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Spectral radius and spanning tree with bounded total k-excess of graphs

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Abstract. The binding number $b(G)$ of a graph G is the minimum value of $|N_G(X)|/|X|$ taken over all non-empty subsets X of $V(G)$ such that $N_G(X) \neq V(G)$. A graph G is called 1-binding if $b(G) \geq 1$. Let $k \geq 2$ be an integer and let T be a spanning tree of a connected graph. The total k-excess $te(T,k)$ is the summation of the k-excesses of all vertices in T, namely, $te(T,k) = \sum_{v \in V(T)} \max\{0, d_T(v) - k\}$. One can see that T is a spanning k-tree if and only if $te(T,k) = 0$. In this paper, we present a tight sufficient condition in terms of the spectral radius for a connected 1-binding graph to contain a spanning tree with bounded total k-excess, which generalizes the result of Fan, Liu and Ao [Linear Algebra Appl. 705 (2025) 1-16] on the existence of a spanning k-tree in 1-binding graphs.

The toughness $\tau(G) = \min\{\frac{|S|}{c(G-S)} : S$ is a cut set of vertices in $G\}$ for $G \neq K_n$. A graph G is called t-tough if $\tau(G) \geq t$. We in this paper also provide a tight sufficient condition based on the spectral radius for a connected $\frac{1}{k-\eta}$-tough graph to contain a spanning tree with bounded total k-excess, where $k \geq 3$ is an integer and $\eta = \{0, 1\}$. It extends the result of Liu, Fan and Shu [Discrete Math. 348 (2025) 114593] on the existence of a spanning k-tree in connected $\frac{1}{k-\eta}$-tough graphs.

1. Introduction

Let G be a finite, undirected and simple graph with vertex set $V(G)$ and edge set $E(G)$. The order and size of G are denoted by $|V(G)| = n$ and $|E(G)| = e(G)$. Let $c(G)$ be the number of components of G. Let G_1 and G_2 be two vertex-disjoint graphs. We denote by $G_1 \cup G_2$ the disjoint union of G_1 and G_2. The join $G_1 \vee G_2$ is the graph obtained from $G_1 \cup G_2$ by adding all possible edges between $V(G_1)$ and $V(G_2)$. For any $v \in V(G)$, let $N_G(v)$ ($N(v)$ for short) denote the neighborhood of v in G, and for $X \subseteq V(G)$, let $N_G(X) = \bigcup_{x \in X} N(x)$. The *binding number* $b(G)$ of a graph G is

$$b(G) = \min\left\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\right\}.$$

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A graph G is called 1-*binding* if $b(G) \geq 1$. The *toughness* of a graph G

$$\tau(G) = \min\left\{\frac{|S|}{c(G-S)} : S \text{ is a cut set of vertices in } G\right\}$$

for $G \neq K_n$. For undefined notions and symbols, one can refer to [5].

For any integer $k \geq 2$, a *spanning k-tree* of a connected graph G is a spanning tree in which every vertex has degree at most k. For a spanning tree T of a connected graph, the k-excess of a vertex v is defined to be $\max\{0, d_T(v) - k\}$. The *total k-excess* $te(T, k)$ is the summation of the k-excesses of all vertices in T, namely,

$$te(T, k) = \sum_{v \in V(T)} \max\{0, d_T(v) - k\}.$$

Note that a spanning tree T satisfies $te(T, k) = 0$ if and only if T is a spanning k-tree. Hence spanning tree with bounded total excess of a connected graph is a natural generalization of spanning k-tree.

Ozeki and Yamashita [17] proved that it is an $\mathcal{NP}$-complete problem to decide whether a given connected graph admits a spanning k-tree. Win [19] provided a toughness-type condition for the existence of a spanning k-tree in a connected graph. Ellingham and Zha [7] presented a short proof to Win's theorem. Using the toughness-type condition, Fan et al. [9] posed a spectral condition for the existence of a spanning k-tree in a connected graph. Recently, Ao et al. [3] extended the above spectral condition by using connectivity and closure theorem for a graph to contain a spanning k-tree [12]. Subsequently, Ao et al. [2] discussed the sufficient conditions for spanning k-trees using the independent number condition [15].

For more results on spanning tree with bounded total k-excess, one can refer to [14, 16, 18]. Very recently, Ao et al. [2] and Fan et al. [9] proved the following spectral condition to ensure that a connected graph contains a spanning tree with bounded total k-excess.

Theorem 1.1 (Ao et al. [1], Fan et al. [9]). *Let G be a connected graph of order $n \geq \max\{5b + 3k + 12, b^2 + 6b + 4k + 3\}$, where $b \geq 0$ and $k \geq 3$. If*

$$\rho(G) \geq \rho(K_1 \vee (K_{n-b-k-1} \cup (b + k)K_1)),$$

then G has a spanning tree T with $te(T, k) \leq b$ unless $G = K_1 \vee (K_{n-b-k-1} \cup (b + k)K_1)$.

Note that $b(K_1 \vee (K_{n-b-k-1} \cup (b+k)K_1) = \frac{1}{b+k} < 1$. Hence $K_1 \vee (K_{n-b-k-1} \cup (b+k)K_1)$ is not 1-binding. Hence, we present the following problem.

Problem 1.2. *What is the sufficient spectral condition for the existence of a spanning tree with bounded total k-excess in 1-binding graphs?*

Concerning Problem 1.2, we prove the following result.

Theorem 1.3. *Let G be a connected 1-binding graph of order $n \geq 16k^2 - 34k + 1$, where $0 \leq b \leq k - 10$. If*

$$\rho(G) \geq \rho(K_1 \vee (K_{n-2b-2k} \cup (b + k - 1)K_2 \cup K_1)),$$

then G has a spanning tree T with $te(T, k) \leq b$ unless $G = K_1 \vee (K_{n-2b-2k} \cup (b + k - 1)K_2 \cup K_1)$.

In fact, one can check that $b(K_1 \vee (K_{n-2b-2k} \cup (b + k - 1)K_2 \cup K_1)) = 1$. By setting $b = 0$ in Theorem 1.3, we can obtain Theorem 1.5 in [8].

Win [19] proved that if $\tau(G) \geq \frac{1}{k-2}$ with $k \geq 3$, then G contains a spanning k-tree. Obviously, if a graph contains a spanning k-tree, then it must contain a spanning tree with bounded total k-excess. Hence, if $\tau(G) \geq \frac{1}{k-2}$ with $k \geq 3$, then G contains a spanning tree with bounded total k-excess. The extremal graph $K_1 \vee (K_{n-b-k-1} \cup (b+k)K_1)$ in Theorem 1.1 is not $\frac{1}{k-\eta}$-tough since $\tau(K_1 \vee (K_{n-b-k-1} \cup (b+k)K_1)) = \frac{1}{b+k+1} \leq \frac{1}{k+1} < \frac{1}{k-\eta}$ with $\eta = \{0, 1\}$. Naturally, we propose the following interesting problem.

Problem 1.4. *What is the sufficient spectral condition to guarantee the existence of a spanning tree with bounded total k-excess among $\frac{1}{k-\eta}$-tough graphs, where $\eta = \{0, 1\}$?*

Basing on Problem 1.4, we prove the following result.

Theorem 1.5. *Let G be a connected $\frac{1}{k-\eta}$-tough graph of order n, where $k \geq 3$, $\eta = \{0, 1\}$, $b \geq 0$ and $n \geq \frac{(2b+9)k^2+(2b^2+13b+15)k-2b^2-21b-48}{2k-4}$. If*

$$\rho(G) \geq \rho(K_{\lceil \frac{b+3}{2-\eta} \rceil} \vee (K_{n-(k-1)\lceil \frac{b+3}{2-\eta} \rceil - b-2} \cup [(k-2)\lceil \frac{b+3}{2-\eta} \rceil + b + 2]K_1)),$$

then G has a spanning tree T with $te(T, k) \leq b$ unless $G = K_{\lceil \frac{b+3}{2-\eta} \rceil} \vee (K_{n-(k-1)\lceil \frac{b+3}{2-\eta} \rceil - b-2} \cup [(k-2)\lceil \frac{b+3}{2-\eta} \rceil + b + 2]K_1)$.

Theorem 1.5 naturally generalizes the following theorem.

Theorem 1.6 (Liu et al. [13]). *Let G be a connected $\frac{1}{k-\eta}$-tough graph of order $n \geq 8k + 12$ with $k \geq 3$ and $\eta = \{0, 1\}$. If*

$$\rho(G) \geq \rho(K_{\eta+2} \vee (K_{n-(k-1)(\eta+2)-2} \cup [(k-2)(\eta+2) + 2]K_1)),$$

then G contains a spanning k-tree unless $G = K_{\eta+2} \vee (K_{n-(k-1)(\eta+2)-2} \cup [(k-2)(\eta+2) + 2]K_1)$.

2. Proof of Theorem 1.3

Enomoto et al. [6] presented a sufficient condition for the existence of a spanning tree with bounded total *k*-excess in a connected graph.

Lemma 2.1 (Enomoto et al. [6]). *Let k and b be integers with $k \geq 2$ and $b \geq 0$, and let G be a connected graph. If $c(G - S) \leq (k-2)|S| + b + 2$ for every $S \subseteq V(G)$, then G has a spanning tree T with $te(T, k) \leq b$.*

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ matrices. For all i and j, we define $A \leq B$ if $a_{ij} \leq b_{ij}$, and define $A < B$ if $A \leq B$ and $A \neq B$.

Lemma 2.2 (Berman and Plemmons [4], Horn and Johnson [10]). *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ matrices with the spectral radii $\lambda(A)$ and $\lambda(B)$, respectively. If $0 \leq A \leq B$, then $\lambda(A) \leq \lambda(B)$. Furthermore, if B is irreducible and $0 \leq A < B$, then $\lambda(A) < \lambda(B)$.*

Lemma 2.3. *Let $n = \sum_{i=1}^{(k-2)s+b+3} n_i + s$ with $s \geq 1$ and $n_1 \geq n_2 \geq \cdots \geq n_{(k-2)s+b+3}$. If $n_i \geq 2$ for $1 \leq i \leq (k-3)s+b+3$ and $n_j \geq 1$ for $(k-3)s + b + 4 \leq j \leq (k-2)s + b + 3$, then*

$$\rho(K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_{(k-2)s+b+3}})) \leq \rho(K_s \vee (K_{n-(2k-4)s-2b-4} \cup ((k-3)s + b + 2)K_2 \cup sK_1)),$$

where equality holds if and only if $(n_1, n_2, \ldots, n_{(k-2)s+b+3}) = (n - (2k-4)s - 2b - 4, \underbrace{2, \ldots, 2}_{(k-3)s+b+2}, \underbrace{1, \ldots, 1}_{s})$.

Proof. Let $G = K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_{(k-2)s+b+3}})$, and let $\mathbf{x}$ be the Perron vector of $A(G)$ corresponding to $\rho(G)$. By symmetry, we can suppose that $x_v = x_i$ for all $v \in V(K_{n_i})$, where $1 \leq i \leq (k-2)s + b + 3$, and $x_u = y$ for all $u \in V(K_s)$. By $A(G)\mathbf{x} = \rho(G)\mathbf{x}$, we have

$$[\rho(G) - (n_1 - 1)]x_1 = sy > 0,$$

which implies that $\rho(G) > n_1 - 1$.

Claim 1. $x_1 \geq x_j$ for $2 \leq j \leq (k-2)s + b + 3$.

Proof. Note that $n_1 \geq n_j$ and $\rho(G) > n_j - 1$. Then

$$\begin{aligned}[\rho(G) - (n_j - 1)](x_1 - x_j) &= [\rho(G) - (n_j - 1)]x_1 - sy \\ &= sy + (n_1 - 1)x_1 - (n_j - 1)x_1 - sy \\ &= (n_1 - n_j)x_1 \geq 0.\end{aligned}$$

Hence $x_1 \geq x_j$ for $2 \leq j \leq (k-2)s + b + 3$. □

Let $G^* = K_s \vee (K_{n-(2k-4)s-2b-4} \cup ((k-3)s + b + 2)K_2 \cup sK_1)$. By Claim 1, $n_i \geq 2$ for $1 \leq i \leq (k-3)s + b + 3$ and $n_j \geq 1$ for $(k-3)s + b + 4 \leq j \leq (k-2)s + b + 3$, we have

$$\begin{aligned}&\rho(G^*) - \rho(G) \geq \mathbf{x}^T(A(G^*) - A(G))\mathbf{x} \\ = \ & 2[\sum_{i=2}^{(k-3)s+b+3} n_1(n_i - 2)x_1x_i + \sum_{i=(k-3)s+b+4}^{(k-2)s+b+3} n_1(n_i - 1)x_1x_i - 2\sum_{i=2}^{(k-3)s+b+3} (n_i - 2)x_i^2 \\ & - \sum_{i=(k-3)s+b+4}^{(k-2)s+b+3} (n_i - 1)x_i^2 + \sum_{i=2}^{(k-3)s+b+2} \sum_{j=i+1}^{(k-3)s+b+3} (n_i - 2)(n_j - 2)x_ix_j \\ & + \sum_{i=2}^{(k-3)s+b+3} \sum_{j=(k-3)s+b+4}^{(k-2)s+b+3} (n_i - 2)(n_j - 1)x_ix_j + \sum_{i=(k-3)s+b+4}^{(k-2)s+b+2} \sum_{j=i+1}^{(k-2)s+b+3} (n_i - 1)(n_j - 1)x_ix_j] \\ = \ & 2[\sum_{i=2}^{(k-3)s+b+3} (n_i - 2)(n_1x_1 - 2x_i)x_i + \sum_{i=(k-3)s+b+4}^{(k-2)s+b+3} (n_i - 1)(n_1x_1 - x_i)x_i \\ & + \sum_{i=2}^{(k-3)s+b+2} \sum_{j=i+1}^{(k-3)s+b+3} (n_i - 2)(n_j - 2)x_ix_j + \sum_{i=2}^{(k-3)s+b+3} \sum_{j=(k-3)s+b+4}^{(k-2)s+b+3} (n_i - 2)(n_j - 1)x_ix_j \\ & + \sum_{i=(k-3)s+b+4}^{(k-2)s+b+2} \sum_{j=i+1}^{(k-2)s+b+3} (n_i - 1)(n_j - 1)x_ix_j] \\ \geq \ & 0,\end{aligned}$$

where all equalities holds if and only if $(n_1, n_2, \ldots, n_{(k-2)s+b+3}) = (n - (2k-4)s - 2b - 4, \underbrace{2, \ldots, 2}_{(k-3)s+b+2}, \underbrace{1, \ldots, 1}_{s})$. □

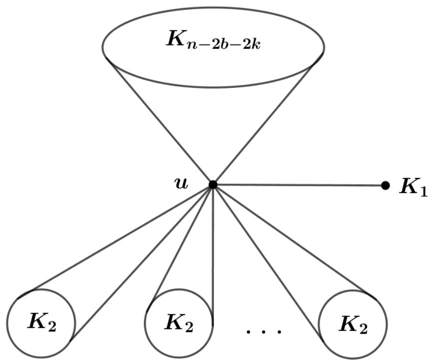

Figure 1: Graph $K_1 \vee (K_{n-2b-2k} \cup (b + k - 1)K_2 \cup K_1)$.

Lemma 2.4. *Let k and b be integers with $k \geq 3$ and $b \geq 0$. Then $K_1 \vee (K_{n-2b-2k} \cup (b + k - 1)K_2 \cup K_1)$ contains no spanning tree T with $te(T, k) \leq b$.*

Proof. Let $G = K_1 \vee (K_{n-2b-2k} \cup (b+k-1)K_2 \cup K_1)$ (see Fig. 1). Let u denote the unique vertex of degree $n-1$ in G. Assume that T is a spanning tree of G. One can observe that $d_T(u) \geq b+k+1$. This implies that $te(T,k) \geq b+1$. Hence G contains no spanning tree T with $te(T,k) \leq b$. □

Now, we are ready to provide the proof of Theorem 1.3.

Proof of Theorem 1.3. Assume that G contains no spanning tree with $te(T,k) \leq b$, where G is a connected 1-binding graph. By Lemma 2.1, there exists some subset $S \subseteq V(G)$ such that $c(G-S) \geq (k-2)|S| + b + 3$. Let $|S| = s$.

Claim 2. $s \geq 1$.

Proof. If $s = 0$, then $S = \emptyset$ and hence $1 = c(G) = c(G-S) \geq b+3$, a contradiction. □

Let $c(G-S) = q$, and let $C_1, C_2, \ldots, C_q$ be the components of $G-S$, where $|C_i| = c_i$ for $1 \leq i \leq q$. Without loss of generality, we can suppose that $c_1 \geq c_2 \geq \cdots \geq c_q$, where $q \geq (k-2)s + b + 3$.

Claim 3. $c_{q-s} \geq 2$.

Proof. If $c_{q-s} < 2$, then $c_{q-s} = c_{q-s+1} = \cdots = c_q = 1$. Take $S' = V(C_{q-s} \cup C_{q-s+1} \cup \cdots \cup C_q)$. Then $N(S') \subseteq S$. Note that $|S'| = s+1$ and $|N(S')| \leq |S| = s$. Then we have

$$b(G) \leq \frac{|N(S')|}{|S'|} \leq \frac{s}{s+1} < 1,$$

which contradicts that G is 1-binding. □

By Claim 3, we have $c_i \geq 2$ for $1 \leq i \leq q-s$ and $c_j \geq 1$ for $q-s+1 \leq j \leq q$. Let $n_1 = \sum_{i=1}^{q-(k-2)s-b-2} c_i$, $n_2 = c_{q-(k-2)s-b-1}$,..., $n_{(k-2)s+b+3} = c_q$. That is to say, $n_i \geq 2$ for $1 \leq i \leq (k-3)s+b+3$ and $n_j \geq 1$ for $(k-3)s+b+4 \leq j \leq (k-2)s+b+3$, where $n_1 \geq n_2 \geq \cdots \geq n_{(k-2)s+b+3}$ are positive integers with $\sum_{i=1}^{(k-2)s+b+3} n_i = n-s$. This implies that $n \geq [2(k-3)s + 2b + 6] + 2s = (2k-4)s + 2b + 6$, and G is a spanning subgraph of $G' = K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_{(k-2)s+b+3}})$. By Lemma 2.2, we have

$$\rho(G) \leq \rho(G'), \tag{1}$$

where equality holds if and only if $G = G'$. Let $G'' = K_s \vee (K_{n-(2k-4)s-2b-4} \cup ((k-3)s+b+2)K_2 \cup sK_1)$. By Lemma 2.3, we have

$$\rho(G') \leq \rho(G''), \tag{2}$$

where equality holds if and only if $(n_1, n_2, \ldots, n_{(k-2)s+b+3}) = (n-(2k-4)s-2b-4, \underbrace{2,\ldots,2}_{(k-3)s+b+2}, \underbrace{1,\ldots,1}_{s})$.

By Claim 2, we divide the proof into the following two cases.

Case 1. $s = 1$.

Then $G'' = K_1 \vee (K_{n-2b-2k} \cup (b+k-1)K_2 \cup K_1)$. By (1) and (2), we have

$$\rho(G) \leq \rho(K_1 \vee (K_{n-2b-2k} \cup (b+k-1)K_2 \cup K_1)),$$

where equality holds if and only if $G = K_1 \vee (K_{n-2b-2k} \cup (b+k-1)K_2 \cup K_1)$. Combining the assumption of Theorem 1.3, we have $\rho(G) = \rho(K_1 \vee (K_{n-2b-2k} \cup (b+k-1)K_2 \cup K_1))$, and hence $G = K_1 \vee (K_{n-2b-2k} \cup (b+k-1)K_2 \cup K_1)$. By Lemma 2.4, $K_1 \vee (K_{n-2b-2k} \cup (b+k-1)K_2 \cup K_1)$ contains no spanning tree with $te(T,k) \leq b$. Hence $G = K_1 \vee (K_{n-2b-2k} \cup (b+k-1)K_2 \cup K_1)$.

Case 2. $s \geq 2$.

Let $G''' = K_1 \vee (K_{n-2b-2k} \cup (b+k-1)K_2 \cup K_1)$. Recall that $G'' = K_s \vee (K_{n-(2k-4)s-2b-4} \cup ((k-3)s+b+2)K_2 \cup sK_1)$. Then vertex set of G'' can be divided into $V(G'') = V(K_s) \cup V(K_{n-(2k-4)s-2b-4}) \cup V(((k-3)s+b+2)K_2) \cup V(sK_1)$, where $V(sK_1) = \{u_1, u_2, \ldots, u_s\}$, $V(K_s) = \{v_1, v_2, \ldots, v_s\}$, $V(((k-3)s+b+2)K_2) = \left\{w_1, w_2, \ldots, w_{2(k-3)s+2b+4}\right\}$ and $V(K_{n-(2k-4)s-2b-4}) = \left\{z_1, z_2, \ldots, z_{n-(2k-4)s-2b-4}\right\}$. Let

$$\begin{aligned} E_1 \;=\;& \{u_i z_j | 2 \le i \le s, 1 \le j \le n-(2k-4)s-2b-4\} \cup \{w_i z_j | 2b+2k-1 \le i \le 2(k-3)s \\ & +2b+4, 1 \le j \le n-(2k-4)s-2b-4\} \cup \{u_i u_j | 2 \le i \le s-1, i+1 \le j \le s\} \cup \\ & \{w_i w_j | 2b+2k-1 \le i \le 2(k-3)s+2b+3, i+1 \le j \le 2(k-3)s+2b+4\}, \end{aligned}$$

and let

$$\begin{aligned} E_2 \;=\;& \{u_1 v_i | 2 \le i \le s\} \cup \{v_i w_j | 2 \le i \le s, 1 \le j \le 2b+2k-2\} \cup \{w_{2b+2k-1}w_{2b+2k}, w_{2b+2k+1} \\ & w_{2b+2k+2}, \ldots, w_{2(k-3)s+2b+3}w_{2(k-3)s+2b+4}\}. \end{aligned}$$

We have $G''' = G'' + E_1 - E_2$. Let $\mathbf{x}$ be the Perron vector of $A(G'')$, and let $\rho'' = \rho(G'')$. By symmetry, $\mathbf{x}$ takes the same value on the vertices of $V(K_s)$, $V(sK_1)$, $V(K_{n-(2k-4)s-2b-4})$ and $V(((k-3)s+b+2)K_2)$, respectively. We always use J to denote the all-one matrix, I to denote the identity square matrix, and O to denote the zero matrix. It is easy to see that $A(G'')$ is equal to

$$\begin{bmatrix} (J-I)_{s\times s} & J_{s\times s} & J_{s\times[n-(2k-4)s-2b-4]} & J_{s\times[2(k-3)s+2b+4]} \\ J_{s\times s} & O_{s\times s} & O_{s\times[n-(2k-4)s-2b-4]} & O_{s\times[2(k-3)s+2b+4]} \\ J_{[n-(2k-4)s-2b-4]\times s} & O_{[n-(2k-4)s-2b-4]\times s} & (J-I)_{[n-(2k-4)s-2b-4]\times[n-(2k-4)s-2b-4]} & O_{[n-(2k-4)s-2b-4]\times[2(k-3)s+2b+4]} \\ J_{[2(k-3)s+2b+4]\times s} & O_{[2(k-3)s+2b+4]\times s} & O_{[2(k-3)s+2b+4]\times[n-(2k-4)s-2b-4]} & B_{[2(k-3)s+2b+4]\times[2(k-3)s+2b+4]} \end{bmatrix},$$

where B equals

$$\begin{bmatrix} (J-I)_{2\times 2} & O_{2\times 2} & \cdots & O_{2\times 2} \\ O_{2\times 2} & (J-I)_{2\times 2} & \cdots & O_{2\times 2} \\ \vdots & \vdots & \ddots & \vdots \\ O_{2\times 2} & O_{2\times 2} & \cdots & (J-I)_{2\times 2} \end{bmatrix}.$$

We denote the entry of $\mathbf{x}$ by x_1, x_2, x_3 and x_4 corresponding to $V(K_s)$, $V(sK_1)$, $V(K_{n-(2k-4)s-2b-4})$ and $V(((k-3)s+b+2)K_2)$, respectively. According to $A(G'')\mathbf{x} = \rho''\mathbf{x}$, we have

$$\begin{aligned} &\rho'' x_2 = s x_1, \\ &\rho'' x_3 = s x_1 + [n-(2k-4)s-2b-5]x_3, \\ &\rho'' x_4 = s x_1 + x_4. \end{aligned}$$

Then $x_1 = \frac{\rho''}{s}x_2$, $x_3 = \frac{\rho''}{\rho''-[n-(2k-4)s-2b-5]}x_2 > x_2$ and $x_4 = \frac{\rho''}{\rho''-1}x_2 > x_2$. Note that G'' contains $K_{n-2ks+5s-2b-4}$ as a proper subgraph and G'' is not a complete graph. So we have $n-2ks+5s-2b-5 < \rho'' < n-1$.

Let $\mathbf{y}$ be the perron vector of $A(G''')$, and let $\rho''' = \rho(G''')$. By symmetry, $\mathbf{y}$ takes the same value (say y_1, y_2, y_3 and y_4) on the vertices of $\{u\}$, $V(K_1)$, $V(K_{n-2b-2k})$ and $V((b+k-1)K_2)$ (see Fig. 1). By $A(G''')\mathbf{y} = \rho'''\mathbf{y}$, we have

$$\begin{aligned} &\rho''' y_2 = y_1, \\ &\rho''' y_3 = y_1 + (n-2b-2k-1)y_3, \\ &\rho''' y_4 = y_1 + y_4. \end{aligned}$$

Then $y_3 = \frac{\rho'''}{\rho'''-(n-2b-2k-1)}y_2 > y_2$ and $y_3 = \frac{y_1}{\rho'''-(n-2b-2k-1)} > \frac{y_1}{\rho'''-1} = y_4$. Notice that $n \ge (2k-4)s+2b+6$. Then $2 \le s \le \frac{n-2b-6}{2k-4}$.

Claim 4. $\rho''' > \rho''$.

Proof. Known that

$$\begin{aligned}
&\mathbf{y}^T(\rho''' - \rho'')\mathbf{x} = \mathbf{y}^T(A(G''') - A(G''))\mathbf{x}\\
= \ & \sum_{i=2}^{s}\sum_{j=1}^{n-(2k-4)s-2b-4}(x_{u_i}y_{z_j} + x_{z_j}y_{u_i}) + \sum_{i=2b+2k-1}^{2(k-3)s+2b+4}\sum_{j=1}^{n-(2k-4)s-2b-4}(x_{w_i}y_{z_j} + x_{z_j}y_{w_i})\\
&+\sum_{i=2}^{s-1}\sum_{j=i+1}^{s}(x_{u_i}y_{u_j} + x_{u_j}y_{u_i}) + \sum_{i=2b+2k-1}^{2(k-3)s+2b+3}\sum_{j=i+1}^{2(k-3)s+2b+4}(x_{w_i}y_{w_j} + x_{w_j}y_{w_i}) - \sum_{i=2}^{s}(x_{v_i}y_{u_1} + x_{u_1}y_{v_i})\\
&-\sum_{i=2}^{s}\sum_{j=1}^{2b+2k-2}(x_{v_i}y_{w_j} + x_{w_j}y_{v_i}) - [x_{w_{2b+2k-1}}y_{w_{2b+2k}} + x_{w_{2b+2k+1}}y_{w_{2b+2k+2}} + \cdots + x_{w_{2(k-3)s+2b+3}}y_{w_{2(k-3)s+2b+4}}\\
&+x_{w_{2b+2k}}y_{w_{2b+2k-1}} + x_{w_{2b+2k+2}}y_{w_{2b+2k+1}} + \cdots + x_{w_{2(k-3)s+2b+4}}y_{w_{2(k-3)s+2b+3}}]\\
= \ & (s-1)[n-(2k-4)s-2b-4](x_2y_3 + x_3y_3) + [2(k-3)s-2k+6][n-(2k-4)s-2b-4]\\
&(x_4y_3 + x_3y_3) + (s-1)(s-2)x_2y_3 + [2(k-3)s-2k+6][2(k-3)s-2k+5]x_4y_3 - (s-1)\\
&(x_1y_2 + x_2y_3) - (s-1)(2b+2k-2)(x_1y_4 + x_4y_3) - [2(k-3)s-2k+6]x_4y_3\\
= \ & (s-1)[(n-(2k-5)s-2b-7)x_2y_3 + (2k-5)(n-(2k-4)s-2b-4)x_3y_3 + [2(k-3)(n\\
&-(2k-4)s-2b-4) + 2(k-3)(2(k-3)(s-1)-1) - (2b+2k-2) - 2(k-3)]x_4y_3 - x_1y_2\\
&-(2b+2k-2)x_1y_4].
\end{aligned}$$

By $n \geq (2k-4)s + 2b + 6$, $0 \leq b \leq k-10$ and $s \geq 2$, we have $f(k) = 2(k-3)(n-(2k-4)s-2b-4) + 2(k-3)(2(k-3)(s-1)-1) - 2b - 4k + 8 \geq 4(k-3) + 2(k-3)(2(k-3)-1) - 2b - 4k + 8 = 4k^2 - 26k - 2b + 38 > 4k^2 - 26k - 2(k-10) + 38 = 4k^2 - 28k + 58 > 0$. Combining $x_3 > x_2$, $x_4 > x_2$, $y_3 > y_2$, $y_3 > y_4$, $x_1 = \frac{\rho''}{s}x_2$, $\rho'' < n-1$ and $2 \leq s \leq \frac{n-2b-6}{2k-4}$, we have

$$\begin{aligned}
&\mathbf{y}^T(\rho''' - \rho'')\mathbf{x}\\
> \ & (s-1)\left[(4k-10)n - 4k^2s - 4k^2 + 12ks - 8bk + 2k + 18b - 3s + 15 - \frac{(2b+2k-1)\rho''}{s}\right]x_2y_3\\
> \ & (s-1)\left[(4k-10)n - 4k^2s - 4k^2 + 12ks - 8bk + 2k + 18b - 3s + 15 - \frac{(2b+2k-1)(n-1)}{2}\right]x_2y_3\\
= \ & (s-1)x_2y_3\left[(3k - b - \frac{19}{2})n - (4k^2 - 12k + 3)s - 4k^2 + 3k - 8bk + 19b + \frac{29}{2}\right]\\
\geq \ & (s-1)x_2y_3\left[(3k - b - \frac{19}{2})n - (4k^2 - 12k + 3)\cdot\frac{(n-2b-6)}{2k-4} - 4k^2 + 3k - 8bk + 19b + \frac{29}{2}\right]\\
= \ & \frac{(s-1)x_2y_3}{2k-4}\left[(2k^2 - 2kb - 19k + 4b + 35)n - 8k^3 + 46k^2 - 8k^2b + 46kb - 55k - 70b - 40\right]\\
\geq \ & \frac{(s-1)x_2y_3}{2k-4}\left[(2k^2 - 2k(k-10) - 19k + 35)n - 8k^3 + 46k^2 - 8k^2(k-10) - 55k - 70(k-10) - 40\right]\\
= \ & \frac{(s-1)x_2y_3}{2k-4}\left[(k+35)n - 16k^3 + 126k^2 - 125k + 660\right].
\end{aligned}$$

By $n \geq 16k^2 - 34k + 1$ and $k \geq 10$, we have

$$\mathbf{y}^T(\rho''' - \rho'')\mathbf{x} > 0.$$

This implies that $\rho''' > \rho''$. □

By Claim 4, (1) and (2), we have

$$\rho(G) \leq \rho(G') \leq \rho(G'') < \rho(G''') = \rho(K_1 \vee (K_{n-2b-2k} \cup (b+k-1)K_2 \cup K_1)),$$

a contradiction. □

3. Proof of Theorem 1.5

Lemma 3.1 (Fan et al. [9]). *Let $n_1 \geq n_2 \geq \cdots \geq n_t \geq 1$ be integers. If $n = \sum_{i=1}^{t} n_i + s$ and $n_1 < n - s - t + 1$, then*

$$\rho(K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_t})) < \rho(K_s \vee (K_{n-s-t+1} \cup (t-1)K_1)).$$

Lemma 3.2 (Hong[11]). *Let G be a graph with n vertices and $e(G)$ edges. Then*

$$\rho(G) \leq \sqrt{2e(G) - n + 1}.$$

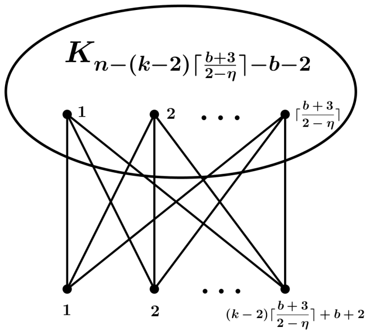

Figure 2: Graph $K_{\lceil \frac{b+3}{2-\eta} \rceil} \vee (K_{n-(k-1)\lceil \frac{b+3}{2-\eta} \rceil - b-2} \cup [(k-2)\lceil \frac{b+3}{2-\eta} \rceil + b + 2]K_1)$.

Lemma 3.3. *Let $k \geq 3$ and $b \geq 0$ be integers and let $\eta = \{0, 1\}$. Then $K_{\lceil \frac{b+3}{2-\eta} \rceil} \vee (K_{n-(k-1)\lceil \frac{b+3}{2-\eta} \rceil - b-2} \cup [(k-2)\lceil \frac{b+3}{2-\eta} \rceil + b + 2]K_1)$ contains no spanning tree T with $te(T, k) \leq b$.*

Proof. Let $G = K_{\lceil \frac{b+3}{2-\eta} \rceil} \vee (K_{n-(k-1)\lceil \frac{b+3}{2-\eta} \rceil - b-2} \cup [(k-2)\lceil \frac{b+3}{2-\eta} \rceil + b + 2]K_1)$ (see Fig. 2). Suppose to the contrary that G contains a spanning tree T with $te(T, k) \leq b$. By Lemma 2.1, for every subset $S \subseteq V(G)$, we have $c(G - S) \leq (k-2)|S| + b + 2$. However, if we take $S = V(K_{\lceil \frac{b+3}{2-\eta} \rceil})$ in G, then $c(G - S) = (k-2)\lceil \frac{b+3}{2-\eta} \rceil + b + 3 > (k-2)|S| + b + 2 = (k-2)\lceil \frac{b+3}{2-\eta} \rceil + b + 2$, a contradiction. □

Now, we are ready to provide the proof of Theorem 1.5.

Proof of Theorem 1.5. Suppose that G is a connected $\frac{1}{k-\eta}$-tough graph with $\eta = \{0, 1\}$ which contains no spanning tree T with $te(T, k) \leq b$. By Lemma 2.1, there exists some subset $S \subseteq V(G)$ such that $c(G - S) \geq (k-2)|S| + b + 3$. By the definition of $\frac{1}{k-\eta}$-tough graphs, we have

$$\frac{|S|}{c(G-S)} \geq \frac{1}{k-\eta}.$$

Then $(k-2)|S| + b + 3 \leq c(G - S) \leq (k - \eta)|S|$, and hence $|S| \geq \frac{b+3}{2-\eta}$.

Let $c(G - S) = q$, $|S| = s$, and let $C_1, C_2, \ldots, C_q$ be the components of $G - S$, where $|C_i| = c_i$ for $1 \leq i \leq q$. Without loss of generality, we can suppose that $c_1 \leq c_2 \leq \cdots \leq c_q$, where $q \geq (k-2)s + b + 3$. Let $n_1 = \sum_{i=(k-2)s+b+3}^{q} c_i$, $n_2 = c_1, \ldots, n_{(k-2)s+b+3} = c_{(k-2)s+b+2}$. This implies that $n \geq (k-1)s + b + 3$. Note that G is a spanning subgraph of $G' = K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_{(k-2)s+b+3}})$. By Lemma 2.2, we have

$$\rho(G) \leq \rho(G'), \tag{3}$$

where equality holds if and only if $G = G'$. Let $G'' = K_s \vee (K_{n-(k-1)s-b-2} \cup [(k-2)s + b + 2]K_1)$. By Lemma 3.1, we have

$$\rho(G') \leq \rho(G''), \tag{4}$$

where equality holds if and only if $(n_1, n_2, \ldots, n_{(k-2)s+b+3}) = (n-(k-1)s-b-2, 1, \ldots, 1)$.

Note that $s \geq \frac{b+3}{2-\eta}$ and s is an integer. Then $s \geq \lceil \frac{b+3}{2-\eta} \rceil$.

Case 1. $s = \lceil \frac{b+3}{2-\eta} \rceil$.

Then $G'' = K_{\lceil \frac{b+3}{2-\eta} \rceil} \vee (K_{n-(k-1)\lceil \frac{b+3}{2-\eta} \rceil - b-2} \cup [(k-2)\lceil \frac{b+3}{2-\eta} \rceil + b + 2]K_1)$. Combining (3) and (4), we obtain

$$\rho(G) \leq \rho(K_{\lceil \frac{b+3}{2-\eta} \rceil} \vee (K_{n-(k-1)\lceil \frac{b+3}{2-\eta} \rceil - b-2} \cup [(k-2)\lceil \frac{b+3}{2-\eta} \rceil + b + 2]K_1)).$$

By the assumption of Theorem 1.5 and Lemma 3.3, we have $G = K_{\lceil \frac{b+3}{2-\eta} \rceil} \vee (K_{n-(k-1)\lceil \frac{b+3}{2-\eta} \rceil - b-2} \cup [(k-2)\lceil \frac{b+3}{2-\eta} \rceil + b + 2]K_1)$.

Case 2. $s \geq \lceil \frac{b+3}{2-\eta} \rceil + 1$.

Recall that $G'' = K_s \vee (K_{n-(k-1)s-b-2} \cup [(k-2)s + b + 2]K_1)$. Then

$$e(G'') = \binom{n-(k-2)s-b-2}{2} + s[(k-2)s + b + 2].$$

By Lemma 3.2, we have

$$\begin{aligned} \rho(G'') &\leq \sqrt{2e(G'') - n + 1} \\ &= \sqrt{[n-(k-2)s+b-2][n-(k-2)s+b-3] + 2s[(k-2)s+b+2] - n + 1} \\ &= \sqrt{f(s)}, \end{aligned}$$

where $f(s) = (k^2-2k)s^2 - [(2k-4)n - (2k-2)b - 5k + 6]s + n^2 - (2b+6)n + b^2 + 5b + 7$. Note that $n \geq (k-1)s + b + 3$. Then $\lceil \frac{b+3}{2-\eta} \rceil + 1 \leq s \leq \frac{n-b-3}{k-1}$. For short, define $d = \lceil \frac{b+3}{2-\eta} \rceil$. We claim that $\max_{d+1 \leq s \leq \frac{n-b-3}{k-1}} f(s) = f(d+1)$. In fact,

$$\begin{aligned} & f(d+1) - f(\frac{n-b-3}{k-1}) \\ = \; & \frac{1}{(k-1)^2}\Big[(k^2-4k+4)n^2 - [(2d+2)k^3 + (2b-8d-3)k^2 - (6b-10d+7)k + 6b - 4d \\ & +14]n + [(d+1)k^3 + (b-3d-1)k^2 - (2b-2d+3)k + 2b + 6][(d+1)k + b - d + 2]\Big] \\ > \; & 0 \end{aligned}$$

for $n \geq \frac{(2b+9)k^2 + (2b^2+13b+15)k - 2b^2 - 21b - 48}{2k-4}$ and $k \geq 3$. This implies that the maximum value of $f(s)$ on $d+1 \leq s \leq \frac{n-b-3}{k-1}$ is attained at $s = d+1$. Hence

$$\rho(G'') \leq \sqrt{f(d+1)} = \sqrt{g(n)},$$

where $g(n) = n^2 - [(2k-4)(d+1) + 2b + 6]n + (k^2-2k)(d+1)^2 + [(2k-2)b + 5k - 6](d+1) + b^2 + 5b + 7 = (n-(k-2)d-b-3)^2 - [(2k-4)n - (2k-4)d^2 - (2k^2+2b-5k+6)d - 2bk - k^2 + 3b - 3k + 8]$. By $n \geq \frac{(2b+9)k^2 + (2b^2+13b+15)k - 2b^2 - 21b - 48}{2k-4} > \frac{(2k-4)d^2 + (2k^2+2b-5k+6)d + 2bk + k^2 - 3b + 3k - 8}{2k-4}$, we have

$$\rho(G'') \leq \sqrt{g(n)} < n - (k-2)d - b - 3. \tag{5}$$

Note that $K_{\lceil \frac{b+3}{2-\eta} \rceil} \vee (K_{n-(k-1)\lceil \frac{b+3}{2-\eta} \rceil - b-2} \cup [(k-2)\lceil \frac{b+3}{2-\eta} \rceil + b + 2]K_1)$ contains $K_{n-(k-2)d-b-2}$ as a proper subgraph. By Lemma 2.2, we have

$$\rho(K_{\lceil \frac{b+3}{2-\eta} \rceil} \vee (K_{n-(k-1)\lceil \frac{b+3}{2-\eta} \rceil - b-2} \cup [(k-2)\lceil \frac{b+3}{2-\eta} \rceil + b + 2]K_1)) > \rho(K_{n-(k-2)d-b-2}) = n - (k-2)d - b - 3.$$

Combining this with (3), (4) and (5), we have

$$\rho(G) \leq \rho(G') \leq \rho(G'') < \rho(K_{\lceil \frac{b+3}{2-\eta} \rceil} \vee (K_{n-(k-1)\lceil \frac{b+3}{2-\eta} \rceil - b-2} \cup [(k-2)\lceil \frac{b+3}{2-\eta} \rceil + b + 2]K_1)),$$

which contradicts the assumption of Theorem 1.5. □

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