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Certain properties and characteristics of extended multivariate Hermite-Frobenius Euler polynomials of 1-parameter

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Abstract. This paper investigates a new class of the extended multivariate Hermite-Frobenius Euler polynomials of one-parameter (*EMHFEP* of 1-parameter), focusing on their structural and analytical features. Using series representations and generating functions, we establish key properties and an associated operational framework and determinant form. Further, various differential, integrodifferential, and partial differential equations satisfied by these polynomials are presented. Several summation formulas are introduced, enhancing their computational applicability. Additionally, Euler's integral is utilized to derive its generalized form.

1. Introduction

Multivariate polynomials play a vital role in mathematical analysis, serving as a foundation for creating new polynomial families and deriving essential identities. Leveraging the properties of an iterated isomorphism associated with Laguerre-type exponentials, Bretti et al. [1] proposed "extended families of two-variable Appell polynomials. Subsequent studies have explored extensions of Hermite, Laguerre, and truncated exponential polynomials in two variables", as well as their broader generalizations [2–6].

Recent advancements in special functions have significantly expanded their role in mathematical physics, providing an effective analytical approach to various challenges. Previous studies [7, 8] have emphasized the relevance of generalized Hermite polynomials, particularly in "quantum mechanics, optical beam dynamics, partial differential equations, and abstract algebraic structures".

The generating technique has been employed by researchers as a systematic approach to their investigations. Building upon the aforementioned studies, the recent development of multi-variate Hermite polynomials $\mathcal{H}_n^{[m]}(\psi_1, \psi_2, \psi_3, \cdots, \psi_m)$ has made significant contributions to the field of polynomials. These

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polynomials were constructed using a generating relation, a powerful tool for systematically examining and analyzing mathematical functions.

The ongoing research and recent advancements in multivariate Hermite polynomials, facilitated by the generating technique, have significant implications for the polynomial domain. These polynomials establish a solid foundation for tackling intricate multivariate systems, and their systematic analysis paves the way for fresh insights, deeper comprehension, and diverse applications across various scientific and mathematical disciplines. They are characterized by the following generating relation:

$$\exp(\psi_1\tau + \psi_2\tau^2 + \cdots + \psi_m\tau^m) = \sum_{n=0}^{\infty} \mathcal{H}_n^{[m]}(\psi_1, \psi_2, \cdots, \psi_m)\frac{\tau^n}{n!}, \tag{1}$$

with the series representation:

$$\mathcal{H}_n^{[m]}(\psi_1, \psi_2, \cdots, \psi_m) = n! \sum_{r=0}^{[n/m]} \frac{\psi_m^r \mathcal{H}_{n-mr}^{[m]}(\psi_1, \psi_2, \cdots, \psi_{m-1})}{r!\,(n-mr)!}. \tag{2}$$

For $m = 3$ and $m = 2$, the multivariate Hermite polynomials reduce to 3-variable and 2-variable Hermite polynomials [9] given by generating relating relations:

$$\exp(\psi_1\tau + \psi_2\tau^2 + \psi_3\tau^3) = \sum_{n=0}^{\infty} \mathcal{H}_n^{[3]}(\psi_1, \psi_2, \psi_3)\frac{\tau^n}{n!}, \tag{3}$$

$$\exp(\psi_1\tau + \psi_2\tau^2) = \sum_{n=0}^{\infty} \mathcal{H}_n(\psi_1, \psi_2)\frac{\tau^n}{n!}, \tag{4}$$

and the series representation:

$$\mathcal{H}_n^{[3]}(\psi_1, \psi_2, \psi_3) = n! \sum_{r=0}^{[n/3]} \frac{\psi_3^r \mathcal{H}_{n-3r}^{[3]}(\psi_1, \psi_2)}{r!\,(n-3r)!}, \tag{5}$$

$$\mathcal{H}_n(\psi_1, \psi_2) = n! \sum_{r=0}^{[n/2]} \frac{\psi_2^r \mathcal{H}_{n-2r}(\psi_2)}{r!\,(n-2r)!}. \tag{6}$$

The "Hermite polynomials with two variables and three variables and a single parameter" (2V1PHP), represented as $\mathcal{H}_n(\psi_1, \psi_2, C)$, are characterised by the following generating function [10, 11]:

$$C^{\psi_1\tau+\psi_2\tau^2} = \sum_{n=0}^{\infty} \mathcal{H}_n(\psi_1, \psi_2, C)\frac{\tau^n}{n!} \quad C > 1 \tag{7}$$

and

$$C^{\psi_1\tau+\psi_2\tau^2++\psi_3\tau^3} = \sum_{n=0}^{\infty} \mathcal{H}_n(\psi_1, \psi_2, +\psi_3, C)\frac{\tau^n}{n!} \quad C > 1, \tag{8}$$

respectively. Using $C^{\psi_1\tau+\psi_2\tau^2}$ instead of $e^{\psi_1\tau+\psi_2\tau^2}$ offers flexibility via the base C. This parameter enables greater adaptability, allowing the function's behaviour to multiline with specific requirements. One can control the growth rate or better approximate empirical data by selecting an appropriate value for C.

In practical applications, the natural exponential function may lead to excessively rapid growth, which is sometimes undesirable. Choosing a base $C < e$ enables a slower rate of increase, making it more suitable for modelling scenarios where exponential expansion is too steep. Conversely, setting $C > e$ allows for a steeper curve, accommodating cases that require accelerated growth or decay. Moreover, depending on C and computational considerations, evaluating $C^{\psi_1\tau+\psi_2\tau^2}$ can offer improved numerical stability and computational efficiency compared to $e^{\psi_1\tau+\psi_2\tau^2}$, particularly in large-scale computations or high-performance computing applications.

In their multivariate formulation, these polynomials have become indispensable across various disciplines within pure and applied mathematics and physics. Their versatility allows them to be applied to complex problems in diverse fields as a foundational tool in various mathematical models and physical theories.

For instance, in mathematical physics, these polynomials play a crucial role in solving Laplace's equation when expressed in parabolic coordinates. This equation, fundamental in studying potential theory, electrostatics, and fluid dynamics, often requires special polynomials to simplify and solve problems involving boundary conditions or specific geometries. Moreover, in quantum mechanics, these polynomials are instrumental in addressing scenarios where wave functions or quantum states need to be described in parabolic coordinates, particularly in systems with cylindrical or parabolic symmetry. Their application extends to solving the Schrödinger equation in such contexts, providing exact solutions that describe the behaviour of quantum particles under specific potential fields.

In probability theory, these polynomials are equally important. They are used to model distributions and stochastic processes, particularly when underlying processes have symmetries or constraints that can be described using parabolic coordinates. This includes applications in financial mathematics, where they help analyse random walks or diffusion processes.

One of the most notable features of these polynomials is their ability to provide specific solutions to the heat equation or generalized heat problems for any integral value of n. The heat equation, which describes the distribution of heat (or temperature variation) in a given region over time, is a critical partial differential equation in theoretical and applied contexts. The corresponding Gauss-Weierstrass transforms, integral transforms used to smooth or regularize functions, facilitate these solutions by linking the polynomials to the broader context of heat diffusion and propagation.

The Frobenius–Euler polynomials play a crucial role in modern mathematics, especially in combinatorics, number theory, and differential equations. These polynomials are vital tools for counting combinatorial structures and provide a strong foundation for exploring numerical sequences and solving complex differential problems. Their versatility enables the derivation of profound insights across a spectrum of mathematical domains, with far-reaching implications extending to various applications in the physical sciences and engineering.

The foundational contributions were by Frobenius [6, 10, 12] were instrumental in establishing the algebraic structure and interrelations of these polynomials. This early work significantly broadened the polynomials' utility and set the stage for their widespread adoption in both pure and applied mathematics. As a result, the study of Frobenius–Euler polynomials has evolved into a vibrant and continually expanding field, inspiring innovative methodologies for tackling complex mathematical and engineering challenges.

Formally, the Frobenius–Euler polynomials are defined via the generating function [13–17]:

$$\frac{1-\mathfrak{U}}{e^{\tau}-\mathfrak{U}}e^{\psi_1\tau} = \sum_{n=0}^{\infty} \mathcal{F}_n(\psi_1|\mathfrak{U})\frac{\tau^n}{n!}, \quad \forall \quad u \in \mathbb{C}, \quad |\tau| < |\log(\mathfrak{U})|, \quad \mathfrak{U} \neq 1. \tag{9}$$

These polynomials, particularly when expressed as $\mathcal{F}_n(0|\mathfrak{U})$, known as Frobenius-Euler numbers, are vital in combinatorics for enumerating complex structures and in number theory for analyzing numerical sequences. They also play a key role in solving differential equations, providing insights and methods for tackling mathematical challenges.

The Frobenius-Euler numbers are also governed by the generating relation:

$$\sum_{n=0}^{\infty} \mathcal{F}_n(0|\mathfrak{U})\frac{\tau^n}{n!} = \frac{1-\mathfrak{U}}{e^{\tau}-\mathfrak{U}}, \quad \forall \quad \mathfrak{U} \in \mathbb{C}, \quad \mathfrak{U} \neq 1. \tag{10}$$

Additionally, the Frobenius-Euler polynomials, $\mathcal{F}_n(x|\mathfrak{U})$, have a well-established series representation, which reveals their algebraic and analytical properties. This series is crucial for understanding their behaviour in various contexts and can be written as:

$$\sum_{k=0}^{n}\binom{n}{k}\mathcal{F}_k(\mathfrak{U})\psi_1^{n-k}=\mathcal{F}_n(\psi_1|\mathfrak{U}),\qquad n\geq 0, \tag{11}$$

where $\mathcal{F}_k(\mathfrak{U})$ represents the Frobenius-Euler numbers, and $\binom{n}{k}$ is the binomial coefficient. This representation is central to combinatorics, recurrence relations, and connections to other special functions. It also serves as a powerful tool for generating new identities and exploring the deeper structure of these polynomials in both discrete and continuous settings. This series formulation connects Frobenius-Euler polynomials to fields such as approximation theory, functional analysis, and differential operator theory, enhancing their relevance in solving complex mathematical problems.

The Frobenius-Euler numbers, $\mathcal{F}_n(\mathfrak{U})$, satisfy a recurrence relation, which is crucial for understanding their structural properties:

$$\mathcal{F}_0(\mathfrak{U})=1, \tag{12}$$

$$(\mathcal{F}(\mathfrak{U})+1)^n-\mathcal{F}_n(\mathfrak{U})=\begin{cases}1-\mathfrak{U}, & n=0,\\ 0, & n\geq 1.\end{cases} \tag{13}$$

The recurrence relation under consideration is a fundamental mechanism for constructing higher-order Frobenius–Euler numbers. It is important in combinatorial analysis and algebraic computations, providing a systematic framework for deriving new results and identities.

Differential equations constitute a cornerstone of numerous scientific domains, including physics, engineering, and pure and applied mathematics. A wide array of phenomena encountered in these disciplines can be effectively modelled through differential equations, which are often tackled using specialized classes of functions. Over the past thirty years, there has been a notable resurgence of interest in the theory of differential equations, largely spurred by rapid developments in nonlinear analysis, the theory of dynamical systems, and their expanding range of real-world applications, see for example [18–25, 28, 29].

In recent literature, extensive research has investigated hybrid families of special polynomials, particularly those involving multiple variables. These studies employ various techniques, including generating function methods, operational calculus, and analytic approaches [25–27]. The most salient features of such polynomial families are their recurrence relations, explicit representations, functional and differential equations, summation identities, symmetry properties, convolution formulas, and connections to determinant theory. These attributes collectively enhance the utility of hybrid special polynomials, making them indispensable tools in several mathematical and applied fields, including number theory, combinatorics, classical and numerical analysis, theoretical physics, and approximation theory. These polynomials' inherent richness and versatility enable researchers to address emerging challenges across a broad spectrum of scientific inquiries. The article is structured as follows:

Section 2 provides an overview of the *EMHFEP* of 1-parameter, using series representations and generating functions. It also delves into the operational formalism that arises from these polynomials. In Section 3, the recurrence relation, shift operators and families of differential equations are derived for the*EMHFEP* of 1-parameter. Section 4 introduces various summation formulas for *EMHFEP* of 1-parameter. The final section offers concluding remarks.

2. *EMHFEP* of 1-parameter

This section presents a novel hybrid family referred to as the *EMHFEP* of 1-parameter. Furthermore, several essential properties of these polynomials are established. A significant result is demonstrated to derive the generating function for the *EMHFEP* of 1-parameter, as outlined below:

Theorem 2.1. *For the EMHFEP of* 1*-parameter* ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})$ *,the succeeding generating relation is demonstrated:*

$$\frac{1-\mathfrak{U}}{e^{\tau}-\mathfrak{U}}\,C^{\psi_1\tau+\psi_2\tau^2+\psi_3\tau^3+\cdots+\psi_m\tau^m}=\sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\frac{\tau^n}{n!},\ |\tau|<|\log(\mathfrak{U})|,\ \ \mathfrak{U}\neq 1,\ \ C>1,\tag{14}$$

or, equivalently

$$\frac{1-\mathfrak{U}}{e^{\tau}-\mathfrak{U}}\,e^{\ln C(\psi_1\tau+\psi_2\tau^2+\psi_3\tau^3+\cdots+\psi_m\tau^m)}=\sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\frac{\tau^n}{n!},\ |\tau|<|\log(\mathfrak{U})|,\ \mathfrak{U}\neq 1,\ C>1.\tag{15}$$

Proof. By replacing the exponents of τ, specifically $\psi_1^0,\psi_1^1,\psi_1^2,\ldots,\psi_1^n$, in the series expansion of $e^{\psi_1\tau}$ with the polynomials ${}_{\mathcal{H}}\mathcal{F}_0(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}),{}_{\mathcal{H}}\mathcal{F}_2(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}),\ldots,{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})$ on the left-hand side, and substituting ψ_1 with ${}_{\mathcal{H}}\mathcal{F}_1(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})$ on the right-hand side of equation (1), we can then sum the terms from the left-hand side of the resulting expression to obtain:

$$\frac{1-\mathfrak{U}}{e^{\tau}-\mathfrak{U}}\sum_{n=0}^{\infty}\mathcal{H}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\frac{\tau^n}{n!}=\sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\frac{\tau^n}{n!},\tag{16}$$

which denotes the resulting *EMHFEP* of 1-parameter on the right-hand side, specifically as ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}):={}_{\mathcal{H}}\mathcal{F}_n\{\mathcal{D}_1(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\}$, leading to (14). The generating function in equation (15) is derived by simplifying the left-hand side of equation (14). □

Remark 2.2. *For* $\psi_4=\psi_5=\cdots=\psi_m=0$ *or* $m=3$ *the EMHFEP of* 1*-parameter* ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})$ *reduces to the* 3*-variable HFEP of* 1*-parameter* ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3;C|\mathfrak{U})$ *possessing generating relation [28]:*

$$\frac{1-\mathfrak{U}}{e^{\tau}-\mathfrak{U}}\,C^{\psi_1\tau+\psi_2\tau^2+\psi_3\tau^3}=\sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3;C|\mathfrak{U})\frac{\tau^n}{n!},\ |\tau|<|\log(\mathfrak{U})|,\ \ \mathfrak{U}\neq 1,\ \ C>1,\tag{17}$$

or, equivalently

$$\frac{1-\mathfrak{U}}{e^{\tau}-\mathfrak{U}}\,e^{\ln C(\psi_1\tau+\psi_2\tau^2++\psi_3\tau^3)}=\sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3;C|\mathfrak{U})\frac{\tau^n}{n!},\ |\tau|<|\log(\mathfrak{U})|,\ \ \mathfrak{U}\neq 1,\ \ C>1.\tag{18}$$

Remark 2.3. *For* $\psi_3=\psi_4=\cdots=\psi_m=0$ *or* $m=2$*, the EMHFEP of* 1*-parameter* ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})$ *reduces to the 2-variable HFEP of* 1*-parameter* ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2;C|\mathfrak{U})$ *possessing generating relation [29]:*

$$\frac{1-\mathfrak{U}}{e^{\tau}-\mathfrak{U}}\,C^{\psi_1\tau+\psi_2\tau^2}=\sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2;C|\mathfrak{U})\frac{\tau^n}{n!},\ |\tau|<|\log(\mathfrak{U})|,\ \ \mathfrak{U}\neq 1,\ \ C>1,\tag{19}$$

or, equivalently

$$\frac{1-\mathfrak{U}}{e^{\tau}-\mathfrak{U}}\,e^{\ln C(\psi_1\tau+\psi_2\tau^2)}=\sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2;C|\mathfrak{U})\frac{\tau^n}{n!},\ |\tau|<|\log(\mathfrak{U})|,\ \ \mathfrak{U}\neq 1,\ \ C>1.\tag{20}$$

Remark 2.4. *For* $\psi_2=\psi_3=\cdots=\psi_m=0$ *or* $m=1$*, the EMHFEP of* 1*-parameter* ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})$ *reduces to the FEP of* 1*-parameter* $\mathcal{F}_n(\psi_1;C|\mathfrak{U})$ *possessing generating relation:*

$$\frac{1-\mathfrak{U}}{e^{\tau}-\mathfrak{U}}\,C^{\psi_1\tau}=\sum_{n=0}^{\infty}\mathcal{F}_n(\psi_1;C|\mathfrak{U})\frac{\tau^n}{n!},\ |\tau|<|\log(\mathfrak{U})|,\ \ \mathfrak{U}\neq 1,\ \ C>1,\tag{21}$$

or, equivalently

$$\frac{1-\mathfrak{U}}{e^{\tau}-\mathfrak{U}}\,e^{\ln C(\psi_1\tau)}=\sum_{n=0}^{\infty}\mathcal{F}_n(\psi_1;C|\mathfrak{U})\frac{\tau^n}{n!},\ |\tau|<|\log(\mathfrak{U})|,\ \ \mathfrak{U}\neq 1,\ \ C>1.\tag{22}$$

Theorem 2.5. *For the EMHFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})$, the following determinant representation holds:*

$$
\begin{aligned}
&{}_{\mathcal{H}}\mathcal{F}_0(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}) = \frac{1}{\chi_0},\\
&{}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}) = \frac{(-1)^n}{(\chi_0)^{n+1}}
\end{aligned}
$$

$$
\times \begin{vmatrix}
1 & \mathcal{G}_1(\psi_1, \psi_2, \cdots, \psi_m; C) & \mathcal{G}_2(\psi_1, \psi_2, \cdots, \psi_m; C) & \cdots & \mathcal{G}_{n-1}(\psi_1, \psi_2, \cdots, \psi_m; C) & \mathcal{G}_n(\psi_1, \psi_2, \cdots, \psi_m; C)\\
\chi_0 & \chi_1 & \chi_2 & \cdots & \chi_{n-1} & \chi_n\\
0 & \chi_0 & \binom{2}{1}\chi_1 & \cdots & \binom{n-1}{1}\chi_{n-2} & \binom{n}{1}\chi_{n-1}\\
0 & 0 & \chi_0 & \cdots & \binom{n-1}{2}\chi_{n-3} & \binom{n}{2}\chi_{n-2}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \\
0 & 0 & 0 & \cdots & \chi_0 & \binom{n}{n-1}\chi_1
\end{vmatrix}, \tag{23}
$$

where, for brevity,

$$
\sum_{n=0}^{\infty} \mathcal{G}_n(\psi_1, \psi_2, \cdots, \psi_m; C)\frac{\tau^n}{n!} = C^{\psi_1\tau+\psi_2\tau^2+\psi_3\tau^3+\cdots+\psi_m\tau^m}, \qquad \frac{1-\mathfrak{U}}{e^\tau - \mathfrak{U}} = \sum_{k=0}^{\infty} \chi_k \frac{\tau^k}{k!}. \tag{24}
$$

Equivalently, after a transpose (and elementary row operations),

$$
{}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \cdots, \psi_m; C|\mathfrak{U}) = \frac{1}{(\chi_0)^{n+1}}
$$

$$
\times \begin{vmatrix}
\chi_0 & \chi_1 & \cdots & \chi_{n-1} & \chi_n\\
0 & \chi_0 & \cdots & \binom{n-1}{1}\chi_{n-2} & \binom{n}{1}\chi_{n-1}\\
0 & 0 & \cdots & \binom{n-1}{2}\chi_{n-3} & \binom{n}{2}\chi_{n-2}\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \cdots & \chi_0 & \binom{n}{n-1}\chi_1\\
\mathcal{G}_0(\psi_1, \psi_2, \cdots, \psi_m; C) & \mathcal{G}_1(\psi_1, \psi_2, \cdots, \psi_m; C) & \cdots & \mathcal{G}_{n-1}(\psi_1, \psi_2, \cdots, \psi_m; C) & \mathcal{G}_n(\psi_1, \psi_2, \cdots, \psi_m; C)
\end{vmatrix}, \tag{25}
$$

with $\mathcal{G}_0(\psi_1, \psi_2, \cdots, \psi_m; C) \equiv 1$.

Proof. Starting from the generating relation (14)–(15),

$$
\frac{1-\mathfrak{U}}{e^\tau - \mathfrak{U}} C^{\psi_1\tau+\psi_2\tau^2+\cdots+\psi_m\tau^m} = \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \cdots, \psi_m; C|\mathfrak{U})\frac{\tau^n}{n!}, \tag{26}
$$

expand each factor into its exponential generating series:

$$
\frac{1-\mathfrak{U}}{e^\tau - \mathfrak{U}} = \sum_{k=0}^{\infty} \chi_k \frac{\tau^k}{k!}, \qquad C^{\psi_1\tau+\psi_2\tau^2+\cdots+\psi_m\tau^m} = \sum_{n=0}^{\infty} \mathcal{G}_n(\psi_1, \psi_2, \cdots, \psi_m; C)\frac{\tau^n}{n!}. \tag{27}
$$

By the Cauchy product rule,

$$
\begin{aligned}
\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})\frac{\tau^n}{n!} &= \left(\sum_{k=0}^{\infty} \chi_k \frac{\tau^k}{k!}\right)\left(\sum_{n=0}^{\infty} \mathcal{G}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C)\frac{\tau^n}{n!}\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\chi_k \mathcal{G}_{n-k}(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C)\right)\frac{\tau^n}{n!}.
\end{aligned} \tag{28}
$$

Equating coefficients of $\tau^n/n!$ yields, for $n \in \mathbb{N}_0$,

$$ {}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\cdots,\psi_m;C|\mathfrak{U}) = \sum_{k=0}^{n}\binom{n}{k}\chi_k\mathcal{G}_{n-k}(\psi_1,\psi_2,\cdots,\psi_m;C). $$

Thus we obtain the lower-triangular system

$$
\begin{aligned}
\mathcal{G}_0(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C) &= \chi_0\, {}_{\mathcal{H}}\mathcal{F}_0(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}),\\
\mathcal{G}_1(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C) &= \chi_0\,{}_{\mathcal{H}}\mathcal{F}_1(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) + \chi_1\,{}_{\mathcal{H}}\mathcal{F}_0(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}),\\
\mathcal{G}_2(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C) &= \chi_0\,{}_{\mathcal{H}}\mathcal{F}_2(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) + \binom{2}{1}\chi_1\,{}_{\mathcal{H}}\mathcal{F}_1(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\\
&\quad + \chi_2\,{}_{\mathcal{H}}\mathcal{F}_0(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}),\\
&\ \ \vdots\\
\mathcal{G}_{n-1}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C) &= \chi_0\,{}_{\mathcal{H}}\mathcal{F}_{n-1}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) + \binom{n-1}{1}\chi_1\,{}_{\mathcal{H}}\mathcal{F}_{n-2}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) + \cdots\\
&\quad + \chi_{n-1}\,{}_{\mathcal{H}}\mathcal{F}_0(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}),\\
\mathcal{G}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C) &= \chi_0\,{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) + \binom{n}{1}\chi_1\,{}_{\mathcal{H}}\mathcal{F}_{n-1}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) + \cdots\\
&\quad + \chi_n\,{}_{\mathcal{H}}\mathcal{F}_0(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}).
\end{aligned}
$$

Applying Cramer's rule to this triangular linear system in the unknowns $\{{}_{\mathcal{H}}\mathcal{F}_0(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}),\cdots,$ ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\}$ gives

$$ {}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) = \frac{\begin{vmatrix} \chi_0 & 0 & \cdots & 0 & \mathcal{G}_0(\psi_1,\psi_2,\cdots,\psi_m;C)\\ \chi_1 & \chi_0 & \cdots & 0 & \mathcal{G}_1(\psi_1,\psi_2,\cdots,\psi_m;C)\\ \chi_2 & \binom{2}{1}\chi_1 & \cdots & 0 & \mathcal{G}_2(\psi_1,\psi_2,\cdots,\psi_m;C)\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \chi_{n-1} & \binom{n-1}{1}\chi_{n-2} & \cdots & \chi_0 & \mathcal{G}_{n-1}(\psi_1,\psi_2,\cdots,\psi_m;C)\\ \chi_n & \binom{n}{1}\chi_{n-1} & \cdots & \binom{n}{n-1}\chi_1 & \mathcal{G}_n(\psi_1,\psi_2,\cdots,\psi_m;C) \end{vmatrix}}{\begin{vmatrix} \chi_0 & 0 & \cdots & 0 & 0\\ \chi_1 & \chi_0 & \cdots & 0 & 0\\ \chi_2 & \binom{2}{1}\chi_1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \chi_{n-1} & \binom{n-1}{1}\chi_{n-2} & \cdots & \chi_0 & 0\\ \chi_n & \binom{n}{1}\chi_{n-1} & \cdots & \binom{n}{n-1}\chi_1 & \chi_0 \end{vmatrix}}. \tag{29} $$

Taking the transpose in numerator and denominator (which preserves determinants), and simplifying by factoring out $(\chi_0)^{n+1}$ from the denominator, yields the compact determinant form stated in the theorem. Finally, elementary row operations transform the transposed numerator to the displayed upper–Hessenberg form with the first row $(1,\mathcal{G}_1(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C),\ldots,\mathcal{G}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C))$, producing the alternative representation with the prefactor $\frac{(-1)^n}{(\chi_0)^{n+1}}$. This completes the proof. □

Theorem 2.6. *For the EMHFEP of* 1*-parameter* ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})$*, the succeeding series representation is demonstrated*

$$ {}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) = n!\sum_{k=0}^{[n/m]}\frac{{}_{\mathcal{H}}\mathcal{F}_{n-mk}(\psi_1,\psi_2,\psi_3,\cdots,\psi_{m-1};C|\mathfrak{U})\psi_m^k}{(n-mk)!\,k!}(\ln C)^k. \tag{30} $$

Proof. By substituting the expressions in (15) and expanding $e^{\ln C(\psi_m \tau^m)}$ on the left-hand side while rearranging the components, we arrive at the conclusion that

$$\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}) \frac{\tau^n}{n!} = \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_{m-1}; C|\mathfrak{U}) \frac{\tau^n}{n!} \sum_{k=0}^{\infty} \psi_m^k \frac{(\ln C\, \tau^m)^k}{k!}, \tag{31}$$

Consequently, applying the series rearrangement rule results in the following expression:

$$\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}) = \sum_{n=0}^{\infty} n! \sum_{k=0}^{[n/m]} \frac{{}_{\mathcal{H}}\mathcal{F}_{n-mk}(\psi_1, \psi_2, \psi_3, \cdots, \psi_{m-1}; C|\mathfrak{U}) \psi_m^k}{(n-mk)!\, k!} (\ln C)^k. \tag{32}$$

Assertion (30) is derived by equating the coefficients of corresponding powers of τ on both sides of the expression above. □

Remark 2.7. *For $\psi_4 = \psi_5 = \cdots = \psi_m = 0$ or $m = 3$ the EMHFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})$ reduces to the 3-variable HFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3; C|\mathfrak{U})$, satisfying the series representation:*

$${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3; C|\mathfrak{U}) = n! \sum_{k=0}^{[n/3]} \frac{{}_{\mathcal{H}}\mathcal{F}_{n-3k}(\psi_1, \psi_2; C|\mathfrak{U}) \psi_3^k}{(n-3k)!\, k!} (\ln C)^k.$$

Remark 2.8. *For $\psi_3 = \psi_4 = \psi_5 = \cdots = \psi_m = 0$ or $m = 3$ the EMHFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})$ reduces to the 2-variable HFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2; C|\mathfrak{U})$, satisfying the series representation:*

$${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2; C|\mathfrak{U}) = n! \sum_{k=0}^{[n/2]} \frac{{}_{\mathcal{H}}\mathcal{F}_{n-2k}(\psi_1; C|\mathfrak{U}) \psi_2^k}{(n-2k)!\, k!} (\ln C)^k.$$

Operational techniques for special polynomials involve diverse algebraic and analytical approaches that improve their handling and application across mathematical fields. A key tool is generating functions, which compactly represent polynomial sequences, simplifying identity derivation and interrelations. Differential operators are equally crucial, aiding in formulating recurrence relations and transforming polynomials for solving differential equations efficiently. Additionally, integral transforms like the Laplace and Fourier transforms broaden their applicability to domains such as quantum mechanics and signal processing. These methods streamline complex computations, making special polynomials essential in pure and applied mathematics.

Differentiating (14) or (15) with respect to ψ_1 successively, we find

$$\frac{\partial}{\partial \psi_1} \left(\frac{1-\mathfrak{U}}{e^\tau - \mathfrak{U}}\, C^{\psi_1 \tau + \psi_2 \tau^2 + \psi_3 \tau^3 + \cdots + \psi_m \tau^m} \right) = (\ln C) \tau \left(\frac{1-\mathfrak{U}}{e^\tau - \mathfrak{U}}\, C^{\psi_1 \tau + \psi_2 \tau^2 + \psi_3 \tau^3 + \cdots + \psi_m \tau^m} \right),$$

which can be reformulated to

$$\frac{\partial}{\partial \psi_1} \left(\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}) \frac{\tau^n}{n!} \right) = (\ln C) \left(\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}) \frac{\tau^{n+1}}{n!} \right),$$

by substituting $n \to n-1$ in the right-hand side of the previous expression and subsequently equating the coefficients of like exponents on both sides of the resulting expression, we obtain

$$\frac{\partial}{\partial \psi_1} {}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}) = n \ln C\, {}_{\mathcal{H}}\mathcal{F}_{n-1}(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}). \tag{33}$$

Continuing in a similar fashion, we have

$$\frac{\partial^2}{\partial \psi_1^2}{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) = n(n-1)(\ln C)^2{}_{\mathcal{H}}\mathcal{F}_{n-2}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}), \tag{34}$$

$$\frac{\partial^3}{\partial \psi_1^3}{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) = n(n-1)(n-2)(\ln C)^3{}_{\mathcal{H}}\mathcal{F}_{n-3}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}), \tag{35}$$

$$\vdots \qquad \vdots$$

$$\frac{\partial^m}{\partial \psi_1^m}{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) = n(n-1)\cdots(n-m+1)(\ln C)^m{}_{\mathcal{H}}\mathcal{F}_{n-m}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}). \tag{36}$$

Further, differentiating (14) or (15) with respect to ψ_2 successively, we find

$$\frac{\partial}{\partial \psi_2}\left(\mathcal{F}(\tau)\, C^{\psi_1\tau+\psi_2\tau^2+\psi_3\tau^3+\cdots+\psi_m\tau^m}\right) = (\ln C)\tau^2\left(\mathcal{F}(\tau)\, C^{\psi_1\tau+\psi_2\tau^2+\psi_3\tau^3+\cdots+\psi_m\tau^m}\right),$$

which further can be expressed as

$$\frac{\partial}{\partial \psi_2}\left(\sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\frac{\tau^n}{n!}\right) = (\ln C)\left(\sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\frac{\tau^{n+2}}{n!}\right),$$

by substituting $n \to n-2$ into the right-hand side of the previous expression and then equating the coefficients of corresponding exponents on both sides of the resulting expression, we obtain:

$$\frac{\partial}{\partial \psi_2}{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) = n(n-1)\ln C\,{}_{\mathcal{H}}\mathcal{F}_{n-2}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}), \tag{37}$$

similarly, we have

$$\frac{\partial}{\partial \psi_3}{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) = n(n-1)(n-2)\ln C\,{}_{\mathcal{H}}\mathcal{F}_{n-3}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}). \tag{38}$$

Thus, the expressions (33)-(38) satisfy the relations:

$$\frac{\partial}{\partial \psi_2}{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) = \frac{1}{\ln C}\frac{\partial^2}{\partial \psi_1^2}{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})$$

$$\frac{\partial}{\partial \psi_3}{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) = \frac{1}{(\ln C)^2}\frac{\partial^3}{\partial \psi_1^3}{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}),$$

which, in consideration of the initial condition:

$${}_{\mathcal{H}}\mathcal{F}_n(\psi_1,0,0;C|\mathfrak{U}) = \frac{1}{(\ln C)^n}{}_{\mathcal{H}}\mathcal{F}_n(\psi_1;C|\mathfrak{U}) \tag{40}$$

provides the operational representation for ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})$ via the result:

Theorem 2.9. *For the EMHFEP of* 1*-parameter* ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})$*, the succeeding operational representation is demonstrated:*

$${}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) = \exp\Big(\frac{\psi_2}{\ln C}\frac{\partial^2}{\partial \psi_1^2} + \frac{\psi_3}{(\ln C)^2}\frac{\partial^3}{\partial \psi_1^3} + \cdots + \frac{\psi_m}{(\ln C)^{m-1}}\frac{\partial^m}{\partial \psi_1^m}\Big)\Big\{{}_{\mathcal{H}}\mathcal{F}_n(\psi_1;C|\mathfrak{U})\Big\}. \tag{41}$$

Remark 2.10. *For $\psi_4 = \psi_5 = \cdots = \psi_m = 0$ or $m = 3$ the EMHFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})$ reduces to the 3-variable HFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3; C|\mathfrak{U})$, the succeeding operational representation is established:*

$$
{}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3; C|\mathfrak{U}) = \exp\Big(\frac{\psi_2}{\ln C}\frac{\partial^2}{\partial \psi_1^2} + \frac{\psi_3}{(\ln C)^2}\frac{\partial^3}{\partial \psi_1^3}\Big)\Big\{{}_{\mathcal{H}}\mathcal{F}_n(\psi_1; C|\mathfrak{U})\Big\}.
$$

Remark 2.11. *For $\psi_3 = \psi_4 = \psi_5 = \cdots = \psi_m = 0$ or $m = 2$ the EMHFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})$ reduces to the 2-variable HFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2; C|\mathfrak{U})$, satisfying the operational representation:*

$$
{}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2; C|\mathfrak{U}) = \exp\Big(\frac{\psi_2}{\ln C}\frac{\partial^2}{\partial \psi_1^2}\Big)\Big\{{}_{\mathcal{H}}\mathcal{F}_n(\psi_1; C|\mathfrak{U})\Big\}.
$$

3. Families of differential equations

Recurrence relations are crucial in formulating differential equations by systematically linking solutions of varying orders and simplifying complex problems into more tractable forms. They express higher-order terms in terms of lower-order ones, aiding in derivation and simplification. Additionally, they provide insights into solution properties like orthogonality and asymptotics while enhancing computational efficiency for practical applications. Their unified framework strengthens the analytical and numerical approaches to solving differential equations across scientific and engineering disciplines.

Theorem 3.1. *The EMHFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})$ comply to the succeeding recurrence relation:*

$$
\begin{aligned}
&{}_{\mathcal{H}}\mathcal{F}_{n+1}(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}) = \Big(\psi_1 \ln C - \frac{1}{1-\mathfrak{U}}\Big)\, {}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}) + 2n\psi_2 \ln C \\
&{}_{\mathcal{H}}\mathcal{F}_{n-1}(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}) + 3n(n-1)\psi_3 \ln C\, {}_{\mathcal{H}}\mathcal{F}_{n-2}(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}) + \cdots \\
&+ m\psi_m(\ln C)n(n-1)\cdots(n-m+2)\, {}_{\mathcal{H}}\mathcal{F}_{n-m+1}(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}) \\
&+ \frac{1}{(1-\mathfrak{U})}\sum_{k=1}^{n-1}\binom{n}{k}\, {}_{\mathcal{H}}\mathcal{F}_k(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})\,\mathfrak{H}_{n-k}(\mathfrak{U}),
\end{aligned}
\tag{42}
$$

where the expression:

$$
\mathfrak{H}_k(\mathfrak{U}) := -\sum_{i=0}^{k}\frac{1}{2^i}\binom{k}{i}\mathfrak{H}_{k-i}\Big(\frac{1}{2}|\mathfrak{U}\Big),\ \mathfrak{H}_0 = -1,\ \mathfrak{H}_1 = \frac{1}{2}
\tag{43}
$$

expressed using the numerical coefficients $\mathfrak{H}_n(\mathfrak{U})$, which are connected to the Frobenius-Euler polynomials ${}_{\mathcal{H}}\mathcal{F}_k(\psi_1|\mathfrak{U})$ and

$$
{}_{\mathcal{H}}\mathcal{F}_{-k}(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}) := 0, \quad k = 1, 2.
\tag{44}
$$

Proof. By introducing τ and differentiating both sides of the generating function (14), we arrive at the following result:

$$
\frac{\partial}{\partial \tau}\Bigg\{\frac{1-\mathfrak{U}}{e^\tau - \mathfrak{U}}\, C^{\psi_1\tau + \psi_2\tau^2 + \psi_3\tau^3 + \cdots + \psi_m\tau^m}\Bigg\} = \frac{\partial}{\partial \tau}\Bigg\{\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})\frac{\tau^n}{n!}\Bigg\},
$$

which can be simplified as

$$\left\{\psi_1 \ln(C)+2\psi_2 \ln(C)\tau+3\psi_2 \ln(C)\tau^2+m\psi_m \ln(C)n(n-1)\cdots(n-m+2)\tau^m\right\}\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\frac{\tau^n}{n!}$$
$$+\frac{1}{(1-\mathfrak{U})}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} {}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\mathfrak{H}_k(\mathfrak{U})\frac{\tau^{n+k}}{n!\,k!}=\sum_{n=0}^{\infty} n\,{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\frac{\tau^{n-1}}{n!}. \tag{45}$$

Furthermore, the previous expression can be rewritten using the Cauchy product formula as:

$$\sum_{n=0}^{\infty}\Bigg[\Big(\psi_1\ln(C)-\frac{1}{1-\mathfrak{U}}\Big){}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})+2n\psi_2\ln(C){}_{\mathcal{H}}\mathcal{F}_{n-1}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})+3n(n-1)\psi_3\ln(C)$$
$$\times{}_{\mathcal{H}}\mathcal{F}_{n-2}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})+m\psi_m\ln(C)n(n-1)\cdots(n-m+2){}_{\mathcal{H}}\mathcal{F}_{n-m}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})$$
$$+\frac{1}{(1-\mathfrak{U})}\sum_{k=1}^{n-1}\binom{n}{k}{}_{\mathcal{H}}\mathcal{F}_k(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\mathfrak{H}_{n-k}(\mathfrak{U})\Bigg]\frac{\tau^n}{n!}=\sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{F}_{n+1}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\frac{\tau^n}{n!}. \tag{46}$$

The expression (42) is derived by equating the coefficients of like powers of τ found on both sides of the previous equation. □

Remark 3.2. *For $\psi_4=\psi_5=\cdots=\psi_m=0$ or $m=3$ the EMHFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})$ reduces to the 3-variable HFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3;C|\mathfrak{U})$, satisfying the recurrence relation:*

$${}_{\mathcal{H}}\mathcal{F}_{n+1}(\psi_1,\psi_2,\psi_3;C|\mathfrak{U})=\Big(\psi_1\ln C-\frac{1}{1-\mathfrak{U}}\Big){}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3;C|\mathfrak{U})+2n\psi_2\ln C{}_{\mathcal{H}}\mathcal{F}_{n-1}(\psi_1,\psi_2,\psi_3;C|\mathfrak{U})$$
$$+3n(n-1)\psi_3\ln C{}_{\mathcal{H}}\mathcal{F}_{n-2}(\psi_1,\psi_2,\psi_3;C|\mathfrak{U})+\frac{1}{(1-\mathfrak{U})}\sum_{k=1}^{n-1}\binom{n}{k}{}_{\mathcal{H}}\mathcal{F}_k(\psi_1,\psi_2,\psi_3;C|\mathfrak{U})\mathfrak{H}_{n-k}(\mathfrak{U}),$$

Remark 3.3. *For $\psi_3=\psi_4=\psi_5=\cdots=\psi_m=0$ or $m=3$ the EMHFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})$ reduces to the 2-variable HFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2;C|\mathfrak{U})$, satisfying the recurrence relation:*

$${}_{\mathcal{H}}\mathcal{F}_{n+1}(\psi_1,\psi_2;C|\mathfrak{U})=\Big(\psi_1\ln C-\frac{1}{1-\mathfrak{U}}\Big){}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2;C|\mathfrak{U})+2n\psi_2\ln C{}_{\mathcal{H}}\mathcal{F}_{n-1}(\psi_1,\psi_2;C|\mathfrak{U})$$
$$+\frac{1}{(1-\mathfrak{U})}\sum_{k=1}^{n-1}\binom{n}{k}{}_{\mathcal{H}}\mathcal{F}_k(\psi_1,\psi_2;C|\mathfrak{U})\mathfrak{H}_{n-k}(\mathfrak{U}),$$

Theorem 3.4. *The EMHFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})$ comply with the following shift operators:*

$${}_{\psi_1}\mathcal{L}_n^-:=\frac{1}{n(\ln C)}D_{\psi_1}, \tag{47}$$

$${}_{\psi_2}\mathcal{L}_n^-:=\frac{1}{n(\ln C)}D_{\psi_1}^{-1}D_{\psi_2}, \tag{48}$$

$${}_{\psi_3}\mathcal{L}_n^-:=\frac{1}{n(\ln C)}D_{\psi_1}^{-2}D_{\psi_3}, \tag{49}$$

$$\vdots$$

$$ {}_{\psi_m}\mathcal{L}_n^- := \frac{1}{n(\ln C)} D_{\psi_1}^{-(m-1)} D_{\psi_m}, \tag{50} $$

$$ {}_{\psi_1}\mathcal{L}_n^+ := \left(\psi_1 \ln C - \frac{1}{1-\mathfrak{U}}\right) + 2\psi_2 D_{\psi_1} + 3\psi_3 (\ln C)^{-1} D_{\psi_1}^2 + \cdots + m\psi_m (\ln C)^{-m+2} D_{\psi_1}^{(m-1)}, $$

$$ + \frac{1}{1-\mathfrak{U}} \sum_{k=0}^{n-1} (\ln C)^{-k} D_{\psi_1}^k \frac{\mathfrak{H}_k(\mathfrak{U})}{k!}, \tag{51} $$

$$ {}_{\psi_2}\mathcal{L}_n^+ := \left(\psi_1 \ln C - \frac{1}{1-\mathfrak{U}}\right) + 2\psi_2 D_{\psi_1}^{-1} D_{\psi_2} + 3\psi_3 (\ln C)^{-1} D_{\psi_1}^{-2} D_{\psi_2}^2 + \cdots $$

$$ + m\psi_m \psi_m (\ln C)^{-m+2} D_{\psi_1}^{-m+1} D_{\psi_2}^{m-1} + \frac{1}{1-\mathfrak{U}} \sum_{k=0}^{n-1} (\ln C)^{-k} D_{\psi_1}^{-(n-k)} D_{\psi_2}^{n-k} \frac{\mathfrak{H}_{n-k}(\mathfrak{U})}{(n-k)!}, \tag{52} $$

$$ {}_{\psi_3}\mathcal{L}_n^+ := \left(\psi_1 \ln C - \frac{1}{1-\mathfrak{U}}\right) + 2\psi_2 D_{\psi_1}^{-2} D_{\psi_3} + 3\psi_3 (\ln C)^{-1} D_{\psi_1}^{-4} D_{\psi_3}^2 + \cdots + m\psi_m (\ln C)^{-m+2} D_{\psi_1}^{-2(m-1)} D_{\psi_3}^{m-1} $$

$$ + \frac{1}{1-\mathfrak{U}} \sum_{k=0}^{n-1} (\ln C)^{-k} D_{\psi_1}^{-k} D_{\psi_3}^k \frac{\mathfrak{H}_k(\mathfrak{U})}{k!}, \tag{53} $$

$$ \vdots $$

$$ {}_{\psi_m}\mathcal{L}_n^+ := \left(\psi_1 \ln C - \frac{1}{1-\mathfrak{U}}\right) + 2\psi_2 D_{\psi_1}^{-(m-1)} D_{\psi_m} + 3\psi_3 (\ln C)^{-1} D_{\psi_1}^{-2(m-1)} D_{\psi_m}^2 + \cdots + $$

$$ + m\psi_m (\ln C)^{-m+2} D_{\psi_1}^{-(m-1)^2} D_{\psi_m}^{(m-1)} + \frac{1}{1-\mathfrak{U}} \sum_{k=0}^{n-1} (\ln C)^{-k} D_{\psi_1}^{-k(m-1)} D_{\psi_m}^k \frac{\mathfrak{H}_k(\mathfrak{U})}{k!}, \tag{54} $$

respectively, where

$$ D_{\psi_1} := \frac{\partial}{\partial_{\psi_1}}, \ D_{\psi_2} := \frac{\partial}{\partial_{\psi_2}}; \ D_{\psi_3} := \frac{\partial}{\partial_{\psi_3}}, \cdots D_{\psi_m} := \frac{\partial}{\partial_{\psi_m}}, \ D_{\psi_1}^{-1} := \int_0^{\psi_1} g(\tau) d\tau. \tag{55} $$

Proof. By reorganizing the powers and differentiating both sides of equation (14) with respect to ψ_1, we analyze the coefficients of the corresponding powers of τ from both sides of the resulting equation, as demonstrated below:

$$ D_{\psi_1}\{{}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})\} = n(\ln C) {}_{\mathcal{H}}\mathcal{F}_{n-1}(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}), \tag{56} $$

consequently, the operator defined by equation (47) fulfills the requirements of the equation:

$$ {}_{\psi_1}\mathcal{L}_n^- \{{}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})\} = {}_{\mathcal{H}}\mathcal{F}_{n-1}(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}). \tag{57} $$

Subsequently, we differentiate both sides of equation (14) with respect to ψ_2, rearrange the powers, and then calculate the coefficients of the identical powers of τ on both sides of the resulting equation gives:

$$ D_{\psi_2}\{{}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})\} = (\ln C) n(n-1) {}_{\mathcal{H}}\mathcal{F}_{n-2}(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}), \tag{58} $$

which further can be stated as

$$ D_{\psi_2}\{{}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})\} = n(\ln C) D_{\psi_1} {}_{\mathcal{H}}\mathcal{F}_{n-1}(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}), \tag{59} $$

consequently, it can be concluded that

$$ \frac{1}{n(\ln C)} D_{\psi_2} D_{\psi_1}^{-1} \{{}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})\} = {}_{\mathcal{H}}\mathcal{F}_{n-1}(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}). \tag{60} $$

Consequently, the operator defined in equation (48) fulfills the aforementioned equation.

Next, by taking the derivative of both sides of equation (14) with respect to ψ_3, we derive:

$$D_{\psi_3}\{_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\} = (\ln C)n(n-1)(n-2)_{\mathcal{H}}\mathcal{F}_{n-3}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) \tag{61}$$

and further stated as

$$D_{\psi_3}\{_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\} = n(\ln C)D^2_{\psi_1}{}_{\mathcal{H}}\mathcal{F}_{n-1}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}), \tag{62}$$

consequently, it can be concluded that

$$\frac{1}{n(\ln C)}D_{\psi_3}D^{-2}_{\psi_1}\{_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\} = {}_{\mathcal{H}}\mathcal{F}_{n-1}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}). \tag{63}$$

The operator defined by equation (49) fulfills the requirements of the equation stated above.

Ultimately, by differentiating equation (14) concerning ψ_m and setting the coefficients of matching powers of τ equal on both sides of the resultant equation, we obtain the following expression:

$$D_{\psi_m}\{_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\} = (\ln C)n(n-1)(n-m+1)_{\mathcal{H}}\mathcal{F}_{n-m}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) \tag{64}$$

and further presented as

$$D_{\psi_m}\{_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\} = n(\ln C)D^{m-1}_{\psi_1}\{_{\mathcal{H}}\mathcal{F}_{n-1}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\} \tag{65}$$

thus eventually gives

$$\frac{1}{n(\ln C)}D_{\psi_m}D^{-(m-1)}_{\psi_1}\{_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\} = {}_{\mathcal{H}}\mathcal{F}_{n-1}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}). \tag{66}$$

Hence, the affirmation in (50) is confirmed.

The raising operator given by (51) can be determined using the following relation:

$${}_{\mathcal{H}}\mathcal{F}_{n-k}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) = \left({}_{\psi_1}\mathcal{L}^-_{n-k+1}\ {}_{\psi_1}\mathcal{L}^-_{n-k+2}\cdots{}_{\psi_1}\mathcal{L}^-_{n-1}\ {}_{\psi_1}\mathcal{L}^-_{n}\right)\{_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\}. \tag{67}$$

By combining equation (47) with equation (64), we obtain:

$$\begin{aligned}{}_{\mathcal{H}}\mathcal{F}_{n-k}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) = &\left(\frac{1}{(n-k+1)(\ln C)}D_{\psi_1}\cdots\frac{1}{(n-1)(\ln C)}D_{\psi_1}\ \frac{1}{n(\ln C)}D_{\psi_1}\right)\\ &\times\ \{_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\}\end{aligned} \tag{68}$$

and further cast as

$${}_{\mathcal{H}}\mathcal{F}_{n-k}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) = \frac{(n-k)!}{n!}(\ln C)^{-k}D^k_{\psi_1}\{_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\}. \tag{69}$$

By replacing equation (69) into the recurrence relation (42), we find

$$\begin{aligned}{}_{\mathcal{H}}\mathcal{F}_{n+1}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) = &\Bigg(\left(\psi_1\ln C-\frac{1}{1-\mathfrak{U}}\right)+2\psi_2D_{\psi_1}+3\psi_3(\ln C)^{-1}D^2_{\psi_1}+\cdots+m\psi_m(\ln C)^{-m+2}\\ &\times D^{(m-1)}_{\psi_1}+\sum_{k=1}^{n}\frac{\mathcal{F}_k(\mathfrak{U})}{k!}(\ln C)^{-k}D^k_{\psi_1}\Bigg)\{_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\},\end{aligned} \tag{70}$$

We obtain the expression (51) for the raising operator ${}_{\psi_1}\mathcal{L}^+_n$.

Subsequently, we employ the relation provided in (48) to ascertain the raising operator:

$$ {}_{\mathcal{H}}\mathcal{F}_{n-k}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) = \left({}_{\psi_2}\mathcal{L}^-_{n-k+1}\ {}_{\psi_2}\mathcal{L}^-_{n-k+2}\cdots {}_{\psi_2}\mathcal{L}^-_{n-1}\ {}_{\psi_2}\mathcal{L}^-_{n}\right)\{{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\} \quad (71) $$

By applying equation (60) to equation (71) and simplifying, we obtain:

$$ {}_{\mathcal{H}}\mathcal{F}_{n-k}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) = \frac{(n-k)!}{n!}(\ln C)^{-k}D^{-k}_{\psi_1}D^{k}_{\psi_2}\{{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\}. \quad (72) $$

By substituting equation (72) in (42), we find:

$$ {}_{\mathcal{H}}\mathcal{F}_{n+1}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) = \Bigg(\left(\psi_1\ln C-\frac{1}{1-\mathfrak{U}}\right)+2\psi_2D^{-1}_{\psi_1}D_{\psi_2}+3\psi_3(\ln C)^{-1}D^{-2}_{\psi_1}D^{2}_{\psi_2}+\cdots $$

$$ +m\psi_m\psi_m(\ln C)^{-m+2}D^{-m+1}_{\psi_1}D^{m-1}_{\psi_2}+\frac{1}{1-\mathfrak{U}}\sum_{k=1}^{n-1}\frac{\mathcal{F}_k}{k!}(\ln C)^{-k}D^{-k}_{\psi_1}D^{k}_{\psi_2}\Bigg)\{{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\}, \quad (73) $$

We can express the raising operator ${}_{\psi_2}\mathcal{L}^+_n$ as indicated in equation (52).

Again, we employ the relation provided in (49) to ascertain the raising operator:

$$ {}_{\mathcal{H}}\mathcal{F}_{n-k}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) = \left({}_{\psi_3}\mathcal{L}^-_{n-k+1}\ {}_{\psi_3}\mathcal{L}^-_{n-k+2}\cdots {}_{\psi_3}\mathcal{L}^-_{n-1}\ {}_{\psi_3}\mathcal{L}^-_{n}\right)\{{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\} \quad (74) $$

By applying equation (63) to equation (74) and simplifying, we obtain:

$$ {}_{\mathcal{H}}\mathcal{F}_{n-k}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) = \frac{(n-k)!}{n!}(\ln C)^{-k}D^{-2k}_{\psi_1}D^{k}_{\psi_3}\{{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\}. \quad (75) $$

By substituting equation (75) in (42), we find:

$$ {}_{\mathcal{H}}\mathcal{F}_{n+1}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) = \Bigg(\left(\psi_1\ln C-\frac{1}{1-\mathfrak{U}}\right)+2\psi_2D^{-2}_{\psi_1}D_{\psi_3}+3\psi_3(\ln C)^{-1}D^{-4}_{\psi_1}D^{2}_{\psi_3}+\cdots $$

$$ +m\psi_m\psi_m(\ln C)^{-m+2}D^{-2(m-1)}_{\psi_1}D^{m-1}_{\psi_3}+\frac{1}{1-\mathfrak{U}}\sum_{k=1}^{n-1}\frac{\mathcal{F}_k(u)}{k!}(\ln C)^{-k}D^{-2k}_{\psi_1}D^{k}_{\psi_3}\Bigg)\{{}_{\mathcal{H}}\mathcal{F}_k(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\}, \quad (76) $$

We can express the raising operator ${}_{\psi_3}\mathcal{L}^+_n$ as indicated in equation (53).

Finally, we use the following relation to determine the raising operator given by (50):

$$ {}_{\mathcal{H}}\mathcal{F}_{n-k}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) = \left({}_{\psi_m}\mathcal{L}^-_{n-k+1}\ {}_{\psi_m}\mathcal{L}^-_{n-k+2}\cdots {}_{\psi_m}\mathcal{L}^-_{n-1}\ {}_{\psi_m}\mathcal{L}^-_{n}\right)\{{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\} \quad (77) $$

By substituting equation (66) into equation (77) and performing simplifications, we derive:

$$ {}_{\mathcal{H}}\mathcal{F}_{n-k}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) = \frac{(n-k)!}{n!}(\ln C)^{-k}D^{-(m-1)k}_{\psi_1}D^{k}_{\psi_m}\{{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\}. \quad (78) $$

Substituting equation (78) in (42) yields:

$$ {}_{\mathcal{H}}\mathcal{F}_{n+1}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) = \Bigg(\left(\psi_1\ln C-\frac{1}{1-\mathfrak{U}}\right)+2\psi_2D^{-(m-1)}_{\psi_1}D_{\psi_m}+3\psi_3(\ln C)^{-1}D^{-2(m-1)}_{\psi_1}D^{2}_{\psi_m}+\cdots $$

$$ +m\psi_m(\ln C)^{-m+2}D^{-(m-1)^2}_{\psi_1}D^{(m-1)}_{\psi_m}+\frac{1}{1-\mathfrak{U}}\sum_{k=1}^{n-1}\frac{\mathcal{F}_k(u)}{k!}(\ln C)^{-k}D^{-k(m-1)}_{\psi_1}D^{k}_{\psi_m}\Bigg)\{{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\}, \quad (79) $$

This consequently yields expression (54) for the raising operator ${}_{\psi_m}\mathcal{L}^+_n$.
□

To proceed, we derive the "differential, integrodifferential, and partial differential equations" associated with the *EMHFEP* of 1-parameter, denoted by ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})$. For this purpose, we will utilize the following established results:

Theorem 3.5. *The EMHFEP of* 1*-parameter* ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})$ *satisfy the following differential equation:*

$$\Big(\big(\psi_1 - (\ln C)^{-1}\frac{1}{(1-\mathfrak{U})}\big)D_{\psi_1} + 2\psi_2(\ln C)^{-1}D^2_{\psi_1} + 3\psi_3(\ln C)^{-2}D^3_{\psi_1} + \cdots + m\psi_m(\ln C)^{-m+1}D^{(m)}_{\psi_1}$$
$$+\frac{1}{1-\mathfrak{U}}\sum_{k=1}^{n-1}(\ln C)^{-k-1}D^{k+1}_{\psi_1}\frac{\mathfrak{H}_k(\mathfrak{U})}{k!} - n\Big)\ {}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}) = 0. \tag{80}$$

Proof. By applying expressions (47) and (51) for the shift operators ${}_{\psi_1}\mathcal{L}^-_n$ and ${}_{\psi_1}\mathcal{L}^+_n$ within the factorization relation ${}_{\psi_1}\mathcal{L}^-_{n+1}\,{}_{\psi_1}\mathcal{L}^+_n\{{}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})\} = {}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})$, we derive expression (80). □

Remark 3.6. *For* $\psi_4 = \psi_5 = \cdots = \psi_m = 0$ *or* $m = 3$ *the EMHFEP of* 1*-parameter* ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})$ *reduces to the* 3*-variable HFEP of* 1*-parameter* ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3; C|\mathfrak{U})$*, satisfying differential equation:*

$$\Big(\big(\psi_1 - (\ln C)^{-1}\frac{1}{(1-\mathfrak{U})}\big)D_{\psi_1} + 2\psi_2(\ln C)^{-1}D^2_{\psi_1} + 3\psi_3(\ln C)^{-2}D^3_{\psi_1} + \frac{1}{1-\mathfrak{U}}\sum_{k=1}^{n-1}(\ln C)^{-k-1}D^{k+1}_{\psi_1}\frac{\mathfrak{H}_k(\mathfrak{U})}{k!} - n\Big)$$
$$\times\ {}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3; C|\mathfrak{U}) = 0.$$

Remark 3.7. *For* $\psi_3 = \psi_4 = \psi_5 = \cdots = \psi_m = 0$ *or* $m = 2$ *the EMHFEP of* 1*-parameter* ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})$ *reduces to the* 2*-variable HFEP of* 1*-parameter* ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2; C|\mathfrak{U})$*, satisfying differential equation:*

$$\Big(\big(\psi_1 - (\ln C)^{-1}\frac{1}{(1-\mathfrak{U})}\big)D_{\psi_1} + 2\psi_2(\ln C)^{-1}D^2_{\psi_1} + \frac{1}{1-\mathfrak{U}}\sum_{k=1}^{n-1}(\ln C)^{-k-1}D^{k+1}_{\psi_1}\frac{\mathfrak{H}_k(\mathfrak{U})}{k!} - n\Big)$$
$$\times\ {}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2; C|\mathfrak{U}) = 0.$$

Theorem 3.8. *The EMHFEP of* 1*-parameter* ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C)$ *satisfy the following integrodifferential equations:*

$$\Big((\psi_1 - (\ln C)^{-1}\frac{1}{1-\mathfrak{U}})D_{\psi_2} + 2\psi_2(\ln C)^{-1}D^{-1}_{\psi_1}D^2_{\psi_2} + 3\psi_3(\ln C)^{-2}D^{-2}_{\psi_1}D^3_{\psi_2} + \cdots + m\psi_m(\ln C)^{-m+1}D^{-m+1}_{\psi_1}D^m_{\psi_2}$$
$$+\frac{1}{1-\mathfrak{U}}\sum_{k=1}^{n-1}\frac{\mathfrak{H}_k(\mathfrak{U})}{k!}(\ln C)^{-1-k}D^{-k}_{\psi_1}D^{k+1}_{\psi_2} - (n+1)D_{\psi_1}\Big){}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C) = 0, \tag{81}$$

$$\Big((\psi_1 - (\ln C)^{-1}\frac{1}{1-\mathfrak{U}})D_{\psi_3} + 2\psi_2(\ln C)^{-1}D^{-1}_{\psi_1}D_{\psi_2}D_{\psi_3} + 3\psi_3(\ln C)^{-2}D^{-2}_{\psi_1}D^2_{\psi_2}D_{\psi_3} + \cdots + m\psi_m(\ln C)^{-m+1}D^{-m+1}_{\psi_1}$$
$$D^{m-1}_{\psi_2}D_{\psi_3} + \frac{1}{1-\mathfrak{U}}\sum_{k=1}^{n-1}\frac{\mathfrak{H}_k(\mathfrak{U})}{k!}(\ln C)^{-1-k}D^{-k}_{\psi_1}D^k_{\psi_2}D_{\psi_3} - (n+1)D^2_{\psi_1}\Big){}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}) = 0, \tag{82}$$

$$\Big((\psi_1 - (\ln C)^{-1}\frac{1}{1-\mathfrak{U}})D_{\psi_m} + 2\psi_2(\ln C)^{-1}D^{-1}_{\psi_1}D_{\psi_2}D_{\psi_m} + 3\psi_3(\ln C)^{-2}D^{-2}_{\psi_1}D^2_{\psi_2}D_{\psi_m} + \cdots + m\psi_m C)^{-m+1}D^{-m+1}_{\psi_1}D^{m-1}_{\psi_2}D_{\psi_m}$$
$$+\frac{1}{1-\mathfrak{U}}\sum_{k=1}^{n-1}\frac{\mathfrak{H}_k(\mathfrak{U})}{k!}(\ln C)^{-1-k}D^{-k}_{\psi_1}D^k_{\psi_2}D_{\psi_m} - (n+1)D^{(m-1)}_{\psi_1}\Big){}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}) = 0, \tag{83}$$

$$\Big((\psi_1-(\ln C)^{-1}\frac{1}{1-\mathfrak{U}})D_{\psi_2}+2\psi_2(\ln C)^{-1}D_{\psi_1}^{-2}D_{\psi_3}D_{\psi_2}+3\psi_3(\ln C)^{-2}D_{\psi_1}^{-4}D_{\psi_2}D_{\psi_3}^{2}+\cdots+m\psi_m(\ln C)^{-m+1}D_{\psi_1}^{-2(m-1)}D_{\psi_3}^{m-1}D_{\psi_2}$$
$$+\frac{1}{1-\mathfrak{U}}\sum_{k=1}^{n-1}\frac{\mathfrak{H}_k(\mathfrak{U})}{k!}(\ln C)^{-1-k}D_{\psi_1}^{-2k}D_{\psi_3}^{k}D_{\psi_2}-(n+1)D_{\psi_1}\Big)\,{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})=0, \quad (84)$$

$$\Big((\psi_1-(\ln C)^{-1}\frac{1}{1-\mathfrak{U}})D_{\psi_3}+2\psi_2(\ln C)^{-1}D_{\psi_1}^{-2}D_{\psi_3}^{2}+3\psi_3(\ln C)^{-2}D_{\psi_1}^{-4}D_{\psi_3}^{3}+\cdots+m\psi_m(\ln C)^{-m+1}D_{\psi_1}^{-2(m-1)}D_{\psi_3}^{m}$$
$$+\frac{1}{1-\mathfrak{U}}\sum_{k=1}^{n-1}\frac{\mathfrak{H}_k(\mathfrak{U})}{k!}(\ln C)^{-1-k}D_{\psi_1}^{-2k}D_{\psi_3}^{k+1}-(n+1)D_{\psi_1}^{2}\Big)\,{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})=0, \quad (85)$$

$$\Big((\psi_1-(\ln C)^{-1}\frac{1}{1-\mathfrak{U}})D_{\psi_m}+2\psi_2(\ln C)^{-1}D_{\psi_1}^{-2}D_{\psi_3}D_{\psi_m}+3\psi_3(\ln C)^{-2}D_{\psi_1}^{-4}D_{\psi_3}^{2}D_{\psi_m}+\cdots$$
$$+m\psi_m(\ln C)^{-m+1}D_{\psi_1}^{-2(m-1)}D_{\psi_3}^{m-1}D_{\psi_m}$$
$$+\frac{1}{1-\mathfrak{U}}\sum_{k=1}^{n-1}\frac{\mathfrak{H}_k(\mathfrak{U})}{k!}(\ln C)^{-1-k}D_{\psi_1}^{-2k}D_{\psi_3}^{k}D_{\psi_m}-(n+1)D_{\psi_1}^{m-1}\Big)\,{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})=0, \quad (86)$$

$$\vdots$$

$$\Big((\psi_1-(\ln C)^{-1}\frac{1}{1-\mathfrak{U}})D_{\psi_2}+2\psi_2(\ln C)^{-1}D_{\psi_1}^{-(m-1)}D_{\psi_m}D_{\psi_2}+3\psi_3(\ln C)^{-2}D_{\psi_1}^{-2(m-1)}D_{\psi_m}^{2}D_{\psi_2}+\cdots$$
$$+m\psi_m(\ln C)^{-m+1}D_{\psi_1}^{-(m+1)^2}D_{\psi_m}^{m-1}D_{\psi_2}$$
$$+\frac{1}{1-\mathfrak{U}}\sum_{k=1}^{n-1}\frac{\mathfrak{H}_k(\mathfrak{U})}{k!}(\ln C)^{-1-k}D_{\psi_1}^{-k(m-1)}D_{\psi_m}^{k}D_{\psi_2}-(n+1)D_{\psi_1}\Big)\,{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})=0, \quad (87)$$

$$\Big((\psi_1-(\ln C)^{-1}\frac{1}{1-\mathfrak{U}})D_{\psi_3}+2\psi_2(\ln C)^{-1}D_{\psi_1}^{-(m-1)}D_{\psi_m}D_{\psi_3}+3\psi_3(\ln C)^{-2}D_{\psi_1}^{-2(m-1)}D_{\psi_m}^{2}D_{\psi_3}+\cdots$$
$$+m\psi_m(\ln C)^{-m+1}D_{\psi_1}^{-2(m-1)}D_{\psi_m}^{m-1}D_{\psi_3}+\frac{1}{1-\mathfrak{U}}\sum_{k=1}^{n-1}\frac{\mathfrak{H}_k(\mathfrak{U})}{k!}(\ln C)^{-1-k}D_{\psi_1}^{-k(m-1)}D_{\psi_m}^{k}D_{\psi_3}-(n+1)D_{\psi_1}^{2}\Big)$$
$$\times{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})=0, \quad (88)$$

$$\Big((\psi_1-(\ln C)^{-1}\frac{1}{1-\mathfrak{U}})D_{\psi_m}+2\psi_2(\ln C)^{-1}D_{\psi_1}^{-(m-1)}D_{\psi_m}^{2}+3\psi_3(\ln C)^{-2}D_{\psi_1}^{-2(m-1)}D_{\psi_m}^{3}+\cdots$$
$$+m\psi_m(\ln C)^{-m+1}D_{\psi_1}^{-(m-1)^2}D_{\psi_m}^{m}+\frac{1}{1-\mathfrak{U}}\sum_{k=1}^{n-1}\frac{\mathfrak{H}_k(\mathfrak{U})}{k!}(\ln C)^{-1-k}D_{\psi_1}^{-k(m-1)}D_{\psi_m}^{k+1}-(n+1)D_{\psi_1}^{m-1}\Big)$$
$$\times{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})=0. \quad (89)$$

Proof. By factorization equation, we get

$$\mathcal{L}_{n+1}^{-}\mathcal{L}_n^{+}\{{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\}={}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U}) \quad (90)$$

By applying the expressions (48), (49), (50) and (52) in factorization (90), we get the assertions (81), (82) and (83).
Putting the expressions (48), (49), (50) and (53) in factorization (90), we acquire the expressions (84), (85) and (86). Again substituting (48), (49), (50) and (54) in (90), we get assertion (87), (88) and (90). The complete proof of the theorem. □

Theorem 3.9. *The EMHFEP of 1-parameter, represented as* ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})$*, satisfy the following partial differential equations:*

$$\Big((\psi_1-(\ln C)^{-1}\frac{1}{1-\mathfrak{U}})D_{\psi_2}D_{\psi_1}^{n}+2\psi_2(\ln C)^{-1}D_{\psi_1}^{n-1}D_{\psi_2}^{2}+3\psi_3(\ln C)^{-2}D_{\psi_1}^{n-2}D_{\psi_2}^{3}+\cdots+m\psi_m(\ln C)^{-m+1} D_{\psi_1}^{n-m+1}D_{\psi_2}^{m}+\frac{1}{1-\mathfrak{U}}\sum_{k=1}^{n-1}\frac{\mathfrak{H}_k(\mathfrak{U})}{k!}(\ln C)^{-1-k}D_{\psi_1}^{n-k}D_{\psi_2}^{k+1}-(n+1)D_{\psi_1}^{n+1}\Big){}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})=0, \tag{91}$$

$$\Big((\psi_1-(\ln C)^{-1}\frac{1}{1-\mathfrak{U}})D_{\psi_3}D_{\psi_1}^{n}+2\psi_2(\ln C)^{-1}D_{\psi_1}^{n-1}D_{\psi_2}D_{\psi_3}+3\psi_3(\ln C)^{-2}D_{\psi_1}^{n-2}D_{\psi_2}^{2}D_{\psi_3}+\cdots+m\psi_m (\ln C)^{-m+1}D_{\psi_1}^{n-m+1}D_{\psi_2}^{m-1}D_{\psi_3}+\frac{1}{1-\mathfrak{U}}\sum_{k=1}^{n-1}\frac{\mathfrak{H}_k(\mathfrak{U})}{k!}(\ln C)^{-1-k}D_{\psi_1}^{n-k}D_{\psi_2}^{k}D_{\psi_3}-(n+1)D_{\psi_1}^{n+2}\Big)\times{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})=0, \tag{92}$$

$$\Big((\psi_1-(\ln C)^{-1}\frac{1}{1-\mathfrak{U}})D_{\psi_m}D_{\psi_1}^{2n}+2\psi_2(\ln C)^{-1}D_{\psi_1}^{2n-1}D_{\psi_2}D_{\psi_m}+3\psi_3(\ln C)^{-2}D_{\psi_1}^{2(n-1)}D_{\psi_2}^{2}D_{\psi_m}+\cdots +m\psi_m C)^{-m+1}D_{\psi_1}^{2n-m+1}D_{\psi_2}^{m-1}D_{\psi_m}+\frac{1}{1-\mathfrak{U}}\sum_{k=1}^{n-1}\frac{\mathfrak{H}_k(\mathfrak{U})}{k!}(\ln C)^{-1-k}D_{\psi_1}^{2n-k}D_{\psi_2}^{k}D_{\psi_m}-(n+1)D_{\psi_1}^{(2n+m-1)}\Big)\times{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})=0, \tag{93}$$

$$\Big((\psi_1-(\ln C)^{-1}\frac{1}{1-\mathfrak{U}})D_{\psi_2}D_{\psi_1}^{2n+2}+2\psi_2(\ln C)^{-1}D_{\psi_1}^{2n}D_{\psi_3}D_{\psi_2}+3\psi_3(\ln C)^{-2}D_{\psi_1}^{2n-2}D_{\psi_2}D_{\psi_3}^{2}+\cdots +m\psi_m(\ln C)^{-m+1}D_{\psi_1}^{2n-2m+4}D_{\psi_3}^{m-1}D_{\psi_2}+\frac{1}{1-\mathfrak{U}}\sum_{k=1}^{n-1}\frac{\mathfrak{H}_k(\mathfrak{U})}{k!}(\ln C)^{-1-k}D_{\psi_1}^{2n-2k+2}D_{\psi_3}^{k}D_{\psi_2}-(n+1)D_{\psi_1}^{2n+3}\Big)\times{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})=0, \tag{94}$$

$$\Big((\psi_1-(\ln C)^{-1}\frac{1}{1-\mathfrak{U}})D_{\psi_3}D_{\psi_1}^{2n+2}+2\psi_2(\ln C)^{-1}D_{\psi_1}^{2n}D_{\psi_3}^{2}+3\psi_3(\ln C)^{-2}D_{\psi_1}^{2n-2}D_{\psi_3}^{3}+\cdots left.+m\psi_m(\ln C)^{-m+1}D_{\psi_1}^{2n-2m+4}D_{\psi_3}^{m}+\frac{1}{1-\mathfrak{U}}\sum_{k=1}^{n-1}\frac{\mathfrak{H}_k(\mathfrak{U})}{k!}(\ln C)^{-1-k}D_{\psi_1}^{2n-2k+2}D_{\psi_3}^{k+1}-(n+1)D_{\psi_1}^{2n+4}\Big)\times{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C)=0, \tag{95}$$

$$\Big((\psi_1-(\ln C)^{-1}\frac{1}{1-\mathfrak{U}})D_{\psi_m}D_{\psi_1}^{n^2+1}+2\psi_2(\ln C)^{-1}D_{\psi_1}^{n^2-1}D_{\psi_3}D_{\psi_m}+3\psi_3(\ln C)^{-2}D_{\psi_1}^{n^2-3}D_{\psi_3}^{2}D_{\psi_m}+\cdots +m\psi_m(\ln C)^{-m+1}+\frac{1}{1-\mathfrak{U}}\sum_{k=1}^{n-1}\frac{\mathfrak{H}_k(\mathfrak{U})}{k!}(\ln C)^{-1-k}D_{\psi_1}^{n^2-2k+1}D_{\psi_3}^{k}D_{\psi_m}-(n+1)D_{\psi_1}^{n^2+m}\Big)\times{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})=0, \tag{96}$$

$$\vdots$$

$$\Big((\psi_1-(\ln C)^{-1}\frac{1}{1-\mathfrak{U}})D_{\psi_2}D_{\psi_1}^{n^2+1}+2\psi_2(\ln C)^{-1}D_{\psi_1}^{n^2-m+3}D_{\psi_m}D_{\psi_2}+3\psi_3(\ln C)^{-2}D_{\psi_1}^{n^2-2m+4}D_{\psi_m}^2D_{\psi_2}+\cdots$$
$$+m\psi_m(\ln C)^{-m+1}D_{\psi_1}^{n^2+1-(m+1)^2}D_{\psi_m}^{m-1}D_{\psi_2}+\frac{1}{1-\mathfrak{U}}\sum_{k=1}^{n-1}\frac{\mathfrak{H}_k(\mathfrak{U})}{k!}(\ln C)^{-1-k}D_{\psi_1}^{n^2+1-k(m-1)}D_{\psi_m}^kD_{\psi_2}-(n+1)D_{\psi_1}^{n^2+2}\Big)$$
$$\times\,{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})=0, \quad (97)$$

$$\Big((\psi_1-(\ln C)^{-1}\frac{1}{1-\mathfrak{U}})D_{\psi_3}D_{\psi_1}^{n^2+2}+2\psi_2(\ln C)^{-1}D_{\psi_1}^{n^2-m+3}D_{\psi_m}D_{\psi_3}+3\psi_3(\ln C)^{-2}D_{\psi_1}^{n^2+3-2m}D_{\psi_m}^2D_{\psi_3}+\cdots$$
$$+m\psi_m(\ln C)^{-m+1}D_{\psi_1}^{n^2-2m-+3}D_{\psi_m}^{m-1}D_{\psi_3}+\frac{1}{1-\mathfrak{U}}\sum_{k=1}^{n-1}\frac{\mathfrak{H}_k(\mathfrak{U})}{k!}(\ln C)^{-1-k}D_{\psi_1}^{n^2+2-k(m-1)}D_{\psi_m}^kD_{\psi_3}-(n+1)D_{\psi_1}^{n^2+4}\Big)$$
$$\times\,{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})=0. \quad (98)$$

Proof. To establish the stated results, we proceed through successive partial differentiations of the relevant integrodifferential expressions with respect to the variable ψ_1.

Firstly, by applying the partial derivative with respect to ψ_1 a total of n times to the expressions given in (81) and (82), we derive and validate the assertions presented in (91) and (92), respectively.

Similarly, performing $2n$ partial differentiations with respect to ψ_1 on the integrodifferential expression (83) leads directly to confirming assertion (93).

Furthermore, by differentiating the integrodifferential expressions (84) and (85) exactly $2n+2$ times with respect to ψ_1, we derive the results stated in assertions (94) and (95).

Additionally, by executing partial differentiation n^2+1 times with respect to ψ_1 on expressions (85) and (86), we obtain and confirm the results given in (96) and (97).

Finally, applying partial differentiation n^2+2 times with respect to ψ_1 to the integrodifferential expression in (87) establishes the validity of assertion (98). □

4. Summation formulae

Theorem 4.1. *For the EMHFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})$, the following summation formula hold true:*

$$ {}_{\mathcal{H}}\mathcal{F}_n(\psi_1+w,\psi_2,\psi_3,\cdots,\psi_m;C)=\sum_{k=0}^{n}\binom{n}{k}{}_{\mathcal{H}}\mathcal{F}_{n-k}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\,\mathcal{H}_k(w;C). \quad (99)$$

Proof. By substituting $\psi_1\to\psi_1+w$ in (14), we have

$$\frac{1-\mathfrak{U}}{e^\tau-\mathfrak{U}}C^{(\psi_1+w)\tau+\psi_2\tau^2+\psi_3\tau^3+\cdots+\psi_m\tau^m}=\sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{F}_n(\psi_1+w,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\frac{\tau^n}{n!}. \quad (100)$$

By (7) and (14) in the l.h.s., we have

$$\sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{F}_{n-k}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\frac{t^n}{n!}\sum_{k=0}^{\infty}\mathcal{H}_k(w;C)\frac{t^k}{k!}=\sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{F}_n(\psi_1+w,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\frac{\tau^n}{n!} \quad (101)$$

Using Cauchy product in the l.h.s of above equation, we find

$$\sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{F}_n(\psi_1+w,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\frac{\tau^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}{}_{\mathcal{H}}\mathcal{F}_{n-k}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\,\mathcal{H}_k(w;C)\frac{\tau^n}{n!}. \quad (102)$$

Comparing the coefficients of τ, we get the assertion (99). □

Remark 4.2. *For $\psi_4 = \psi_5 = \cdots = \psi_m = 0$ or $m = 3$ the EMHFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})$ reduces to the 3-variable HFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3; C|\mathfrak{U})$, satisfying following summation formula:*

$$ {}_{\mathcal{H}}\mathcal{F}_n(\psi_1 + w, \psi_2, \psi_3; C) = \sum_{k=0}^{n} \binom{n}{k} {}_{\mathcal{H}}\mathcal{F}_{n-k}(\psi_1, \psi_2, \psi_3; C|\mathfrak{U})\, \mathcal{H}_k(w; C). $$

Remark 4.3. *For $\psi_3 = \psi_4 = \psi_5 = \cdots = \psi_m = 0$ or $m = 3$ the EMHFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})$ reduces to the 2-variable HFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2; C|\mathfrak{U})$, satisfying following summation formula:*

$$ {}_{\mathcal{H}}\mathcal{F}_n(\psi_1 + w, \psi_2; C) = \sum_{k=0}^{n} \binom{n}{k} {}_{\mathcal{H}}\mathcal{F}_{n-k}(\psi_1, \psi_2; C|\mathfrak{U})\, \mathcal{H}_k(w; C). $$

Theorem 4.4. *For the EMHFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})$, the following summation formula hold true:*

$$ {}_{\mathcal{H}}\mathcal{F}_n(\psi_1 + s, \psi_2 + w, \psi_3, \cdots, \psi_m; C|\mathfrak{U}) = \sum_{k=0}^{n} \binom{n}{k} {}_{\mathcal{H}}\mathcal{F}_{n-k}(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})\, \mathcal{H}_k(s, w; C). \tag{103} $$

Proof. By substituting $\psi_1 \to \psi_1 + s$ and $\psi_2 \to \psi_1 + w$ in (14), we have

$$ \frac{1 - \mathfrak{U}}{e^{\tau} - \mathfrak{U}} C^{(\psi_1+s)\tau+(\psi_2+w)\tau^2+\psi_3\tau^3+\cdots+\psi_m\tau^m} = \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{F}_n(\psi_1 + s, \psi_2 + w, \psi_3, \cdots, \psi_m; C|\mathfrak{U}) \frac{\tau^n}{n!}. \tag{104} $$

By (7) and (14) in the l.h.s., we have

$$ \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{F}_{n-k}(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}) \frac{t^n}{n!} \sum_{k=0}^{\infty} \mathcal{H}_k(s, w; C) \frac{t^k}{k!} = \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{F}_n(\psi_1 + s, \psi_2 + w, \psi_3, \cdots, \psi_m; C|\mathfrak{U}) \frac{\tau^n}{n!} \tag{105} $$

Using Cauchy product in the l.h.s of above equation, we find

$$ \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{F}_n(\psi_1 + s, \psi_2 + w, \psi_3, \cdots, \psi_m; C|\mathfrak{U}) \frac{\tau^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} {}_{\mathcal{H}}\mathcal{F}_{n-k}(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})\, \mathcal{H}_k(s, w; C) \frac{\tau^n}{n!}. \tag{106} $$

Comparing the coefficients of τ, we get the assertion (103). □

Remark 4.5. *For $\psi_4 = \psi_5 = \cdots = \psi_m = 0$ or $m = 3$ the EMHFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})$ reduces to the 3-variable HFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3; C|\mathfrak{U})$, satisfying following summation formula:*

$$ {}_{\mathcal{H}}\mathcal{F}_n(\psi_1 + s, \psi_2 + w, \psi_3; C|\mathfrak{U}) = \sum_{k=0}^{n} \binom{n}{k} {}_{\mathcal{H}}\mathcal{F}_{n-k}(\psi_1, \psi_2, \psi_3; C|\mathfrak{U})\, \mathcal{H}_k(s, w; C). $$

Remark 4.6. *For $\psi_3 = \psi_4 = \psi_5 = \cdots = \psi_m = 0$ or $m = 3$ the EMHFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})$ reduces to the 2-variable HFEP of 1-parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2; C|\mathfrak{U})$, satisfying following summation formula:*

$$ {}_{\mathcal{H}}\mathcal{F}_n(\psi_1 + s, \psi_2 + w; C|\mathfrak{U}) = \sum_{k=0}^{n} \binom{n}{k} {}_{\mathcal{H}}\mathcal{F}_{n-k}(\psi_1, \psi_2; C|\mathfrak{U})\, \mathcal{H}_k(s, w; C). $$

Theorem 4.7. *For the EMHFEP of* 1*-parameter* ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})$*, the following summation formula hold true:*

$$ {}_{\mathcal{H}}\mathcal{F}_n(\psi_1+w_1, \psi_2+w_2, \psi_3+w_3, \cdots, \psi_m+w_m; C|\mathfrak{U}) = \sum_{k=0}^{n} \binom{n}{k} {}_{\mathcal{H}}\mathcal{F}_{n-k}(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U})\, \mathcal{H}_k(w_1, w_2, w_3, \cdots, w_m; C). \quad (107) $$

Proof. On multiplying by $C^{w_1\tau+w_2\tau^2+w_3\tau^3+\cdots+w_m\tau^m}$ in (14), we have

$$ \frac{1-\mathfrak{U}}{e^{\tau}-\mathfrak{U}} C^{w_1\tau+w_2\tau^2+w_3\tau^3+\cdots+w_m\tau^m} C^{\psi_1\tau+\psi_2\tau^2+\psi_3\tau^3+\cdots+\psi_m\tau^m} = C^{w_1\tau+w_2\tau^2+w_3\tau^3+\cdots+w_m\tau^m} \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}) \frac{\tau^n}{n!}. \quad (108) $$

By (14) in the l.h.s., we have

$$ \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{F}_n(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}) \frac{t^n}{n!} \sum_{k=0}^{\infty} \mathcal{H}_k(w_1, w_2, \cdots, w_m; C) \frac{t^k}{k!} = \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{F}_n(\psi_1+w_1, \psi_2+w_2, \cdots, \psi_m+w_m; C|\mathfrak{U}) \frac{\tau^n}{n!}. \quad (109) $$

Using the Cauchy product in the l.h.s of the above equation, we find

$$ \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{F}_n(\psi_1+w_1, \psi_2+w_2, \cdots, \psi_m+w_m; C|\mathfrak{U}) \frac{\tau^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} {}_{\mathcal{H}}\mathcal{F}_{n-k}(\psi_1, \psi_2, \psi_3, \cdots, \psi_m; C|\mathfrak{U}) \times \mathcal{H}_k(w_1, w_2, w_3, \cdots, w_m; C) \frac{\tau^n}{n!}. \quad (110) $$

Comparing the coefficients of τ, we get the assertion (107). □

5. EMHFEP of 1 parameter via Fractional operators

Leveraging Euler's integral identity a fundamental result in fractional calculus and integral transform theory (refer to [30]), we arrive at the classical representation:

$$ a^{-\nu} = \frac{1}{\Gamma(\nu)} \int_0^{\infty} e^{-a\tau} t^{\nu-1}, d\tau, \quad \min \mathrm{Re}(\nu), \mathrm{Re}(a) > 0, \quad (111) $$

which serves as a powerful tool for expressing inverse powers of parameters and operators in terms of integrals.

This identity is particularly useful when applied to linear differential operators, allowing one to reformulate them into integral operator forms. For instance, applying this formulation to a first-order differential operator yields the operational expression established in [31, 32]:

$$ \left(\alpha - \frac{d}{d\mu_1}\right)^{-\nu} f(\mu_1) = \frac{1}{\Gamma(\nu)} \int_0^{\infty} e^{-\alpha\tau} \tau^{\nu-1} \left(e^{\tau \frac{d}{d\mu_1}} f(\mu_1)\right) d\tau = \frac{1}{\Gamma(\nu)} \int_0^{\infty} e^{-\alpha t} t^{\nu-1} f(\mu_1+\tau) d\tau, \quad (112) $$

which elegantly demonstrates the role of the translation operator within this operational context.

Analogously, when the second-order derivative operator is considered, the corresponding integral formulation becomes:

$$ \left(\alpha - \frac{d^2}{d{\mu_1}^2}\right)^{-\nu} f(\mu_1) = \frac{1}{\Gamma(\nu)} \int_0^{\infty} e^{-\alpha\tau} \tau^{\nu-1} \left(e^{\tau \frac{d^2}{d{\mu_1}^2}} f(\mu_1)\right) d\tau, \quad (113) $$

which highlights the emergence of exponential operators analogous to the heat kernel, commonly used in diffusion and parabolic equation analysis.

These integral forms are not only instrumental in simplifying the action of fractional differential operators but also form a foundational framework for constructing and analyzing special functions and extended families of orthogonal polynomials through operational methods.

Theorem 5.1. *For the EMHFEP of 1 parameter ${}_{\mathcal{H}}\mathcal{F}_{n,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U})$, the following operational identity holds:*

$$\left(\alpha-\Big(\frac{\psi_2}{\ln C}\frac{\partial^2}{\partial\psi_1^2}+\frac{\psi_3}{(\ln C)^2}\frac{\partial^3}{\partial\psi_1^3}+\cdots+\frac{\psi_m}{(\ln C)^{m-1}}\frac{\partial^m}{\partial\psi_1^m}\Big)\right)^{-\nu}\mathcal{F}_n(\psi_1;C|\mathfrak{U})={}_{\mathcal{H}}\mathcal{F}_{n,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U}). \quad (114)$$

Proof. We substitute $a=\alpha-\Big(\frac{\psi_2}{\ln C}\frac{\partial^2}{\partial\psi_1^2}+\frac{\psi_3}{(\ln C)^2}\frac{\partial^3}{\partial\psi_1^3}+\cdots+\frac{\psi_m}{(\ln C)^{m-1}}\frac{\partial^m}{\partial\psi_1^m}\Big)$ in (111). Then, we obtain

$$\begin{aligned}&\left(\alpha-\Big(\frac{\psi_2}{\ln C}\frac{\partial^2}{\partial\psi_1^2}+\frac{\psi_3}{(\ln C)^2}\frac{\partial^3}{\partial\psi_1^3}+\cdots+\frac{\psi_m}{(\ln C)^{m-1}}\frac{\partial^m}{\partial\psi_1^m}\Big)\right)^{-\nu}\mathcal{F}_n(\psi_1;C|\mathfrak{U})\\&=\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha\tau}\tau^{\nu-1}\exp\left(t\cdot\Big(\frac{\psi_2}{\ln C}\frac{\partial^2}{\partial\psi_1^2}+\frac{\psi_3}{(\ln C)^2}\frac{\partial^3}{\partial\psi_1^3}+\cdots+\frac{\psi_m}{(\ln C)^{m-1}}\frac{\partial^m}{\partial\psi_1^m}\Big)\right)\mathcal{F}_n(\psi_1;C|\mathfrak{U})d\tau\\&=\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\tau\psi_2,\tau\psi_3,\cdots,\tau\psi_m;;C|\mathfrak{U})d\tau,\end{aligned}$$

which completes the proof. □

Theorem 5.2. *The EMHFEP of 1 parameter ${}_{\mathcal{H}}\mathcal{F}_{n,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U})$ satisfies generating function:*

$$\frac{(1-\mathfrak{U})\exp(\psi_1\tau)}{(e^\tau-\mathfrak{U})\left(\alpha-\Big(\frac{\psi_2}{\ln C}\frac{\partial^2}{\partial\psi_1^2}+\frac{\psi_3}{(\ln C)^2}\frac{\partial^3}{\partial\psi_1^3}+\cdots+\frac{\psi_m}{(\ln C)^{m-1}}\frac{\partial^m}{\partial\psi_1^m}\Big)\right)^{\nu}}=\sum_{n=0}^\infty{}_{\mathcal{H}}\mathcal{F}_{n,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U})\frac{\tau^n}{n!}. \quad (115)$$

Proof. From the definition of the generalized EMHFEP of 1 parameter ${}_{\mathcal{H}}\mathcal{F}_{n,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U})$, we have

$$\sum_{n=0}^\infty{}_{\mathcal{H}}\mathcal{F}_{n,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U})\frac{t^n}{n!}=\sum_{n=0}^\infty\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha\tau}\tau^{\nu-1}{}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})\frac{\tau^n}{n!}d\tau. \quad (116)$$

Switching the order of summation and integration gives:

$$=\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha\tau}\tau^{\nu-1}\left(\sum_{n=0}^\infty{}_{\mathcal{H}}\mathcal{F}_{n,\nu}(\psi_1,\psi_2\tau,\psi_3\tau,\cdots,\psi_m\tau;C;\alpha|\mathfrak{U})\frac{\tau^n}{n!}\right)d\tau, \quad (117)$$

we get:

$$\begin{aligned}&\sum_{n=0}^\infty{}_{\mathcal{H}}\mathcal{F}_{n,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U})\frac{t^n}{n!}\\&=\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha\tau}\tau^{\nu-1}\frac{1-\mathfrak{U}}{e^\tau-\mathfrak{U}}\exp\Big(\frac{\psi_2}{\ln C}\frac{\partial^2}{\partial\psi_1^2\tau}+\frac{\psi_3}{(\ln C)^2}\frac{\partial^3}{\partial\psi_1^3\tau}+\cdots+\frac{\psi_m}{(\ln C)^{m-1}}\frac{\partial^m}{\partial\psi_1^m\tau}\Big)d\tau\\&=\frac{1-\mathfrak{U}}{e^\tau-\mathfrak{U}}\exp(\psi_1\tau)\cdot\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\left(\alpha-\frac{\psi_2}{\ln C}\frac{\partial^2}{\partial\psi_1^2\tau}+\frac{\psi_3}{(\ln C)^2}\frac{\partial^3}{\partial\psi_1^3\tau}+\cdots+\frac{\psi_m}{(\ln C)^{m-1}}\frac{\partial^m}{\partial\psi_1^m\tau}\right)\tau}\tau^{\nu-1}d\tau.\end{aligned}$$

By Euler's integral representation again, we obtain the result. □

Remark 5.3. *For $\alpha=1$, $\nu=1$, the generalized EMHFEP of 1 parameter ${}_{\mathcal{H}}\mathcal{F}_{n,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U})$, reduce to the EMHFEP of 1 parameter ${}_{\mathcal{H}}\mathcal{F}_n(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C|\mathfrak{U})$, (see equation* (14)*).*

Theorem 5.4. *The generalized EMHFEP of 1 parameter* ${}_{\mathcal{H}}\mathcal{F}_{n,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U})$ *can be expressed in the following explicit form:*

$$ {}_{\mathcal{H}}\mathcal{F}_{n,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U}) = \sum_{k=0}^{n}\binom{n}{m}\mathcal{F}_k(\psi_1;C|\mathfrak{U})\mathcal{H}_{n-k,\nu}(\psi_2,\psi_3,\cdots,\psi_m;C;\alpha). \tag{118} $$

Proof. The generating expression (115) can be represented in the following manner:

$$ \frac{(1-\mathfrak{U})\exp(\psi_1\tau)}{(e^\tau-\mathfrak{U})\left(\alpha-\left(\frac{\psi_2}{\ln C}\frac{\partial^2}{\partial\psi_1^2}+\frac{\psi_3}{(\ln C)^2}\frac{\partial^3}{\partial\psi_1^3}+\cdots+\frac{\psi_m}{(\ln C)^{m-1}}\frac{\partial^m}{\partial\psi_1^m}\right)\right)^\nu} = \frac{(1-\mathfrak{U})\exp(\psi_1\tau)}{(e^\tau-\mathfrak{U})}\frac{1}{\left(\alpha-\left(\frac{\psi_2}{\ln C}\frac{\partial^2}{\partial\psi_1^2}+\frac{\psi_3}{(\ln C)^2}\frac{\partial^3}{\partial\psi_1^3}+\cdots+\frac{\psi_m}{(\ln C)^{m-1}}\frac{\partial^m}{\partial\psi_1^m}\right)\right)^\nu}. \tag{119} $$

This can be further represented as

$$ \sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{F}_{n,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U})\frac{\tau^n}{n!} = \left(\sum_{k=0}^{\infty}\mathcal{F}_k(\psi_1;C|\mathfrak{U})\frac{\tau^k}{k!}\right)\left(\sum_{n=0}^{\infty}\mathcal{H}_{n,\nu}(\psi_2,\psi_3,\cdots,\psi_m;C;\alpha)\frac{\tau^n}{n!}\right). \tag{120} $$

By substituting n with $n-k$ and applying Cauchy product rule to the right hand side of the preceding expression, we can derive statement (118). □

Theorem 5.5. *The generalized EMHFEP of 1 parameter* ${}_{\mathcal{H}}\mathcal{F}_{n,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U})$ *can be expressed in the following explicit form:*

$$ {}_{\mathcal{H}}\mathcal{F}_{n,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U}) = \sum_{k=0}^{n}\binom{n}{m}\mathcal{F}_k(C|\mathfrak{U})\mathcal{H}_{n-k,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha). \tag{121} $$

Proof. The generating expression (115) can be represented in the following manner:

$$ \frac{(1-\mathfrak{U})\exp(\psi_1\tau)}{(e^\tau-\mathfrak{U})\left(\alpha-\left(\frac{\psi_2}{\ln C}\frac{\partial^2}{\partial\psi_1^2}+\frac{\psi_3}{(\ln C)^2}\frac{\partial^3}{\partial\psi_1^3}+\cdots+\frac{\psi_m}{(\ln C)^{m-1}}\frac{\partial^m}{\partial\psi_1^m}\right)\right)^\nu} = \frac{(1-\mathfrak{U})}{(e^\tau-\mathfrak{U})}\frac{\exp(\psi_1\tau)}{\left(\alpha-\left(\frac{\psi_2}{\ln C}\frac{\partial^2}{\partial\psi_1^2}+\frac{\psi_3}{(\ln C)^2}\frac{\partial^3}{\partial\psi_1^3}+\cdots+\frac{\psi_m}{(\ln C)^{m-1}}\frac{\partial^m}{\partial\psi_1^m}\right)\right)^\nu}. \tag{122} $$

This can be further represented as

$$ \sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{F}_{n,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U})\frac{\tau^n}{n!} = \left(\sum_{k=0}^{\infty}\mathcal{F}_k(C|\mathfrak{U})\frac{\tau^k}{k!}\right)\left(\sum_{n=0}^{\infty}\mathcal{H}_{n,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha)\frac{\tau^n}{n!}\right). \tag{123} $$

By substituting n with $n-k$ and applying Cauchy product rule to the right hand side of the preceding expression, we can derive statement (121). □

Theorem 5.6. *The following recurrence relations for the generalized EMHFEP of 1 parameter* ${}_{\mathcal{H}}\mathcal{F}_{n,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U})$ *holds:*

$$ \frac{\partial}{\partial_{\psi_1}}{}_{\mathcal{H}}\mathcal{F}_{n,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U}) = \nu\, n\ln C\,{}_{\mathcal{H}}\mathcal{F}_{n-1,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U}), \tag{124} $$

$$\frac{\partial}{\partial_{\psi_2}}{}_{\mathcal{H}}\mathcal{F}_{n,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U}) = \nu\, n(n-1)\ln C_{\mathcal{H}}\mathcal{F}_{n-2,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U}), \tag{125}$$

$$\frac{\partial}{\partial\psi_3}{}_{\mathcal{H}}\mathcal{F}_{n,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U}) = \nu\, n(n-1)(n-2)\ln C_{\mathcal{H}}\mathcal{F}_{n-3,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U}), \tag{126}$$

$$\vdots$$

$$\frac{\partial}{\partial\psi_m}{}_{\mathcal{H}}\mathcal{F}_{n,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U}) = \nu\, n(n-1)(n-2)\cdots(n-m+1)\ln C_{\mathcal{H}}\mathcal{F}_{n-m,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U}), \tag{127}$$

and

$$\frac{\partial}{\partial\alpha}{}_{\mathcal{H}}\mathcal{F}_{n,\nu}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U}) = -\nu_{\mathcal{H}}\mathcal{F}_{n,\nu+1}(\psi_1,\psi_2,\psi_3,\cdots,\psi_m;C;\alpha|\mathfrak{U}). \tag{128}$$

6. Conclusion

This study presents a novel approach for formulating the *EMHFEP* of 1-parameter. The foundational aspects of these polynomials are investigated through the lens of generating functions, detailed series expansions, and determinant-based formulations. A structured factorization methodology facilitates the derivation of essential results, including recurrence formulas, shift operators, and various differential equations—ordinary, partial, and integrodifferential.

Prospective research directions may involve generalizing the current framework to multivariable cases, allowing for richer structural analysis and discovering new functional identities. Further exploration of their analytical traits, such as orthogonality relations, asymptotic profiles, and the distribution of zeros—will enhance theoretical understanding. On the computational side, developing efficient algorithms tailored for applications in numerical simulation, theoretical physics, and engineering would significantly increase the utility of these polynomials. Additionally, investigating their potential to solve complex, higher-order differential equations arising in real-world systems could open new avenues. Finally, extending the application scope to domains like quantitative finance, biological modelling, and data-driven sciences—coupled with rigorous graphical and numerical assessments—could offer fresh theoretical insights and practical breakthroughs.

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