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Topologicality, completeness and cocompleteness of the category of structural spaces

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Abstract. After introducing the notion of operant relative to a base, we present the concepts of structural space and structural morphism with respect to a given operant. We prove, under mild conditions, that the category of structural spaces and structural morphisms is a topological category. Concluding that when the underlying category is complete or cocomplete, so is the category of structural spaces, and that limits or colimits are concrete. Several illustrative examples are furnished that show the diversity of the concept of a structural space.

1. Introduction

Grothendieck's topology and the concept of a topos were introduced in the 1960s. Around 1962, Grothendieck developed the initial concepts of a topos within the context of algebraic geometry. He introduced these ideas as a tool for studying algebraic geometry and similar structures, ultimately leading to the formation of topos theory. Subsequently, these ideas were further developed by F. W. Lawvere and M. Tierney, who provided a more formal and general framework for topos theory, laying the foundation for its application in various mathematical fields. In the 1960s, as the result of Lawvere's bold project for the categorical foundation of all mathematics, which led to the publication of the paper "An Elementary Theory of the Category of Sets", see [13], many authors sought to express various fields of mathematics, including topology, in categorical terms. For instance, one can refer to the work of A. Kock and G. C. Wraith, see [12], who provided a definition of a topological space object based on a fundamental system of neighborhoods, as well as MacFarlane's work on closure operators, which included a preliminary report in the Abstracts of the Sydney Category Seminar, see [11]. In their theses, L. N. Stout and A. I. MacFarlane by internalization of structures, introduced the concept of a topological space object and an open topological object in an elementary topos, respectively, in 1974 and 1975, see [19] and [14]. In this paper, as a generalization of the works of Stout and MacFarlane, see Examples 6.6 and 6.7, and inspired by the foundational ideas of S. Mac Lane and S. Eilenberg, see [2], emphasizing that arrows are as important as objects, we shift our focus from considering the necessary properties a category must possess for topology to be meaningful to

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concentrating on a base $\mathbb{B}$ consisting of a triple of functors and an operant $\mathbb{O}$ with respect to the base $\mathbb{B}$, consisting of quadruple of natural transformations, between specific functors derived from the base, that satisfy certain conditions. This approach is sufficiently general to encompass all the examples one would wish to consider as an $\mathbb{O}$-structural space. There have been some attempts to unify the concepts of topology and fuzzy topology by several authors, see [22], [23], [3], [8] and [9], to mention a few. In [9], the concept of topology and five different fuzzy topological concepts existing in the literature, are unified on a categorical basis. In [10], the category of structural topological spaces is formed and it is shown that the category has (finite) limits whenever the base category does. In this article, we use the approach introduced in [24], which serves as a generalization of the one presented in [9] and [10], by replacing the squaring functor in the base with a general functor. This slight generalization allows one to consider many new examples, see Examples 6.4 and 6.5. We show that under mild conditions, the so called category of structural spaces is indeed a topological category. We conclude that when the underlying category is complete or cocomplete, so is the category of structural spaces. Several illustrative examples are provided. More specifically, the paper is divided into several sections as follows.

In Section 2, we give the definition of a base $\mathbb{B}$, an operant $\mathbb{O}$ relative to a base and an $\mathbb{O}$-structure on an object X of a category, leading to the notion of $\mathbb{O}$-structural space. The main result proved in this section is that the collection of the $\mathbb{O}$-structures on an object X is a complete lattice.

In Section 3, we introduce the notion of structural morphism. We show that the identity morphism and the composition of structural morphisms are structural morphisms. Other results related to $\mathbb{O}$-structural morphisms are also provided.

In Section 4, we form the category of $\mathbb{O}$-structural spaces and $\mathbb{O}$-structural morphisms and show that it is a concrete category. We prove, under mild hypothesis, the existence of the initial (respectively, the final) $\mathbb{O}$-structure induced by a morphism or a source (respectively, a sink).

In Section 5, we characterize the initial sources and final sinks and we prove that the category of $\mathbb{O}$-structural spaces is, under mild conditions, a topological category. We conclude that if the underlying category is complete or cocomplete, then so is the category of $\mathbb{O}$-structural spaces.

In Section 6, we provide several examples that show the diversity of the introduced concept of structural spaces and morphisms, and proves the topologicality, completeness or cocompleteness of several categories.

For categorical concepts we refer the reader to [1].

2. Structural spaces in a category

In this section, on a categorical basis, we introduce the notions of a base $\mathbb{B}$, an operant $\mathbb{O}$ relative to $\mathbb{B}$ and an $\mathbb{O}$-structure on an object X leading to the concept of $\mathbb{O}$-structural space. We prove the intersection of $\mathbb{O}$-structures on X is an $\mathbb{O}$-structure on X as well as some other related results. We conclude that the collection of $\mathbb{O}$-structures on X is a complete lattice.

To this end we recall that when a category $\mathcal{C}$ is strongly finitely complete (i.e., it has intersections and finite limits), it is (*ExtEpi*, *Mono*)-structured, where *ExtEpi* and *Mono* stand for the collections of extremal epimorphisms and monomorphisms, respectively, see [1]. We say a category has *Mono*-pullbacks, when pullbacks of monomorphisms along every map exist. We recall from [24] that:

Definition 2.1. Let $\mathcal{E}$ and $\mathcal{C}$ be categories.

a) A triple of functors $\mathcal{E}^{op} \xrightarrow{\mathsf{T}} \mathcal{C} \underset{\mathsf{Q}}{\overset{\mathsf{P}}{\rightrightarrows}} \mathcal{C}$ is called a base on $(\mathcal{E}, \mathcal{C})$ and is denoted by $\mathbb{B}$. A base is said to be:

- an I-base if $\mathcal{C}$ has intersections,
- an F-base if $\mathcal{C}$ has finite limits,
- an M-base if $\mathcal{C}$ has *Mono*-pullbacks,
- an E-base if P and Q preserve extremal epimorphisms,

- an IF-base if it is both an I-base and an F-base, etc.

b) An operant relative to a base $\mathbb{B} = (\mathsf{T}, \mathsf{P}, \mathsf{Q})$ or a $\mathbb{B}$-operant is a quadruple $\mathbb{O} = (\beta, \tau, \mu, \nu)$, where

$$1 \xrightarrow{\beta} \mathsf{T},$$
$$1 \xrightarrow{\tau} \mathsf{T},$$
$$\mathsf{P} \circ \mathsf{T} \xrightarrow{\mu} \mathsf{T} \text{ and}$$
$$\mathsf{Q} \circ \mathsf{T} \xrightarrow{\nu} \mathsf{T}$$

are natural transformations. Here $\mathcal{E}^{op} \xrightarrow{1} \mathcal{C}$ denotes the constant functor with terminal object 1 as its value.

c) A structure with respect to an operant $\mathbb{O}$ or an $\mathbb{O}$-structure, on an object X of $\mathcal{E}$, is a $\mathcal{C}$-monomorphism $T_X \rightarrowtail^{t_X} \mathsf{T}(X)$, such that the morphisms β_X, τ_X, μ_X and ν_X restrict to T_X in the sense that there exist morphisms $1 \xrightarrow{\beta_X} T_X$, $1 \xrightarrow{\tau_X} T_X$, $\mathsf{P}(T_X) \xrightarrow{\mu_X} T_X$ and $\mathsf{Q}(T_X) \xrightarrow{\nu_X} T_X$ rendering commutative the following diagrams.

$$\begin{array}{ccc} & & T_X \\ & \overset{\beta_X}{\nearrow} & \downarrow{\scriptstyle t_X} \\ 1 & \xrightarrow[\beta_X]{} & \mathsf{T}(X) \end{array} \qquad \begin{array}{ccc} & & T_X \\ & \overset{\tau_X}{\nearrow} & \downarrow{\scriptstyle t_X} \\ 1 & \xrightarrow[\tau_X]{} & \mathsf{T}(X) \end{array}$$

$$\begin{array}{ccc} \mathsf{P}(T_X) & \xrightarrow{\mu_X} & T_X \\ {\scriptstyle \mathsf{P}(t_X)}\downarrow & & \downarrow{\scriptstyle t_X} \\ \mathsf{P}(\mathsf{T}(X)) & \xrightarrow[\mu_X]{} & \mathsf{T}(X) \end{array} \qquad \begin{array}{ccc} \mathsf{Q}(T_X) & \xrightarrow{\nu_X} & T_X \\ {\scriptstyle \mathsf{Q}(t_X)}\downarrow & & \downarrow{\scriptstyle t_X} \\ \mathsf{Q}(\mathsf{T}(X)) & \xrightarrow[\nu_X]{} & \mathsf{T}(X) \end{array}$$

When t_X is an $\mathbb{O}$-structure on X, we call the pair (X, t_X) an $\mathbb{O}$-structural space.

We recall that since for an I-base the category $\mathcal{C}$ has intersections, the collection $Sub(A)$ of subobjects of $A \in \mathcal{C}$ is a complete lattice, in which meet (denoted by $\wedge$) is the same as intersection and the existence of join (denoted by $\vee$) is implied by that of meet. Also for morphisms $a : A \longrightarrow C$ and $b : B \longrightarrow C$, we write $a \leq b$, if there is a morphism $h : A \longrightarrow B$ such that $a = bh$. We have:

Proposition 2.2. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on an I-base $\mathbb{B}$. For an object X of $\mathcal{E}$, the intersection of any collection of $\mathbb{O}$-structures on X is an $\mathbb{O}$-structure on X.*

Proof. Let t_i, $i \in I$ be $\mathbb{O}$-structures on X. So for each $i \in I$, we have $T_i \rightarrowtail^{t_i} \mathsf{T}(X)$ and morphisms β_i, τ_i, μ_i and ν_i making commutative the following diagrams.

$$\begin{array}{ccc} & & T_i \\ & \overset{\beta_i}{\nearrow} & \downarrow{\scriptstyle t_i} \\ 1 & \xrightarrow[\beta_X]{} & \mathsf{T}(X) \end{array} \qquad \begin{array}{ccc} & & T_i \\ & \overset{\tau_i}{\nearrow} & \downarrow{\scriptstyle t_i} \\ 1 & \xrightarrow[\tau_X]{} & \mathsf{T}(X) \end{array}$$

$$\begin{array}{ccc} \mathsf{P}(T_i) & \xrightarrow{\mu_i} & T_i \\ {\scriptstyle \mathsf{P}(t_i)}\downarrow & & \downarrow{\scriptstyle t_i} \\ \mathsf{P}(\mathsf{T}(X)) & \xrightarrow[\mu_X]{} & \mathsf{T}(X) \end{array} \qquad \begin{array}{ccc} \mathsf{Q}(T_i) & \xrightarrow{\nu_i} & T_i \\ {\scriptstyle \mathsf{Q}(t_i)}\downarrow & & \downarrow{\scriptstyle t_i} \\ \mathsf{Q}(\mathsf{T}(X)) & \xrightarrow[\nu_X]{} & \mathsf{T}(X) \end{array}$$

Let $T \rightarrowtail^{t} \mathsf{T}(X)$ be the intersection of t_i's. So for each i we have $t \leq t_i$, i.e., there is a morphism g_i making the following triangle commutative.

$$\begin{array}{ccc} T & \xrightarrow{g_i} & T_i \\ & {\scriptstyle t}\searrow \quad \swarrow{\scriptstyle t_i} & \\ & \mathsf{T}(X) & \end{array}$$

Since for each i, $\beta_X = t_i\beta_i$, $\beta_X \leq t_i$. Thus $\beta_X \leq t$. Similarly $\tau_X \leq t$. Therefore there are β and τ making the corresponding triangles commutative as desired. Now for each i we have $\mu_X\mathsf{P}(t) = \mu_X\mathsf{P}(t_i)\mathsf{P}(g_i) = t_i\mu_i\mathsf{P}(g_i)$, implying for all i, $\mu_X\mathsf{P}(t) \leq t_i$. So $\mu_X\mathsf{P}(t) \leq t$. Similarly $\nu_X\mathsf{Q}(t) \leq t$. Therefore there are μ and ν making the corresponding squares commute as required. Hence t is an $\mathbb{O}$-structure on X. □

The intersection t in the above proposition is denoted by $\bigwedge_{i\in I} t_i$.

Proposition 2.3. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on an* I*-base $\mathbb{B}$. For $a \in Sub(\mathsf{T}(X))$, there exists a smallest $\mathbb{O}$-structure t on X containing a, i.e.,*

- **a)** *$a \leq t$ and*
- **b)** *if t' is an $\mathbb{O}$-structure on X with $a \leq t'$, then $t \leq t'$.*

Proof. Let $\mathfrak{F} = \{s : s$ is an $\mathbb{O}$-structure on X containing $a\}$ and set $t = \bigwedge_{s\in\mathfrak{F}} s$. By Proposition 2.2, t is an $\mathbb{O}$-structure on X. Since for all $s \in \mathfrak{F}$, $a \leq s$, we get $a \leq t$, proving **(a)**. Now let t' be an $\mathbb{O}$-structure on X with $a \leq t'$. Thus $t' \in \mathfrak{F}$ and so $t \leq t'$, proving **(b)**. □

The $\mathbb{O}$-structure in the above proposition is called the $\mathbb{O}$-structure induced by a and is denoted by $\langle a\rangle$.

Proposition 2.4. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on an* I*-base $\mathbb{B}$. For $\mathbb{O}$-structures t_i on X, $i \in I$, there is a smallest $\mathbb{O}$-structure t containing t_i for all $i \in I$.*

Proof. Let $a = \bigvee_{i\in I} t_i$ in $Sub(\mathsf{T}(X))$. By Proposition 2.3, $t = \langle a\rangle$ is the required $\mathbb{O}$-structure. □

The $\mathbb{O}$-structure in the above proposition is called the $\mathbb{O}$-structure induced by the $\mathbb{O}$-structures t_i and is denoted by $\langle\bigvee_{i\in I} t_i\rangle$.

Definition 2.5. $\mathbb{O}$-structures $S_X \rightarrowtail^{s_X} \mathsf{T}(X)$ and $T_X \rightarrowtail^{t_X} \mathsf{T}(X)$ are said to be isomorphic if there is a $\mathcal{C}$-isomorphism $\phi : S_X \longrightarrow T_X$ such that $s_X = t_X\phi$. In this case we write $s_X \cong t_X$. The isomorphism class of t_X is denoted by $[t_X]$.

For an object X in $\mathcal{E}$, setting

$\mathbb{O}$-Str(X)=$\{t_X : t_X$ is an $\mathbb{O}$-structure on $X\}$ and $\mathbb{O}$-[Str](X)=$\{[t_X] : t_X$ is an $\mathbb{O}$-structure on $X\}$

we have:

Theorem 2.6. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on an* I*-base $\mathbb{B}$. For each $X \in \mathcal{E}$, $\mathbb{O}$-Str(X) and $\mathbb{O}$-[Str](X) are complete lattices.*

Proof. The existence of meet and join in $\mathbb{O}$-Str(X) are implied by Proposition 2.2 and **??**, respectively. It can easily be verified that $\bigwedge_{i\in I}[t_i] = [\bigwedge_{i\in I} t_i]$ and $\bigvee_{i\in I}[t_i] = [\langle\bigvee_{i\in I} t_i\rangle]$ give respectively, the meet and join in $\mathbb{O}$-[Str](X). □

3. Structural morphisms in a category

In this section, we recall from [24] the notion of $\mathbb{O}$-structural morphism and we show that the identity morphism as well as the composition of $\mathbb{O}$-structural morphisms are $\mathbb{O}$-structural. We also prove that $\mathbb{O}$-structurality of a morphism is preserved when the $\mathbb{O}$-structure on domain (respectively, codomain) is replaced by a bigger domain (respectively, smaller codomain). We conclude that when the $\mathbb{O}$-structure on domain and/or codomain of a morphism is replaced by an isomorphic one, the $\mathbb{O}$-structurality of the morphism stays intact.

Definition 3.1. Given $\mathbb{O}$-structural spaces (X, t_X) and (Y, t_Y), an $\mathcal{E}$-morphism $f : X \longrightarrow Y$ is said to be an $\mathbb{O}$-structural

morphism, if there exists a (necessarily unique) $\mathcal{C}$-morphism $T_f : T_Y \longrightarrow T_X$ such that the following diagram commutes.

$$\begin{array}{ccc} T_Y & \overset{t_Y}{\rightarrowtail} & \mathsf{T}(Y) \\ {\scriptstyle T_f}\downarrow & & \downarrow{\scriptstyle \mathsf{T}(f)} \\ T_X & \underset{t_X}{\rightarrowtail} & \mathsf{T}(X) \end{array}$$

In this case we write $f : (X, t_X) \longrightarrow (Y, t_Y)$.

Theorem 3.2. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on a base $\mathbb{B}$. Given $\mathbb{O}$-structural spaces (X, t_X), (Y, t_Y) and (Z, t_Z),*

1. *the identity morphism $(X, t_X) \xrightarrow{1_X} (X, t_X)$ is an $\mathbb{O}$-structural morphism.*
2. *the composition of the $\mathbb{O}$-structural morphisms*

$$(X, t_X) \xrightarrow{f} (Y, t_Y) \xrightarrow{g} (Z, t_Z)$$

is an $\mathbb{O}$-structural morphism.

Proof. Follows in a manner analogous to the proof in [9]. □

Theorem 3.3. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on a base $\mathbb{B}$ and $(X, t_X) \xrightarrow{f} (Y, t_Y)$ be an $\mathbb{O}$-structural morphism. If t'_X is an $\mathbb{O}$-structure on X such that $t_X \le t'_X$ and t'_Y is an $\mathbb{O}$-structure on Y such that $t'_Y \le t_Y$, then*

a) $(X, t'_X) \xrightarrow{f} (Y, t_Y)$ *and*

b) $(X, t_X) \xrightarrow{f} (Y, t'_Y)$

are $\mathbb{O}$-structural morphisms.

Proof. By the assumptions and the fact that f is an $\mathbb{O}$-structural morphism, the result follows. □

Corollary 3.4. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on a base $\mathbb{B}$. Consider the $\mathbb{O}$-structures $S_X \overset{s_X}{\rightarrowtail} \mathsf{T}(X)$, $T_X \overset{t_X}{\rightarrowtail} \mathsf{T}(X)$, $S_Y \overset{s_Y}{\rightarrowtail} \mathsf{T}(Y)$, and $T_Y \overset{t_Y}{\rightarrowtail} \mathsf{T}(Y)$. Assume that $s_X \cong t_X$ and $s_Y \cong t_Y$. If a morphism $f : (X, t_X) \longrightarrow (Y, t_Y)$ is $\mathbb{O}$-structural, so is $f : (X, s_X) \longrightarrow (Y, s_Y)$.*

Proof. Follows from Theorem 3.3. □

4. The concrete category of structural spaces, initial and final structures

In this section we introduce the categories $\mathbb{OSS}$ and $[\mathbb{OSS}]$ of $\mathbb{O}$-structures and $\mathbb{O}$-structural morphisms (in two senses). We prove both $\mathbb{OSS}$ and $[\mathbb{OSS}]$ are concrete categories over $\mathcal{E}$ and that they are concretely equivalent. We show the existence of initial (respectively, final) $\mathbb{O}$-structures for both morphisms and sources (respectively, sinks).

By abuse of language calling the isomorphism class $[t_X]$ also an $\mathbb{O}$-structure on X and saying a morphism $f : (X, [t_X]) \longrightarrow (Y, [t_Y])$ is $\mathbb{O}$-structural if the morphism $f : (X, t_X) \longrightarrow (Y, t_Y)$ is, we have:

Theorem 4.1. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on a base $\mathbb{B}$. The collections of $\mathbb{O}$-structural spaces and $\mathbb{O}$-structural morphisms (in both senses) form a category.*

Proof. When using the strict structures, the proof follows by Theorem 3.2 and when the isomorphism classes are used, the proof follows by Theorem 3.2 and Corollary 3.4. □

The categories in the above theorem are denoted respectively by, $\mathbb{OSS}$ and $[\mathbb{OSS}]$.

Theorem 4.2. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on a base $\mathbb{B}$. The categories $\mathbb{OSS}$ and $[\mathbb{OSS}]$ are concrete categories over $\mathcal{E}$ and they are concretely equivalent.*

Proof. The mapping $U : \mathbb{OSS} \longrightarrow \mathcal{E}$ that takes $f : (X, t_X) \longrightarrow (Y, t_Y)$ in $\mathbb{OSS}$ to $f : X \longrightarrow Y$ in $\mathcal{E}$ is easily seen to be a faithful functor. A similar faithful functor also denoted by $U : [\mathbb{OSS}] \longrightarrow \mathcal{E}$ exists. The last assertion holds by a straight verification that the mapping $G : \mathbb{OSS} \longrightarrow [\mathbb{OSS}]$ sending $f : (X, t_X) \longrightarrow (Y, t_Y)$ to $f : (X, [t_X]) \longrightarrow (Y, [t_Y])$ is the desired concrete equivalence. □

For simplicity, we prove many of our results for the category $\mathbb{OSS}$. One can easily verify that results similar to those proved for $\mathbb{OSS}$ hold for $[\mathbb{OSS}]$.

Now we prove the existence of initial $\mathbb{O}$-structures induced by morphisms and we use that to prove the same for sources.

Proposition 4.3. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on a base $\mathbb{B}$. Let $f : X \longrightarrow Y$ be an $\mathcal{E}$-morphism and t_Y be an $\mathbb{O}$-structure on Y. There is a smallest $\mathbb{O}$-structure t_X on X making f an $\mathbb{O}$-structural morphism, obtained as:*

- **a)** *the meet $\bigwedge_{t \in \mathfrak{F}} t$, where*

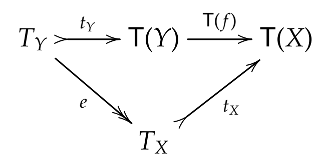
$$\mathfrak{F} = \{t : t \text{ is an } \mathbb{O}\text{-structure on } X \text{ making } f\ \mathbb{O}\text{-structural}\}$$

provided that $\mathbb{B}$ is an I*-base, and*

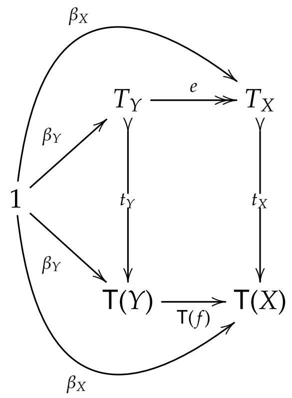
- **b)** *the mono part of the $(ExtEpi, Mono)$-factorization of $\mathsf{T}(f)t_Y$, provided that $\mathbb{B}$ is an* IFE*-base.*

Proof. **a)** It follows from the $\mathbb{O}$-structurality of the members of $\mathfrak{F}$ and Proposition 2.2.

b) We take the $(ExtEpi, Mono)$-factorization of $\mathsf{T}(f)t_Y$ to get t_X, as the following diagram shows.



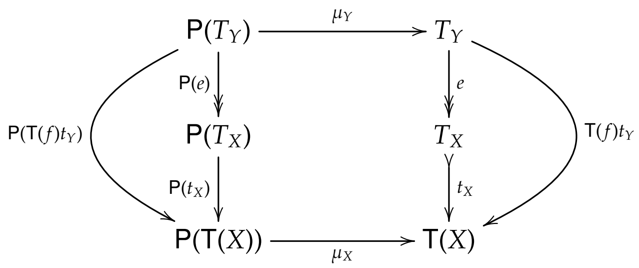
First we show t_X is an $\mathbb{O}$-structure on X. Define $\beta_X = e\beta_Y$. We have the commutative diagram,



So we have $t_X\beta_X = \beta_X$. Similarly by defining $\tau_X = e\tau_Y$, we can show $t_X\tau_X = \tau_X$. To show the existence of $\mu_X : \mathsf{P}(T_X) \longrightarrow T_X$, we have:

$$\mathsf{T}(f)t_Y\mu_Y = \mathsf{T}(f)\mu_Y\mathsf{P}(t_Y) = \mu_X\mathsf{P}(\mathsf{T}(f))\mathsf{P}(t_Y) = \mu_X\mathsf{P}(\mathsf{T}(f)t_Y)$$

This shows that in the following diagram, the outer square commutes.

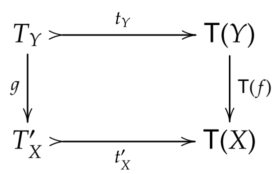


In the above diagram, the right triangle commutes by definition of t_X and the left one by functoriality of P. Since the base is assumed to be an IFE-base, P preserves extremal epimorphisms and so $\mathsf{P}(e)$ is an extremal epimorphism. Thus there exists a morphism $\mu_X : \mathsf{P}(T_X) \longrightarrow T_X$ making the newly formed squares in the above diagram commute.

If in the above diagram, we replace P by Q, and μ by ν, similar arguments show the existence of $\nu_X : \mathsf{Q}(T_X) \longrightarrow T_X$ making the corresponding squares commute. This proves t_X is an $\mathbb{O}$-structure on X.

Next we show that $f : (X, t_X) \longrightarrow (Y, t_Y)$ is an $\mathbb{O}$-structural morphism. Setting $T_f = e$, the defining (*ExtEpi*, *Mono*)-factorization triangle shows that f is an $\mathbb{O}$-structural morphism.

Finally we show that t_X is the smallest $\mathbb{O}$-structure on X making f an $\mathbb{O}$-structural morphism. Let $t'_X : T'_X \rightarrowtail \mathsf{T}(X)$ be an $\mathbb{O}$-structure on X making f an $\mathbb{O}$-structural morphism. So we have a morphism g making the following diagram commute.



Since $t_Xe = \mathsf{T}(f)t_Y$, we get the following commutative square,

$$\begin{array}{ccc} T_Y & \overset{e}{\twoheadrightarrow} & T_X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle t_X} \\ T'_X & \underset{t'_X}{\rightarrowtail} & \mathsf{T}(X) \end{array}$$

with e extremal epi and t'_X mono. So we have a diagonal $d : T_X \longrightarrow T'_X$ making the newly formed triangles in the above diagram commute. In particular $t'_X d = t_X$, i.e., $t_X \leq t'_X$ as desired. □

The $\mathbb{O}$-structure in part (a) of the above proposition is called the initial $\mathbb{O}$-structure on X induced by the morphism f and the $\mathbb{O}$-structure t_Y, and is denoted by $t_X^{(f)}$.

Let us's remark that under the stronger condition that $\mathbb{B}$ is an IFE-base, the $\mathbb{O}$-structures obtained in parts (a) and (b) of the above proposition are obviously isomorphic.

Corollary 4.4. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on an* I*-base $\mathbb{B}$. Let $f : X \longrightarrow Y$ be an $\mathcal{E}$-morphism, t_X and t_Y be $\mathbb{O}$-structures on X and Y, respectively. The morphism $f : (X, t_X) \longrightarrow (Y, t_Y)$ is $\mathbb{O}$-structural if and only if $t_X^{(f)} \leq t_X$.*

Proof. The direct implication follows by Proposition 4.3 and the converse by Proposition 4.3 and Theorem 3.3. □

Theorem 4.5. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on an* I*-base $\mathbb{B}$. Let for each $i \in I$, the morphism $f_i : X \longrightarrow Y_i$ be an $\mathcal{E}$-morphism and t_i be an $\mathbb{O}$-structure on Y_i. There is a smallest $\mathbb{O}$-structure on X making f_i an $\mathbb{O}$-structural morphism, for all i.*

Proof. By Proposition 4.3, for each i there is a smallest $\mathbb{O}$-structure s_i on X, making f_i an $\mathbb{O}$-structural morphism. Set $t_X = \langle \bigvee_{i\in I} s_i \rangle$. By **??**, t_X is an $\mathbb{O}$-structure on X. Since $f_i : (X, s_i) \longrightarrow (Y_i, t_i)$ is an $\mathbb{O}$-structural morphism and $s_i \leq t_X$, by Theorem 3.3 $f_i : (X, t_X) \longrightarrow (Y_i, t_i)$ is an $\mathbb{O}$-structural morphism.

To show that t_X is the smallest $\mathbb{O}$-structure on X such that for all i, the morphism f_i is $\mathbb{O}$-structural, let t'_X be another such $\mathbb{O}$-structure. So for each i, the morphism $f_i : (X, t'_X) \longrightarrow (Y_i, t_i)$ is $\mathbb{O}$-structural. Since s_i is the smallest $\mathbb{O}$-structure making $f_i : (X, s_i) \longrightarrow (Y_i, t_i)$ $\mathbb{O}$-structural, $s_i \leq t'_X$. So $s_i \leq t'_X$ for all i, thus $\bigvee_{i\in I} s_i \leq t'_X$. t_X is the smallest $\mathbb{O}$-structure containing $\bigvee_{i\in I} s_i$, so $t_X \leq t'_X$ and this concludes the proof. □

The $\mathbb{O}$-structure in the above theorem is called the initial $\mathbb{O}$-structure on X induced by the morphisms f_i (or by the source $(f_i)_I$) and the $\mathbb{O}$-structures t_i and is denoted by $t_X^{(f_i)_I}$.

Corollary 4.6. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on an* I*-base $\mathbb{B}$. Let for each $i \in I$, the morphism $f_i : X \longrightarrow Y_i$ be an $\mathcal{E}$-morphism, t_X and t_i be $\mathbb{O}$-structures on X and Y_i, respectively. Then $f_i : (X, t_X) \longrightarrow (Y_i, t_i)$ is $\mathbb{O}$-structural for all i if and only if $t_X^{(f_i)_I} \leq t_X$.*

Proof. The direct implication follows by Theorem 4.5 and the converse by Theorem 4.5 and Theorem 3.3. □

Finally we prove the existence of final $\mathbb{O}$-structures coinduced by morphisms and we use that to prove the same for sinks.

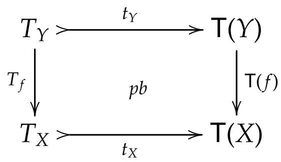
Proposition 4.7. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on a base $\mathbb{B}$. Let $f : X \longrightarrow Y$ be an $\mathcal{E}$-morphism and $T_X \overset{t_X}{\rightarrowtail} \mathsf{T}(X)$ be an $\mathbb{O}$-structure on X. There is a largest $\mathbb{O}$-structure t_Y on Y making f an $\mathbb{O}$-structural morphism, obtained as:*

- **a)** *the pullback of t_X along $\mathsf{T}(f)$, provided that $\mathbb{B}$ is an* M*-base, and*
- **b)** *the join $\langle \bigvee_{t\in\mathfrak{F}} t \rangle$, where*

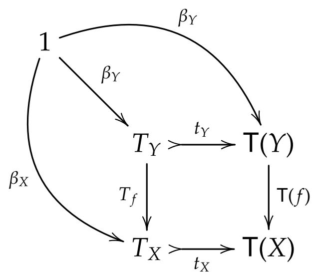
$$\mathfrak{F} = \{t : t \textit{ is an } \mathbb{O}\textit{-structure on } Y \textit{ making } f \ \mathbb{O}\textit{-structural}\}$$

provided that $\mathbb{B}$ *is an* IM-*base.*

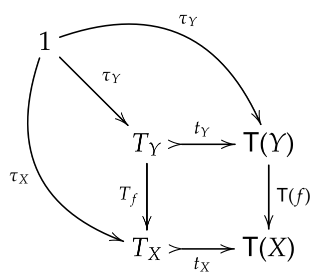
Proof. **a)** We take the pullback of $T_X \xrightarrow{t_X} \mathsf{T}(X)$ along $\mathsf{T}(f) : \mathsf{T}(Y) \longrightarrow \mathsf{T}(X)$ to get $T_Y \xrightarrow{t_Y} \mathsf{T}(Y)$ as the following diagram shows.



First we show t_Y is an $\mathbb{O}$-structure on Y. In the following diagram since the outer square commutes, there is a unique morphism $1 \xrightarrow{\beta_Y} T_Y$ making the triangles commute.



The commutativity of the upper triangle gives $t_Y\beta_Y = \beta_Y$ as desired. Similarly we get a morphism $1 \xrightarrow{\tau_Y} T_Y$ rendering commutative the triangles in the following diagram.

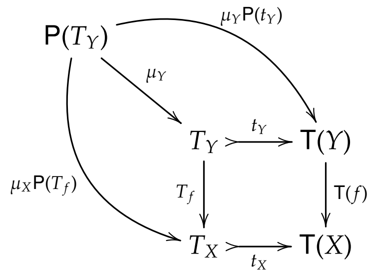


Thus we have $t_Y\tau_Y = \tau_Y$. To show the existence of $\mu_Y : \mathsf{P}(T_Y) \longrightarrow T_Y$, we have:

$$\mathsf{T}(f)\mu_Y\mathsf{P}(t_Y) = \mu_X\mathsf{P}(\mathsf{T}(f))\mathsf{P}(t_Y) = \mu_X\mathsf{P}(\mathsf{T}(f)t_Y) = \mu_X\mathsf{P}(t_XT_f)$$

$$= \mu_X\mathsf{P}(t_X)\mathsf{P}(T_f) = t_X\mu_X\mathsf{P}(T_f)$$

This shows that in the following diagram, the outer square commutes and so there is a unique morphism $\mu_Y : \mathsf{P}(T_Y) \longrightarrow T_Y$ making the triangles commute.



The commutativity of the upper triangle results in the commutativity of the desired square.

Replacing P by Q, and μ by ν, similar arguments show the existence of a morphism $\nu_Y : \mathsf{Q}(T_Y) \longrightarrow T_Y$ making the corresponding square commutative. This proves t_Y is an $\mathbb{O}$-structure on Y.

The pullback square that defines t_Y shows that $f : (X, t_X) \longrightarrow (Y, t_Y)$ is an $\mathbb{O}$-structural morphism and one can verify that t_Y is the largest $\mathbb{O}$-structure on Y making f an $\mathbb{O}$-structural morphism.

b) With t_Y as in part (*a*), we have for all $t \in \mathfrak{F}$, $t \leq t_Y$. So $\bigvee_{t\in\mathfrak{F}} t \leq t_Y$. By Proposition 2.3, $\langle \bigvee_{t\in\mathfrak{F}} t\rangle$ is the smallest $\mathbb{O}$-structure on Y containing $\bigvee_{t\in\mathfrak{F}} t$, implying $\langle \bigvee_{t\in\mathfrak{F}} t\rangle \leq t_Y$. On the other hand, by part (a), $t_Y \in \mathfrak{F}$ and so $t_Y \leq \bigvee_{t\in\mathfrak{F}} t$. Thus $t_Y \leq \langle \bigvee_{t\in\mathfrak{F}} t\rangle$. Hence $\langle \bigvee_{t\in\mathfrak{F}} t\rangle \cong t_Y$ and therefore is the largest $\mathbb{O}$-structure on Y making f $\mathbb{O}$-structural. □

The $\mathbb{O}$-structure in part (*a*) of the above theorem is called the final $\mathbb{O}$-structure on Y coinduced by the morphism f and the $\mathbb{O}$-structure t_X and is denoted by $t_Y^{(\!(f)\!)}$.

Let us remark that under the stronger condition that $\mathbb{B}$ is an IM-base, the $\mathbb{O}$-structures obtained in parts (a) and (b) of the above proposition are obviously isomorphic, in which case $\langle \bigvee_{t\in\mathfrak{F}} t\rangle$ is the pullback of t_X along $\mathsf{T}(f)$.

Corollary 4.8. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on an* M*-base $\mathbb{B}$. Let $X \xrightarrow{f} Y$ be an $\mathcal{E}$-morphism, t_X and t_Y be $\mathbb{O}$-structures on X and Y, respectively. $(X, t_X) \xrightarrow{f} (Y, t_Y)$ is $\mathbb{O}$-structural if and only if $t_Y \leq t_Y^{(\!(f)\!)}$.*

Proof. The direct implication follows by Proposition 4.7 and the converse by Proposition 4.7 and Theorem 3.3. □

Theorem 4.9. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on an* IM*-base $\mathbb{B}$. Let for each $i \in I$, $f_i : X_i \longrightarrow Y$ be an $\mathcal{E}$-morphism and t_i be an $\mathbb{O}$-structure on X_i. There is a largest $\mathbb{O}$-structure on Y making f_i an $\mathbb{O}$-structural morphism, for all i.*

Proof. By Proposition 4.7, for each i there is a largest $\mathbb{O}$-structure s_i on Y, making f_i an $\mathbb{O}$-structural morphism. Set $t_Y = \bigwedge_{i\in I} s_i$. By Proposition 2.2, t_Y is an $\mathbb{O}$-structure on Y. Since for each i $f_i : (X_i, t_i) \longrightarrow (Y, s_i)$ is an $\mathbb{O}$-structural morphism and $t_Y \leq s_i$, by Theorem 3.3 $f_i : (X_i, t_i) \longrightarrow (Y, t_Y)$ is an $\mathbb{O}$-structural morphism. To show t_Y is the largest such, suppose t'_Y is another $\mathbb{O}$-structure on Y making $f_i : (X_i, t_i) \longrightarrow (Y, t'_Y)$ $\mathbb{O}$-structural for all $i \in I$. Since for each i, $f_i : (X_i, t_i) \longrightarrow (Y, s_i)$ is $\mathbb{O}$-structural, by Proposition 4.7 we get $t'_Y \leq s_i$. This is the case for all i, therefore $t'_Y \leq \bigwedge_{i\in I} s_i = t_Y$. □

The $\mathbb{O}$-structure in the above theorem is called the final $\mathbb{O}$-structure on Y coinduced by the morphisms f_i (or by the sink $(\!(f_i)\!)_I$) and the $\mathbb{O}$-structures t_i and is denoted by $t_Y^{(\!(f_i)\!)_I}$.

Corollary 4.10. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on an* IM*-base $\mathbb{B}$. Let for each $i \in I$, $f_i : X_i \longrightarrow Y$ be an $\mathcal{E}$-morphism, t_i and t_Y be $\mathbb{O}$-structures on X_i and Y, respectively. The morphism $f_i : (X_i, t_i) \longrightarrow (Y, t_Y)$ is $\mathbb{O}$-structural for all i if and only if $t_Y \leq t_Y^{(\!(f_i)\!)_I}$.*

Proof. The direct implication follows by Theorem 4.9 and the converse by Theorem 4.9 and Theorem 3.3. □

5. Initial sources, final sinks and topologicality of the category of structural spaces

In this section, we give a characterization of initial sources and final sinks. The main result of this section is that the concrete category [$\mathbb{OSS}$] is topological.

Proposition 5.1. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on an* I*-base $\mathbb{B}$. If the source*

$$\Big(f_i : (X, t_X) \longrightarrow (Y_i, t_i)\Big)_I$$

is an initial source in $\mathbb{OSS}$, then $t_X \cong t_X^{(f_i)_I}$.

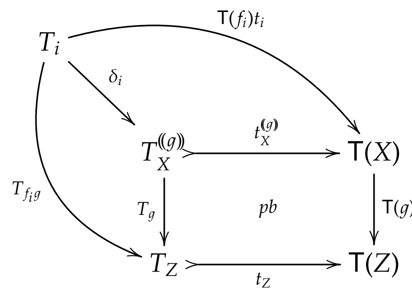
Proof. Let $\Big(f_i : (X, t_X) \longrightarrow (Y_i, t_i)\Big)_I$ be an initial source. Consider the morphism $1_X : (X, t_X^{(f_i)_I}) \longrightarrow (X, t_X)$. Since by Theorem 4.5, for each i, the morphism $f_i : (X, t_X^{(f_i)}) \longrightarrow (Y_i, t_i)$ is $\mathbb{O}$-structural, by initiality of the source $(f_i)_I$, we get $1_X : (X, t_X^{(f_i)_I}) \longrightarrow (X, t_X)$ is $\mathbb{O}$-structural. It follows that $t_X \leq t_X^{(f_i)_I}$. On the other hand $f_i : (X, t_X) \longrightarrow (Y_i, t_i)$ is $\mathbb{O}$-structural for all i, so by Theorem 4.5, $t_X^{(f_i)_I} \leq t_X$. The result follows. □

Proposition 5.2. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on an* IM*-base $\mathbb{B}$. If the source*

$$\Big(f_i : (X, t_X) \longrightarrow (Y_i, t_i)\Big)_I$$

is in $\mathbb{OSS}$ with $t_X \cong t_X^{(f_i)_I}$, then it is an initial source.

Proof. Suppose $\Big(f_i : (X, t_X) \longrightarrow (Y_i, t_i)\Big)_I$ with $t_X \cong t_X^{(f_i)_I}$ is given in $\mathbb{OSS}$. Let (Z, t_Z) be an object in $\mathbb{OSS}$ and $g : Z \longrightarrow X$ be a morphism in $\mathcal{E}$ such that $f_i g : (Z, t_Z) \longrightarrow (Y_i, t_i)$ is $\mathbb{O}$-structural. We need to show $g : (Z, t_Z) \longrightarrow (X, t_X)$ is $\mathbb{O}$-structural. In the following diagram, the inner square is by Proposition 4.7 a pullback and since for each i, $f_i g : (Z, t_Z) \longrightarrow (Y_i, t_i)$ is $\mathbb{O}$-structural, there is a morphism $T_{f_i g}$ making the outer square commutative. So there is a unique morphism δ_i rendering commutative the two triangles.



Commutativity of the upper triangle shows that $f_i : (X, t_X^{(\!(g)\!)}) \longrightarrow (Y_i, t_i)$ is $\mathbb{O}$-structural. Since by Theorem 4.5, $t_X \cong t_X^{(f_i)_I}$ is the smallest $\mathbb{O}$-structure making f_i $\mathbb{O}$-structural for all i, we get $t_X \leq t_X^{(\!(g)\!)}$. So by Theorem 3.3 $g : (Z, t_Z) \longrightarrow (X, t_X)$ is $\mathbb{O}$-structural, as desired. □

Proposition 5.3. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on an* IM*-base $\mathbb{B}$. If the sink*

$$\Big(\!\Big(f_i : (X_i, t_i) \longrightarrow (Y, t_Y)\Big)\!\Big)_I$$

is a final sink in $\mathbb{OSS}$, then $t_Y \cong t_Y^{(\!(f_i)\!)_I}$.

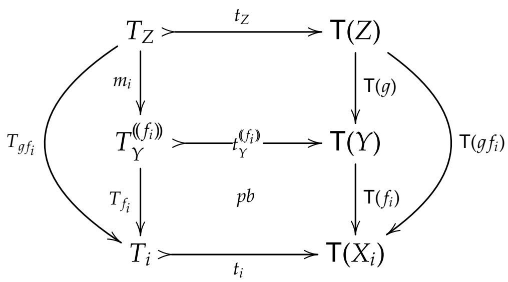
Proof. Let $\Big(\!\!\Big(f_i : (X_i, t_i) \longrightarrow (Y, t_Y)\Big)\!\!\Big)_I$ be a final sink in $\mathbb{OSS}$. Consider the morphism $1_Y : (Y, t_Y) \longrightarrow (Y, t_Y^{(\!(f_i)\!)_I})$. By Proposition 4.7, for each i, the morphism $f_i : (X_i, t_i) \longrightarrow (Y, t_Y^{(\!(f_i)\!)})$ is $\mathbb{O}$-structural. So by finality of the sink $(\!(f_i)\!)_I$, the morphism $1_Y : (Y, t_Y) \longrightarrow (Y, t_Y^{(\!(f_i)\!)_I})$ is $\mathbb{O}$-structural. It follows that $t_Y^{(\!(f_i)\!)_I} \leq t_Y$. On the other hand $f_i : (X_i, t_i) \longrightarrow (Y, t_Y)$ is $\mathbb{O}$-structural for all i, so by Theorem 4.9, $t_Y \leq t_Y^{(\!(f_i)\!)_I}$. The result follows. □

Proposition 5.4. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on an* IM*-base $\mathbb{B}$. If the sink*

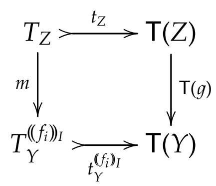
$$\Big(\!\!\Big(f_i : (X_i, t_i) \longrightarrow (Y, t_Y)\Big)\!\!\Big)_I$$

is in $\mathbb{OSS}$ with $t_Y \cong t_Y^{(\!(f_i)\!)_I}$, then it is a final sink.

Proof. Let (Z, t_Z) be an object in $\mathbb{OSS}$ and $g : Y \longrightarrow Z$ be a morphism in $\mathcal{E}$ such that $gf_i : (X_i, t_i) \longrightarrow (Z, t_Z)$ is $\mathbb{O}$-structural for all i. We need to show that $g : (Y, t_Y) \longrightarrow (Z, t_Z)$ is an $\mathbb{O}$-structural morphism. In the following diagram, the lower square is by Proposition 4.7 a pullback, the right triangle commutes by functoriality of T and since for each i, gf_i is $\mathbb{O}$-structural, there is a morphism T_{gf_i} making the outer square commutative. So there is a morphism m_i rendering commutative the left triangle and the upper square.



The commutativity of the upper square gives $\mathsf{T}(g)t_Z \leq t_Y^{(\!(f_i)\!)}$. Since this is so for all $i \in I$, by Proposition 2.2 we get $\mathsf{T}(g)t_Z \leq \bigwedge_{i\in I} t_Y^{(\!(f_i)\!)}$. By Theorem 4.9, $\bigwedge_{i\in I} t_Y^{(\!(f_i)\!)} = t_Y^{(\!(f_i)\!)_I}$ and therefore $\mathsf{T}(g)t_Z \leq t_Y^{(\!(f_i)\!)_I}$. So there is a morphism m such that $\mathsf{T}(g)t_Z = t_Y^{(\!(f_i)\!)_I} m$, i.e., the following square commutes.



Hence $g : (Y, t_Y^{(\!(f_i)\!)_I}) \longrightarrow (Z, t_Z)$ is $\mathbb{O}$-structural. Since $t_Y \cong t_Y^{(\!(f_i)\!)_I}$, by Corollary 3.4 $g : (Y, t_Y) \longrightarrow (Z, t_Z)$ is $\mathbb{O}$-structural and that concludes the proof. □

Recalling that the existence of unique U-initial lifts of U-structured sources implies the existence of unique U-final lifts of U-structured sinks, see [1], one can use either to show that $[\mathbb{OSS}]$ is a topological category. We give a proof based on initial sources.

Theorem 5.5. *Let $\mathbb{O}$ be a $\mathbb{B}$-operant on an* IM*-base $\mathbb{B}$. The concrete category* $([\mathbb{OSS}], U)$ *is topological.*

Proof. Let $\Big(f_i : X \longrightarrow Y_i = U(Y_i, [t_i])\Big)_I$ be a U-structured source in $\mathcal{E}$, where for each $i \in I$, $t_i : T_i \rightarrowtail \mathsf{T}(Y_i)$ is an $\mathbb{O}$-structure on Y_i. By Proposition 5.2, the source $\Big(f_i : (X, t_X^{(f_i)_I}) \longrightarrow (Y_i, t_i)\Big)_I$ is initial in $\mathbb{OSS}$. It is easy to verify that the source $\left(f_i : (X, [t_X^{(f_i)_I}]) \longrightarrow (Y_i, [t_i])\right)_I$ is initial in $[\mathbb{OSS}]$. To show that this U-initial lift is unique, let $\left(f_i' : (X, [t_X]) \longrightarrow (Y_i, [t_i])\right)_I$ be another U-initial lift in $[\mathbb{OSS}]$. It is easy to verify that $\left(f_i' : (X, t_X) \longrightarrow (Y_i, t_i)\right)_I$ is initial in $\mathbb{OSS}$. Faithfulness of U yields $f_i' = f_i$ and so $\left(f_i : (X, t_X) \longrightarrow (Y_i, t_i)\right)_I$ is initial in $\mathbb{OSS}$. Now by Proposition 5.1, we get $t_X \cong t_X^{(f_i)_I}$. Therefore $[t_X] = [t_X^{(f_i)_I}]$ and so the source $\left(f_i' : (X, [t_X]) \longrightarrow (Y_i, [t_i])\right)_I$ is equal to the source $\left(f_i : (X, [t_X^{(f_i)_I}]) \longrightarrow (Y_i, [t_i])\right)_I$ as desired. □

Corollary 5.6. *The category* $[\mathbb{OSS}]$ *is complete and cocomplete if the base category is. The limits and colimits are concrete.*

Proof. Since $[\mathbb{OSS}]$ is topological over $\mathcal{E}$, the result follows by [1], Theorem 21.16. □

6. Examples

By considering different bases $\mathbb{B}$ and taking various $\mathbb{B}$-operants $\mathbb{O}$, one will get several categories of $\mathbb{O}$-structural spaces. In this section we provide several examples of such categories, some of which are known categories and some not so familiar.

In some of the following examples we use the squaring functor $\mathsf{S} : \mathcal{C} \longrightarrow \mathcal{C}$ that takes a morphism $f : X \longrightarrow Y$ to $f \times f : X \times X \longrightarrow Y \times Y$.

Example 6.1. The operants that give raise to the category of topological spaces and the categories of certain fuzzy topological spaces (types one to five) are considered in [9], with a slightly different setting and notation. Here we only mention the operants that lead to the category of topological spaces and types one and four fuzzy topological spaces and refer the reader to [9] for the remaining categories of fuzzy topological spaces.

a) Let $\mathbb{B}$ be the base $Set^{op} \xrightarrow{\mathsf{T}} Set \underset{\mathsf{Q}}{\overset{\mathsf{P}}{\rightrightarrows}} Set$, where T and P are respectively, the contravariant and covariant powerset functors and Q is the squaring functor S, and let $\mathbb{O} = (\beta, \tau, \mu, \nu)$ be the operant relative to $\mathbb{B}$, called the topology operant, defined for each X by,

$$1 \xrightarrow{\beta_X} \mathcal{P}(X) \text{ , taking } 1 \longmapsto \emptyset$$

$$1 \xrightarrow{\tau_X} \mathcal{P}(X) \text{ , taking } 1 \longmapsto X$$

$$\mathcal{P}(\mathcal{P}(X)) \xrightarrow{\mu_X} \mathcal{P}(X) \text{ , taking } E \longmapsto \bigcup_{A \in E} A \text{ and}$$

$$\mathcal{P}(X) \times \mathcal{P}(X) \xrightarrow{\nu_X} \mathcal{P}(X) \text{ , taking } (A, B) \longmapsto A \cap B$$

It is not hard to verify that (X, t_X), where $t_X : T_X \hookrightarrow \mathcal{P}(X)$ is the set inclusion, is an $\mathbb{O}$-structural space if and only if T_X is a topology on X, so that the category $[\mathbb{OSS}]$ with respect to this topology operant is (isomorphic to) the category *Top* of topological spaces.

Since $\mathcal{C} = Set$ is strongly complete, the base $\mathbb{B}$ is an IM-base (actually an IFE-base) and so by Theorem 5.5, the category *Top* is topological over *Set*. Since $\mathcal{E} = Set$ is complete and cocomplete, by Corollary 5.6, the category *Top* is concretely complete and concretely cocomplete.

b) Let $\mathbb{B}$ be the base $Set^{op} \xrightarrow{\mathsf{T}} Set \underset{\mathsf{Q}}{\overset{\mathsf{P}}{\rightrightarrows}} Set$, where T takes $f : X \longrightarrow Y$ to the "composition with f" function $- \circ f : I^Y \longrightarrow I^X$ with I being the unit interval, P is the covariant powerset functor and Q is the squaring functor S and let $\mathbb{O} = (\beta, \tau, \mu, \nu)$ be the operant relative to $\mathbb{B}$, called the type I fuzzy topology operant, defined for each X by,

$$1 \xrightarrow{\beta_X} I^X \text{ , taking } 1 \longmapsto \underline{0}$$

$$1 \xrightarrow{\tau_X} I^X \text{ , taking } 1 \longmapsto \underline{1}$$

$$\mathcal{P}(I^X) \xrightarrow{\mu_X} I^X \text{ , taking } E \longmapsto \bigvee_{A\in E} A \text{ and}$$

$$I^X \times I^X \xrightarrow{\nu_X} I^X \text{ , taking } (A,B) \longmapsto A \wedge B$$

The functions $\underline{0}$ and $\underline{1}$ are the constant functions with values 0 and 1, respectively.

It is not hard to verify that (X, t_X), where $t_X : T_X \hookrightarrow I^X$ is the set inclusion, is an $\mathbb{O}$-structural space if and only if T_X is a type one fuzzy topology on X, so that the category $[\mathbb{OSS}]$ with respect to this Type I fuzzy topology operant is (isomorphic to) the category *I-FTop* of type I fuzzy topological spaces.

Since the categories $\mathcal{C}$ and $\mathcal{E}$ are as in part (*a*) and satisfy the mentioned conditions, the category *I-FTop* is topological over *Set* and it is concretely complete and concretely cocomplete.

c) Let $\mathbb{B}$ be the base $(I^U)^{op} \xrightarrow{\mathsf{T}} Set \underset{\mathsf{Q}}{\overset{\mathsf{P}}{\rightrightarrows}} Set$, where U is a set, I is the unit interval, the partially ordered set I^U is considered as a category, T takes X to the set F_X of all the fuzzy subsets of X and the morphism $X \leq Y$ to $- \wedge X$, P is the covariant powerset functor and Q is the squaring functor S, and let $\mathbb{O} = (\beta, \tau, \mu, \nu)$ be the operant relative to $\mathbb{B}$, called the type IV fuzzy topology operant, defined for each X by,

$$1 \xrightarrow{\beta_X} F_X \text{ , taking } 1 \longmapsto \underline{0}$$

$$1 \xrightarrow{\tau_X} F_X \text{ , taking } 1 \longmapsto X$$

$$\mathcal{P}(F_X) \xrightarrow{\mu_X} F_X \text{ , taking } E \longmapsto \bigvee_{A\in E} A \text{ and}$$

$$F_X \times F_X \xrightarrow{\nu_X} F_X \text{ , taking } (A,B) \longmapsto A \wedge B$$

The function $\underline{0}$ is the constant function with values 0.

It is not hard to verify that (X, t_X), where $t_X : T_X \hookrightarrow F_X$ is the set inclusion, is an $\mathbb{O}$-structural space if and only if T_X is a type four fuzzy topology on X, so that the category $[\mathbb{OSS}]$ with respect to this Type IV fuzzy topology operant is (isomorphic to) the category *IV-FTop* of type four fuzzy topological spaces.

As $\mathcal{C} = Set$ is strongly complete, it follows that the base $\mathbb{B}$ is an IM-base (or, more precisely, an IFE-base), and hence by Theorem 5.5, the category *IV-FTop* is topological over I^U. Since $\mathcal{E} = I^U$ is both complete and cocomplete, see [5], the category *IV-FTop* is, by Corollary 5.6, both concretely complete and concretely cocomplete.

Let us remark that replacing the unit interval I above by a complete lattice L, we get similar results for the category *LTop* of L-topological spaces, see [7].

Example 6.2. In this example, we propose an operant $\mathbb{O}$ such that the category of $\mathbb{O}$-structures corresponds to the category Top_p of topological pairs, see [18]. Let $\mathbb{B}$ be the base $Set_p^{op} \xrightarrow{\mathsf{T}} Set \underset{\mathsf{Q}}{\overset{\mathsf{P}}{\rightrightarrows}} Set$, where Set_p is the category of set pairs, see [6], T is the functor taking a pair (X, A) to the powerset $\mathcal{P}(X)$ and a morphism $f : (X, A) \longrightarrow (Y, B)$ to the function $\mathsf{T}(f) = f^{-1}(-) : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$, P is the covariant powerset functor, and Q is the squaring functor S. Let $\mathbb{O} = (\beta, \tau, \mu, \nu)$ be the operant relative to $\mathbb{B}$ defined for each (X, A) as follows,

$$1 \xrightarrow{\beta_{(X,A)}} \mathcal{P}(X) \text{ , taking } 1 \longmapsto \emptyset$$

$$1 \xrightarrow{\tau_{(X,A)}} \mathcal{P}(X) \text{ , taking } 1 \longmapsto X$$

$$\mathcal{P}(\mathcal{P}(X)) \xrightarrow{\mu_{(X,A)}} \mathcal{P}(X) \text{ , taking } M \longmapsto \bigcup_{E\in M} E \quad \text{and}$$

$$\mathcal{P}(X) \times \mathcal{P}(X) \xrightarrow{\nu_{(X,A)}} \mathcal{P}(X) \text{ , taking } (M,N) \longmapsto M \cap N$$

Obviously $\mathbb{O}$ is an operant, which we call the topology pair operant. It can be easily verified that $t_{(X,A)} : E \hookrightarrow \mathcal{P}(X)$ is an $\mathbb{O}$-structure on (X, A), i.e., $((X, A), t_{(X,A)})$ is an $\mathbb{O}$-structural space, with respect to the topology pair operant if and only if E is a topology on X. Also $f : ((X, A), t_{(X,A)}) \longrightarrow ((X', A'), t_{(X',A')})$, where $t_{(X,A)} : E \hookrightarrow \mathcal{P}(X)$ and $t_{(X',A')} : E' \hookrightarrow \mathcal{P}(X')$ are set inclusions, is seen to be $\mathbb{O}$-structural if and only if $f : (X, E) \longrightarrow (X', E')$ is continuous.

It follows that the objects of $[\mathbb{OSS}]$ correspond to (X, A, E), with A a subset of X and E a topology on X and the morphisms $f : ((X, A), E) \longrightarrow ((Y, B), F)$ correspond to continuous functions $f : (X, E) \longrightarrow (Y, F)$ with $f(A) \subseteq B$. By putting the subspace topologies on A and B, one can see that the category $[\mathbb{OSS}]$ is (isomorphic to) the category Top_p of topological pairs.

Since $\mathcal{C} = Set$ is strongly complete, the base $\mathbb{B}$ is an IM-base (actually an IFE-base) and so by Theorem 5.5, Top_p is topological over Set_p. Since $\mathcal{E} = Set_p$ is both complete and cocomplete, see [6], by Corollary 5.6, Top_p is both concretely complete and concretely cocomplete.

Example 6.3. In this example we define an operant $\mathbb{O}$ such that the category of $\mathbb{O}$-structures coincides with the category *TopMLat* of topological molecular lattices and generalized order-homomorphisms, as discussed in [23]. Let $\mathbb{B}$ be the base $MLat^{op} \xrightarrow{\mathsf{T}} Set \overset{\mathsf{P}}{\underset{\mathsf{Q}}{\rightrightarrows}} Set$, where *MLat* is the category of molecular lattices and generalized-order homomorphisms, see [23], T is the functor taking a molecular lattice $(L, \leq)$ to its underlying set L and a generalized-order homomorphism $f : (L, \leq) \longrightarrow (L', \leq')$ to the function $\mathsf{T}(f) = f^{-1}(-) : L' \longrightarrow L$ $(f^{-1}(b) = \bigvee_{f(a) \leq b} a)$, P is the covariant powerset functor, and Q is the squaring functor S, see [24]. Let $\mathbb{O} = (\beta, \tau, \mu, \nu)$ be the operant relative to $\mathbb{B}$ defined for each $(L, \leq)$ as follows,

$$1 \xrightarrow{\beta_{(L,\leq)}} L \text{ , taking } 1 \longmapsto 0$$
$$1 \xrightarrow{\tau_{(L,\leq)}} L \text{ , taking } 1 \longmapsto 1$$
$$\mathcal{P}(L) \xrightarrow{\mu_{(L,\leq)}} L \text{ , taking } A \longmapsto \bigwedge_{a \in A} a \quad \text{and}$$
$$L \times L \xrightarrow{\nu_{(L,\leq)}} L \text{ , taking } (a, b) \longmapsto a \vee b$$

The naturality of β, τ, μ and ν follows from the fact that for a generalized order-homomorphism f, one has $f^{-1}(-)$ preserves 0, 1, meet and join, respectively. So $\mathbb{O}$ is indeed an operant relative to the above given base $\mathbb{B}$, which we call molecular lattice cotopology operant.

It can be routinely verified that $t_{(L,\leq)} : E \hookrightarrow L$ is an $\mathbb{O}$-structure on $(L, \leq)$ with respect to the molecular lattice cotopology operant if and only if E is a molecular lattice cotopology on $(L, \leq)$, so that $((L, \leq), E)$ is a molecular lattice cotopological space (or a topological molecular lattice in the sense of [23]). Also it is easy to show that $f : (L, t_{(L,\leq)}) \longrightarrow (L', t_{(L',\leq')})$, where $t_{(L,\leq)} : E \hookrightarrow L$ and $t_{(L',\leq')} : E' \hookrightarrow L'$ are set inclusions, is an $\mathbb{O}$-structural morphism if and only if the morphism $f : ((L, \leq), E) \longrightarrow ((L', \leq'), E')$ is continuous. It follows that the category *TopMLat* is (isomorphic to) the category $[\mathbb{OSS}]$ for the molecular lattice cotopology operant $\mathbb{O}$.

Given that $\mathcal{C} = Set$ is strongly complete, the base $\mathbb{B}$ qualifies as an IM-base (actually an IFE-base). Consequently, by Theorem 5.5, *TopMLat* is topological over *MLat*. Moreover, since $\mathcal{E} = MLat$ is both complete and cocomplete, see [23], by Corollary 5.6, the category *TopMLat* of topological molecular lattices is concretely complete and concretely cocomplete.

Example 6.4. In this example, we introduce an operant $\mathbb{O}$ such that the category of $\mathbb{O}$-structures coincides with the category *Meas* of measurable spaces and measurable functions, see [4]. Let $\mathbb{B}$ be the base $Set^{op} \xrightarrow{\mathsf{T}} Set \overset{\mathsf{P}}{\underset{\mathsf{Q}}{\rightrightarrows}} Set$, where T is the functor taking a set X to its powerset $\mathcal{P}(X)$ and a function f to

$\mathcal{P}(f) = f^{-1}(-)$, P is the functor taking a set X to the set of sequences $X^{\mathbb{N}}$ on X, and a function f to $\mathsf{P}(f) = f(-)$, and $\mathsf{Q} = 1_{Set}$ is the identity functor. Let $\mathbb{O} = (\beta, \tau, \mu, \nu)$ be the operant relative to $\mathbb{B}$ defined for each X as follows,

$$1 \xrightarrow{\beta_X} \mathcal{P}(X) \text{ , taking } 1 \longmapsto \emptyset$$
$$1 \xrightarrow{\tau_X} \mathcal{P}(X) \text{ , taking } 1 \longmapsto X$$
$$\mathcal{P}(X)^{\mathbb{N}} \xrightarrow{\mu_X} \mathcal{P}(X) \text{ , taking } (E_i)_{i=1}^{\infty} \longmapsto \bigcup_{i=1}^{\infty} E_i \text{ and}$$
$$\mathcal{P}(X) \xrightarrow{\nu_X} \mathcal{P}(X) \text{ , taking } E \longmapsto X \setminus E$$

where $X \setminus E$ is the complement of E in X. It is not hard to verify that $\mathbb{O}$ is indeed an operant on $\mathbb{B}$, which we call the σ-algebra operant.

It can be easily shown that $t_X : M \hookrightarrow \mathcal{P}(X)$ is an $\mathbb{O}$-structure on X with respect to the σ-algebra operant if and only if M is a σ-algebra on X, so that (X, M) is a measurable space. Also one can show that a morphism $f : (X, t_X) \longrightarrow (Y, t_Y)$, where $t_X : M \hookrightarrow \mathcal{P}(X)$ and $t_Y : N \hookrightarrow \mathcal{P}(Y)$ are inclusions, is $\mathbb{O}$-structural if and only if $f : X \longrightarrow Y$ is a measurable function. It follows that the category *Meas* is (isomorphic to) the category $[\mathbb{OSS}]$ for the σ-algebra operant $\mathbb{O}$.

Again the categories $\mathcal{C}$ and $\mathcal{E}$ are both the category *Set*, thus the category *Meas* of measurable spaces is topological over *Set* and is both concretely complete and concretely cocomplete.

Example 6.5. Here, we define an operant $\mathbb{O}$ such that the category of $\mathbb{O}$-structures is the category $RMod_p$ of left R-module pairs over a commutative ring R. Let $\mathbb{B}$ be the base $RMod \xrightarrow{\mathsf{T}} RMod \overset{\mathsf{P}}{\underset{\mathsf{Q}}{\rightrightarrows}} RMod$, where $\mathsf{T} = 1_{RMod}$ is the identity functor, $\mathsf{P} = \mathsf{S}$ is the squaring functor, and $\mathsf{Q} = R \times -$ is the functor taking a module M to $R \times M$ and a morphism f to $1_R \times f$. Let $\mathbb{O} = (\beta, \tau, \mu, \nu)$ be the operant relative to $\mathbb{B}$ defined for each module M as follows,

$$1 \xrightarrow{\beta_M} M \text{ , taking } 1 \longmapsto 0$$
$$1 \xrightarrow{\tau_M} M \text{ , taking } 1 \longmapsto 0$$
$$M \times M \xrightarrow{\mu_M} M \text{ , taking } (x, y) \longmapsto x + y \text{ and}$$
$$R \times M \xrightarrow{\nu_M} M \text{ , taking } (r, x) \longmapsto r \cdot x$$

where $+$ and $\cdot$ are the module operations. Note that τ is taken to be equal to β. It is not hard to verify that $\mathbb{O}$ is indeed an operant on $\mathbb{B}$, which we call the module pair operant.

It is straightforward to demonstrate that $t_M : N \hookrightarrow M$ is an $\mathbb{O}$-structure on M with respect to the module pair operant if and only if N is a submodule of M, so that (M, N) is an R-module pair.

Additionally, it can be proven that $f : (M, t_M) \longrightarrow (M', t_{M'})$ with R-mod inclusions $t_M : N \hookrightarrow M$ and $t_{M'} : N' \hookrightarrow M'$ is $\mathbb{O}$-structural if and only if the R-module homomorphism $f : M \longrightarrow M'$ restricts to an R-module homomorphism $f' : N \longrightarrow N'$. This implies that the category $RMod_p$ is (isomorphic to) the category $[\mathbb{OSS}]$ for the module pair operant $\mathbb{O}$.

Because $\mathcal{C} = RMod$ is strongly complete, as indicated in [16], the base $\mathbb{B}$ can be identified as an IM-base (and specifically an IFE-base). Thus, $RMod_p$ is topological over $RMod^{op}$ by Theorem 5.5. Since $\mathcal{E} = RMod^{op}$ is both complete and cocomplete, see [16], $RMod_p$ is concretely complete and cocomplete, according to Corollary 5.6.

Example 6.6. As seen in this example, we introduce an operant $\mathbb{O}$ such that the category of $\mathbb{O}$-structures coincides with the category $\underline{Top}(\mathcal{E})$ of internal topological space objects in an elementary topos $\mathcal{E}$, see [20] and [21]. Let $\mathbb{B}$ be the base $\mathcal{E}^{op} \xrightarrow{\mathsf{T}} \mathcal{E} \overset{\mathsf{P}}{\underset{\mathsf{Q}}{\rightrightarrows}} \mathcal{E}$, where T is the internal contravariant power object functor, P is $\mathcal{E} \xrightarrow{\exists} \mathcal{E}$, the internal existential quantifier functor, taking an object X to PX, the power object of X, and a morphism $X \xrightarrow{f} Y$ to the $\exists_f : PX \longrightarrow PY$, see [15], and Q is the squaring functor S, and let $\mathbb{O} = (\beta, \tau, \mu, \nu)$, which we call the internal topology object operant, be defined as follows for each $X \in \mathcal{E}$.

$$1 \xrightarrow{\beta_X} PX \text{ , be the } 1 \xrightarrow{\ulcorner false_X \urcorner} PX$$
$$1 \xrightarrow{\tau_X} PX \text{ , be the } 1 \xrightarrow{\ulcorner true_X \urcorner} PX$$
$$P^2X \xrightarrow{\mu_X} PX \text{ , be the } P^2X \xrightarrow{\underline{\bigcup}} PX \text{ , the internal arbitrary union and}$$
$$PX \times PX \xrightarrow{\nu_X} PX \text{ , be the } PX \times PX \xrightarrow{\wedge_X} PX \text{ , the internal binary meet}$$

In which $\ulcorner false_X \urcorner$ and $\ulcorner true_X \urcorner$ are the P-transposes of $X \times 1 \xrightarrow{t!_{X\times 1}} \Omega$ and $X \times 1 \xrightarrow{f!_{X\times 1}} \Omega$, or briefly $\beta_X := \widetilde{f!_{X\times 1}}$ and $\tau_X := \widetilde{t!_{X\times 1}}$ respectively, and $\underline{\bigcup}$ is the P-transpose of the characteristic map of the subobject

$$\exists_{pr_2}(X \times \varepsilon_{PX}) \wedge (\varepsilon_X \times P^2X) \overset{m}{\rightarrowtail} X \times P^2X$$

In which $\varepsilon_X \overset{e_X}{\rightarrowtail} X \times PX$ is the subobject that corresponds to the evaluation map $X \times PX \xrightarrow{ev_X} \Omega$, the projection is $X \times PX \times P^2X \xrightarrow{pr_2} X \times P^2X$ and $m = \exists_{pr_2}\big[(1_X \times e_{PX}) \wedge (e_X \times 1_{P^2X})\big]$. Also $\wedge_X$ is the P-transpose of the characteristic map of the subobject,

$$\varepsilon_X^{\pi_1} \wedge \varepsilon_X^{\pi_2} \overset{e_{\pi_1} \wedge e_{\pi_2}}{\rightarrowtail} X \times PX \times PX$$

where $\varepsilon_X^{\pi_1} \overset{e_{\pi_1}}{\rightarrowtail} X \times PX \times PX$ and $\varepsilon_X^{\pi_2} \overset{e_{\pi_2}}{\rightarrowtail} X \times PX \times PX$ are the subobjects corresponding to the P-transpose of the projection morphisms $PX \times PX \xrightarrow{\pi_i} PX$, for $i = 1, 2$. For details see chapter 1 of [19] and see section 4.16, on page 324 of [17]. One can show that the quadruple $\mathbb{O} = (\beta, \tau, \mu, \nu)$ is a $\mathbb{B}$-operant. It can be verified that (X, k_X), where $T_X \overset{k_X}{\rightarrowtail} PX$ is a monomorphism, is $\mathbb{O}$-structural with respect to the internal topology object operant if and only if (X, T_X) is an internal topological space object in $\mathcal{E}$ in the sense of [19]. Next one can prove that a morphism $f : (X, k_X) \longrightarrow (Y, k_Y)$ is $\mathbb{O}$-structural if and only if the morphism $f : X \longrightarrow Y$ between two internal topological space objects (X, T_X) and (Y, T_Y) is continuous in the sense of [19]. It can be observed that the category $\underline{Top}(\mathcal{E})$ is (isomorphic to) the category $\mathbb{OSS}$ for the internal topology object operant $\mathbb{O}$.

If topos $\mathcal{E}$ has arbitrary intersection, then the base $\mathbb{B}$ is an IM-base (actually an IFE-base) and so by Theorem 5.5, $\underline{Top}(\mathcal{E})$ is topological over $\mathcal{E}$. Since every topos is both finitely complete and finitely cocomplete, see [15], by Corollary 5.6, $\underline{Top}(\mathcal{E})$ is both concretely finitely complete and concretely finitely cocomplete.

Example 6.7. In this example, we present an operant $\mathbb{O}$ such that the category of $\mathbb{O}$-structures corresponds exactly to the category $Top_o(\mathcal{E})$ of open topological objects in an elementary topos $\mathcal{E}$, see [14] and [11]. Let $\mathbb{B}$ be the base $\mathcal{E}^{op} \xrightarrow{\mathsf{T}} \mathcal{E} \overset{\mathsf{P}}{\underset{\mathsf{Q}}{\rightrightarrows}} \mathcal{E}$, where T and P are respectively, the internal contravariant power object and the internal existential quantifier functors and Q is the squaring functor S, and let $\mathbb{O} = (\beta, \tau, \mu, \nu)$, which we call the internal open topology object operant, be defined as follows for each $X \in \mathcal{E}$.

$$1 \xrightarrow{\beta_X} PX \text{ , be the } 1 \xrightarrow{\ulcorner true_X \urcorner} PX$$
$$1 \xrightarrow{\tau_X} PX \text{ , be the } 1 \xrightarrow{\ulcorner true_X \urcorner} PX$$
$$P^2X \xrightarrow{\mu_X} PX \text{ , be the } \exists PX \xrightarrow{\bigcup} PX \text{ , the internal union and}$$
$$PX \times PX \xrightarrow{\nu_X} PX \text{ , be the } PX \times PX \xrightarrow{\wedge_X} PX \text{ , the internal binary meet}$$

In which $\ulcorner true_X \urcorner$ and $\wedge_X$ are the same as in Example 6.6 and $\bigcup$ is the internal union in the sense of [14]. Note that we have considered β equal to τ. By Corollary II.1.4, on page 20 of [14], a simple proof establishes that the quadruple $\mathbb{O} = (\beta, \tau, \mu, \nu)$ is a $\mathbb{B}$-operant. Also it is not hard to verify that (X, k_X), where $T_X \rightarrowtail^{k_X} PX$ is a monomorphism, is an $\mathbb{O}$-structural space with respect to the open topology object operant if and only if (X, T_X) is an open topological object in $\mathcal{E}$ in the sense of [14] and a morphism $f : (X, k_X) \longrightarrow (Y, k_Y)$ is $\mathbb{O}$-structural if and only if the morphism $f : X \longrightarrow Y$ between two open topological objects (X, T_X) and (Y, T_Y) is continuous in the sense of [14]. We conclude that the category $Top_o(\mathcal{E})$ is (isomorphic to) the category $\mathbb{OSS}$ for the internal open topology object operant $\mathbb{O}$.

Moreover, if the topos $\mathcal{E}$ admits arbitrary intersections, then the base $\mathbb{B}$ qualifies as an IM-base (more precisely IFE-base) and consequently, by Theorem 5.5, $Top_o(\mathcal{E})$ is topological over $\mathcal{E}$. Since every topos is both finitely complete and finitely cocomplete, see [15], it follows from Corollary 5.6 that $Top_o(\mathcal{E})$ is both concretely finitely complete and concretely finitely cocomplete.

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