Filomat 40:8 (2026), 3043–3059
https://doi.org/10.2298/FIL2608043A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

q-almost Riordan arrays

Yasemin Alp[a,*], Fatma Yesil Baran[b], E. Gokcen Kocer[c], Naim Tuglu[d]

[a]*Department of Education of Mathematics and Science, Faculty of Education, Selcuk University, Konya-Türkiye*
[b]*Amasya University, Faculty of Engineering, Department of Computer Engineering, Amasya-Türkiye*
[c]*Necmettin Erbakan University, Faculty of Science, Department of Mathematics and Computer Sciences, Konya-Türkiye*
[d]*Gazi University, Faculty of Science, Department of Mathematics, 06500 Teknikokullar, Ankara-Türkiye*

Abstract. Our focus is on examining a group of Riordan arrays in which each member is represented by a triple of power series that is called an almost-Riordan array. If we take a triple of power series that is chosen as a q-function, we identify the q-analogue of the almost-Riordan arrays called q-almost Riordan arrays. In addition, we obtain the fundamental theorem for q-almost Riordan arrays (FTqARA). For suitably chosen pairs of q-almost Riordan arrays, new formulas for the multiplication of any q-almost Riordan arrays are obtained. Finally, with the help of the FTqARA, the generating functions for some row sums of q-almost Riordan matrices are derived.

1. Introduction and Preliminaries

In combinatorics, Riordan arrays are crucial for deriving combinatorial identities. Furthermore, Riordan arrays are useful in a wide range of mathematical domains. A Riordan array is an infinite lower triangular matrix defined by formal power series. Let us consider the following formal power series

$$g(t) = \sum_{k=0}^{\infty} g_k t^k$$

and

$$f(t) = \sum_{k=1}^{\infty} f_k t^k$$

with $g_0 f_1 \neq 0$. The generating function of the jth column of the Riordan matrix is defined as follows

$$g(t) f^j(t), \quad j = 0, 1, 2 \ldots.$$

2020 *Mathematics Subject Classification*. Primary 05A30; Secondary 11C20, 15B99, 05A15.
Keywords. Riordan arrays, almost-Riordan arrays, q-analogue.
Received: 20 January 2025; Revised: 13 December 2025; Accepted: 22 January 2026
Communicated by Paola Bonacini
* Corresponding author: Yasemin Alp
Email addresses: yaseminalp66@gmail.com (Yasemin Alp), ftm1618ysl@gmail.com (Fatma Yesil Baran), ekocer@erbakan.edu.tr (E. Gokcen Kocer), naimtuglu@gazi.edu.tr (Naim Tuglu)
ORCID iDs: https://orcid.org/0000-0002-4146-7374 (Yasemin Alp), https://orcid.org/0000-0001-8613-2706 (Fatma Yesil Baran), https://orcid.org/0000-0002-7154-9063 (E. Gokcen Kocer), https://orcid.org/0000-0002-7277-0034 (Naim Tuglu)

$D = (g(t), f(t))$ is another way to express a Riordan matrix as a pair of formal power series. The multiplication of two Riordan matrices is defined by

$$(g(t), f(t))(u(t), v(t)) = (g(t)(u \circ f)(t), (v \circ f)(t)) \tag{1}$$

where $\circ$ denotes the composition of the functions. The set of Riordan matrices is a group under matrix multiplication with (1) and denoted by $\mathcal{R}$ in [17].

The Fundamental Theorem of Riordan Arrays (FTRA) is an essential theorem in the study of Riordan arrays. Let $A(t) = \sum_{k=0}^{\infty} a_k t^k$ and $B(t) = \sum_{k=0}^{\infty} b_k t^k$ be any formal series and $(g(t), f(t))$ be a Riordan array. Then, FTRA is

$$(g(t), f(t)) A(t) = B(t) \Leftrightarrow g(t) A(f(t)) = B(t)$$

in [18]. Many topics have been studied associated with Riordan arrays. One of them is the characterization of Riordan arrays. The Riordan arrays are characterized by the A and $Z-$sequences in [15, 16]. More information related to Riordan arrays can be found in [4, 13, 19, 21].

The q-calculus has a crucial role in many fields, including mathematics and physics. Especially, q-calculus has wide applications in number theory, quantum group theory, and combinatorics [8]. The Riordan group notation is closely associated with the Lagrange inversion formula. Numerous generalizations of the Lagrange inversion formula with the q-analogue have been studied in [12].

Let $F(t) = \sum_{k=0}^{\infty} F_k t^k$ and $f(t) = \sum_{k=1}^{\infty} f_k t^k$ be any formal series. Garsia defined

$$F \overline{\circ} f(t) = \sum_{n \geq 0} F_n \prod_{k=0}^{n-1} f\left(q^k t\right) \tag{2}$$

and

$$F \underline{\circ} f(t) = \sum_{n \geq 0} F_n \prod_{k=0}^{n-1} f\left(t/q^k\right) \tag{3}$$

which is named by q-analogue of composition of functions [9]. Detailed information on the q-calculus can be found in [10, 11].

Many studies have been associated with the q analog of Riordan arrays. The q-analogue of Riordan arrays has been obtained by defining two binary operations $*_q$ and $*_{1/q}$ in [22]. Consider the following generating functions:

$$g(t) = \sum_{k=0}^{\infty} g_k(q) t^k$$

and

$$f(t) = \sum_{k=1}^{\infty} f_k(q) t^k$$

with $g_0(q) f_1(q) \neq 0$. Two binary operations $*_q$ and $*_{1/q}$ are provided with the following expressions

$$g(t) *_q f(t) = g(t) f(qt) \tag{4}$$

and

$$g(t) *_{1/q} f(t) = g(t) f(t/q). \tag{5}$$

Furthermore, the q-analogue of the Riordan arrays is indicated by $(g(t), f(t))_q$ and $(g(t), f(t))_{1/q}$. The generating functions of their jth columns are, respectively, as follows:

$$g(t) *_q f^{[\bar{j}]}(t), \quad for \quad j = 0, 1, 2, \ldots \tag{6}$$

and

$$g(t) *_{1/q} f^{[\underline{j}]}(t), \quad for \quad j = 0, 1, 2, \ldots \tag{7}$$

where the jth power of $f(t)$ is defined by

$$f^{[\bar{j}]}(t) = f(t) *_q f^{[\overline{j-1}]}(t) \tag{8}$$

and

$$f^{[\underline{j}]}(t) = f(t) *_{1/q} f^{[\underline{j-1}]}(t) \tag{9}$$

for $j \geq 1$ and $f^{[\bar{0}]}(t) = f^{[\underline{0}]}(t) = 1$. Using (2) and (3), the q-analogue of the fundamental theorem of Riordan arrays has been obtained

$$(g(t), f(t))_q A(t) = B(t) \Leftrightarrow g(t) *_q (A \bar{\circ} f)(t) = B(t) \tag{10}$$

and

$$(g(t), f(t))_{1/q} A(t) = B(t) \Leftrightarrow g(t) *_{1/q} (A \underline{\circ} f)(t) = B(t) \tag{11}$$

[22]. In addition, the q-Riordan array has been defined by using the Eulerian generating functions in [6]. Furthermore, some applications of q-Riordan arrays have been provided in [7]. The multiplications of any q-Riordan arrays and q-double Riordan arrays have been found in [2, 24].

The generalizations of Riordan arrays have been the subject of recent research. One of them is the almost-Riordan arrays. Barry has considered the following formal power series

$$k(t) = \sum_{s=0}^{\infty} k_s t^s, \quad g(t) = \sum_{s=0}^{\infty} g_s t^s, \quad f(t) = \sum_{s=1}^{\infty} f_s t^s$$

with $k_0 g_0 f_1 \neq 0$. $(k(t)|g(t), f(t))$ is an almost-Riordan array of order 1. The generating function for the jth column of the almost-Riordan array of order 1 is

$$\begin{array}{rlcl} k(t) & \text{for } j & = & 0 \\ tg(t) f^{j-1}(t) & \text{for } k & = & 1, 2, \ldots \end{array}$$

Furthermore, the multiplication of two almost-Riordan arrays is given

$$(k(t)|g(t), f(t))(\ell(t)|u(t), v(t)) = ((k(t)|g(t), f(t))\, \ell(t)\, |g(t)(u \circ f)(t), (v \circ f)(t)) \tag{12}$$

where $(k(t)|g(t), f(t))\, \ell(t)$ is

$$(k(t)|g(t), f(t))\, \ell(t) = \ell_0 k(t) + tg(t) \frac{(\ell \circ f)(t) - \ell_0}{f(t)}.$$

The set of almost-Riordan arrays is a group with multiplication (12) and is denoted by $a\mathcal{R}$ in [3]. The sequence characterizations of almost-Riordan arrays have been provided in [1, 3]. The involution and total positivity of the almost-Riordan group have been obtained in [5, 14, 20, 23].

Based on the preceding studies, the main aim of this paper is to examine the q-analogue of the almost-Riordan arrays. The definitions of the q-analogue of the almost-Riordan arrays are provided. In addition, the fundamental theorem for q-almost Riordan arrays is obtained. Using the q-analogue of the fundamental theorem, the row, the alternating row, the weighted row and the alternating weighted row sums are given. Furthermore, the multiplication of any q-almost Riordan arrays is found.

2. Main Results

The section provides a definition of the q-almost Riordan arrays as well as their fundamental theorem. In addition, we obtain the generating functions for some row sums of the q-almost Riordan arrays.

Definition 2.1. *Consider the following generating functions*

$$k(t) = \sum_{s=0}^{\infty} k_s(q)t^s \tag{13}$$

$$g(t) = \sum_{s=0}^{\infty} g_s(q)t^s \tag{14}$$

$$f(t) = \sum_{s=1}^{\infty} f_s(q)t^s \tag{15}$$

with $k_0(q)g_0(q)f_1(q) \neq 0$. *The q-almost Riordan arrays are represented as* $(k(t)|g(t), f(t))_q$. *The generating function for the jth column of the q-almost Riordan arrays is*

$$k(t), \text{ for } j = 0, \tag{16}$$

$$tg(t) *_q f^{\overline{[j-1]}}(t), \quad \text{for } j = 1, 2, 3, \ldots \tag{17}$$

Theorem 2.2 (FTqARA). *Let* $h(t) = \sum_{s=0}^{\infty} h_s(q)t^s$ *be a generating function and be* $(k(t)|g(t), f(t))_q$ *a q-almost Riordan array. Then,*

$$(k(t)|g(t), f(t))_q\, h(t) = h_0(q)k(t) + tg(t) *_q (\widetilde{h \overline{\circ}} f)(t) \tag{18}$$

where

$$\widetilde{h}(t) = \frac{h(t) - h_0(q)}{t}.$$

Proof. Taking into account the product of $(k(t)|g(t), f(t))_q$ and $h(t)$, we have

$$(k(t)|g(t), f(t))_q\, h(t) = h_0(q)k(t) + h_1(q)\left(tg(t) *_q f^{\overline{[0]}}(t)\right) + h_2(q)\left(tg(t) *_q f^{\overline{[1]}}(t)\right) + \cdots.$$

From (4) and (8), we get

$$(k(t)|\, g(t), f(t))_q\, h(t) = h_0(q)k(t) + tg(t)\left(h_1(q) + h_2(q)f(qt) + h_3(q)f(qt)f\left(q^2t\right) + \cdots\right).$$

Considering (2), (4) and $\widetilde{h}(t) = \frac{h(t)-h_0(q)}{t}$, the result is obtained. □

If we take $h(t) = \frac{1}{1-t}$ and $h(t) = \frac{1}{1+t}$ in (18), we obtain the generating functions for the row sums and the alternating row sums of the q-almost Riordan arrays in the following corollary.

Corollary 2.3. *The row and the alternating row sums of* $(k(t)|g(t), f(t))_q$ *have the following generating functions*

$$R_q^+(t) = k(t) + tg(t)\left(1 + f(qt) + f(qt)f\left(q^2t\right) + \cdots\right)$$

and

$$R_q^-(t) = k(t) - tg(t)\left(1 - f(qt) + f(qt)f\left(q^2t\right) - \cdots\right).$$

Example 2.4. *Consider* $\left(\frac{1}{1-t-t^2}|\frac{1}{1-t},\frac{t}{1-t}\right)_q$. *This matrix is*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \vdots \\ 1 & 1 & 0 & 0 & 0 & \vdots \\ 2 & 1 & q & 0 & 0 & \vdots \\ 3 & 1 & q(q+1) & q^3 & 0 & \vdots \\ 5 & 1 & q\left(q^2+q+1\right) & q^3\left(q^2+q+1\right) & q^6 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The generating functions for the row and the alternating row sums of $\left(\frac{1}{1-t-t^2}|\frac{1}{1-t},\frac{t}{1-t}\right)_q$ *are*

$$R_q^+(t) = \frac{1}{1-t-t^2} + \frac{t}{1-t}\left(1 + \frac{qt}{1-qt} + \frac{qt}{1-qt}\frac{q^2t}{1-q^2t} + \frac{qt}{1-qt}\frac{q^2t}{1-q^2t}\frac{q^3t}{1-q^3t} + \cdots\right)$$

and

$$R_q^-(t) = \frac{1}{1-t-t^2} - \frac{t}{1-t}\left(1 - \frac{qt}{1-qt} + \frac{qt}{1-qt}\frac{q^2t}{1-q^2t} - \frac{qt}{1-qt}\frac{q^2t}{1-q^2t}\frac{q^3t}{1-q^3t} + \cdots\right).$$

Hence, the sequences of row and the alternating row sums are as follows:

$$\{1, 2, q+3, q^3+q^2+q+4, q^6+q^5+q^4+2q^3+q^2+q+6, \ldots\}$$

and

$$\{1, 0, q+1, -q^3+q^2+q+2, q^6-q^5-q^4+q^2+q+4, \ldots\}.$$

Taking $h(t) = \frac{1}{(1-t)^2}$ and $h(t) = \frac{1}{(1+t)^2}$ in (18), the generating functions for the weighted row and the alternating weighted row sums of the q-almost Riordan arrays are obtained as the following corollary.

Corollary 2.5. *The weighted row and the alternating weighted row sums of* $(k(t)|g(t), f(t))_q$ *have the following generating functions*

$$W_q^+(t) = k(t) + tg(t)\left(2 + 3f(qt) + 4f(qt)f\left(q^2t\right) + \cdots\right)$$

and

$$W_q^-(t) = k(t) - tg(t)\left(2 - 3f(qt) + 4f(qt)f\left(q^2t\right) - \cdots\right).$$

Example 2.6. *Let us take* $\left(\frac{2-t}{1-t-t^2}|\frac{1}{1+t},\frac{t}{1+t}\right)_q$. *This matrix is*

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & \vdots \\ 1 & 1 & 0 & 0 & 0 & \vdots \\ 3 & -1 & q & 0 & 0 & \vdots \\ 4 & 1 & -q(q+1) & q^3 & 0 & \vdots \\ 7 & -1 & q\left(q^2+q+1\right) & -q^3\left(q^2+q+1\right) & q^6 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The generating functions for the weighted row and the alternating weighted row sums are

$$W_q^+(t) = \frac{2-t}{1-t-t^2} + \frac{t}{1+t}\left(2 + 3\frac{qt}{1+qt} + 4\frac{qt}{1+qt}\frac{q^2t}{1+q^2t} + 5\frac{qt}{1+qt}\frac{q^2t}{1+q^2t}\frac{q^3t}{1+q^3t} + \cdots\right)$$

and

$$W_q^-(t) = \frac{2-t}{1-t-t^2} - \frac{t}{1+t}\left(2 - 3\frac{qt}{1+qt} + 4\frac{qt}{1+qt}\frac{q^2t}{1+q^2t} - 5\frac{qt}{1+qt}\frac{q^2t}{1+q^2t}\frac{q^3t}{1+q^3t} + \cdots\right).$$

The sequences of $W_q^+(t)$ and $W_q^-(t)$ are given as

$$\{2, 3, 3q+1, 4q^3-3q^2-3q+6, 5q^6-4q^5-4q^4-q^3+3q^2+3q+5, \ldots\}$$

and

$$\{2, -1, 3q+5, -4q^3-3q^2-3q+2, 5q^6+4q^5+4q^4+7q^3+3q^2+3q+9, \ldots\}.$$

Definition 2.7. *For the generating functions* (13)-(15)*, the q-almost Riordan array's other notation is* $(k(t)|g(t), f(t))_{1/q}$. *The generating function for the jth column of the q-almost Riordan arrays is*

$$k(t), \qquad for\ \ j = 0, \tag{19}$$

$$tg(t) *_{1/q} f^{\underline{[j-1]}}(t), \qquad for\ \ j = 1, 2, 3, \ldots. \tag{20}$$

Theorem 2.8 (FTqARA). *Let $(k(t)|g(t), f(t))_{1/q}$ be a q-almost Riordan array and $h(t) = \sum_{s=0}^{\infty} h_s(q)\, t^s$. Then*

$$(k(t)|g(t), f(t))_{1/q}\, h(t) = h_0(q)k(t) + tg(t) *_{1/q} (\widetilde{h} \underline{\circ} f)(t) \tag{21}$$

where

$$\widetilde{h}(t) = \frac{h(t) - h_0(q)}{t}.$$

Proof. Considering the product $(k(t)|g(t), f(t))_{1/q}$ and $h(t)$, we get

$$(k(t)|g(t), f(t))_{1/q}\, h(t) = h_0(q)k(t) + h_1(q)\left(tg(t) *_{1/q} f^{\underline{[0]}}(t)\right) + h_2(q)\left(tg(t) *_{1/q} f^{\underline{[1]}}(t)\right) + \cdots.$$

From (5) and (9), we have

$$(k(t)|g(t), f(t))_{1/q}\, h(t) = h_0(q)k(t) + tg(t)\left(h_1(q) + h_2(q)f(t/q) + h_3(q)f(t/q)f\left(t/q^2\right) + \cdots\right).$$

Using (3), (5) and $\widetilde{h}(t) = \frac{h(t)-h_0(q)}{t}$, the result is clear. □

Now, we give the generating functions for the row sum, the alternating row sum, the weighted row sum and the alternating weighted row sum of $(k(t)|g(t), f(t))_{1/q}$.

Corollary 2.9. *The generating functions of the row, the alternating row, the weighted and the alternating weighted row sums for $(k(t)|g(t), f(t))_{1/q}$ are*

$$R_{1/q}^+(t) = k(t) + tg(t)\left(1 + f(t/q) + f(t/q)f\left(t/q^2\right) + \cdots\right)$$
$$R_{1/q}^-(t) = k(t) - tg(t)\left(1 - f(t/q) + f(t/q)f\left(t/q^2\right) - \cdots\right)$$
$$W_{1/q}^+(t) = k(t) + tg(t)\left(2 + 3f(t/q) + 4f(t/q)f\left(t/q^2\right) + \cdots\right)$$
$$W_{1/q}^-(t) = k(t) - tg(t)\left(2 - 3f(t/q) + 4f(t/q)f\left(t/q^2\right) - \cdots\right)$$

respectively.

Example 2.10. *Consider* $\left(\frac{1}{1-t-t^2}|\frac{1}{1-t},\frac{t}{1-t}\right)_{1/q}$. *This matrix is*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \vdots \\ 1 & 1 & 0 & 0 & 0 & \vdots \\ 2 & 1 & \frac{1}{q} & 0 & 0 & \vdots \\ 3 & 1 & \frac{1}{q}\left(\frac{1}{q}+1\right) & \frac{1}{q^3} & 0 & \vdots \\ 5 & 1 & \frac{1}{q}\left(\frac{1}{q^2}+\frac{1}{q}+1\right) & \frac{1}{q^3}\left(\frac{1}{q^2}+\frac{1}{q}+1\right) & \frac{1}{q^6} & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The generating functions for the row and the alternating row sums of $\left(\frac{1}{1-t-t^2}|\frac{1}{1-t},\frac{t}{1-t}\right)_{1/q}$ are

$$R^+_{1/q}(t)=\frac{1}{1-t-t^2}+\frac{t}{1-t}\left(1+\frac{t/q}{1-t/q}+\frac{t/q}{1-t/q}\frac{t/q^2}{1-t/q^2}+\frac{t/q}{1-t/q}\frac{t/q^2}{1-t/q^2}\frac{t/q^3}{1-t/q^3}+\cdots\right)$$

and

$$R^-_{1/q}(t)=\frac{1}{1-t-t^2}-\frac{t}{1-t}\left(1-\frac{t/q}{1-t/q}+\frac{t/q}{1-t/q}\frac{t/q^2}{1-t/q^2}-\frac{t/q}{1-t/q}\frac{t/q^2}{1-t/q^2}\frac{t/q^3}{1-t/q^3}+\cdots\right).$$

These sums are

$$\left\{1,2,\frac{1}{q}+3,\frac{1}{q}+\frac{1}{q^2}+\frac{1}{q^3}+4,\frac{1}{q^6}\left(6q^6+q^5+q^4+2q^3+q^2+q+1\right),\ldots\right\}$$

and

$$\left\{1,0,\frac{1}{q}+1,\frac{1}{q}+\frac{1}{q^2}-\frac{1}{q^3}+2,\frac{1}{q^6}\left(4q^6+q^5+q^4-q^2-q+1\right),\ldots\right\}.$$

Similarly, the generating functions for the weighted row and the alternating weighted row sums of $\left(\frac{1}{1-t-t^2}|\frac{1}{1-t},\frac{t}{1-t}\right)_{1/q}$ are

$$W^+_{1/q}(t)=\frac{1}{1-t-t^2}+\frac{t}{1-t}\left(2+3\frac{t/q}{1-t/q}+4\frac{t/q}{1-t/q}\frac{t/q^2}{1-t/q^2}+5\frac{t/q}{1-t/q}\frac{t/q^2}{1-t/q^2}\frac{t/q^3}{1-t/q^3}+\cdots\right)$$

and

$$W^-_{1/q}(t)=\frac{1}{1-t-t^2}-\frac{t}{1-t}\left(2-3\frac{t/q}{1-t/q}+4\frac{t/q}{1-t/q}\frac{t/q^2}{1-t/q^2}-5\frac{t/q}{1-t/q}\frac{t/q^2}{1-t/q^2}\frac{t/q^3}{1-t/q^3}+\cdots\right).$$

The first few elements of these sums are

$$\left\{1,3,\frac{3}{q}+4,\frac{3}{q}+\frac{3}{q^2}+\frac{4}{q^3}+5,\frac{1}{q^6}\left(7q^6+3q^5+3q^4+7q^3+4q^2+4q+5\right),\ldots\right\}$$

and

$$\left\{1,-1,\frac{3}{q},\frac{3}{q}+\frac{3}{q^2}-\frac{4}{q^3}+1,\frac{1}{q^6}\left(3q^6+3q^5+3q^4-q^3-4q^2-4q+5\right),\ldots\right\}.$$

In this part of the paper, we consider the multiplications of any q-almost Riordan arrays. The following four theorems give a new method for the multiplication of any q-almost Riordan matrices.

Theorem 2.11. *Let $D = (k(t)\,|g(t), f(t))_q$ and $R = (\ell(t)\,|u(t), v(t))_q$ be q-almost Riordan arrays. Then, as shown below, the generating function is given for the jth column of DR .*

$$(k(t)\,|g(t), f(t))_q\,\ell(t), \quad \text{for } j = 0 \tag{22}$$

$$\frac{tg(t)}{f(t)}\left(\left(tu *_q v^{[\overline{j-1}]}\right)\bar{\circ} f\right)(t), \quad \text{for } j = 1, 2, \ldots . \tag{23}$$

The following equation represents the multiplication of two q-almost Riordan arrays:

$$(k(t)\,|g(t), f(t))_q\,(\ell(t)\,|u(t), v(t))_q = \left((k(t)\,|g(t), f(t))_q\,\ell(t)\,|g(t), h(t/q)\right)_q$$

where

$$h^{[\bar{j}]}(t) = \frac{\left(\left(tu *_q v^{[\bar{j}]}\right)\bar{\circ} f\right)(t)}{f(t)}.$$

Proof. Taking into account the matrix product, the generating function for the 0th column of DR is

$$r_{0,0}k(t) + r_{1,0}\left(tg(t) *_q f^{[\bar{0}]}(t)\right) + r_{2,0}\left(tg(t) *_q f^{[\bar{1}]}(t)\right) + r_{3,0}\left(tg(t) *_q f^{[\bar{2}]}(t)\right) + \cdots .$$

From (4) and (8), we have

$$r_{0,0}\,k(t) + tg(t)\left(r_{1,0} + r_{2,0}f(qt) + r_{3,0}f(qt)\,f\left(q^2t\right) + \cdots\right).$$

Using (2) and (18), the generating function for the 0th column of DR is obtained as

$$(k(t)\,|g(t), f(t))_q\,\ell(t).$$

For the first column of DR , the generating function is

$$r_{1,1}\left(tg(t) *_q f^{[\bar{0}]}(t)\right) + r_{2,1}\left(tg(t) *_q f^{[\bar{1}]}(t)\right) + r_{3,1}\left(tg(t) *_q f^{[\bar{2}]}(t)\right) + \cdots .$$

Considering (4) and (8), we have

$$tg(t)\left(r_{1,1} + r_{2,1}f(qt) + r_{3,1}f(qt)\,f\left(q^2t\right) + \ldots\right).$$

Using (2), the generating function for the 1st column of DR is given as

$$tg(t)\left(\frac{(tu\bar{\circ} f)(t)}{f(t)}\right).$$

Similarly, the generating function for the jth column of DR is

$$r_{j,j}\left(tg(t) *_q f^{[\overline{j-1}]}(t)\right) + r_{j+1,j}\left(tg(t) *_q f^{[\bar{j}]}(t)\right) + r_{j+2,j}\left(tg(t) *_q f^{[\overline{j+1}]}(t)\right) + \ldots .$$

From (4), we get

$$tg(t)\left(r_{j,j}f^{[\overline{j-1}]}(qt) + r_{j+1,j}f^{[\bar{j}]}(qt) + r_{j+2,j}f^{[\overline{j+1}]}(qt) + \ldots\right).$$

Using (2), (4) and (8), the generating function for the jth column of DR is

$$\frac{tg(t)}{f(t)}\left(\left(tu *_q v^{[\overline{j-1}]}\right)\bar{\circ} f\right)(t).$$

□

Example 2.12. *Let D and R be the q-almost Riordan arrays as follows*

$$D = \left(\frac{1}{1-t-t^2}\middle|\frac{1}{1-t}, \frac{t}{1-t}\right)_q \quad and \quad R = \left(\frac{1}{1+t}\middle|\frac{1}{1+t}, t\right)_q .$$

Namely, the q-almost Riordan matrices D and R are

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \vdots \\ 1 & 1 & 0 & 0 & 0 & \vdots \\ 2 & 1 & q & 0 & 0 & \vdots \\ 3 & 1 & q(q+1) & q^3 & 0 & \vdots \\ 5 & 1 & q\left(q^2+q+1\right) & q^3\left(q^2+q+1\right) & q^6 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \vdots \\ -1 & 1 & 0 & 0 & 0 & \vdots \\ 1 & -1 & q & 0 & 0 & \vdots \\ -1 & 1 & -q & q^3 & 0 & \vdots \\ 1 & -1 & q & -q^3 & q^6 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Considering (22)*, the generating function for the* 0*th column of DR is*

$$\frac{1}{1-t-t^2} - \frac{t}{1-t}\left(1 - \frac{qt}{1-qt} + \frac{qt}{1-qt}\frac{q^2t}{1-q^2t} - \frac{qt}{1-qt}\frac{q^2t}{1-q^2t}\frac{q^3t}{1-q^3t} + \dots\right).$$

Hence, the first few elements of the 0*th column are*

$$\left\{1, 0, q+1, -q^3+q^2+q+2, q^6-q^5-q^4+q^2+q+4, \dots\right\}.$$

From (23)*, the generating function for the* 1*st column of DR is*

$$\frac{t}{1-t}\left(1 - \frac{qt}{1-qt} + \frac{qt}{1-qt}\frac{q^2t}{1-q^2t} - \frac{qt}{1-qt}\frac{q^2t}{1-q^2t}\frac{q^3t}{1-q^3t} + \cdots\right)$$

and the first terms of the 1*st column are*

$$\left\{0, 1, 1-q, (q-1)^2(q+1), -(q-1)^3(q^3+2q^2+2q+1), \dots\right\}.$$

Using (23)*, the generating function for the* 2*nd column of DR is*

$$\frac{qt}{1-t}\frac{qt}{1-qt}\left(1 - \frac{q^2t}{1-q^2t} + \frac{q^2t}{1-q^2t}\frac{q^3t}{1-q^3t} - \frac{q^2t}{1-q^2t}\frac{q^3t}{1-q^3t}\frac{q^4t}{1-q^4t} + \cdots\right)$$

and the elements of the 2*nd column are*

$$\left\{0, 0, q^2, q^2(-q^2+q+1), q^2(q^5-q^4-q^3+q+1), \dots\right\}.$$

Using (23)*, the generating functions for the other columns can be obtained in a similar way. Finally, the matrix DR is obtained as follows*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \vdots \\ 0 & 1 & 0 & 0 & 0 & \vdots \\ q+1 & 1-q & q^2 & 0 & 0 & \vdots \\ -q^3+q^2+q+2 & (q-1)^2(q+1) & q^2(-q^2+q+1) & q^6 & 0 & \vdots \\ q^6-q^5-q^4+q^2+q+4 & -(q-1)^3(q^3+2q^2+2q+1) & q^2\left(q^5-q^4-q^3+q+1\right) & q^6(-q^3+q^2+q+1) & q^{12} & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 2.13. *Let $D = (k(t)\,|g(t), f(t))_q$ and $R = (\ell(t)\,|u(t), v(t))_{1/q}$ be q-almost Riordan arrays. Then, the following equation is the generating function for the jth column of DR*

$$(k(t)\,|g(t), f(t))_q\, \ell(t), \quad \text{for } j = 0 \tag{24}$$

$$\frac{tg(t)}{f(t)}\left(\left(tu *_{1/q} v^{\left[\overline{j-1}\right]}\right)\overline{\circ} f\right)(t), \quad \text{for } j = 1, 2, \ldots. \tag{25}$$

The multiplication of two q-almost Riordan arrays is

$$(k(t)\,|g(t), f(t))_q\,(\ell(t)\,|u(t), v(t))_{1/q} = \left((k(t)\,|g(t), f(t))_q\,\ell(t)\,|g(t), h(qt)\right)_{1/q}$$

where

$$h^{\left[\overline{j}\right]}(t) = \frac{\left(\left(tu *_q v^{\left[\overline{j}\right]}\right)\overline{\circ} f\right)(t)}{f(t)}.$$

Proof. Taking into account the matrix product, the generating function for the 0th column of DR is

$$r_{0,0}k(t) + r_{1,0}\left(tg(t) *_q f^{\left[\overline{0}\right]}(t)\right) + r_{2,0}\left(tg(t) *_q f^{\left[\overline{1}\right]}(t)\right) + r_{3,0}\left(tg(t) *_q f^{\left[\overline{2}\right]}(t)\right) + \cdots.$$

From (4) and (8), we have

$$r_{0,0}k(t) + tg(t)\left(r_{1,0} + r_{2,0}f(qt) + r_{3,0}f(qt)f\left(q^2t\right) + \ldots\right).$$

Using (2) and (18), we obtain the generating function for the 0th column of DR as follows

$$(k(t)\,|g(t), f(t))_q\,\ell(t).$$

The generating function for the 1st column of DR is

$$r_{1,1}\left(tg(t) *_q f^{\left[\overline{0}\right]}(t)\right) + r_{2,1}\left(tg(t) *_q f^{\left[\overline{1}\right]}(t)\right) + r_{3,1}\left(tg(t) *_q f^{\left[\overline{2}\right]}(t)\right) + \cdots.$$

Considering (2), (4) and (8), the generating function for the 1st column of DR is obtained

$$tg(t)\left(\frac{(tu(t))\,\overline{\circ} f(t)}{f(t)}\right).$$

Similarly, the generating function for the jth column of DR is

$$r_{j,j}\left(tg(t) *_q f^{\left[\overline{j-1}\right]}(t)\right) + r_{j+1,j}\left(tg(t) *_q f^{\left[\overline{j}\right]}(t)\right) + r_{j+2,j}\left(tg(t) *_q f^{\left[\overline{j+1}\right]}(t)\right) + \cdots.$$

From (4), we get

$$tg(t)\left(r_{j,j}f^{\left[\overline{j-1}\right]}(qt)+r_{j+1,j}f^{\left[\overline{j}\right]}(qt)+r_{j+2,j}f^{\left[\overline{j+1}\right]}(qt)+\cdots\right).$$

Using (2), (4), and (8), the generating function for the jth column of DR is

$$\frac{tg(t)}{f(t)}\left(\left(tu *_{1/q} v^{\left[\underline{j-1}\right]}\right)\overline{\circ}f\right)(t).$$

□

Example 2.14. *Let D and R be the q-almost Riordan arrays as follows*

$$D=\left(\frac{1}{1-t-t^2}\middle|\frac{1}{1-t},\frac{t}{1-t}\right)_q \quad \text{and} \quad R=\left(\frac{1}{1+t}\middle|\frac{1}{1+t},t\right)_{1/q}.$$

Hence, we get

$$D=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \vdots \\ 1 & 1 & 0 & 0 & 0 & \vdots \\ 2 & 1 & q & 0 & 0 & \vdots \\ 3 & 1 & q(q+1) & q^3 & 0 & \vdots \\ 5 & 1 & q\left(q^2+q+1\right) & q^3\left(q^2+q+1\right) & q^6 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$R=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \vdots \\ -1 & 1 & 0 & 0 & 0 & \vdots \\ 1 & -1 & \frac{1}{q} & 0 & 0 & \vdots \\ -1 & 1 & -\frac{1}{q} & \frac{1}{q^3} & 0 & \vdots \\ 1 & -1 & \frac{1}{q} & -\frac{1}{q^3} & \frac{1}{q^6} & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

From (24)*, the generating function for the* 0*th column of DR is*

$$\frac{1}{1-t-t^2}+\frac{t}{1-t}\left(-1+\frac{qt}{1-qt}-\frac{qt}{1-qt}\frac{q^2t}{1-q^2t}+\frac{qt}{1-qt}\frac{q^2t}{1-q^2t}\frac{q^3t}{1-q^3t}-\cdots\right)$$

and the first few elements of the 0*th column are*

$$\left\{1,0,q+1,-q^3+q^2+q+2,q^6-q^5-q^4+q^2+q+4,\ldots\right\}.$$

From (25)*, the generating function for the* 1*st column of DR is obtained as*

$$\frac{t}{1-t}\left(1-\frac{qt}{1-qt}+\frac{qt}{1-qt}\frac{q^2t}{1-q^2t}-\frac{qt}{1-qt}\frac{q^2t}{1-q^2t}\frac{q^3t}{1-q^3t}+\cdots\right)$$

and the first terms of the 1*st column are*

$$\left\{0,1,1-q,(q-1)^2(q+1),-(q-1)^3(q^3+2q^2+2q+1),\ldots\right\}.$$

Using (25)*, the generating function and the first elements for the* 2*nd column of DR are*

$$\frac{t^2}{(1-t)(1-qt)}\left(1-\frac{q^2t}{1-q^2t}+\frac{q^2t}{1-q^2t}\frac{q^3t}{1-q^3t}-\frac{q^2t}{1-q^2t}\frac{q^3t}{1-q^3t}\frac{q^4t}{1-q^4t}+\cdots\right)$$

and

$$\left\{0,0,1,-q^2+q+1,q^5-q^4-q^3+q+1,\ldots\right\}.$$

Namely, the matrix DR is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \vdots \\ 0 & 1 & 0 & 0 & 0 & \vdots \\ q+1 & 1-q & 1 & 0 & 0 & \vdots \\ -q^3+q^2+q+2 & (q-1)^2(q+1) & -q^2+q+1 & 1 & 0 & \vdots \\ q^6-q^5-q^4+q^2+q+4 & -(q-1)^3(q^3+2q^2+2q+1) & q^5-q^4-q^3+q+1 & -q^3+q^2+q+1 & 1 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 2.15. *Let* $D=(k(t)\,|g(t),f(t))_{1/q}$ *and* $R=(\ell(t)\,|u(t),v(t))_q$ *be q-almost Riordan arrays. Then, the generating function for the jth column of DR is*

$$(k(t)\,|g(t),f(t))_{1/q}\,\ell(t), \quad \text{for } j=0 \tag{26}$$

$$\frac{tg(t)}{f(t)}\left(\left(tu *_q v^{[\underline{j-1}]}\right)\underline{\circ}f\right)(t), \quad \text{for } j=1,2,\ldots. \tag{27}$$

The multiplication of D and R is

$$(k(t)\,|g(t),f(t))_{1/q}\,(\ell(t)\,|u(t),v(t))_q=\left((k(t)\,|g(t),f(t))_{1/q}\,\ell(t)\,|g(t),h(t/q)\right)_q$$

where

$$h^{[\underline{j}]}(t)=\frac{\left(\left(tu *_q v^{[\underline{j}]}\right)\underline{\circ}f\right)(t)}{f(t)}.$$

Proof. Considering the matrix product, the generating function for the 0th column of *DR* is

$$r_{0,0}a(t)+r_{1,0}\left(tg(t) *_{1/q} f^{[\underline{0}]}(t)\right)+r_{2,0}\left(tg(t) *_{1/q} f^{[\underline{1}]}(t)\right)+r_{3,0}\left(tg(t) *_{1/q} f^{[\underline{2}]}(t)\right)+\cdots.$$

From (5) and (9), we have

$$r_{0,0}a(t)+tg(t)\left(r_{1,0}+r_{2,0}f(t/q)+r_{3,0}f(t/q)f\left(t/q^2\right)+\cdots\right).$$

Using (3) and (21), we obtain the generating function for the *DR*'s 0th column

$$(a(t)\,|g(t),f(t))_{1/q}\,\ell(t).$$

The generating function for the first column of DR is

$$r_{1,1}\left(tg\left(t\right)*_{1/q}f^{[\underline{0}]}\left(t\right)\right)+r_{2,1}\left(tg\left(t\right)*_{1/q}f^{[\underline{1}]}\left(t\right)\right)+r_{3,1}\left(tg\left(t\right)*_{1/q}f^{[\underline{2}]}\left(t\right)\right)+\cdots.$$

Using (3), (5) and (9), we get

$$tg\left(t\right)\left(\frac{\left(tu\left(t\right)\right)\underline{\circ}f\left(t\right)}{f\left(t\right)}\right).$$

Similarly, the generating function for the jth column of DR is

$$r_{j,j}\left(tg\left(t\right)*_{1/q}f^{[\underline{j-1}]}\left(t\right)\right)+r_{j+1,j}\left(tg\left(t\right)*_{1/q}f^{[\underline{j}]}\left(t\right)\right)+r_{j+2,j}\left(tg\left(t\right)*_{1/q}f^{[\underline{j+1}]}\left(t\right)\right)+\cdots.$$

From (5), we have

$$tg(t)\left(r_{j,j}f^{[\underline{j-1}]}\left(t/q\right)+r_{j+1,j}f^{[\underline{j}]}\left(t/q\right)+r_{j+2,j}f^{[\underline{j+1}]}\left(t/q\right)+\cdots\right).$$

Considering (3), (5), and (9), we find the generating function for the jth column of DR as

$$\frac{tg\left(t\right)}{f(t)}\left(\left(tu*_{q}v^{[\underline{j-1}]}\right)\underline{\circ}f\right)(t).$$

□

Example 2.16. *Let D and R be the q-almost Riordan arrays as follows*

$$D=\left(\frac{1}{1+t}\middle|\frac{1}{1+t},t\right)_{1/q}\quad \text{and}\quad R=\left(\frac{1}{1-t-t^{2}}\middle|\frac{1}{1-t},\frac{t}{1-t}\right)_{q}.$$

Hence, we get

$$D=\begin{pmatrix}1&0&0&0&0&\vdots\\-1&1&0&0&0&\vdots\\1&-1&\frac{1}{q}&0&0&\vdots\\-1&1&-\frac{1}{q}&\frac{1}{q^{3}}&0&\vdots\\1&-1&\frac{1}{q}&-\frac{1}{q^{3}}&\frac{1}{q^{6}}&\vdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}$$

and

$$R=\begin{pmatrix}1&0&0&0&0&\vdots\\1&1&0&0&0&\vdots\\2&1&q&0&0&\vdots\\3&1&q\left(q+1\right)&q^{3}&0&\vdots\\5&1&q\left(q^{2}+q+1\right)&q^{3}\left(q^{2}+q+1\right)&q^{6}&\vdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}.$$

Using (26)*, we get the generating function for the* 0*th column of DR as follows*

$$\frac{1}{1+t}+\frac{t}{1+t}\left(1+2\frac{t}{q}+3\frac{t^2}{q^3}+5\frac{t^3}{q^6}+8\frac{t^4}{q^{10}}+\cdots\right)$$

and the first few elements of the 0*th column are*

$$\left\{1,0,\frac{2}{q},-\frac{1}{q^3}(2q^2-3),\frac{1}{q^6}(2q^5-3q^3+5),\ldots\right\}.$$

From (27)*, the generating function for the first column of DR is*

$$\frac{t}{1+t}\left(1+\frac{t}{q}+\frac{t^2}{q^3}+\frac{t^3}{q^6}+\frac{t^4}{q^{10}}+\frac{t^5}{q^{14}}+\cdots\right)$$

and the first terms of the first column are

$$\left\{0,1,-\frac{1}{q}(q-1),\frac{1}{q^3}(q^3-q^2+1),-\frac{1}{q^6}(q^6-q^5+q^3-1),\ldots\right\}.$$

Similarly, the generating function and the first few elements for the second column of DR are obtained as

$$\frac{t^2}{1+t}\left(1+\frac{(q+1)t}{q^2}+\frac{(q^2+q+1)t^2}{q^5}+\frac{(q^3+q^2+q+1)t^3}{q^9}+\cdots\right)$$

and

$$\left\{0,0,1,\frac{1}{q^2}(-q^2+q+1),\frac{1}{q^5}(q^5-q^4-q^3+q^2+q+1),\ldots\right\}.$$

Hence, the matrix DR is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \vdots \\ 0 & 1 & 0 & 0 & 0 & \vdots \\ \frac{2}{q} & -\frac{1}{q}(q-1) & 1 & 0 & 0 & \vdots \\ -\frac{1}{q^3}(2q^2-3) & \frac{1}{q^3}(q^3-q^2+1) & \frac{1}{q^2}(-q^2+q+1) & 1 & 0 & \vdots \\ \frac{1}{q^6}(2q^5-3q^3+5) & -\frac{1}{q^6}(q^6-q^5+q^3-1) & \frac{1}{q^5}\left(q^5-q^4-q^3+q^2+q+1\right) & \frac{1}{q^3}(-q^3+q^2+q+1) & 1 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 2.17. *Let* $D=(k(t)\,|g(t),f(t))_{1/q}$ *and* $R=(\ell(t)\,|u(t),v(t))_{1/q}$ *be q-almost Riordan arrays. Then, the generating function for the jth column of DR is*

$$(k(t)\,|g(t),f(t))_{1/q}\,\ell(t),\quad \text{for } j=0 \tag{28}$$

$$\frac{tg(t)}{f(t)}\left(\left(tu *_{1/q} v^{\underline{[j-1]}}\right)\underline{\circ}f\right)(t),\quad \text{for } j=1,2,\ldots. \tag{29}$$

The following equation represents the multiplication of two q-almost Riordan arrays:

$$(k(t)\,|g(t),f(t))_{1/q}\,(\ell(t)\,|u(t),v(t))_{1/q}=\left((k(t)\,|g(t),f(t))_{1/q}\,\ell(t)\,|g(t),h(qt)\right)_{1/q}$$

where

$$h^{\underline{[j]}}(t)=\frac{\left(\left(tu *_{1/q} v^{\underline{[j]}}\right)\underline{\circ}f\right)(t)}{f(t)}.$$

Proof. Considering the matrix product, the generating function for the 0th column of DR is

$$r_{0,0}a(t) + r_{1,0}\left(tg(t) *_{1/q} f^{\underline{[0]}}(t)\right) + r_{2,0}\left(tg(t) *_{1/q} f^{\underline{[1]}}(t)\right) + r_{3,0}\left(tg(t) *_{1/q} f^{\underline{[2]}}(t)\right) + \cdots .$$

From (5) and (9), we have

$$r_{0,0}a(t) + tg(t)\left(r_{1,0} + r_{2,0}f(t/q) + r_{3,0}f(t/q)f\left(t/q^2\right) + \cdots\right).$$

Using (3) and (21), we have the generating function for DR's 0th column

$$(a(t)\,|g(t), f(t))_{1/q}\,\ell(t).$$

The generating function for the first column of DR is

$$r_{1,1}\left(tg(t) *_{1/q} f^{\underline{[0]}}(t)\right) + r_{2,1}\left(tg(t) *_{1/q} f^{\underline{[1]}}(t)\right) + r_{3,1}\left(tg(t) *_{1/q} f^{\underline{[2]}}(t)\right) + \cdots .$$

Considering (3), (5) and (9), we find the generating function for the first column of DR as follows

$$tg(t)\left(\frac{(tu(t)) \underline{\circ} f(t)}{f(t)}\right).$$

Similarly, the generating function for the jth column of DR is

$$r_{j,j}\left(tg(t) *_{1/q} f^{\underline{[j-1]}}(t)\right) + r_{j+1,j}\left(tg(t) *_{1/q} f^{\underline{[j]}}(t)\right) + r_{j+2,j}\left(tg(t) *_{1/q} f^{\underline{[j+1]}}(t)\right) + \cdots .$$

From (5), we have

$$tg(t)\left(r_{j,j}f^{\underline{[j-1]}}(t/q) + r_{j+1,j}f^{\underline{[j]}}(t/q) + r_{j+2,j}f^{\underline{[j+1]}}(t/q) + \cdots\right).$$

Considering (3), (5), and (9), the generating function for the jth column of DR is found as

$$\frac{tg(t)}{f(t)}\left(\left(tu *_{1/q} v^{\underline{[j-1]}}\right)\underline{\circ} f\right)(t).$$

□

Example 2.18. *Assume that D and R are the q-almost Riordan arrays defined by*

$$D = \left(\frac{1}{1+t}\middle|\frac{1}{1+t}, t\right)_{1/q} \quad \text{and} \quad R = \left(\frac{1}{1-t}\middle|\frac{1}{1-t}, \frac{t}{1-t}\right)_{1/q}.$$

Hence, we get

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \vdots \\ -1 & 1 & 0 & 0 & 0 & \vdots \\ 1 & -1 & \frac{1}{q} & 0 & 0 & \vdots \\ -1 & 1 & -\frac{1}{q} & \frac{1}{q^3} & 0 & \vdots \\ 1 & -1 & \frac{1}{q} & -\frac{1}{q^3} & \frac{1}{q^6} & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \vdots \\ 1 & 1 & 0 & 0 & 0 & \vdots \\ 1 & 1 & \frac{1}{q} & 0 & 0 & \vdots \\ 1 & 1 & \frac{1}{q}\left(\frac{1}{q}+1\right) & \frac{1}{q^3} & 0 & \vdots \\ 1 & 1 & \frac{1}{q}\left(\frac{1}{q^2}+\frac{1}{q}+1\right) & \frac{1}{q^3}\left(\frac{1}{q^2}+\frac{1}{q}+1\right) & \frac{1}{q^6} & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Using (28)*, we get the generating function for the* 0*th column of DR*

$$\frac{1}{1+t} + \frac{t}{1+t}\left(1 + \frac{t}{q} + \frac{t^2}{q^3} + \frac{t^3}{q^6} + \frac{t^4}{q^{10}} + \frac{t^5}{q^{15}} + \cdots\right).$$

The first few elements of the 0*th column are*

$$\left\{1, 0, \frac{1}{q}, -\frac{1}{q^3}(q^2-1), \frac{1}{q^6}(q^5-q^3+1), \dots\right\}.$$

The generating function and the first few elements for the first column of DR are

$$\frac{t}{1+t}\left(1 + \frac{t}{q} + \frac{t^2}{q^3} + \frac{t^3}{q^6} + \frac{t^4}{q^{10}} + \frac{t^5}{q^{14}} + \cdots\right)$$

and

$$\left\{0, 1, -\frac{1}{q}(q-1), \frac{1}{q^3}(q^3-q^2+1), -\frac{1}{q^6}(q^6-q^5+q^3-1), \dots\right\}.$$

Similarly, the generating function and the first few elements for the second column of DR are obtained as

$$\frac{t^2}{q^2(1+t)}\left(1 + \frac{t}{q^2}\left(\frac{1}{q}+1\right) + \frac{t^2}{q^5}\left(\frac{1}{q^2}+\frac{1}{q}+1\right) + \frac{t^3}{q^{10}}\left(\frac{1}{q^3}+\frac{1}{q^2}+\frac{1}{q}+1\right) + \cdots\right)$$

and

$$\left\{0, 0, \frac{1}{q^2}, \frac{1}{q^5}(-q^3+q+1), \frac{1}{q^9}(q^7-q^5-q^4+q^2+q+1), \dots\right\}.$$

Finally, the matrix DR is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \vdots \\ 0 & 1 & 0 & 0 & 0 & \vdots \\ \frac{1}{q} & -\frac{1}{q}(q-1) & \frac{1}{q^2} & 0 & 0 & \vdots \\ -\frac{1}{q^3}(q^2-1) & \frac{1}{q^3}(q^3-q^2+1) & \frac{1}{q^5}(-q^3+q+1) & \frac{1}{q^6} & 0 & \vdots \\ \frac{1}{q^6}(q^5-q^3+1) & -\frac{1}{q^6}(q^6-q^5+q^3-1) & \frac{1}{q^9}\left(q^7-q^5-q^4+q^2+q+1\right) & \frac{1}{q^{11}}(-q^5+q^2+q+1) & \frac{1}{q^{12}} & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

3. Conclusion

In this paper, we have considered q-almost Riordan arrays. We have provided the fundamental theorem for q-almost Riordan arrays (FTqAR). Using FTqAR, we have obtained four different multiplications of q-almost Riordan arrays. All results obtained in this paper are reduced to almost-Riordan arrays when $q \to 1^-$. Additionally, the results for Riordan arrays can be derived by taking $q \to 1^-$, $k(t) = g(t)$ and $g(t) = \frac{g(t)f(t)}{t}$.

References

[1] Y. Alp and E. G. Koçer, *Sequence characterization of almost-Riordan arrays*, Linear Algebra Appl. **664** (2023), 1–23.
[2] F. Y. Baran and N. Tuğlu, *q-Riordan representation*, Linear Algebra Appl. **525** (2017), 105–117.
[3] P. Barry, *On the group of almost-Riordan arrays*, arXiv:1606.05077.
[4] P. Barry, *Riordan Arrays: A Primer*, Logic Press, Raleigh, 2017.
[5] P. Barry and N. Pantelidis, *On pseudo-involutions, involutions and quasi-involutions in the group of almost-Riordan arrays*, J. Algebraic Combin. **54** (2021), 399–423.
[6] G. S. Cheon, J. H. Jung, and Y. Lim, *A q-analogue of the Riordan group*, Linear Algebra Appl. **439** (2013), 4119–4129.
[7] G. S. Cheon and J. H. Jung, *Some combinatorial applications of the q-Riordan matrix*, Linear Algebra Appl. **482** (2015), 241–260.
[8] T. Ernst, *A method for q-calculus*, J. Nonlinear Math. Phys. **10** (2003), 487–525.
[9] A. M. Garsia, *A q-analogue of the Lagrange inversion formula*, Houston J. Math. **7** (1981), 205–237.
[10] A. M. Garsia and J. Remmel, *A novel form of q-Lagrange inversion*, Houston J. Math. **12** (1986), 503–523.
[11] A. M. Garsia and M. Haiman, *A remarkable q, t-Catalan sequence and q-Lagrange inversion*, J. Algebraic Combin. **5** (1996), 191–244.
[12] I. Gessel, *A noncommutative generalization and q-analog of the Lagrange inversion formula*, Trans. Amer. Math. Soc. **257** (1980), 455–482.
[13] T. X. He and R. Sprugnoli, *Sequence characterization of Riordan arrays*, Discrete Math. **309** (2009), 3962–3974.
[14] T. X. He and R. Słowik, *Total positivity of almost-Riordan arrays*, arXiv:2406.03774.
[15] D. Merlini, D. G. Rogers, R. Sprugnoli, and M. C. Verri, *On some alternative characterizations of Riordan arrays*, Canad. J. Math. **49** (1997), 301–320.
[16] D. G. Rogers, *Pascal triangles, Catalan numbers and renewal arrays*, Discrete Math. **22** (1978), 301–310.
[17] L. W. Shapiro, S. Getu, W. J. Woan, and L. C. Woodson, *The Riordan group*, Discrete Appl. Math. **34** (1991), 229–239.
[18] L. W. Shapiro, *A survey of the Riordan group*, Amer. Math. Soc. Meeting, Richmond, VA, 1994.
[19] L. Shapiro, R. Sprugnoli, P. Barry, G. S. Cheon, T. X. He, D. Merlini, and W. Wang, *The Riordan Group and Applications*, Springer, Cham, 2022.
[20] R. Słowik, *More about involutions in the group of almost-Riordan arrays*, Linear Algebra Appl. **624** (2021), 247–258.
[21] R. Sprugnoli, *Riordan arrays and combinatorial sums*, Discrete Math. **132** (1994), 267–290.
[22] N. Tuğlu, F. Yeşil, M. Dziemiańczuk, and E. G. Koçer, *q-Riordan array for q-Pascal matrix and its inverse matrix*, Turkish J. Math. **40** (2016), 1038–1048.
[23] Y. Wang, J. Zhang, and H. Liang, *Some properties of combinatorial triangles related to Horadam polynomials*, Linear Multilinear Algebra (2024), 1–17.
[24] L. Zhang and X. Zhao, *q-double Riordan matrices*, Linear Algebra Appl. **603** (2020), 212–225.