}}(A', r))' = \overline{\mathscr{G}}(A, r)$. □

[1)] A completely distributive De Morgan algebra L is called regular if $a' = a \to \bot_L$ for each $a \in L$. If L is regular, then $a \to b = a' \vee b$ for each $a, b \in L$. Indeed, $\forall x \in \{x \mid a \wedge x \leq b\}$, we have $x = x \wedge 1 = x \wedge (a \vee a') = (x \wedge a) \vee (x \wedge a') \leq b \vee a'$. Hence, $a \to b = \bigvee\{x \in L \mid a \wedge x \leq b\} \leq b \vee a'$. Conversely, since $a \wedge (b \vee a') = a \wedge b \leq b$, we have $b \vee a' \leq a \to b$. Therefore, $a \to b = a' \vee b$.

Theorem 4.2. *Let $\mathscr{R} : L^X \times M_{\perp_M} \longrightarrow L^X$ be a convex (L, M)-fuzzy remote neighborhood operator. Then the upper rough approximation operator $\overline{\mathscr{R}}$ satisfies the following properties: for $A, B \in L^X$, and $s, r \in M_{\perp_M}$,*

(1) $\overline{\mathscr{R}}(\perp_L^X, r) = \perp_L^X$.

(2) $A \leq \overline{\mathscr{R}}(A, r)$.

(3) *If $A \leq B$, then $\overline{\mathscr{R}}(A, r) \leq \overline{\mathscr{R}}(B, r)$.*

(4) *If $r \leq s$, then $\overline{\mathscr{R}}(A, r) \leq \overline{\mathscr{R}}(A, s)$.*

(5) *If $\mathscr{R}$ is transitive, that is,*

$$\bigwedge_{B \in L^X} \left\{ \mathscr{R}'(B, r, x) \mid B'(y) \leq \mathscr{R}(A, r, y), \forall y \in X \right\} \leq \overline{\mathscr{R}}(A, r)(x).$$

Then, (i) $\overline{\mathscr{R}}(\overline{\mathscr{R}}(A, r), r) = \overline{\mathscr{R}}(A, r)$.

(ii) If $\{B_j : j \in J\} \subseteq L^X$ be nonempty and totally ordered by inclusion, then

$$\overline{\mathscr{R}}\Big(\bigvee_{j \in J} B_j, r\Big) = \bigvee_{j \in J} \overline{\mathscr{R}}(B_j, r).$$

(iii) If $r = \bigvee\{s \in M_{\perp_M} : A = \overline{\mathscr{R}}(A, s)\}$, then $\overline{\mathscr{R}}(A, r) = A$.

Proof. $\forall x \in X$.

(1) By (CFR1), we have

$$\overline{\mathscr{R}}(\perp_L^X, r)(x) = \bigwedge_{B \in L^X} \Big(\mathscr{R}(B, r, x) \to \widetilde{\sqcap}(B', \perp_L^X)\Big) = \Big[\bigvee_{B \in L^X} \mathscr{R}(B, r, x)\Big]' \leq \Big[\mathscr{R}(\perp_L^X, r, x)\Big]' = \perp_L.$$

Hence, $\overline{\mathscr{R}}(\perp_L^X, r) = \perp_L^X$.

(2) By (CFR2), we have

$$\begin{aligned}
\overline{\mathscr{R}}(A, r)(x) &= \bigwedge_{B \in L^X} \Big(\mathscr{R}(B, r, x) \to \widetilde{\sqcap}(B', A)\Big) \\
&= \bigwedge_{B \in L^X} \Big(\mathscr{R}(B, r, x) \to \bigvee_{y \in X} (B'(y) \wedge A(y))\Big) \\
&\geq \bigwedge_{B \in L^X} \Big(\mathscr{R}(B, r, x) \to (B'(x) \wedge A(x))\Big) \\
&\geq \bigwedge_{B \in L^X} \Big(B'(x) \to (B'(x) \wedge A(x))\Big) \\
&\geq A(x).
\end{aligned}$$

Hence, $A \leq \overline{\mathscr{R}}(A, r)$.

(3) If $A \leq B$, we have

$$\overline{\mathscr{R}}(A, r)(x) = \bigwedge_{D \in L^X} \Big(\mathscr{R}(D, r, x) \to \widetilde{\sqcap}(D', A)\Big) \leq \bigwedge_{D \in L^X} \Big(\mathscr{R}(D, r, x) \to \widetilde{\sqcap}(D', B)\Big) = \overline{\mathscr{R}}(B, r)(x).$$

Hence, $\overline{\mathscr{R}}(A, r) \leq \overline{\mathscr{R}}(B, r)$.

(4) If $r \leq s$, by (CFR4), we have

$$\overline{\mathscr{R}}(A, r)(x) = \bigwedge_{B \in L^X} \Big(\mathscr{R}(B, r, x) \to \widetilde{\sqcap}(B', A)\Big) \leq \bigwedge_{B \in L^X} \Big(\mathscr{R}(B, s, x) \to \widetilde{\sqcap}(B', A)\Big) = \overline{\mathscr{R}}(A, s)(x),$$

Hence, $\overline{\mathscr{R}}(A, r) \leq \overline{\mathscr{R}}(A, s)$.

(5) (i) Noticing that $\mathscr{R}_{\mathscr{C}}(\mathscr{R}'_{\mathscr{C}}(A,b),a)=\mathscr{R}_{\mathscr{C}}(A,b)$ (see [Theorem 3.3, [37]]) and $\mathscr{R}_{\mathscr{C}_{\mathscr{R}}}=\mathscr{R}$ (see [Proposition 3.5, [37]]), we have $\mathscr{R}(\mathscr{R}'(A,r),r,x)=\mathscr{R}(A,r,x)$. By (CFR5), we have

$$\mathscr{R}'(A,r,x)=\bigwedge_{C'\leq\mathscr{R}(A,r)}\mathscr{R}'(C,r,x)\leq\overline{\mathscr{R}}(A,r)(x)=\bigwedge_{B\in L^X}\Big(\mathscr{R}(B,r,x)\to\widetilde{\cap}(B',A)\Big)\leq\mathscr{R}'(A,r,x).$$

Hence,

$$\begin{array}{rcl}\overline{\mathscr{R}}(A,r)(x) & \leq & \overline{\mathscr{R}}(\overline{\mathscr{R}}(A,r),r)(x)\\ & = & \bigwedge_{B\in L^X}\Big(\mathscr{R}(B,r,x)\to\widetilde{\cap}(B',\overline{\mathscr{R}}(A,r))\Big)\\ & \leq & \mathscr{R}(\mathscr{R}'(A,r),r,x)\to\widetilde{\cap}(\mathscr{R}(A,r),\mathscr{R}'(A,r))\\ & = & \mathscr{R}(A,r,x)\to\perp_L\\ & = & \overline{\mathscr{R}}(A,r)(x).\end{array}$$

Therefore, $\overline{\mathscr{R}}(\overline{\mathscr{R}}(A,r),r)=\overline{\mathscr{R}}(A,r)$.

(ii) Let $\{B_j : j\in J\}\subseteq L^X$ be nonempty and totally ordered by inclusion. We easily obtain the following inequality:

$$\begin{array}{rcl}\overline{\mathscr{R}}\Big(\bigvee_{j\in J}B_j,r\Big)(x) & = & \bigwedge_{A\in L^X}\Big(\mathscr{R}(A,r,x)\to\widetilde{\cap}\Big(A',\bigvee_{j\in J}B_j\Big)\Big)\\ & \geq & \bigwedge_{A\in L^X}\Big(\mathscr{R}(A,r,x)\to\bigvee_{j\in J}\widetilde{\cap}\Big(A',B_j\Big)\Big)\\ & \geq & \bigvee_{j\in J}\bigwedge_{A\in L^X}\Big(\mathscr{R}(A,r,x)\to\widetilde{\cap}(A',B_j)\Big)\\ & = & \bigvee_{j\in J}\overline{\mathscr{R}}(B_j,r)(x).\end{array}$$

Conversely, by (i) and (CFR6), we have

$$\begin{array}{rcl}\overline{\mathscr{R}}\Big(\bigvee_{j\in J}B_j,r\Big)(x) & = & \bigwedge_{A\in L^X}\Big(\mathscr{R}(A,r,x)\to\widetilde{\cap}\Big(A',\bigvee_{j\in J}B_j\Big)\Big)\\ & \leq & \mathscr{R}\Big(\bigvee_{j\in J}B_j,r,x\Big)\to\widetilde{\cap}\Big(\bigwedge_{j\in J}B'_j,\bigvee_{j\in J}B_j\Big)\\ & = & \mathscr{R}'\Big(\bigvee_{j\in J}B_j,r,x\Big)=\bigvee_{j\in J}\mathscr{R}'(B_j,r,x)\\ & = & \bigvee_{j\in J}\overline{\mathscr{R}}(B_j,r,x).\end{array}$$

Therefore, $\overline{\mathscr{R}}(\bigvee_{j\in J}B_j,r)=\bigvee_{j\in J}\overline{\mathscr{R}}(B_j,r)$.

(iii) Let $r=\bigvee\{s\in M_{\perp_M} : A=\overline{\mathscr{R}}(A,s)\}$. By (CFR7), we have

$$\overline{\mathscr{R}}(A,r)(x)=\overline{\mathscr{R}}\Big(A,\bigvee_{\overline{\mathscr{R}}(A,s)=A}s\Big)(x)=\Big[\mathscr{R}\Big(A,\bigvee_{\forall x\in X,A'(x)=\mathscr{R}(A,s,x)}s,x\Big)\Big]'=A(x).$$

Therefore, $\overline{\mathscr{R}}(A,r)=A$. □

Theorem 4.3. *Let $\mathscr{R}: L^X\times M_{\perp_M}\times X\longrightarrow L$ be a convex (L,M)-fuzzy remote neighborhood operator. Then the lower rough approximation operator $\underline{\mathscr{R}}$ satisfies the following properties: for $A,B\in L^X$, and $s,r\in M_{\perp_M}$,*

(1) $\underline{\mathscr{R}}(\top_L^X,r)=\top_L^X$.

(2) $\underline{\mathscr{R}}(A,r)\leq A$.

(3) *If $A\leq B$, then $\underline{\mathscr{R}}(A,r)\leq\underline{\mathscr{R}}(B,r)$.*

(4) *If $r\leq s$, then $\underline{\mathscr{R}}(A,r)\geq\underline{\mathscr{R}}(A,s)$.*

(5) *If $\mathscr{R}$ is transitive, Then,*

(i) $\underline{\mathscr{R}}(\underline{\mathscr{R}}(A,r),r) = \underline{\mathscr{R}}(A,r)$.

(ii) If $\{A_i : i \in J\} \overset{cdir}{\subseteq} L^X$, then $\underline{\mathscr{R}}\Big(\bigwedge_{i\in J} A_i, r\Big) = \bigwedge_{i\in J} \underline{\mathscr{R}}(A_i, r)$.

(iii) $\underline{\mathscr{R}}(B, \bigvee\{r \in M_{\perp_M} : B = \underline{\mathscr{R}}(B,r)\}) = B$.

Proof. The proof of this theorem can be derived from Lemma 4.1 and Theorem 4.2, so we omit it. □

Remark 4.4. Let (L,M)-**CFR** denote the category of all transitive convex (L,M)-fuzzy remote neighborhood spaces and all their (L,M)-RNP mappings as morphisms. Define a functor $\overline{\mathbf{R}}$ as follows:

$$\overline{\mathbf{R}} : \begin{cases} (L,M)\text{-}\mathbf{CFR} \longrightarrow (L,M)\text{-}\mathbf{FCH}, \\ (X,\mathscr{R}) \longmapsto (X,\overline{\mathscr{R}}), \\ g \longmapsto g. \end{cases}$$

Then, by Theorem 4.2, if $\mathscr{R}$ is a transitive convex (L,M)-fuzzy remote neighborhood operator, then the upper rough approximation operator $\overline{\mathscr{R}}$ is a convex (L,M)-fuzzy hull operator. In addition, let $(X,\mathscr{R}_X)$ and $(Y,\mathscr{R}_Y)$ be transitive convex (L,M)-fuzzy remote neighborhood spaces, and $g : X \longrightarrow Y$ be an (L,M)-RNP mapping. Then, for each $x \in X$, $B \in L^Y$ and $r \in M_{\perp_M}$, we have $\mathscr{R}_Y(B,r,g(x)) \le \mathscr{R}_X(g^{\leftarrow}(B),r,x)$. So,

$$\begin{aligned}
\overline{\mathscr{R}_Y}(g^{\rightarrow}(A),r)(g(x)) &= \bigwedge\nolimits_{B\in L^Y}\Big(\mathscr{R}_Y(B,r,g(x)) \to \widetilde{\sqcap}(B',g^{\rightarrow}(A))\Big) \\
&= \bigwedge\nolimits_{B\in L^Y}\Big(\mathscr{R}_Y(B,r,g(x)) \to \bigvee\nolimits_{y\in Y}\Big(B'(y)\wedge g^{\rightarrow}(A)(y)\Big)\Big) \\
&\ge \bigwedge\nolimits_{B\in L^Y}\Big(\mathscr{R}_Y(B,r,g(x)) \to \bigvee\nolimits_{z\in X}\Big(B'(g(z))\wedge g^{\rightarrow}(A)(g(z))\Big)\Big) \\
&\ge \bigwedge\nolimits_{B\in L^Y}\Big(\mathscr{R}_Y(B,r,g(x)) \to \bigvee\nolimits_{z\in X}\Big(g^{\leftarrow}(B')(z)\wedge A(z)\Big)\Big) \\
&\ge \bigwedge\nolimits_{B\in L^Y}\Big(\mathscr{R}_X(g^{\leftarrow}(B),r,x) \to \bigvee\nolimits_{z\in X}\Big(g^{\leftarrow}(B')(z)\wedge A(z)\Big)\Big) \\
&\ge \bigwedge\nolimits_{C\in L^X}\Big(\mathscr{R}_X(C,r,x) \to \bigvee\nolimits_{z\in X}(C'(z)\wedge A(z))\Big) \\
&= \bigwedge\nolimits_{C\in L^X}\Big(\mathscr{R}_X(C,r,x) \to \widetilde{\sqcap}(C',A)\Big) \\
&= \overline{\mathscr{R}_X}(A,r)(x).
\end{aligned}$$

Hence, $g^{\rightarrow}(\overline{\mathscr{R}_X}(A,r)) \le g^{\rightarrow}(g^{\leftarrow}(\overline{\mathscr{R}_Y}(g^{\rightarrow}(A),r))) \le \overline{\mathscr{R}_Y}(g^{\rightarrow}(A),r)$.

From the above proof, we can see the category (L,M)-**CFR** can be embedded into the category (L,M)-**FCH**. The relationship between concave (L,M)-fuzzy interior operators and the lower rough approximation operators can be similarly studied, and will not be repeated here.

The set of all convex (L,M)-fuzzy remote neighborhood operators on X is denoted by **CFR**(X,L,M). Define a relation $\le$ on **CFR**(X,L,M) as follows: $\forall x \in X, A \in L^X$ and $r \in M_{\perp_M}$, $\mathscr{R}_1 \le \mathscr{R}_2 \Longleftrightarrow \mathscr{R}_1(A,r,x) \le \mathscr{R}_2(A,r,x)$. Then $(\mathbf{CFR}(X,L,M),\le)$ is a complete lattice ([37]). Moreover, let $\emptyset \ne \{\mathscr{R}_i\}_{i\in I} \subseteq (\mathbf{CFR}(X,L,M),\le)$. Then $\mathscr{R}_{\mathscr{C}^*}$ is the infimum of $\{\mathscr{R}_i\}_{i\in I}$, where $\mathscr{C}^* : L^X \longrightarrow M$ is an (L,M)-fuzzy convex structure on X, and is defined by $\mathscr{C}^*(C) = \bigwedge\limits_{i\in I} \mathscr{C}_{\mathscr{R}_i}(C)$([38]). In this case, we denote it as $\bigwedge_{i\in I}\mathscr{R}_i$, i.e.,

$$\mathscr{R}_{\mathscr{C}^*}(A,r,x) = \Big(\bigwedge\nolimits_{i\in I}\mathscr{R}_i\Big)(A,r,x) = \bigvee\Big\{B'(x) \in L \mid A \le B,\ \bigwedge\nolimits_{i\in I} \mathscr{C}_{\mathscr{R}_i}(B) \ge r\Big\}.$$

Proposition 4.5. *If $\mathscr{R}_1 \le \mathscr{R}_2$, then $(\underline{\mathscr{R}_1},\overline{\mathscr{R}_1}) \lesssim (\underline{\mathscr{R}_2},\overline{\mathscr{R}_2})$.*

Proof. Firstly, let $\{\mathscr{R}_i\}_{i\in I}$ be a family of convex (L,M)-fuzzy remote neighborhood operators. Then, for any $(A,r) \in L^X \times M_{\perp_M}$ and $x \in X$,

$$\underline{\bigwedge\nolimits_{i\in I}\mathscr{R}_i}(A,r)(x) \le \bigwedge\nolimits_{i\in I}\underline{\mathscr{R}_i}(A,r)(x),\quad \overline{\bigwedge\nolimits_{i\in I}\mathscr{R}_i}(A,r)(x) \ge \bigvee\nolimits_{i\in I}\overline{\mathscr{R}_i}(A,r)(x).$$

Secondly, if $\mathscr{R}_1 \leq \mathscr{R}_2$, then $\mathscr{C}_{\mathscr{R}_1}(C) \wedge \mathscr{C}_{\mathscr{R}_2}(C) = \mathscr{C}_{\mathscr{R}_1}(C)$. So,

$$\Big(\mathscr{R}_1 \wedge \mathscr{R}_2\Big)(B,r,x) = \bigvee_{B\leq C,\ \mathscr{C}_{\mathscr{R}_1}(C)\wedge\mathscr{C}_{\mathscr{R}_2}(C)\geq r} C'(x) = \mathscr{R}_{\mathscr{C}_{\mathscr{R}_1}}(B,r,x) = \mathscr{R}_1(B,r,x),$$

i.e., $\mathscr{R}_1 \wedge \mathscr{R}_2 = \mathscr{R}_1$. Thus,

$$\underline{\mathscr{R}_1}(A,r)(x) = \underline{\Big(\mathscr{R}_1 \wedge \mathscr{R}_2\Big)}(A,r)(x) \leq \underline{\mathscr{R}_1}(A,r)(x) \wedge \underline{\mathscr{R}_2}(A,r)(x) \leq \underline{\mathscr{R}_2}(A,r)(x),$$

and

$$\overline{\mathscr{R}_1}(A,r)(x) = \overline{\Big(\mathscr{R}_1 \wedge \mathscr{R}_2\Big)}(A,r)(x) \geq \overline{\mathscr{R}_1}(A,r)(x) \vee \overline{\mathscr{R}_2}(A,r)(x) \geq \overline{\mathscr{R}_2}(A,r)(x).$$

It follows that $(\underline{\mathscr{R}_1}, \overline{\mathscr{R}_1}) \lesssim (\underline{\mathscr{R}_2}, \overline{\mathscr{R}_2})$. □

Proposition 4.6. *Let $\emptyset \neq \{\mathscr{R}_i\}_{i\in I} \subseteq (\mathbf{CFR}(X,L,M),\leq)$. If every $\mathscr{R}_i$ is transitive, then $\mathscr{R}_{\mathscr{C}^*}$ is transitive.*

Proof. Define the mapping $\mathscr{R}_{\mathscr{C}^*} : L^X \times M_{\perp_M} \longrightarrow L^X$ as follows: for any $A \in L^X$, $r \in M_{\perp_M}$, and $x \in X$,

$$\mathscr{R}_{\mathscr{C}^*}(A,r)(x) = \mathscr{R}_{\mathscr{C}^*}(A,r,x).$$

We first claim that $\mathscr{R}'_{\mathscr{C}^*}$ is the infimum of $\{\overline{\mathscr{R}_i}\}_{i\in I}$ in $\mathbf{FH}(X,L,M)$ (the set of all convex (L,M)-fuzzy hull operators on X).

Indeed, by Remark 4.4 and [Theorem 3.7,[35]], the mapping $\mathscr{C}\mathscr{O} : L^X \times M_{\perp_M} \longrightarrow L^X$ defined by

$$\mathscr{C}\mathscr{O}(A,a) = \bigwedge\Big\{B \in L^X : A \leq B,\ \bigwedge\nolimits_{i\in I} \bigvee_{\overline{\mathscr{R}_i}(B,r)=B} r \geq a\Big\}$$

is the infimum of $\{\overline{\mathscr{R}_i}\}_{i\in I}$ in $\mathbf{FH}(X,L,M)$.

Noticing that

$$\begin{aligned}
\mathscr{C}\mathscr{O}(A,a)(x) &= \bigwedge\Big\{B(x) \in L : A \leq B,\ \bigwedge\nolimits_{i\in I} \bigvee_{\overline{\mathscr{R}_i}(B,r)=B} r \geq a\Big\}\\
&= \bigwedge\Big\{B(x) \in L : A \leq B,\ \bigwedge\nolimits_{i\in I} \bigvee_{\mathscr{R}_i(B,r,y)=B'(y),\forall y\in X} r \geq a\Big\}\\
&= \Big[\bigvee\Big\{B'(x) \in L \mid A \leq B,\ \bigwedge\nolimits_{i\in I} \mathscr{C}_{\mathscr{R}_i}(B) \geq a\Big\}\Big]'\\
&= \mathscr{R}'_{\mathscr{C}^*}(A,a,x)\\
&= \mathscr{R}'_{\mathscr{C}^*}(A,a)(x),
\end{aligned}$$

we confirm the above claim that $\mathscr{R}'_{\mathscr{C}^*}$ is the infimum of $\{\overline{\mathscr{R}_i}\}_{i\in I}$ in $\mathbf{FH}(X,L,M)$.

As is known from [Theorem 3.7, [35]] and Proposition 4.5, $\overline{\mathscr{R}_\bigstar} : L^X \times M_{\perp_M} \longrightarrow L^X$ is the maximum element of $\mathbf{FH}(X,L,M)$, where $\mathscr{R}_\bigstar : L^X \times M_{\perp_M} \times X \longrightarrow L$ defined by

$$\mathscr{R}_\bigstar(A,r,x) = A'(x) \quad (\forall x \in X, A \in L^X, r \in M_{\perp_M})$$

is the maximum element of $(\mathbf{CFR}(X,L,M),\leq)$ and is transitive.

For the upper approximation operators, we adopt the default partial order on $\mathbf{FH}(X,L,M)$:

$$\overline{\mathscr{R}_1} \leq \overline{\mathscr{R}_2} \iff \overline{\mathscr{R}_1}(A,r)(x) \geq \overline{\mathscr{R}_2}(A,r)(x),$$

for all $A \in L^X$, $r \in M_{\perp_M}$ and $x \in X$. With this partial order, the monotonicity of upper approximation operators holds: $\mathscr{R}_1 \leq \mathscr{R}_2 \Rightarrow \overline{\mathscr{R}_1} \leq \overline{\mathscr{R}_2}$. Combining this with the definition of infimum, we obtain the inequality chain:

$$\mathscr{R}'_{\mathscr{C}^*} \leq \overline{\mathscr{R}_{\mathscr{C}^*}} = \overline{\bigwedge_{i\in I} \mathscr{R}_i} \leq \overline{\mathscr{R}_i} \leq \overline{\mathscr{R}_\bigstar}.$$

By the uniqueness of the infimum in a poset, the above inequality directly implies $\mathscr{R}_{\mathscr{C}^*}$ is transitive, and accordingly we have $\overline{\mathscr{R}_{\mathscr{C}^*}} = \mathscr{R}'_{\mathscr{C}^*}$. $\square$

Corollary 4.7. *Let* $\mathbf{TCFR}(X, L, M)$ *denote the set of all transitive convex* (L, M)*-fuzzy remote neighborhood operators on* X. *Then* $\left(\{\overline{\mathscr{R}} \mid \mathscr{R} \in \mathbf{TCFR}(X, L, M)\}, \leq\right)$ *is a sub-complete lattice of* $(\boldsymbol{FH}(X, L, M), \leq)$.

It is well known that, $\forall E, F \in L^X$, the M-valued inclusion measure (cf. [7]) $\widetilde{\subset}^M : L^X \times L^X \longrightarrow M$ of the L-subset E into the L-subset F is given by

$$\widetilde{\subset}^M(E, F) = \phi(\widetilde{\subset}(E, F)) = \bigwedge_{x\in X} \phi(E(x) \to F(x)) = \bigwedge_{x\in X} \phi(E'(x) \vee F(x)),$$

where $\phi : L \longrightarrow M$ is a mapping that preserves bottom and top elements of the lattices and satisfies the following condition: $\phi\left(\bigwedge_{j\in J} x_j\right) = \bigwedge_{j\in J} \phi(x_j)$ for any $\{x_j\}_{j\in J} \subseteq L$.

Let $\mathscr{R} : L^X \times M_{\perp_M} \times X \longrightarrow L$ be a convex (L, M)-fuzzy remote neighborhood operator. To measure the quality of rough approximation of an L-fuzzy set $A \in L^X$, we can define the measure of its upper rough approximation by $\mathscr{U}_{\mathscr{R}}(A, r)$ and the measure of its lower rough approximation by $\mathscr{L}_{\mathscr{R}}(A, r)$, respectively, where $\mathscr{U}_{\mathscr{R}} : L^X \times M_{\perp_M} \longrightarrow M$ and $\mathscr{L}_{\mathscr{R}} : L^X \times M_{\perp_M} \longrightarrow M$ are defined by $\mathscr{U}_{\mathscr{R}}(A, r) = \widetilde{\subset}^M(\overline{\mathscr{R}}(A, r), A)$, $\mathscr{L}_{\mathscr{R}}(A, r) = \widetilde{\subset}^M(A, \underline{\mathscr{R}}(A, r))$ for any $A \in L^X$, and $r \in M_{\perp_M}$.

Proposition 4.8. *Let* $A \in L^X$, *and* $r \in M_{\perp_M}$, *then*

$$\mathscr{U}_{\mathscr{R}}(A', r) = \mathscr{L}_{\mathscr{R}}(A, r), \ \ \mathscr{L}_{\mathscr{R}}(A', r) = \mathscr{U}_{\mathscr{R}}(A, r).$$

Proof. We only prove $\mathscr{U}_{\mathscr{R}}(A', r) = \mathscr{L}_{\mathscr{R}}(A, r)$ as an example. Indeed,

$$\mathscr{U}_{\mathscr{R}}(A', r) = \widetilde{\subset}^M(\overline{\mathscr{R}}(A', r), A') = \widetilde{\subset}^M((\underline{\mathscr{R}}(A, r))', A') = \phi(\widetilde{\subset}(A, \underline{\mathscr{R}}(A, r))) = \mathscr{L}_{\mathscr{R}}(A, r).$$

$\square$

Theorem 4.9. *Let* $\mathscr{R} : L^X \times M_{\perp_M} \times X \longrightarrow L$ *be a transitive convex* (L, M)*-fuzzy remote neighborhood operator on* X. *Then,* $\forall a \in M_{\perp_M}$, $(X, [\mathscr{L}_{\mathscr{R}}]^a)$ *is an* (L, M)*-fuzzy concave space on* X, *and* $(X, [\mathscr{U}_{\mathscr{R}}]^a)$ *is an* (L, M)*-fuzzy convex space on* X, *where* $[\mathscr{L}_{\mathscr{R}}]^a : L^X \longrightarrow M$ *and* $[\mathscr{U}_{\mathscr{R}}]^a : L^X \longrightarrow M$ *are two mappings defined by* $[\mathscr{L}_{\mathscr{R}}]^a(A) = \mathscr{L}_{\mathscr{R}}(A, a)$ *and* $[\mathscr{U}_{\mathscr{R}}]^a(A) = \mathscr{U}_{\mathscr{R}}(A, a)$, *respectively.*

Proof. (LMA1)

$$[\mathscr{L}_{\mathscr{R}}]^a(\perp_L^X) = \mathscr{L}_{\mathscr{R}}(\perp_L^X, a) = \widetilde{\subset}^M(\perp_L^X, \underline{\mathscr{R}}(\perp_L^X, a)) = \bigwedge\nolimits_{x\in X} \phi(\perp_L^X(x) \to \underline{\mathscr{R}}(\perp_L^X, a)(x)) = \bigwedge\nolimits_{x\in X} \phi(\perp_L \to \perp_L) = \top_M,$$

$$[\mathscr{L}_{\mathscr{R}}]^a(\top_L^X) = \mathscr{L}_{\mathscr{R}}(\top_L^X, a) = \widetilde{\subset}^M(\top_L^X, \underline{\mathscr{R}}(\top_L^X, a)) = \bigwedge\nolimits_{x\in X} \phi(\top_L^X(x) \to \underline{\mathscr{R}}(\top_L^X, a)(x)) = \bigwedge\nolimits_{x\in X} \phi(\top_L \to \top_L) = \top_M.$$

(LMA2) $\forall \{U_i\}_{i\in I} \subseteq L^X$, we have

$$\begin{aligned}
[\mathscr{L}_{\mathscr{R}}]^a\Big(\bigvee_{i\in I} U_i\Big) &= \widetilde{\subset}^M\Big(\bigvee\nolimits_{i\in I} U_i, \underline{\mathscr{R}}\Big(\bigvee\nolimits_{i\in I} U_i, a\Big)\Big) \\
&= \bigwedge\nolimits_{x\in X} \phi\Big(\bigvee\nolimits_{i\in I} U_i(x) \to \underline{\mathscr{R}}\Big(\bigvee\nolimits_{i\in I} U_i, a\Big)(x)\Big) \\
&\geq \bigwedge\nolimits_{x\in X} \phi\Big(\bigvee\nolimits_{i\in I} U_i(x) \to \bigvee\nolimits_{i\in I} \underline{\mathscr{R}}(U_i, a)(x)\Big) \\
&\geq \bigwedge\nolimits_{x\in X} \bigwedge\nolimits_{i\in I} \phi\big(U_i(x) \to \underline{\mathscr{R}}(U_i, a)(x)\big) \\
&= \bigwedge\nolimits_{i\in I} \bigwedge\nolimits_{x\in X} \phi\big(U_i(x) \to \underline{\mathscr{R}}(U_i, a)(x)\big) \\
&= \bigwedge\nolimits_{i\in I} [\mathscr{L}_{\mathscr{R}}]^a(U_i).
\end{aligned}$$

(LMA3) If $\{A_j : j \in J\} \overset{cdir}{\subseteq} L^X$, we obtain

$$\begin{aligned}
[\mathscr{L}_{\mathscr{R}}]^a\Big(\bigwedge_{j\in J} A_j\Big) &= \widetilde{\subset}^M\Big(\bigwedge_{j\in J} A_j, \underline{\mathscr{R}}(\bigwedge_{j\in J} A_j, a)\Big) \\
&= \bigwedge_{x\in X} \phi\Big(\bigwedge_{j\in J} A_j(x) \to \underline{\mathscr{R}}(\bigwedge_{j\in J} A_j, a)(x)\Big) \\
&= \bigwedge_{x\in X} \phi\Big(\bigwedge_{j\in J} A_j(x) \to \bigwedge_{j\in J} \underline{\mathscr{R}}(A_j, a)(x)\Big) \\
&\geq \bigwedge_{x\in X} \bigwedge_{j\in J} \phi\Big(A_j(x) \to \underline{\mathscr{R}}(A_j, a)(x)\Big) \\
&= \bigwedge_{j\in J} \bigwedge_{x\in X} \phi\Big(A_j(x) \to \underline{\mathscr{R}}(A_j, a)(x)\Big) \\
&= \bigwedge_{j\in J} [\mathscr{L}_{\mathscr{R}}]^a(A_j).
\end{aligned}$$

From the above proof, we can see that $[\mathscr{L}_{\mathscr{R}}]^a$ is an (L, M)-fuzzy concave structure on X. Similarly, by Proposition 4.8, we can obtain $[\mathscr{U}_{\mathscr{R}}]^a$ is an (L, M)-fuzzy convex structure on X. □

5. Conclusions

In this paper, a pair of lower and upper rough approximation operators based on (L, M)-G-fuzzy remote neighborhood operators was proposed. As a special kind of (L, M)-G-fuzzy remote neighborhood operators, convex (L, M)-fuzzy remote neighborhood operators play an important role in (L, M)-fuzzy convex structures. In fact, they are categorically isomorphic to (L, M)-fuzzy convex structures. Therefore, approximation operators based on convex (L, M)-fuzzy remote neighborhood operators were also discussed, and the measure of their rough approximation was given. A realization of the measures of lower and upper rough approximations based on convex (L, M)-fuzzy remote neighborhood operators in terms of (L, M)-fuzzy convex structures was presented, respectively.

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