



## Some results on 3-monochromatic vertex index

Yingbin Ma<sup>a</sup>, Wei Zhao<sup>a</sup>, Mingli Wang<sup>a</sup>, Xiaoxue Zhang<sup>a,\*</sup>

<sup>a</sup>School of Mathematics and Statistics  
Henan Normal University, Xixiang 453007, P.R. China

**Abstract.** A tree  $T$  in a vertex colored graph  $G$  is called a vertex-monochromatic tree if all the internal vertices of  $T$  have the same color. For  $S \subseteq V(G)$ , a vertex-monochromatic  $S$ -tree in  $G$  is a vertex-monochromatic tree of  $G$  containing the vertices of  $S$ . For a connected graph  $G$  and a given integer  $k$  with  $2 \leq k \leq |V(G)|$ , the  $k$ -monochromatic vertex-index  $mvx_k(G)$  of  $G$  is the maximum number of colors needed such that for each subset  $S \subseteq V(G)$  of  $k$  vertices, there exists a vertex-monochromatic  $S$ -tree. In this paper, we give an upper bound of  $mvx_3(G)$ . We present all graphs with  $mvx_3(G)$  of 3, 4, and verify that almost all simple graphs satisfy  $mvx_3(G) = n$ . We investigate the 3-monochromatic vertex-index of a graph  $G$  of order  $n$  and  $\omega(G) = n - i$  for  $1 \leq i \leq 3$ .

### 1. Introduction

We only consider simple, finite, undirected and connected graphs. Throughout this paper, let  $n$  and  $m$  denote the order and size of a graph. We refer to [1] for notation and terminology that are not defined here. In recent years, colored notions of connectivity in graphs become a new and active subject in graph theory. Starting from rainbow connection, rainbow vertex connection and total rainbow connection appeared later. Many researchers are working in this field, see [5–7, 13, 15, 20] for further details. The reader also can see [17] for a dynamic survey and [16] for a new monograph on this topic.

Actually, the concepts of rainbow connection number  $rc(G)$  and rainbow vertex connection number  $rvc(G)$  are natural opposite concepts of monochromatic connection number  $mc(G)$  and monochromatic vertex connection number  $mvc(G)$ . Caro et al. [4] introduced the concept of monochromatic connection. A graph  $G$  is called *monochromatically connected* if for any two vertices of  $G$  there exists a monochromatic path connecting them. As a natural generation, Cai et al. [3] introduced the concept of vertex monochromatic connection. The graph  $G$  is called *monochromatically vertex-connected*, if for any two vertices of  $G$  there exists a vertex monochromatic path connecting them. For a connected graph  $G$ , the *monochromatic vertex-connection*

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\* Corresponding author: Xiaoxue Zhang

*Email addresses*: mayingbinw@htu.cn (Yingbin Ma), zhaowei2022913@stu.htu.edu.cn (Wei Zhao), wangmingli@stu.htu.edu.cn (Mingli Wang), zhang\_xx1209@163.com (Xiaoxue Zhang)

ORCID iDs: <https://orcid.org/0000-0002-0380-9468> (Yingbin Ma), <https://orcid.org/0009-0004-7904-9364> (Wei Zhao), <https://orcid.org/0009-0006-1221-9255> (Mingli Wang), <https://orcid.org/0009-0005-5725-8058> (Xiaoxue Zhang)

number of  $G$ , denoted by  $mvc(G)$ , is the maximum number of colors that are needed in order to make  $G$  monochromatically vertex-connected. A vertex-coloring of  $G$  is a *monochromatically vertex-connecting coloring* (MVC-coloring) if it makes  $G$  monochromatically vertex-connected. Many researchers are working in this field, they prove some upper, lower bounds, random graphs, Erdős-Gallai-type Problem, the Nordhaus-Gaddum Problem and so on for  $mc(G)$  and  $mvc(G)$ , see [2, 9, 11]. We refer the reader to [19] for more details.

The concept of monochromatic path was generalized to monochromatic tree by Li et al. [18]. A tree  $T$  in an edge colored graph  $G$  is called a *monochromatic tree* if all the edges of  $T$  have the same color. For an  $S \subseteq V(G)$ , a *monochromatic  $S$ -tree* in  $G$  is a monochromatic tree of  $G$  containing the vertices of  $S$ . Given an integer  $k$  with  $2 \leq k \leq |V(G)|$ , the graph  $G$  is called  *$k$ -monochromatically connected* if for any set  $S$  of  $k$  vertices of  $G$ , there exists a monochromatic  $S$ -tree in  $G$ .

As a natural counterpart of the  $k$ -monochromatic index, Li et al. introduced the concept of  *$k$ -vertex monochromatic index* in [18]. A tree  $T$  in a vertex colored graph  $G$  is called a *vertex-monochromatic tree* if its internal vertices have the same color. For an  $S \subseteq V(G)$ , a *vertex-monochromatic  $S$ -tree* in  $G$  is a vertex-monochromatic tree of  $G$  containing the vertices of  $S$ . Given an integer  $k$  with  $2 \leq k \leq |V(G)|$ , the graph  $G$  is called  *$k$ -monochromatically vertex-connected* if for any set  $S$  of  $k$  vertices of  $G$ , there exists a vertex-monochromatic  $S$ -tree in  $G$ . For a connected graph  $G$  and a given integer  $k$  such that  $2 \leq k \leq |V(G)|$ , the  *$k$ -monochromatic vertex-index*  $mvx_k(G)$  of  $G$  is the maximum number of colors that are needed in order to make  $G$   $k$ -monochromatically vertex-connected. A vertex-coloring of  $G$  is called a  *$k$ -monochromatically vertex-connecting coloring* ( $MVX_k$ -coloring) if it makes  $G$   $k$ -monochromatically vertex-connected. An *extremal  $MVX_k$ -coloring* is an  $MVX_k$ -coloring that uses  $mvx_k(G)$  colors. When  $k = 2$ , we have  $mvx_2(G) = mvc(G)$ . Obviously, we have  $mvx_{|V(G)|}(G) \leq \dots \leq mvx_3(G) \leq mvc(G)$ .

In this paper, we focus on 3-monochromatic vertex index, where  $k = 3$  and  $|S| = 3$ . The rest of this paper is organized as follows: In Section 2, we give some definitions and determine the values of  $mvx_3(G)$  for some special graphs. In Section 3, we obtain an upper bound of  $mvx_3(G)$ . In Section 4, we characterize all graphs with  $mvx_3(G)$  of 3, 4, and verify that almost all simple graphs satisfy  $mvx_3(G) = n$ . In Section 5, we investigate the 3-monochromatic vertex-index of a graph  $G$  of order  $n$ ,  $diam(G) \geq 3$  and  $\omega(G) = n - i$  for  $1 \leq i \leq 3$ .

## 2. Definitions and some basic results

Let  $G$  be a connected graph. If  $H$  is a connected spanning subgraph of  $G$ , then  $mvx_3(G) \geq mvx_3(H)$ . We denote a spanning tree of  $G$  by  $T$ , the leaf set of  $T$  by  $L(T)$ , the number of leaves of  $T$  by  $l(T)$  and the inner vertices of  $T$  by  $I(T)$ . Let  $l(G) = \max\{l(T) \mid T \text{ is a spanning tree of } G\}$ . Observe that  $mvx_3(G) \geq l(G) + 1$ . We denote a star with  $k + 1$  vertices by  $S_k$ . Let  $K_n^- = K_n - e$  and  $[r] = \{1, 2, \dots, r\}$ .

**Proposition 2.1 ([18]).** *Let  $G$  be a connected graph of order  $n$  with a cut vertex. Then  $mvx_k(G) = l(G) + 1$  for each  $k$  with  $2 \leq k \leq n$ .*

**Proposition 2.2 ([3]).** *For a graph  $G$  with order  $n$  and diameter  $d$ , we have  $mvc(G) = n$  if and only if  $d \leq 2$ ; If  $d \geq 3$ , then  $mvc(G) \leq n - d + 2$ .*

A  $(a, b, c)$ -graph is constructed from three paths  $P_{a+1}, P_{b+1}, P_{c+1}$ , where the three paths are internally disjoint and have the same end vertices. In [8], Gao et al. determined the vertex-monochromatic connection number of a  $(a, b, c)$ -graph. In this section, we study the 3-monochromatic vertex index of a  $(a, b, c)$ -graph.

**Proposition 2.3 ([8]).** *Given a  $(a, b, c)$ -graph  $G$  with order  $n$ . If  $n \leq 5$ , or  $G$  is a  $(2, 2, 3)$ -graph, then  $mvc(G) = n$ . Otherwise,  $mvc(G) = l(G) + 1$ .*

**Theorem 2.4.** *Let  $G$  be a  $(a, b, c)$ -graph with  $n$  vertices. If  $n = 4$ , or  $G$  is a  $(2, 2, 2)$ -graph, then  $mvx_3(G) = n$ . Otherwise,  $mvx_3(G) = l(G) + 1$ .*

*Proof.* Since  $G$  is a simple graph, we can deduce that  $n \geq 4$ . If  $n = 4$ , then  $G$  is a  $(1, 2, 2)$ -graph, it is not difficult to verify  $mvx_3(G) = n$ . If  $n = 5$ , then  $G$  is a  $(2, 2, 2)$ -graph or  $(1, 2, 3)$ -graph. Let  $G$  be a  $(2, 2, 2)$ -graph. Assign a different color to each vertex of  $G$ , we can verify that for any three vertices in  $V(G)$ , there exists a vertex-monochromatic tree containing them, then  $mvx_3(G) = 5 = n$ . Let  $G$  be a  $(1, 2, 3)$ -graph, where  $P_1 = uv, P_2 = ux_1v$ , and  $P_3 = ux_2x_3v$ . Suppose  $mvx_3(G) \geq 5$  and  $f$  is an extremal coloring of  $G$ . There does not exist a vertex-monochromatic  $\{x_1, x_2, x_3\}$ -tree, and hence  $mvx_3(G) \leq 4$ . On the other hand, we have that  $mvx_3(G) \geq l(G) + 1 = 4$ . Thus  $mvx_3(G) = 4 = l(G) + 1$ .

Let  $G$  be a  $(2, 2, 3)$ -graph, where  $P_1 = ux_1v, P_2 = ux_2v$  and  $P_3 = ux_3x_4v$ . To the contrary, suppose  $mvx_3(G) \geq 5$  and  $f$  is an extremal  $MVX_3$ -coloring of  $G$ . For the three vertices  $x_1, u, x_4$ , there exists a vertex-monochromatic  $\{x_1, u, x_4\}$ -tree  $T$ . If  $I(T) = \{x_1, v\}$ , then  $f(x_1) = f(v)$ , and there does not exist a vertex-monochromatic  $\{x_1, v, x_3\}$ -tree. If  $I(T) = \{u, x_3\}$ , then  $f(u) = f(x_3)$ , and there does not exist a vertex-monochromatic  $\{x_1, v, x_3\}$ -tree. If  $I(T) = \{v, x_2\}$ , then  $f(v) = f(x_2)$ , and there does not exist a vertex-monochromatic  $\{x_1, v, x_3\}$ -tree. Note that  $mvx_3(G) \geq l(G) + 1 = 4$ , and so  $mvx_3(G) = 4$ .

For other graphs, we have  $l(G) + 1 \leq mvx_3(G) \leq mvc(G) = l(G) + 1$  by Proposition 2.3.  $\square$

In [10], the Brush graph  $B_n$  is constructed from path graph  $P_n$  by joining the star graph  $S_1$  at each vertex of the path. It is clear to see that  $B_n$  has a spanning tree with  $n$  leaves, and then  $mvc(B_n) = mvx_3(B_n) = n + 1$ . In [14], we know square of Brush graph  $B_n^2$  is obtained from  $B_n$  by adding new edges between every two vertices having distance two in  $B_n$ , see Figure 1(a). In [21], the glue graph of  $G_g(B_n)$  is constructed by  $B_n$  with vertex set  $V(G_g(B_n)) = V(B_n)$ , and edge set  $E(G_g(B_n)) = E(B_n) \cup \{(u, v) | e_{B_n}(u) = e_{B_n}(v)\}$ , where  $e_{B_n}(u)$  is the eccentricity of  $u$  in  $B_n$ , see Figure 1(b).

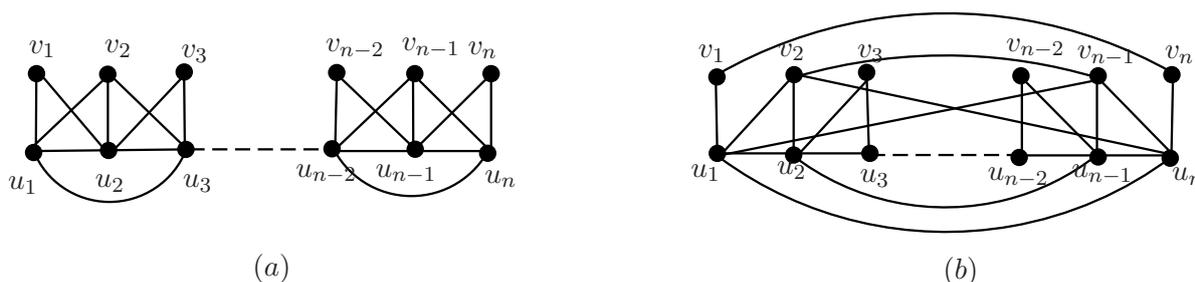


Figure 1. The graph  $B_n^2$  and  $G_g(B_n)$ .

**Theorem 2.5.** *If  $n \geq 4$ , then  $mvc(B_n^2) = mvx_3(B_n^2) = 2n - \lfloor \frac{n}{2} \rfloor + 1$ ; If  $n = 2$  or  $n = 3$ , then  $mvc(B_n^2) = mvx_3(B_n^2) = 2n$ .*

*Proof.* Note that  $diam(B_n^2) = d = \lfloor \frac{n}{2} \rfloor + 1$ . If  $n \leq 3$ , then  $diam(B_n^2) \leq 2$ , and so  $mvc(B_n^2) = 2n$ . If  $n \geq 4$ , then  $mvc(B_n^2) \leq 2n - d + 2 = 2n - \lfloor \frac{n}{2} \rfloor + 1$  by Proposition 2.2. Now we prove  $mvc(B_n^2) \geq 2n - \lfloor \frac{n}{2} \rfloor + 1$ . Define a vertex-coloring  $f$  of  $G$  as follows: assign a color to  $u_i$  where  $i$  is even and  $1 \leq i \leq n$ , assign a new color to each of the remaining vertices. We can verify that there exists a monochromatic path between any two vertices. Hence  $mvc(B_n^2) \geq 2n - \lfloor \frac{n}{2} \rfloor + 1$ .

If  $n \geq 4$ , we can find that  $f$  is also an  $MVX_3$ -coloring of  $B_n^2$ , and so  $mvx_3(B_n^2) \geq 2n - \lfloor \frac{n}{2} \rfloor + 1$ . On the other hand, we have  $mvx_3(B_n^2) \leq mvc(B_n^2) = 2n - \lfloor \frac{n}{2} \rfloor + 1$ . If  $n = 2$ , then  $B_n^2 \cong K_4^-$ . Thus  $mvx_3(B_n^2) = 4 = 2n$ . If  $n = 3$ , there exists a spanning subgraph which is a star with maximum degree vertex  $u_2$ . Hence  $mvx_3(B_n^2) = mvx_3(S_5) = 2n$ .  $\square$

**Theorem 2.6.** *If  $n$  is even, then  $mvc(G_g(B_n)) = mvx_3(G_g(B_n)) = \frac{3n}{2}$ ; If  $n$  is odd, then  $mvc(G_g(B_n)) = mvx_3(G_g(B_n)) = \frac{3n+1}{2}$ .*

*Proof.* Let  $n$  be even. Since  $diam(G_g(B_n)) = d = \frac{n}{2} + 1$ , we obtain  $mvc(G) \leq 2n - d + 2 = \frac{3n}{2} + 1$ . Note that  $d(v_1, u_{\frac{n}{2}+1}) = \frac{n}{2} + 1$ , and there are only three paths between  $v_1$  and  $u_{\frac{n}{2}+1}$ :  $P_1 = v_1u_1u_2 \cdots u_{\frac{n}{2}}u_{\frac{n}{2}+1}$ ,

$P_2 = v_1 u_1 u_n u_{n-1} \cdots u_{\frac{n}{2}+1}$ , and  $P_3 = v_1 v_n u_n u_{n-1} \cdots u_{\frac{n}{2}+1}$ . To the contrary, assume that  $mvc(G_g(B_n)) \geq \frac{3}{2}n + 1$ . Let the inner vertices of  $P_i$  use the same color for some  $i \in [3]$ , and each of the remaining vertices of  $G_g(B_n)$  be assigned a new color. In any way, there does not exist a monochromatic path between  $v_n$  and  $v_{\frac{n}{2}}$ . Hence  $mvc(G_g(B_n)) \leq \frac{3}{2}n$ . On the other hand, we can find a spanning tree  $T$  with inner vertices  $v_1, u_1, u_2, \dots, u_{\frac{n}{2}}$ , and then  $|L(T)| = \frac{3}{2}n - 1$ . Hence  $mvc(G_g(B_n)) \geq \frac{3}{2}n - 1 + 1 = \frac{3}{2}n$ . We have  $mvx_3(G_g(B_n)) \geq l(G) + 1 = \frac{3}{2}n$ . Since  $mvx_3(G_g(B_n)) \leq mvc(G_g(B_n))$ , we obtain  $mvx_3(G_g(B_n)) = \frac{3}{2}n$ .

Let  $n$  be odd. Since  $diam(G_g(B_n)) = d = \frac{n+1}{2}$ , we obtain  $mvc(G_g(B_n)) \leq n - d + 2 = \frac{3}{2}(n + 1)$ . Note that  $d(v_1, u_{\frac{n+1}{2}}) = \frac{n+1}{2}$ , there is only one path  $P$  between  $v_1$  and  $u_{\frac{n+1}{2}}$ , and let  $P = v_1 u_1 u_2 \cdots u_{(\frac{n+1}{2}-1)} u_{\frac{n+1}{2}}$ . To the contrary, assume that  $mvx_3(G_g(B_n)) \geq \frac{3}{2}(n+1)$ . Let the inner vertices of  $P$  use the same color and the remaining vertices of  $G_g(B_n)$  be assigned by different colors. In any way, there does not exist a monochromatic path between  $v_n$  and  $v_{\frac{n+1}{2}}$ . Hence  $mvc(G) \leq \frac{1}{2}(3n + 1)$ . On the other hand, we can find a spanning tree  $T$  with inner vertices  $v_1, u_1, u_2, \dots, u_{\frac{n+1}{2}}$ , and so  $|L(T)| = \frac{1}{2}(3n - 1)$ . Therefore,  $mvc(G) \geq \frac{1}{2}(3n - 1) + 1 = \frac{1}{2}(3n + 1)$ . Note that  $mvx_3(G_g(B_n)) \geq l(G) + 1 = \frac{1}{2}(3n + 1)$ . Since  $mvx_3(G_g(B_n)) \leq mvc(G_g(B_n))$ , we obtain  $mvx_3(G_g(B_n)) = \frac{1}{2}(3n + 1)$ .  $\square$

### 3. Sharp upper bound of $mvx_3(G)$

**Theorem 3.1.** *Let  $G$  be a connected graph. Then  $mvx_3(G) \leq l(G) + \delta(G)$ , and the bound is sharp.*

*Proof.* Let  $v$  be a minimum degree vertex of  $G$  and  $B = V(G) \setminus N[v]$ . Suppose  $f$  is an extremal  $MVX_3$ -coloring of  $G$ . Without loss of generality, let  $f(N(v)) = \{1, 2, \dots, r\}$ . Denote  $A_i = \{x \in N(v) \mid f(x) = i\}$ . If  $\Delta(G) = n - 1$ , then  $G$  has a spanning tree with  $n - 1$  leaves, and so  $mvx_3(G) = l(G) + 1 \leq l(G) + \delta(G)$ . Now we discuss  $\Delta(G) \leq n - 2$ .

**Case 1.** There is only one vertex in  $B$ . Note that the minimum degree of  $G$  is  $n - 2$ . Then  $G$  has a spanning tree that has  $n - 2$  leaves, and so  $mvx_3(G) \leq n \leq 2n - 4 \leq l(G) + \delta(G)$  for  $n \geq 4$ . If  $n = 3$ , then  $mvx_3(G) = 3 = l(G) + \delta(G)$ .

**Case 2.** There is at least two vertices in  $B$ . For any two vertices  $w_1, w_2 \in B$ , since  $f$  is an extremal  $MVX_3$ -coloring of  $G$ , there exists a vertex-monochromatic  $S$ -tree where  $S = \{w_1, w_2, v\}$ . Denote this tree by  $T_{w_1, w_2}$ . If the inner vertices of  $T_{w_1, w_2}$  color  $i$ , we denote this tree by  $T'_i$ . For any  $i \in [r]$ , we have  $V_i = \{w \in V(T'_i) \cap B \mid w \in I(T'_i), \text{ or } w \in L(T'_i) \text{ and } w \notin I(T'_i) \text{ for } i \neq j\}$ . Let  $U_1 = V_1$  and  $U_i = V_i \setminus (\cup_{j \in [i-1]} V_j)$ , for  $2 \leq i \leq r$ . Then  $U_1, U_2, \dots, U_r$  are pairwise disjoint. For any  $i \in [r]$ , let  $H_i = G[U_i \cup A_i \cup \{v\}]$ . Since any vertex  $w \in V_i$ , there exists a path between  $w$  and  $v$ . Hence for any vertex  $w \in U_i$ , there exists a path between  $w$  and  $v$ . This means that there is a path connecting any two vertices in  $H_i$ . Thus every  $H_i$  is connected and there exists a spanning tree  $T_i$  in  $H_i$ . If  $i \neq j$ , then  $V(T_i) \cap V(T_j) = \{v\}$ . Hence  $T = \cup_{i \in [r]} T_i$  is a spanning tree of  $G$ . Since  $|f(T_i)| \leq l(T_i) + 1$ , we have  $mvx_3(G) = f(T) \leq \sum_{i \in [r]} f(T_i) - (r - 1) \leq \sum_{i \in [r]} (l(T_i) + 1) - r + 1 = \sum_{i \in [r]} l(T_i) + 1 \leq l(\cup_{i \in [r]} T_i) + r + 1 \leq l(G) + \delta(G) + 1$ . The equal holds if and only if the following conditions must occur.

- For  $x \in V(T_i) \setminus \{v\}$  and  $y \in V(T_j) \setminus \{v\}$ , we have  $f(x) \neq f(y)$ .
- $v$  is a leaf of  $T_i$  for  $i \in [r]$ .
- $|A_i| = 1$  for  $i \in [r]$ .
- $|L(T)| = l(G)$ .

**Subcase 2.1.** For any  $w \in V_i \cap V_j$  with  $1 \leq i \neq j \leq r$ , we have  $U_r = \emptyset$ . Since  $H_r = G[U_r \cup A_r \cup \{v\}]$ , we know that  $H_r$  is  $K_2$ . Thus  $|f(T_r)| = 2 < l(T_r) + 1$ , contradicting the above conditions.

**Subcase 2.2.** There exists a vertex  $w$  satisfying  $w \in V_i$  and  $w \notin V_j$ , where  $j \neq i$ . Denote  $A_i = \{x_i\}$ . If the vertex  $w$  is a leaf of  $T_i$ , there exists a vertex-monochromatic  $\{w, v, w'\}$ -tree whose inner vertices color  $i$  and  $w' \in V(G) \setminus V_i \cup \{v\}$ . Hence  $w'$  has a neighbor in  $I(T_i)$ . We can find a spanning tree  $T'$  with root  $x_i$  such that  $S(T_i) \subset S(T')$ . Note that  $v \in L(T') \setminus L(T)$ , contradicting the above conditions. If  $w$  is an inner vertex of

$T_i$ , there exists a vertex-monochromatic  $\{w, v, w'\}$ -tree whose inner vertices color  $i$ . Hence  $w$  is a leaf of  $T_j$  with  $j \neq i$ , and so  $w$  has at least one neighbor in  $I(T_j)$ . Assume that there exists a vertex  $x \in I(T_k)$ , such that  $xy \notin E(G)$ , where  $y \in I(T_j)$  and  $j \neq k$ . We can find that there does not exist a vertex-monochromatic  $\{x, v, y\}$ -tree, a contradiction. Therefore for any vertex  $x \in I(T_k)$ , there must exist a vertex  $y \in I(T_j)$  satisfying  $xy \in E(G)$ . We can find a spanning tree  $T'$  with root  $x_i$  such that  $L(T_i) \subset L(T')$ . Note that  $v \in L(T') \setminus L(T)$ , contradicting the above conditions. Thus  $mvx_3(G) \leq l(G) + \delta(G)$ .

**Tight example:** Let  $G$  be a connected graph with  $\delta(G) = 1$ . Then  $G$  must exist a cut vertex, and so  $mvx_3(G) = l(G) + 1$  by Proposition 2.1.  $\square$

#### 4. Graphs with 3-monochromatic vertex index 3, 4 and $n$

In this section, we will characterize all graphs with  $mvx_3(G)$  of 3 and 4. Moreover we prove that almost all simple graphs satisfy  $mvx_3(G) = n$ . We call a connected graph  $G$  *unicycle*, *bicyclic*, or *tricyclic* if  $m = n$ ,  $m = n + 1$  or  $m = n + 2$ , respectively. Let  $\Theta_3$  denote the set of the trees with  $l(G) = 3$ . Note that if  $G$  is a connected graph with  $l(G) = 2$ , then  $G$  is either a path or a cycle.

**Theorem 4.1.** *Let  $G$  be a connected graph. Then  $mvx_3(G) = 3$  if and only if  $G$  is a path or a cycle except for  $C_4$ .*

*Proof.* If  $G$  is a path then  $mvx_3(G) = 3$ . If  $G$  is a cycle except for  $C_4$ , we can prove that  $mvx_3(G) = 3$ . Conversely, let  $G$  be a connected graph with  $mvx_3(G) = 3$ . Since  $mvx_3(G) \geq l(G) + 1$  and  $l(G) \geq 2$ , it follows that  $l(G) = 2$ . Thus  $G$  is either a path or a cycle except for  $C_4$ .  $\square$

**Theorem 4.2.** *Let  $G$  be a connected graph. Then  $mvx_3(G) = 4$  if and only if  $G \in \Theta_3$ , or  $G \in \{K_4, K_4^-, C_4\}$ , or  $G = \Omega_i$  for  $i \in [12]$ , where the graphs  $\Omega_i$  are shown in Figure 2.*

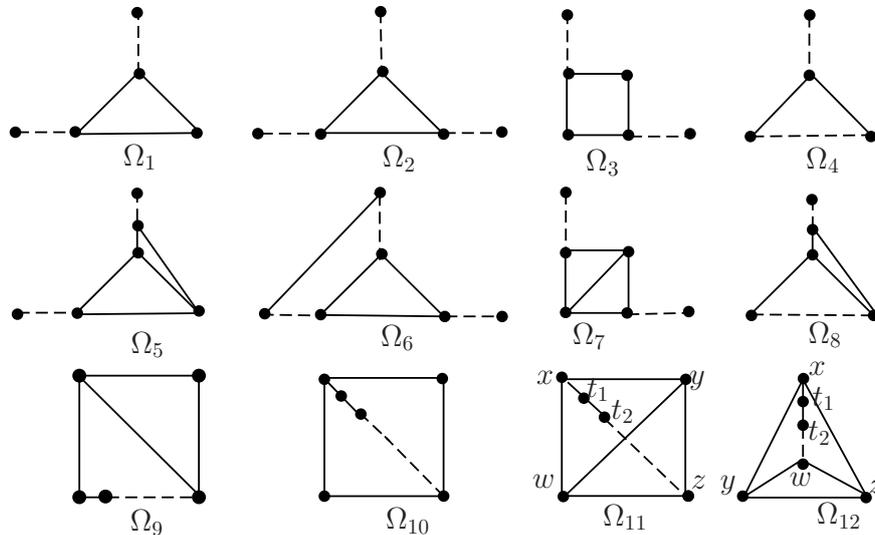


Figure 2. The graphs in Theorem 4.2.

*Proof.* If  $G \in \Theta_3$ , or  $G \in \{K_4, K_4^-, C_4\}$ , then  $mvx_3(G) = 4$ . If  $G = \Omega_i$  where  $1 \leq i \leq 8$ , then  $G$  is a connected graph with a cut vertex. Thus  $mvx_3(G) = l(T) + 1 = 4$  by Proposition 2.1. If  $G = \Omega_9$ , then  $G$  is a  $(a, b, c)$ -graph with  $n \geq 5$ . If  $G = \Omega_{10}$ , then  $G$  is a  $(a, b, c)$ -graph with  $n \geq 6$ . Thus  $mvx_3(G) = l(G) + 1 = 4$  by Theorem 2.4.

Let  $f$  be an  $MVX_3$ -coloring of  $\Omega_{11}$ ,  $P = xt_1t_2 \cdots t_{n-4}z$  and  $S \subseteq V(P)$ . Assume that  $n = 6$ . Suppose  $mvx_3(\Omega_{11}) = 5$ . Let  $T$  be a vertex-monochromatic  $\{t_1, t_2, w\}$ -tree. If  $T$  is a spanning tree of  $\Omega_{11}$ , then  $mvx_3(\Omega_{11}) \leq 4$ . If  $I(T) = \{x, y, z\}$  or  $I(T) = \{x, z, w\}$  or  $I(T) = \{x, t_1, t_2\}$  or  $I(T) = \{z, t_1, t_2\}$ , then  $mvx_3(\Omega_{11}) \leq 4$ , a contradiction. If  $I(T) = \{x, t_1\}$  or  $I(T) = \{z, t_2\}$ , then there does not exist a vertex-monochromatic  $\{z, w, t_i\}$ -tree,

a contradiction. Hence  $mvx_3(\Omega_{11}) \leq 4$ . Assume that  $n \geq 7$ . Suppose there exists a vertex-monochromatic  $S$ -tree in  $P$  for any  $S \subseteq V(P)$ . Then  $f(t_1) = f(t_2) = \dots = f(t_{n-4})$ . Note that there exists a vertex-monochromatic  $\{y, w, t_i\}$ -tree such that  $f(x) = f(t_i)$  or  $f(z) = f(t_i)$  or  $f(x) = f(z) = f(t_i)$ , where  $2 \leq i \leq n - 5$ . In any way, we obtain  $mvx_3(\Omega_{11}) \leq 4$ . Now we suppose that all the vertex-monochromatic  $S$ -trees must through  $y$  or  $w$  for some  $S \subseteq V(P)$ . Then  $f(x) = f(w) = f(z)$  or  $f(x) = f(y) = f(z)$  or  $f(x) = f(y) = f(z) = f(w)$ . If  $mvx_3(\Omega_{11}) = 5$ , then there exist three vertices  $t_i, t_j, t_k$  such that  $f(t_i) \neq f(t_j) \neq f(t_k) \neq f(x)$ , where  $i < j < k$ . Thus there does not exist a vertex-monochromatic  $\{x, y, t_j\}$ -tree, a contradiction. Hence  $mvx_3(\Omega_{11}) \leq 4$ . Note that  $mvx_3(\Omega_{11}) \geq l(\Omega_{11}) + 1 = 4$ , and so  $mvx_3(\Omega_{11}) = 4$ .

Let  $f$  be an  $MVX_3$ -coloring of  $\Omega_{12}$ ,  $P = xt_1t_2 \dots t_{n-4}w$  and  $S \subseteq V(P)$ . Assume that  $n = 6$ . Suppose  $mvx_3(\Omega_{12}) = 5$ . Let  $T$  be a vertex-monochromatic  $\{t_1, y, w\}$ -tree. If  $T$  is a spanning tree of  $\Omega_{12}$ , then  $mvx_3(\Omega_{12}) \leq 4$ . If  $I(T) = \{x, y\}$  or  $I(T) = \{t_2, w\}$  or  $I(T) = \{x, z\}$ , then there does not exist a vertex-monochromatic  $\{t_2, y, x\}$ -tree, a contradiction. If  $I(T) = \{z, w, t_2\}$  or  $I(T) = \{x, z, t_1\}$  or  $I(T) = \{x, y, w\}$  or  $I(T) = \{w, y, t_2\}$  or  $I(T) = \{x, t_1, t_2\}$ , then  $mvx_3(\Omega_{12}) \leq 4$ , a contradiction. Hence  $mvx_3(\Omega_{12}) \leq 4$ . Assume that  $n \geq 7$ . Suppose there exists a vertex-monochromatic  $S$ -tree in  $P$  for any  $S \subseteq V(P)$ . Then  $f(t_1) = f(t_2) = \dots = f(t_{n-4})$ . Note that there exists a vertex-monochromatic  $\{y, w, t_i\}$ -tree. If the vertex-monochromatic  $\{y, w, t_i\}$ -tree is a spanning tree of  $\Omega_{12}$ , then  $mvx_3(\Omega_{12}) \leq 4$ . If the vertex-monochromatic  $\{y, w, t_i\}$ -tree satisfying  $f(w) = f(t_i)$  or  $f(x) = f(y)$  or  $f(x) = f(z)$  or  $f(z) = f(w)$ , where  $2 \leq i \leq n - 5$ . Then  $mvx_3(\Omega_{12}) \leq 4$ . Now we suppose that all the vertex-monochromatic  $S$ -trees must through  $y$  or  $z$  for some  $S \subseteq V(P)$ . We have  $f(x) = f(y) = f(w)$  or  $f(x) = f(z) = f(w)$  or  $f(x) = f(y) = f(z) = f(w)$ . If  $mvx_3(\Omega_{11}) = 5$ , then there exist three vertices  $t_i, t_j, t_k$  such that  $f(t_i) \neq f(t_j) \neq f(t_k) \neq f(x)$ , where  $i < j < k$ . We can find that there does not exist a vertex-monochromatic  $\{x, y, t_j\}$ -tree, a contradiction. Thus  $mvx_3(\Omega_{12}) \leq 4$ . On the other hand, we have  $mvx_3(\Omega_{12}) \geq l(\Omega_{12}) + 1 = 4$ , and so  $mvx_3(\Omega_{12}) = 4$ .

Conversely, let  $G$  be a connected graph with  $mvx_3(G) = 4$ . Since  $mvx_3(G) \geq l(G) + 1$ , it follows that  $2 \leq l(G) \leq 3$ . Assume that  $l(G) = 2$ . We must have  $mvx_3(C_4) = 4$  from Theorem 4.1. Now we consider  $l(G) = 3$ . Suppose  $m = n - 1$ . Then  $G \in \Theta_3$ . Suppose  $m = n$ . Then  $G$  is a unicyclic graph. If the length of the unique cycle is three, then  $G \in \{\Omega_1, \Omega_2, \Omega_4\}$  (see Figure 2). If the length of the unique cycle is four, then  $G \in \{\Omega_3, \Omega_4\}$ . If the length of the unique cycle is at least five, then  $G = \Omega_4$ . Otherwise, we have  $l(G) = 4$  or  $mvx_3(G) = 5$ , a contradiction. Suppose  $m = n + 1$ . The graph obtained by adding an edge to  $\Omega_i$  for  $1 \leq i \leq 4$  is isomorphic to  $K_4^-$  or  $\Omega_{10}$  or a bicyclic graph  $\Omega_i$  for  $5 \leq i \leq 10$ . If  $G \notin \Omega_i$  for  $5 \leq i \leq 10$ , then we add an edge to the vertices in  $\Omega_i$  respectively, where  $1 \leq i \leq 4$ . We find that  $l(G) = 4$  or  $mvx_3(G) = 5$ , a contradiction. Suppose  $m = n + 2$ . Then  $G$  is isomorphic to  $K_4$  or a tricyclic graph, and we add one edge to  $\Omega_i$ , where  $5 \leq i \leq 10$ . We observe that  $G$  is isomorphic to  $\Omega_{11}$  or  $\Omega_{12}$  (see Figure 2). Suppose  $m = n + 3$ . It is not difficult to see that no matter how we add edges, we always have  $l(G) \geq 4$ .  $\square$

**Theorem 4.3.** For almost all simple graphs  $G$ , we have  $mvx_3(G) = n$ .

*Proof.* If any three vertices have a common neighbor in  $G$ , then for any  $S \subseteq V(G)$ , there must exist a vertex-monochromatic  $S$ -tree that does not waste color, and so  $mvx_3(G) = n$ . Incident  $A$  denotes that  $w$  is the common vertex of  $x, y, z$ , and incident  $\bar{A}$  denotes that  $w$  is not the common vertex of  $x, y, z$ , where  $w \in V(G) \setminus \{x, y, z\}$ . For a random graph  $G(n, \frac{1}{2})$ , we just need to prove that  $x, y, z$  have a common vertex  $w$ . Note that  $P_r(A) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ , and  $P_r(\bar{A}) = 1 - \frac{1}{8} = \frac{7}{8}$ . Let  $\Gamma_{x,y,z}$  denote a set of random events that  $x, y, z$  do not have a common vertex, and  $\Gamma = \bigcup_{x,y,z \in V(G)} \Gamma_{x,y,z}$ . Then  $P_r(\Gamma_{x,y,z}) = (\frac{7}{8})^{n-3}$ . Hence  $P_r(\Gamma) < \binom{n}{3} (\frac{7}{8})^{n-3} < n^3 (\frac{7}{8})^{n-3} \rightarrow 0$ , where  $n \rightarrow \infty$ . Thus it is almost true that any three vertices of  $G$  have at least one common neighbor, and so  $mvx_3(G) = n$  is true for almost all simple graphs.  $\square$

### 5. Monochromatic connection, 3-monochromatic vertex index and clique number

Kemnitz and Schiermeyer [12] considered the rainbow connection number of graph  $G$  of order  $n$  and  $\omega(G) = n - i$  for  $1 \leq i \leq 3$ . In this section, we investigate the monochromatic vertex connection number, 3-monochromatic vertex index of graph  $G$  of order  $n$ , and  $\omega(G) = n - i$  for  $1 \leq i \leq 3$ . For a connected graph  $G$ , let  $Q$  be a subgraph induced by a maximum clique of  $G$ . We say  $N_Q(u)$  is the set of neighbors of  $u$  in  $Q$ ,  $d_Q(u) = |N_Q(u)|$  and  $V(Q) = \{q_1, q_2, \dots, q_{n-i}\}$ , where  $1 \leq i \leq 3$ .

**Theorem 5.1.** Let  $G$  be a connected graph of order  $n$ , and  $\omega(G) = n - 1$ . Then  $mvc(G) = mvx_3(G) = n$ .

*Proof.* Note that  $diam(G) = 2$ . Then  $mvc(G) = n$  by Proposition 2.2. We can find a spanning tree with  $l(G) = n - 1$ , and so  $mvx_3(G) = n$ .  $\square$

**Theorem 5.2.** Let  $G$  be a connected graph of order  $n$ , and  $\omega(G) = n - 2$ . Let  $Q$  be a subgraph induced by a maximum clique of  $G$  and  $V(G) \setminus V(Q) = \{u, v\}$ .

- (i) If  $uv \notin E(G)$  and  $N(u) \cap N(v) = \emptyset$ , then  $mvc(G) = mvx_3(G) = n - 1$ .
- (ii) If  $uv \notin E(G)$  and  $N(u) \cap N(v) \neq \emptyset$ , then  $mvc(G) = mvx_3(G) = n$ .
- (iii) If  $uv \in E(G)$ ,  $\max\{|d_Q(u)|, |d_Q(v)|\} \geq 1$  and  $\min\{|d_Q(u)|, |d_Q(v)|\} = 0$ , then  $mvc(G) = mvx_3(G) = n - 1$ .
- (iv) If  $uv \in E(G)$ ,  $\max\{|d_Q(u)|, |d_Q(v)|\} \geq 1$ ,  $\min\{|d_Q(u)|, |d_Q(v)|\} \geq 1$  and  $N_Q(u) \cup N_Q(v) = V(Q)$ , then  $mvc(G) = n$  and  $mvx_3(G) = n - 1$ .
- (v) If  $uv \in E(G)$ ,  $\max\{|d_Q(u)|, |d_Q(v)|\} \geq 1$ ,  $\min\{|d_Q(u)|, |d_Q(v)|\} \geq 1$ ,  $N_Q(u) \cup N_Q(v) \neq V(Q)$  and  $N_Q(u) \cap N_Q(v) = \emptyset$ , then  $mvc(G) = n$  and  $mvx_3(G) = n - 1$ .
- (vi) If  $uv \in E(G)$ ,  $\max\{|d_Q(u)|, |d_Q(v)|\} \geq 1$ ,  $\min\{|d_Q(u)|, |d_Q(v)|\} \geq 1$ ,  $N_Q(u) \cup N_Q(v) \neq V(Q)$  and  $N_Q(u) \cap N_Q(v) \neq \emptyset$ , then  $mvc(G) = mvx_3(G) = n$ .

*Proof.* Assume that  $uv \notin E(G)$ . Then  $d_Q(u) \geq 1$  and  $d_Q(v) \geq 1$ . If  $N_Q(u) \cap N_Q(v) = \emptyset$ , then  $d(u, v) = 3$ . The vertex monochromatic path connecting  $u$  and  $v$  wastes at least one color. Hence  $mvx_3(G) \leq mvc(G) \leq n - 1$ . On the other hand, we can find a spanning tree with  $n - 2$  leaves, and so  $mvc(G) \geq mvx_3(G) \geq n - 1$ . Thus  $mvc(G) = mvx_3(G) = n - 1$ . If  $N_Q(u) \cap N_Q(v) \neq \emptyset$ , we can find a spanning tree with  $n - 1$  leaves, and so  $mvc(G) = mvx_3(G) = n$ .

Assume that  $uv \in E(G)$ . Then  $\max\{|d_Q(u)|, |d_Q(v)|\} \geq 1$ . Suppose  $\min\{|d_Q(u)|, |d_Q(v)|\} = 0$ . Without loss of generality, let  $d_Q(u) \geq 1$  and  $d_Q(v) = 0$ . Then there exists a vertex  $q'$  such that  $d(v, q') = 3$ . The vertex monochromatic path connecting  $v$  and  $q'$  wastes at least one color. Thus  $mvx_3(G) \leq mvc(G) \leq n - 1$ . On the other hand, we can find a spanning tree with  $n - 2$  leaves, and so  $mvc(G) \geq mvx_3(G) \geq n - 1$ . Hence  $mvc(G) = mvx_3(G) = n - 1$ . Suppose  $\min\{|d_Q(u)|, |d_Q(v)|\} \geq 1$ . Note that  $diam(G) = 2$ . Then  $mvc(G) = n$  by Proposition 2.2. If  $N_Q(u) \cup N_Q(v) = V(Q)$ , then let the vertices of  $G$  be assigned by different colors. There exists a vertex-monochromatic  $S$ -tree for any  $S \subseteq V(G)$ . Thus  $mvx_3(G) = n$ . If  $N_Q(u) \cup N_Q(v) \neq V(Q)$  and  $N_Q(u) \cap N_Q(v) = \emptyset$ , then there exists a vertex  $q'$  such that  $q' \notin N_Q(u) \cup N_Q(v)$ . We assign the vertices of  $G$  by different colors, and find that there does not exist a vertex-monochromatic  $\{q', u, v\}$ -tree. Hence  $mvx_3(G) \leq n - 1$ . On the other hand, we can find a spanning tree with  $n - 2$  leaves. We have  $mvx_3(G) \geq l(G) + 1 = n - 1$ , and so  $mvx_3(G) = n - 1$ . If  $N_Q(u) \cup N_Q(v) \neq V(Q)$  and  $N_Q(u) \cap N_Q(v) \neq \emptyset$ , then there exists a spanning tree with  $n - 1$  leaves. Hence  $mvx_3(G) = n$ .  $\square$

**Theorem 5.3.** Let  $G$  be a connected graph of order  $n$ , and  $\omega(G) = n - 3$ . Let  $Q$  be a subgraph induced by a maximum clique of  $G$  and  $V(G) \setminus V(Q) = \{u, v, w\}$ .

- (i) Suppose  $d_Q(u) \geq 1$  and  $d_Q(v) = d_Q(w) = 0$ . If  $uv \in E(G), vw \in E(G)$  and  $uw \notin E(G)$ , then  $mvc(G) = mvx_3(G) = n - 2$ . Otherwise,  $mvc(G) = mvx_3(G) = n - 1$ .
- (ii) Suppose  $d_Q(u) \geq 1, d_Q(v) \geq 1$  and  $d_Q(w) = 0$ . If  $vw \in E(G), uv \notin E(G), uw \notin E(G)$  and  $N_Q(u) \cap N_Q(v) \neq \emptyset$ , then  $mvc(G) = mvx_3(G) = n - 2$ . Otherwise,  $mvc(G) = mvx_3(G) = n - 1$ .
- (iii) Suppose  $d_Q(u) \geq 1, d_Q(v) \geq 1, d_Q(w) = 0, uv \in E(G), vw \in E(G)$  and  $uw \notin E(G)$ . Then  $mvc(G) = mvx_3(G) = n - 1$ .
- (iv) Suppose  $d_Q(u) \geq 1, d_Q(v) \geq 1, d_Q(w) = 0, uw \in E(G), vw \in E(G)$  and  $uv \notin E(G)$ . If  $N_Q(u) \cup N_Q(v) = V(Q)$ , then  $mvc(G) = n$  and  $mvx_3(G) = n - 2$  for  $N_Q(u) \cap N_Q(v) = \emptyset$ ,  $mvc(G) = n$  and  $mvx_3(G) = n - 1$  for  $N_Q(u) \cap N_Q(v) \neq \emptyset$ . If  $N_Q(u) \cup N_Q(v) \neq V(Q)$ , then  $mvc(G) = n - 1$  and  $mvx_3(G) = n - 2$  for  $N_Q(u) \cap N_Q(v) = \emptyset$ , and  $mvc(G) = mvx_3(G) = n - 1$  for  $N_Q(u) \cap N_Q(v) \neq \emptyset$ .
- (v) Suppose  $d_Q(u) \geq 1, d_Q(v) \geq 1, d_Q(w) \geq 1, vw \notin E(G), uv \notin E(G)$  and  $uw \notin E(G)$ . If  $N_Q(u) \cap N_Q(v) \cap N_Q(w) \neq \emptyset$ , then  $mvc(G) = mvx_3(G) = n$ . If  $\max\{|N_Q(u) \cap N_Q(v)|, |N_Q(v) \cap N_Q(w)|, |N_Q(u) \cap N_Q(w)|\} = 0$ , then  $mvc(G) = mvx_3(G) = n - 2$ . If  $\max\{|N_Q(u) \cap N_Q(v)|, |N_Q(v) \cap N_Q(w)|, |N_Q(u) \cap N_Q(w)|\} \geq 1$ , then  $mvc(G) = mvx_3(G) = n - 1$  for  $\min\{|N_Q(u) \cap N_Q(v)|, |N_Q(v) \cap N_Q(w)|, |N_Q(u) \cap N_Q(w)|\} = 0$ ,  $mvc(G) = n$  and  $mvx_3(G) = n - 1$  for  $\min\{|N_Q(u) \cap N_Q(v)|, |N_Q(v) \cap N_Q(w)|, |N_Q(u) \cap N_Q(w)|\} \geq 1$ .

(vi) Suppose  $d_Q(u) \geq 1, d_Q(v) \geq 1, d_Q(w) \geq 1, uv \in E(G), vw \notin E(G)$  and  $uw \notin E(G)$ . If  $N_Q(u) \cap N_Q(v) \cap N_Q(w) \neq \emptyset$ , then  $mvc(G) = mvx_3(G) = n$ . If  $\max\{|N_Q(u) \cap N_Q(v)|, |N_Q(v) \cap N_Q(w)|, |N_Q(u) \cap N_Q(w)|\} = 0$ , then  $mvc(G) = mvx_3(G) = n - 2$ . If  $\max\{|N_Q(u) \cap N_Q(v)|, |N_Q(v) \cap N_Q(w)|, |N_Q(u) \cap N_Q(w)|\} \geq 1$ , then  $mvc(G) = mvx_3(G) = n - 1$  for  $\min\{|N_Q(u) \cap N_Q(v)|, |N_Q(v) \cap N_Q(w)|, |N_Q(u) \cap N_Q(w)|\} = 0$ ,  $mvc(G) = n$  and  $mvx_3(G) = n - 1$  for  $\min\{|N_Q(u) \cap N_Q(v)|, |N_Q(v) \cap N_Q(w)|, |N_Q(u) \cap N_Q(w)|\} \geq 1$ .

(vii) Suppose  $d_Q(u) \geq 1, d_Q(v) \geq 1, d_Q(w) \geq 1, uv \in E(G), vw \in E(G)$  and  $uw \notin E(G)$ . If  $\min\{|N_Q(u) \cap N_Q(v)|, |N_Q(v) \cap N_Q(w)|, |N_Q(u) \cap N_Q(w)|\} \geq 1$ , then  $mvc(G) = mvx_3(G) = n$ .

(xiii) Suppose  $d_Q(u) \geq 1, d_Q(v) \geq 1, d_Q(w) \geq 1, uv \in E(G), vw \in E(G), uw \notin E(G), N_Q(u) \cup N_Q(v) \cup N_Q(w) = V(Q)$  and  $\min\{|N_Q(u) \cap N_Q(v)|, |N_Q(v) \cap N_Q(w)|, |N_Q(u) \cap N_Q(w)|\} = 0$ . If  $N_Q(u) \cap N_Q(w) \neq \emptyset, N_Q(u) \cap N_Q(v) \neq \emptyset, N_Q(w) \subset N_Q(u)$  and  $N_Q(u) \setminus N_Q(w) \subset N_Q(v)$ , then  $mvc(G) = mvx_3(G) = n$ . If  $N_Q(u) \cap N_Q(w) \neq \emptyset, N_Q(u) \cap N_Q(v) = \emptyset$  and  $N_Q(u) = N_Q(w)$ , then  $mvc(G) = mvx_3(G) = n$ . If  $N_Q(u) \cap N_Q(w) \neq \emptyset, N_Q(u) \cap N_Q(v) \neq \emptyset, N_Q(u) \subset N_Q(w)$  and  $N_Q(w) \setminus N_Q(u) \subset N_Q(v)$ , then  $mvc(G) = mvx_3(G) = n$ . Otherwise,  $mvc(G) = n$  and  $mvx_3(G) = n - 1$ .

(ix) Suppose  $d_Q(u) \geq 1, d_Q(v) \geq 1, d_Q(w) \geq 1, uv \in E(G), vw \in E(G)$  and  $uw \in E(G)$ . If  $N_Q(u) \cup N_Q(v) \cup N_Q(w) \neq V(Q)$  and  $\min\{|N_Q(u) \cap N_Q(v)|, |N_Q(v) \cap N_Q(w)|, |N_Q(u) \cap N_Q(w)|\} = 0$ , then  $mvc(G) = n$  and  $mvx_3(G) = n - 1$ . Otherwise,  $mvc(G) = mvx_3(G) = n$ .

*Proof.* If  $d_Q(u) \geq 1$ , without loss of generality, let  $q_1 \in N_Q(u)$ . Similarly, let  $q_2 \in N_Q(v), q_3 \in N_Q(w), x \in N_Q(u) \cap N_Q(v), y \in N_Q(v) \cap N_Q(w)$  and  $z \in N_Q(u) \cap N_Q(w)$ .

Assume that  $d_Q(u) \geq 1, d_Q(v) = 0$  and  $d_Q(w) = 0$ . If  $uv \in E(G), vw \in E(G)$  and  $uw \notin E(G)$ , then  $diam(G) = 4$ . Hence  $mvc(G) \leq n - 2$  by Proposition 2.2. There exists a spanning tree  $T$  with  $I(T) = \{q_1, u, v\}$ . Thus  $mvc(G) = mvx_3(G) = n - 2$ . If  $uv \in E(G), uw \in E(G)$ , then  $diam(G) = 3$ . Hence  $mvc(G) \leq n - 1$  by Proposition 2.2. There exists a spanning tree  $T$  with  $I(T) = \{q_1, u\}$ . Thus  $mvc(G) = mvx_3(G) = n - 1$ .

Assume that  $d_Q(u) \geq 1, d_Q(v) \geq 1$  and  $d_Q(w) = 0$ . Suppose  $vw \in E(G), uv \notin E(G)$  and  $uw \notin E(G)$ . If  $N_Q(u) \cap N_Q(v) = \emptyset$ , then  $diam(G) = 4$ . Hence  $mvc(G) \leq n - 2$  by Proposition 2.2. There exists a spanning tree  $T$  with  $I(T) = \{q_1, q_2, v\}$ . Then  $mvc(G) \geq mvx_3(G) \geq n - 2$ , and so  $mvc(G) = mvx_3(G) = n - 2$ . If  $N_Q(u) \cap N_Q(v) \neq \emptyset$ , then  $diam(G) = 3$ . Hence  $mvc(G) \leq n - 1$  by Proposition 2.2. There exists a spanning tree  $T$  with  $I(T) = \{x, v\}$ . Hence  $mvc(G) \geq mvx_3(G) \geq n - 1$ , and so  $mvc(G) = mvx_3(G) = n - 1$ .

Suppose  $uv \in E(G), vw \in E(G)$  and  $uw \notin E(G)$ . Note that  $diam(G) = 3$ . Then  $mvc(G) \leq n - 1$  by Proposition 2.2. There exists a spanning tree  $T$  with  $I(T) = \{q_2, v\}$ . Hence  $mvc(G) \geq mvx_3(G) \geq n - 1$ , and so  $mvc(G) = mvx_3(G) = n - 1$ . Suppose  $uw \in E(G), vw \in E(G)$  and  $uv \notin E(G)$ . If  $N_Q(u) \cup N_Q(v) = V(Q)$ , then  $diam(G) = 2$ . Hence  $mvc(G) = n$  by Proposition 2.2. We consider  $N_Q(u) \cap N_Q(v) = \emptyset$ . There exists a spanning tree  $T$  and  $I(T) = \{q_1, u, w\}$ , then  $mvx_3(G) \geq n - 2$ . Next we prove that  $mvx_3(G) \leq n - 2$ . Otherwise, we have either  $f(u) = f(w)$  or  $f(q_2) = f(v)$ , since the vertex-monochromatic  $\{q_1, w, v\}$ -tree wastes at most one color. Note that there does not exist a vertex-monochromatic  $\{q_1, q_2, w\}$ -tree or vertex-monochromatic  $\{u, w, q_2\}$ -tree, a contradiction. Hence  $mvx_3(G) = n - 2$ . We consider  $N_Q(u) \cap N_Q(v) \neq \emptyset$ . There exists a spanning tree  $T$  and  $I(T) = \{x, u\}$ , then  $mvx_3(G) \geq n - 1$ . We assign different colors to the vertices of  $G$ , and find that there does not exist a vertex-monochromatic  $\{q', x_2, w\}$ -tree where  $q' \in N_Q(u) \setminus N_Q(v)$ . Thus  $mvx_3(G) = n - 1$ . If  $N_Q(u) \cup N_Q(v) \neq V(Q)$ , then  $diam(G) = 3$ . We have  $mvx_3(G) \leq mvc(G) \leq n - 1$ . We consider  $N_Q(u) \cap N_Q(v) = \emptyset$ . Let  $f(x) = f(u)$ . We assign different colors to the remaining vertices of  $G$ , and we can find that any two vertices have a vertex monochromatic path connecting them. Hence  $mvc(G) = n - 1$ . Note that there exists a spanning  $T$  and  $I(T) = \{q_1, q_2, u\}$ . Hence  $mvx_3(G) \geq n - 2$ . Now we prove that  $mvx_3(G) \leq n - 2$ . Otherwise we have  $f(q_1) = f(u)$ , since the vertex-monochromatic  $\{q', u, w\}$ -tree wastes at most one color where  $q' \notin N_Q(u) \cup N_Q(v)$ . Note that there does not exist a vertex-monochromatic  $\{q', v, w\}$ -tree, a contradiction. Hence  $mvx_3(G) = n - 2$ . We consider  $N_Q(u) \cap N_Q(v) \neq \emptyset$ . Note that there exists a spanning tree  $T$  and  $I(T) = \{x, u\}$ . Thus  $mvc(G) = mvx_3(G) = n - 1$ .

Suppose  $uv \in E(G), vw \in E(G)$  and  $uw \in E(G)$ . If  $N_Q(u) \cup N_Q(v) = V(Q)$ , then  $diam(G) = 2$ . Hence  $mvc(G) = n$  by Proposition 2.2. There exists a spanning tree  $T$  and  $I(T) = \{q_1, u\}$ , then  $mvx_3(G) \geq n - 1$ . We consider  $mvx_3(G) = n$ . There does not exist a vertex-monochromatic  $\{q_1, q_2, w\}$ -tree, where  $q_1 \in N_Q(u), q_1 \notin N_Q(v)$  and  $q_2 \in N_Q(v), q_2 \notin N_Q(u)$ . Thus  $mvx_3(G) = n - 1$ . If  $N_Q(u) \cup N_Q(v) \neq V(Q)$ , then there exists a vertex  $q'$  such that  $d(w, q') = 3$ . Hence  $mvc(G) \leq n - 1$ . There exists a spanning tree with  $n - 2$  leaves. Hence  $mvc(G) \geq mvx_3(G) \geq n - 1$ , and so  $mvc(G) = mvx_3(G) = n - 1$ .

Assume that  $d_Q(u) \geq 1, d_Q(v) \geq 1$  and  $d_Q(w) \geq 1$ . Suppose  $uv \notin E(G), uw \notin E(G), vw \notin E(G)$ . If

$N_Q(u) \cap N_Q(v) \cap N_Q(w) \neq \emptyset$ , then there exists a spanning tree with  $n - 1$  leaves. Hence  $mvc(G) = mvx_3(G) = n$ . Next we consider  $N_Q(u) \cap N_Q(v) \cap N_Q(w) = \emptyset$ . If  $\max\{|N_Q(u) \cap N_Q(v)|, |N_Q(v) \cap N_Q(w)|, |N_Q(u) \cap N_Q(w)|\} = 0$ , then  $d(u, v) = d(u, w) = d(v, w) = 3$ . Hence  $mvc(G) \leq n - 2$ . Note that there exists a spanning tree  $T$  with  $I(T) = \{q_1, q_2, q_3\}$ . Then  $mvc(G) \geq mvx_3(G) \geq n - 2$ , and so  $mvc(G) = mvx_3(G) = n - 2$ . If  $\max\{|N_Q(u) \cap N_Q(v)|, |N_Q(v) \cap N_Q(w)|, |N_Q(u) \cap N_Q(w)|\} \geq 1$  and  $\min\{|N_Q(u) \cap N_Q(v)|, |N_Q(v) \cap N_Q(w)|, |N_Q(u) \cap N_Q(w)|\} = 0$ , then  $diam(G) = 3$ . Hence  $mvc(G) \leq n - 1$  by Proposition 2.2. There exists a spanning tree  $T$  with  $I(T) = \{x, q_3\}$  or  $I(T) = \{y, q_1\}$  or  $I(T) = \{z, q_2\}$ . Thus  $mvc(G) = mvx_3(G) = n - 1$ . If  $\max\{|N_Q(u) \cap N_Q(v)|, |N_Q(v) \cap N_Q(w)|, |N_Q(u) \cap N_Q(w)|\} \geq 1$  and  $\min\{|N_Q(u) \cap N_Q(v)|, |N_Q(v) \cap N_Q(w)|, |N_Q(u) \cap N_Q(w)|\} \geq 1$ , then  $diam(G) = 2$ . Hence  $mvc(G) = n$  by Proposition 2.2. There exists a spanning tree  $T$  with  $I(T) = \{x, q_3\}$ . Thus  $mvx_3(G) \geq n - 1$ . Assign each vertex of  $G$  by different colors. We can find that there does not exist a vertex-monochromatic  $\{u, v, w\}$ -tree. Hence  $mvx_3(G) = n - 1$ .

Suppose  $uv \in E(G), uv \notin E(G)$  and  $vw \notin E(G)$ . If  $N_Q(u) \cap N_Q(v) \cap N_Q(w) \neq \emptyset$ , there exists a spanning tree with  $n - 1$  leaves. Hence  $mvc(G) = mvx_3(G) = n$ . Next we consider  $N_Q(u) \cap N_Q(v) \cap N_Q(w) = \emptyset$ . If  $\max\{|N_Q(u) \cap N_Q(v)|, |N_Q(v) \cap N_Q(w)|, |N_Q(u) \cap N_Q(w)|\} = 0$ , then  $d(u, w) = d(v, w) = 3$ . Hence  $mvc(G) \leq n - 2$ . There exists a spanning tree  $T$  with  $I(T) = \{q_1, q_2, q_3\}$ . Thus  $mvc(G) \geq mvx_3(G) \geq n - 2$ , and so  $mvc(G) = mvx_3(G) = n - 2$ . If  $\max\{|N_Q(u) \cap N_Q(v)|, |N_Q(v) \cap N_Q(w)|, |N_Q(u) \cap N_Q(w)|\} \geq 1$  and  $\min\{|N_Q(u) \cap N_Q(w)|, |N_Q(v) \cap N_Q(w)|\} \geq 1$ . We have  $diam(G) = 2$ . Hence  $mvc(G) = n$  by Proposition 2.2. There exists a spanning tree  $T$  with  $I(T) = \{y, z\}$ . Thus  $mvx_3(G) \geq n - 1$ . We assign each vertex of  $G$  by different colors. There does not exist a vertex-monochromatic  $\{u, v, w\}$ -tree. Hence  $mvx_3(G) = n - 1$ . Thus  $mvx_3(G) = n - 1$ . If  $\max\{|N_Q(u) \cap N_Q(v)|, |N_Q(v) \cap N_Q(w)|, |N_Q(u) \cap N_Q(w)|\} \geq 1, \min\{|N_Q(u) \cap N_Q(w)|, |N_Q(v) \cap N_Q(w)|\} = 0$ , then  $diam(G) = 3$ . Hence  $mvc(G) \leq n - 1$  by Proposition 2.2. There exists a spanning tree  $T$  with  $I(T) = \{x, q_3\}$  or  $I(T) = \{y, q_1\}$  or  $I(T) = \{z, q_2\}$ . Thus  $mvc(G) \geq mvx_3(G) \geq n - 1$ , and so  $mvc(G) = mvx_3(G) = n - 1$ .

Suppose  $uv \in E(G), vw \in E(G)$  and  $uw \notin E(G)$ . Note that  $diam(G) = 2$ , and hence  $mvc(G) = n$  by Proposition 2.2. There exists a spanning tree  $T$  with  $I(T) = \{q_2, v\}$ , and so  $mvx_3(G) \geq n - 1$ . If  $\min\{|N_Q(u) \cap N_Q(w)|, |N_Q(v) \cap N_Q(w)|, |N_Q(u) \cap N_Q(v)|\} \geq 1$ , then we assign each vertex of  $G$  by different colors. There exists a vertex-monochromatic  $S$ -tree for any  $S \subseteq V(G)$ . Hence  $mvx_3(G) = n$ . If  $\min\{|N_Q(u) \cap N_Q(w)|, |N_Q(v) \cap N_Q(w)|, |N_Q(u) \cap N_Q(v)|\} = 0$  and  $N_Q(u) \cap N_Q(w) = \emptyset$ , then  $mvx_3(G) = n - 1$ . Otherwise, let the vertices of  $G$  be assigned by different colors. We can find that there does not exist a vertex-monochromatic  $\{q_i, u, w\}$ -tree, where  $q_i \in N_Q(u) \setminus N_Q(v)$  or  $q_i \in N_Q(w) \setminus N_Q(v)$  or  $q_i \notin N_Q(u) \cup N_Q(v) \cup N_Q(w)$ . If  $\min\{|N_Q(u) \cap N_Q(w)|, |N_Q(v) \cap N_Q(w)|, |N_Q(u) \cap N_Q(v)|\} = 0$  and  $N_Q(u) \cap N_Q(w) \neq \emptyset$ , then for  $N_Q(u) \cup N_Q(v) \cup N_Q(w) \neq V(Q)$ , there exists a vertex  $q'$  such that  $q' \notin N_Q(u) \cup N_Q(v) \cup N_Q(w)$ . Assign each vertex of  $G$  by different colors. We can find that there does not exist a vertex-monochromatic  $\{q', u, v\}$ -tree or vertex-monochromatic  $\{q', v, w\}$ -tree. Hence  $mvx_3(G) = n - 1$ . Next we consider  $N_Q(u) \cup N_Q(v) \cup N_Q(w) = V(Q)$ .

**Case 1.**  $N_Q(u) \cap N_Q(v) \neq \emptyset$ . If  $N_Q(w) \subset N_Q(u)$  and  $N_Q(u) \setminus N_Q(w) \subset N_Q(v)$ , then we assign each vertex of  $G$  by different colors. There exists a vertex-monochromatic  $S$ -tree for any  $S \subseteq V(G)$ . Hence  $mvx_3(G) = n$ . Otherwise, there exists a vertex  $q'$  such that  $q' \in N_Q(u) \setminus (N_Q(v) \cup N_Q(w))$  or  $q' \in N_Q(w) \setminus N_Q(u)$ . We assign each vertex of  $G$  by different colors. We can find that there does not exist a vertex-monochromatic  $\{q', v, w\}$ -tree. Hence  $mvx_3(G) = n - 1$ .

**Case 2.**  $N_Q(u) \cap N_Q(v) = \emptyset$ . Suppose  $N_Q(v) \cap N_Q(w) \neq \emptyset$ . We have  $mvx_3(G) = n$  for  $N_Q(u) \subset N_Q(w)$  and  $N_Q(w) \setminus N_Q(u) \subset N_Q(v)$ , and  $mvx_3(G) = n - 1$  for other cases by the same proof of Case 1. Suppose  $N_Q(v) \cap N_Q(w) = \emptyset$ . If  $N_Q(u) = N_Q(w)$ , then we assign each vertex of  $G$  by different colors. There exists a vertex-monochromatic  $S$ -tree for any  $S \subseteq V(G)$ . Hence  $mvx_3(G) = n$ . Otherwise, there exists a vertex  $q'$  such that  $q' \in N_Q(w) \setminus N_Q(u)$  or  $q' \in N_Q(u) \setminus N_Q(w)$ . We assign each vertex of  $G$  by different colors. We can find that there does not exist a vertex-monochromatic  $\{q', u, v\}$ -tree. Hence  $mvx_3(G) = n - 1$ .

Suppose  $uv \in E(G), uv \in E(G)$  and  $vw \in E(G)$ . Note that  $diam(G) = 2$ , hence  $mvc(G) = n$  by Proposition 2.2. If  $N_Q(u) \cup N_Q(v) \cup N_Q(w) = V(Q)$ , then we assign each vertex of  $G$  by different colors. There exists a vertex-monochromatic  $S$ -tree for any  $S \subseteq V(G)$ . Hence  $mvx_3(G) = n$ . If  $N_Q(u) \cup N_Q(v) \cup N_Q(w) \neq V(Q)$ , then there exists a vertex  $q'$  such that  $q' \notin N_Q(u) \cup N_Q(v) \cup N_Q(w)$ . Consider  $\min\{|N_Q(u) \cap N_Q(v)|, |N_Q(u) \cap N_Q(w)|, |N_Q(v) \cap N_Q(w)|\} \geq 1$ . We assign each vertex of  $G$  by different colors. There exists a vertex-monochromatic  $S$ -tree for any  $S \subseteq V(G)$ . Hence  $mvx_3(G) = n$ . Consider  $\min\{|N_Q(u) \cap N_Q(v)|, |N_Q(u) \cap N_Q(w)|, |N_Q(v) \cap N_Q(w)|\} = 0$ . There exists a spanning tree  $T$  with  $I(T) = \{q_2, v\}$ , and so  $mvx_3(G) \geq n - 1$ . On the other hand, let the vertices of  $G$  be assigned by different colors. There does not exist a vertex-monochromatic  $\{q', u, v\}$ -tree or

vertex-monochromatic  $\{q', u, w\}$ -tree or vertex-monochromatic  $\{q', v, w\}$ -tree. Hence  $mvx_3(G) \leq n - 1$ , and so  $mvx_3(G) = n - 1$ .  $\square$

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