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Wilf classes for descent sequences avoiding a pattern or a pair of patterns of length three

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Abstract. A descent sequence is a word $\pi = \pi_1\pi_2\cdots\pi_n$ over the set of nonnegative integers such that $\pi_1 = 0$ and $\pi_i \le 1 + \mathrm{des}(\pi_1\pi_2\cdots\pi_{i-1})$ for $i = 2, 3, \ldots, n$, where $\mathrm{des}(\pi_1\pi_2\cdots\pi_m)$ is the number of descents in the word $\pi_1\pi_2\cdots\pi_m$, that is, the number of two entry factors $\pi_j\pi_{j+1}$ such that $\pi_j > \pi_{j+1}$. In this paper, we obtain some enumerative results for descent sequences avoiding patterns of length 3 and 4. In particular, we determine the number of Wilf equivalence classes among single patterns of length 3 and among pairs of patterns of length 3, and state the corresponding result for a set of k patterns of length 3 when $3 \le k \le 13$. We also consider single patterns of length 4. The main tool is the use of generating trees.

1. Introduction

Ascent sequences have received a lot of attention in recent years due to their connections with (2 + 2)-free posets, various other combinatorial structures, and pattern avoidance (see, for example, [3, 5–9] and references therein). A natural generalization, weak ascent sequences (see, for example, [1, 4]), further broadens these connections while preserving links to permutation patterns and related structures.

Given the central role of ascents and weak ascents in these constructions, it is natural to ask for the descent analogue. *Descent sequences* were introduced by Callan [2]. A *descent* in a sequence of integers $w_1w_2\cdots w_m$ is a pair of adjacent entries w_jw_{j+1} such that $w_j > w_{j+1}$. A *descent sequence* $\pi_1\pi_2\cdots\pi_n$ of length n is a sequence of n non-negative integers satisfying $\pi_1 = 0$, and for all i with $1 < i \le n$, it holds that $\pi_i \le \mathrm{des}(\pi_1\pi_2\cdots\pi_{i-1}) + 1$, where $\mathrm{des}(\pi_1\pi_2\cdots\pi_m)$ denotes the number of descents in the sequence $\pi_1\pi_2\cdots\pi_m$. For example, the sequence 010213 is a descent sequence, while 01003 is not.

Let $w = w_1w_2\cdots w_n$ be any sequence, and let $\tau = \tau_1\cdots\tau_m$ be any *pattern*, that is, a word in $\{0,\ldots,\ell\}^m$ which contains each letter $0, 1, \ldots, \ell$ for some $m \ge 1$ and $\ell \ge 0$. We say the sequence w *contains* τ if w has a subsequence that is order isomorphic to τ. That is, there is a subsequence $w_{f_1}, w_{f_2}, \ldots, w_{f_m}$, where $1 \le f_1 < f_2 < \cdots < f_m \le n$, such that $w_{f_i}Xw_{f_j}$ if and only if $\tau_iX\tau_j$, for all $X \in \{<, >, =\}$ and $1 \le i, j \le m$. Otherwise, we say that w *avoids* τ. For instance, the descent sequence 01021042 contains three occurrences of the pattern 101, namely, the subsequences 101, 212, and 202, but it avoids the pattern 201. We denote

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the set of all descent sequences that avoid a list of patterns $\tau^{(1)},\dots,\tau^{(s)}$ by $\mathcal{DS}_n(\{\tau^{(1)},\dots,\tau^{(s)}\})$ or simply $\mathcal{DS}_n(\tau^{(1)},\dots,\tau^{(s)})$. Two sets of patterns P and Q are said to be *D-Wilf-equivalent*, denoted by $P \overset{d}{\sim} Q$, if $|\mathcal{DS}_n(P)| = |\mathcal{DS}_n(Q)|$ for every $n \geq 0$.

There are 13 patterns of length 3: 000, 001, 010, 100, 011, 101, 110, 012, 021, 102, 120, 201, and 210. In [2], it is shown that the number of descent sequences of length $n \geq 1$ that avoid the pattern 021 is given by

$$\frac{1 - x - \sqrt{\frac{1-3x-x^2-x^3}{1-x}}}{2x},$$

see Section 8 in [2]. In this paper, we recover this result and prove the following theorem.

Theorem 1.1. *We have*

(1) The number of D-Wilf-equivalence classes among single patterns of length three is 9.

(2) The number of D-Wilf-equivalence classes among pairs of patterns of length three is 23.

In Section 2, we present the strategy for proving Theorem 1.1, which is based on generating trees, and provide two detailed examples. Sections 3 and 4 are devoted to the proofs of Theorem 1.1(1) and Theorem 1.1(2), respectively. We conclude the paper by presenting, without proofs, the number of D-Wilf-equivalence classes among sets of k patterns of length three, for all $k \geq 3$. In the course of the proofs, we derive several generating functions enumerating descent sequences of length $n \geq 1$ that avoid a pattern or a pair of patterns of length three. Moreover, in the last section, we show that the number of D-Wilf-equivalence classes among patterns of length four is 69.

2. Generating trees and the strategy for the proofs

By making simple modifications to the generating trees for ascent sequences as presented in [3], we obtain the generating tree $\mathcal{T}(P)$ for the class of pattern-avoiding descent sequences $\mathcal{DS}(P)$. The tree $\mathcal{T}(P)$ begins with the root labeled 0, which is placed at level 1, and the children of a sequence $\pi_1\cdots\pi_{n-1} \in \mathcal{DS}_{n-1}(P)$ are constructed from the set

$$\{\pi_1\pi_2\cdots\pi_{n-1}\pi_n \mid \pi_n = 0, 1, \dots, \mathrm{des}(\pi_1\cdots\pi_{n-1}) + 1\}$$

by applying the pattern-avoidance restrictions corresponding to the patterns in P.

Let $\mathcal{DS}(P)$ be the set of all nodes of $\mathcal{T}(P)$. For any $\pi \in \mathcal{DS}(P)$, we denote by $\mathcal{T}(P;a)$ the subtree of $\mathcal{T}(P)$ rooted at π and consisting of all its descendants. For any two nodes $\pi, \pi' \in \mathcal{DS}(P)$, we say that the subtrees $\mathcal{T}(P;\pi)$ and $\mathcal{T}(P;\pi')$ are *isomorphic*, and write $\mathcal{T}(P;\pi) \cong \mathcal{T}(P;\pi')$, if they are isomorphic as plane (i.e., ordered) trees. Let $\mathcal{T}[P]$ be the tree obtained from $\mathcal{T}(P)$ by replacing each node π with the first node $\pi' \in \mathcal{DS}(P)$ (according to a top-to-bottom and left-to-right traversal of $\mathcal{T}(P)$) such that $\mathcal{T}(P;\pi) \cong \mathcal{T}(P;\pi')$. From now on, we identify $\mathcal{T}(P)$ with $\mathcal{T}[P]$.

Example 2.1. *Let $P = \{021, 110\}$. Clearly, the children of $0 \in \mathcal{T}(P)$ are 00 and 01. Note that any descent sequence of the form 00π belongs to $\mathcal{DS}_n(P)$ if and only if $0\pi \in \mathcal{DS}_{n-1}(P)$. Hence, we have $00 \overset{d}{\sim} 0$, which leads to the succession rule $0 \rightsquigarrow 0, 01$.*

Clearly, the children of $01 \in \mathcal{T}(P)$ are $d_1 = 010$ and 011, so we have $01 \rightsquigarrow 010, 011$. The only child of 011 is $0111 \overset{d}{\sim} 011$, so we obtain $011 \rightsquigarrow 011$.

Now, define the following sequences:

$$d_m = 01020\cdots m0, \qquad c_m = 0102\cdots 0m,$$
$$b_m = 0102\cdots 0(m-1)0(m-1)m, \qquad a_m = 0102\cdots 0m0m.$$

For $m \geq 1$, the children of $d_m \in \mathcal{T}(P)$ are: $d_m0 \overset{d}{\sim} d_m$, $a_m = d_mm$, and $c_{m+1} = d_m(m+1)$, which gives the rule $d_m \rightsquigarrow d_m, a_m, c_{m+1}$.

Moreover, for $m \geq 2$, the children of $c_m \in \mathcal{T}(P)$ are: $c_m0 = d_m$ and $d_mm \overset{d}{\sim} b_m$, so we obtain the rule $c_m \rightsquigarrow d_m, b_m$. Similarly, for $m \geq 2$, we have $b_m \rightsquigarrow b_m$, and for $m \geq 1$, $a_m \rightsquigarrow a_m, b_{m+1}$.

For a given set P, we aim to obtain an explicit formula for the generating function

$$F_P(x) = \sum_{n\geq 1} (\text{number of nodes in the } n\text{th level of } \mathcal{T}(P))\, x^n.$$

To do so, we refine this definition by introducing

$$F_{P;\pi}(x) = \sum_{n\geq 1} (\text{number of nodes in the } n\text{th level of } \mathcal{T}(P;\pi))\, x^n,$$

where $\pi \in D(p)$. Clearly, we have $F_P(x) = F_{P;0}(x)$. Moreover, any succession rule of the form

$$v \rightsquigarrow v^{(1)}, \ldots, v^{(s)} \quad \text{with } v, v^{(1)}, \ldots, v^{(s)} \in D(P)$$

is equivalent to the equation

$$F_{P;v}(x) = x + x\sum_{i=1}^{s} F_{P;v^{(i)}}(x).$$

Example 2.2. *Let $P = \{021, 110\}$. By Example 2.1, we have*

$$\begin{aligned}
F_{P;0}(x) &= x + xF_{P;0}(x) + xF_{P;01}(x),\\
F_{P;01}(x) &= x + xF_{P;d_1}(x) + xF_{P;011}(x),\\
F_{P;011}(x) &= x + xF_{P;011}(x),\\
F_{P;d_m}(x) &= x + xF_{P;d_m}(x) + xF_{P;a_m}(x) + xF_{P;c_{m+1}}(x), \quad m \geq 1,\\
F_{P;c_m}(x) &= x + xF_{P;d_m}(x) + xF_{P;b_m}(x), \quad m \geq 2,\\
F_{P;b_m}(x) &= x + xF_{P;b_m}(x), \quad m \geq 2,\\
F_{P;a_m}(x) &= x + xF_{P;a_m}(x) + xF_{P;b_{m+1}}(x), \quad m \geq 1.
\end{aligned}$$

Thus, $F_{P;b_m}(x) = \frac{x}{1-x}$ for all $m \geq 2$, which implies $F_{P;a_m}(x) = \frac{x}{(1-x)^2}$ for all $m \geq 1$. So,

$$F_{P;d_m}(x) = \frac{x}{(1-x)^2} + xF_{P;d_m}(x) + x^2F_{P;d_{m+1}}(x),$$

which leads to

$$F_{P;d_m}(x) = \frac{x}{(1-x)^3} + \frac{x^2}{1-x}F_{P;d_{m+1}}(x),$$

for all $m \geq 1$. By iterating this equation infinitely many times under the assumption that $|x| < 1$, we obtain

$$F_{P;d_m}(x) = \sum_{j\geq 0} \frac{x^{2j+1}}{(1-x)^{j+3}} = \frac{x}{(1-x)^2(1-x-x^2)}.$$

Hence,

$$F_P(x) = F_{P;0}(x) = \frac{x((1-x)^2 + x^3)}{(1-x)^3(1-x-x^2)},$$

which implies that the number of descent sequences of length $n \geq 1$ that avoid P is given by $F_{n+4} - \frac{n(n+1)}{2} - 3$, where F_m is the mth Fibonacci number (see Sequence A000128 in [12]).

Now, let us focus on the generating tree for all descent sequences without any restriction. It is not hard to see that the generating tree $\mathcal{T}$ for the set of all nonempty descent sequences can be given by a root $a_{0,0}$ and the following succession rule:

$$a_{m,j} \rightsquigarrow a_{m+1,0}, a_{m+1,1}, \ldots, a_{m+1,j-1}, a_{m,j}, a_{m,j+1}, \ldots, a_{m,m+1},$$

for all $0 \le j \le m+1$ and $m \ge 0$, where $a_{m,j} = 01020\cdots m0j$.

Define $A_{m,j}(x)$ to be the generating function for the number of nodes at the nth level in the subtree of $\mathcal{T}$ rooted at $a_{m,j}$, where this root stays at level 1. Thus,

$$A_{m,j}(x) = x + x\sum_{i=0}^{j-1} A_{m+1,i}(x) + x\sum_{i=j}^{m+1} A_{m,i}(x),$$

for all $0 \le j \le m+1$ and $m \ge 0$.

Define $A(u,v) = A(x;u,v) = \sum_{m\ge0}\sum_{j=0}^{m+1} A_{m,j}(x)v^{m+1-j}u^m$. Thus, by multiplying the above recurrence relation by $v^{m+1-j}u^m$ and summing over $0 \le j \le m+1$ and $m \ge 0$, we obtain the following result.

Proposition 2.3. *The generating function $A(u,v)$ satisfies*

$$\begin{aligned} A(u,v) &= \frac{x(1+v-vu)}{(1-u)(1-vu)} + \frac{x}{vu(1-v)}(vA(u,1) - A(u,v) + (1-v)A(u,0)) \\ &\quad + \frac{x}{1-v}(A(u,v) - v^2A(vu,1)). \end{aligned}$$

We cannot solve this functional equation to obtain an explicit expression for the generating function $A(0,0)$, which counts the number of descent sequences of length $n \ge 1$. However, the functional equation can be used to compute the first n terms of the generating function $A(0,0)$ for any positive integer n. The first 20 terms are 1, 2, 4, 9, 23, 67, 222, 832, 3501, 16412, 85062, 484013, 3004342, 20226212, 146930527, 1146389206, 9566847302, 85073695846, 803417121866, and 8032911742979.

Now, let us present another way to find the generating function for the number of descent sequences of length n. Define $g_{d,\ell,n}(r)$ to be the number of descent sequences of length n that begin with r 0s followed by some nonzero element, have d descents, and last letter ℓ. Also, set

$$G_r(t,u,v) = \sum_{n\ge r+1,\, d\ge0,\, \ell\ge0} g_{d,\ell,n}(r)t^n u^d v^\ell.$$

Clearly, the generating function for the number of descent sequences according to the length marked by t, number of descents marked by u, and last letter marked by v is given by

$$G(t,u,v) = \frac{1}{1-t} + \sum_{r\ge1} G_r(t,u,v),$$

where $\frac{1}{1-t}$ counts the descent sequences where all elements are 0 (including the empty sequence).

Note that $G_r(t,u,v) = t^{r-1}G_1(t,u,v)$. So,

$$G(t,u,v) = \frac{1}{1-t} + \frac{1}{1-t}G_1(t,u,v).$$

Lemma 2.4. *We have*

$$\begin{aligned} G_1(y,1) &= -t^2\sum_{j\ge1} \frac{\rho_j(y)-1}{\rho_j(y)(1+t(\rho_j(y)-1))} \frac{\prod_{i=1}^{j-1}\rho_i(y)}{\prod_{i=1}^{j-1}(1+t(\rho_i(y)-1))^2} \\ &= t^2 + (y+1)t^3 + (4y+1)t^4 + (3y^2+10y+1)t^5 + \cdots, \end{aligned}$$

where $\rho_j(y) = \underbrace{h\circ h\circ\cdots\circ h}_{j}(y)$ (the j-fold composition of h) with

$$h(y) = \frac{-1+t+\sqrt{(1-t)^2+4yt}}{2t}.$$

Proof. Let $\pi = \pi_1\pi_2\cdots\pi_{n-1}$ be a descent sequence beginning with 01, with d descents and $\pi_{n-1} = \ell$. Then $\pi' = \pi_1\pi_2\cdots\pi_{n-1}\pi_n$ is a descent sequence if and only if $0 \le \pi_n \le d+1$. Thus,

$$G_1(u,v) = vt^2 + \sum_{d\ge0,\ell\ge0,n\ge2} g_{d,\ell,n}(1)t^{n+1}\left(u^{d+1}v^0 + \sum_{i=1}^{\ell-1} u^{d+1}v^i + \sum_{i=\ell}^{d+1} u^d v^i\right)$$
$$= vt^2 + utG_1(u,1) + \frac{ut}{1-v}(vG_1(u,1) - G_1(u,v)) + \frac{t}{1-v}(G_1(u,v) - v^2G_1(uv,1)).$$

By taking $v = 1 + t(u-1)$, we obtain

$$G_1(u,1) = \frac{(1+t(u-1))^2}{u}G_1(u(1+t(u-1)),1) + \frac{(u-1)(1+t(u-1))t^2}{u}.$$

Let $y = u(1+t(u-1))$ so that by taking $u = u_0 = \frac{-1+t+\sqrt{(1-t)^2+4yt}}{2t}$, we obtain

$$G_1(y,1) = \frac{u_0}{(1+t(u_0-1))^2}G_1(u_0,1) - \frac{(u_0-1)t^2}{u_0(1+t(u_0-1))}.$$

Iterating the last equation (assume $|y| < 1$), we obtain

$$G_1(y,1) = -t^2\sum_{j\ge1}\frac{\rho_j(y)-1}{\rho_j(y)(1+t(\rho_j(y)-1))}\frac{\prod_{i=1}^{j-1}\rho_i(y)}{\prod_{i=1}^{j-1}(1+t(\rho_i(y)-1))^2},$$

as required. □

Hence, we can state the following result.

Theorem 2.5. *The generating function for the number of descent sequences of length n is given by* $\frac{1}{1-t}(1+G_1(1,1))$.

Note that we cannot just take $y = 1$ in the expression of $G_1(y,1)$. First, we expand

$$-t^2\sum_{j=1}^{n}\frac{\rho_j(y)-1}{\rho_j(y)(1+t(\rho_j(y)-1))}\frac{\prod_{i=1}^{j-1}\rho_i(y)}{\prod_{i=1}^{j-1}(1+t(\rho_i(y)-1))^2}$$

as a power series up to coefficient of t^n (for fixed n). Second, we remove all the terms y^j in the coefficient of t^n, for all $j \ge n$. Then, we can set $y = 1$.

3. Descent sequences avoiding a pattern of length three

By computing the number of descent sequences of length $n = 1, 2, \ldots, 11$ that avoid a pattern τ of length three, we are led to conjecture the nontrivial Wilf-equivalences (Wilf-equivalences with more than one pattern) are $001 \overset{d}{\sim} 010$ and $012 \overset{d}{\sim} 100 \overset{d}{\sim} 101 \overset{d}{\sim} 102$. The goal of this section is to prove these two claims, which are equivalent to proving Theorem 1.1(1). Moreover, we determine the generating function $F_{\{\tau\}}(x)$ for several cases where τ is a pattern of length three.

3.1. *Pattern* 000

Let $P = \{000\}$. Note that the generating tree $\mathcal{T}(\{000\})$ is given by

$0 \rightsquigarrow 00, 01,$ $\quad$ $00 \rightsquigarrow 001,$
$001 \rightsquigarrow 0011,$ $\quad$ $01 \rightsquigarrow e_1, 011,$
$011 \rightsquigarrow b_1,$ $\quad$ $b_m \rightsquigarrow d_{m+1}, \quad m \ge 1,$
$c_m \rightsquigarrow e_m, f_m, \quad m \ge 2,$ $\quad$ $d_m \rightsquigarrow a_m, \quad m \ge 2,$
$e_m \rightsquigarrow b_m, c_{m+1}, \quad m \ge 1,$ $\quad$ $f_m \rightsquigarrow b_m, \quad m \ge 2,$

where $a_m = e_{m-1}(m-1)mm$, $b_m = e_m m$, $c_m = e_{m-1}m$, $d_m = e_{m-1}(m-1)m$, $e_m = 01021\cdots m(m-1)$, and $f_m = e_{m-1}mm$. By translating these succession rules to equations, we obtain

$$\begin{aligned}
&F_{P;0}(x) = x + xF_{P;00}(x) + xF_{P;01}(x), && F_{P;00}(x) = x + xF_{P;001}(x),\\
&F_{P;001}(x) = x + xF_{P;0011}(x), && F_{P;01}(x) = x + xF_{P;e_1}(x) + xF_{P;011}(x),\\
&F_{P;011}(x) = x + xF_{P;b_1}(x), && F_{P;b_m}(x) = x + xF_{P;d_{m+1}}(x), \quad m \ge 1,\\
&F_{P;c_m}(x) = x + xF_{P;e_m}(x) + xF_{P;f_m}(x), \quad m \ge 2, && F_{P;d_m}(x) = x + xF_{P;a_m}(x), \quad m \ge 2,\\
&F_{P;e_m}(x) = x + xF_{P;b_m}(x) + xF_{P;c_{m+1}}(x), \quad m \ge 1, && F_{P;f_m}(x) = x + xF_{P;b_m}(x), \quad m \ge 2,\\
&F_{P;a_m}(x) = x, \quad m \ge 2.
\end{aligned}$$

Thus, $F_{P;c_m}(x) = x + 2x^2(1 + x + x^2 + x^3) + x^2 c_{m+1}$ for all $m \ge 2$. Hence, by iterating this equation under the assumption $|x| < 1$, we obtain $F_{P;c_m}(x) = \frac{x+2x^2(1+x+x^2+x^3)}{1-x^2}$ for all $m \ge 2$. So,

$$F_{P;e_1}(x) = x + xF_{P;b_1}(x) + xF_{P;c_2}(x) = x(1 + x + x^2 + x^3) + \frac{x^2 + 2x^3(1 + x + x^2 + x^3)}{1 - x^2}.$$

Therefore,

$$F_P(x) = F_{P;0}(x) = x(1 + x + x^2 + x^3 + x^4) + xF_{P;e_1}(x),$$

which implies

$$F_P(x) = \frac{x(1 + 2x + 2x^2 + 2x^3 + x^4 + x^5)}{1 - x^2}.$$

So, the number of descent sequences of length n that avoid P is $\frac{9+(-1)^n}{2}$, for all $n \ge 5$.

3.2. *Patterns* 001 *and* 010

Note that any descent sequence of length n that avoids 001 can be written in the form either $00\cdots0100\cdots0$ or $00\cdots0$. Thus $|\mathcal{DS}_n(\{001\})| = n$ for all $n \ge 1$.

Also, any descent sequence of length n that avoids 010 can be written in the form either $00\cdots0111\cdots1$ or $00\cdots0$. Thus $|\mathcal{DS}_n(\{010\})| = n$ for all $n \ge 1$.

Hence, $|\mathcal{DS}_n(\{001\})| = |\mathcal{DS}_n(\{010\})| = n$, for all $n \ge 1$. Thus, $001 \overset{d}{\sim} 010$.

3.3. *Pattern* 011

Let $P = \{011\}$. The generating tree $\mathcal{T}(P)$ satisfies the following succession rules: $0 \rightsquigarrow 0, a_1$; $a_m \rightsquigarrow b_m$; $b_m \rightsquigarrow b_m, a_{m+1}$, for all $m \ge 1$, where $a_m = 0102\cdots0m$ and $b_m = a_m0$. By translating these rules to equations, we obtain

$$\begin{aligned}
&F_{P;0}(x) = x + xF_{P;0}(x) + xF_{P;a_1}(x),\\
&F_{P;a_m}(x) = x + xF_{P;b_m}(x), \quad m \ge 1,\\
&F_{P;b_m}(x) = x + xF_{P;b_m}(x) + xF_{P;a_{m+1}}(x), m \ge 1.
\end{aligned}$$

Thus, $F_{P;a_m}(x) = \frac{x}{1-x} + \frac{x^2}{1-x}F_{P;a_{m+1}}(x)$, for all $m \ge 1$. Hence, by iterating this equation under the assumption $|x| < 1$, we obtain $F_{P;a_1}(x) = \frac{x}{1-x-x^2}$, which implies

$$F_{P;0}(x) = \frac{x(1 + x)}{1 - x - x^2}.$$

So, the number of descent sequences of length n that avoid P is F_{n+1}, for all $n \ge 1$, where F_m is the mth Fibonacci number.

3.4. Patterns 012, 100, 101, *and* 102

Note that any descent sequence $\pi = \pi_1\pi_2\cdots\pi_n$ of length n that avoid 012 satisfies $\pi_1 = 0$ and $\pi_j \in \{0,1\}$ for all $j = 2,3,\ldots,n$. Thus $|\mathcal{DS}_n(\{012\})| = 2^{n-1}$ for all $n \geq 1$.

Let $\tau \in \{100, 101, 102\}$. It is not hard to see that the generating tree $\mathcal{T}(\{\tau\})$ has the following succession rule: $0 \rightsquigarrow 0, 0$, that is, this generating tree is presented by a rooted binary tree in which every node is labeled by 0. So, $|\mathcal{DS}_n(\{\tau\})| = 2^{n-1}$ for all $n \geq 1$.

Hence, $012 \overset{d}{\sim} 100 \overset{d}{\sim} 101 \overset{d}{\sim} 102$.

3.5. Pattern 021

Let $P = \{021\}$ and let $a_{m,j} = (01)^m 0j0$ and $b_{m,j} = (01)^m 0j$, where $1 \leq j \leq m+1$. The generating tree $\mathcal{T}(P)$ satisfies the following succession rules: $0 \rightsquigarrow 0, b_{0,1}$; $b_{m,j} \rightsquigarrow a_{m,j}, b_{m,j}, b_{m,j+1}, \ldots, b_{m,m+1}$; $a_{m,j} \rightsquigarrow a_{m,j}, b_{m+1,j}, b_{m+1,j+1}, \ldots, b_{m+1,m+2}$, for all $m \geq 0$ and $1 \leq j \leq m+1$. By translating these rules to equations, we obtain

$$F_{P;0}(x) = x + xF_{P;0}(x) + xF_{P;b_{0,1}}(x),$$
$$F_{P;b_{m,j}}(x) = x + xF_{P;a_{m,j}}(x) + x\sum_{i=j}^{m+1} F_{P;b_{m,i}}(x),$$
$$F_{P;a_{m,j}}(x) = x + xF_{P;a_{m,j}}(x) + x\sum_{i=j}^{m+2} F_{P;b_{m+1,i}}(x),$$

Define

$$A(u,v) = \sum_{m\geq 0}\sum_{j=1}^{m+1} F_{P;a_{m,j}}(x)v^m u^{m+1-j} \text{ and } B(u,v) = \sum_{m\geq 0}\sum_{j=1}^{m+1} F_{P;b_{m,j}}(x)v^m u^{m+1-j}.$$

By multiplying the above recurrence by $v^m u^{m+1-j}$ and summing over $m \geq 0$ and $1 \leq j \leq m+1$, we obtain

$$A(u,v) = \frac{x}{(1-v)(1-uv)} + xA(u,v) + \frac{x}{uv}\left(\frac{1}{1-u}(B(u,v) - uB(1,uv)) - B(0,v)\right),$$
$$B(u,v) = \frac{x}{(1-v)(1-uv)} + xA(u,v) + \frac{x}{1-u}(B(u,v) - uB(1,uv)).$$

By eliminating $A(u,v)$ from the second equation and substituting its expression into the first equation, we obtain

$$\frac{u^2vx + uvx^2 - u^2v - 2uvx + uv - x^2}{uvx(1-u)}B(u,v) = \frac{uvx - uv - x}{v(1-u)}B(1,uv) - \frac{x}{uv}B(0,v) + \frac{1}{(1-v)(1-uv)}. \tag{1}$$

In order to solve (1), we assume that

$$B(0,v) := \frac{1 - 2x - x^2 - \sqrt{(1-x)(1-3x-x^2-x^3)}}{2x^2(1-vx)}. \tag{2}$$

Now, let us solve the system (1) and (2). So, (1) with replacing v with v/u gives

$$\frac{uvx + vx^2 - uv - 2vx + v - x^2}{vx(1-u)}B(u,v/u) = \frac{u(vx - v - x)}{v(1-u)}B(1,v) - \frac{1 - 2x - x^2 - \sqrt{(1-x)(1-3x-x^2-x^3)}}{2xv(1-vx/u)} + \frac{1}{(1-v/u)(1-v)}. \tag{3}$$

Thus, by taking $u=\frac{vx^2-2vx-x^2+v}{v(1-x)}$, we obtain an explicit formula for $B(1,v)$. So, by using the expression of $B(1,v)$ and solving (3) for $B(u,v)$, we obtain

$$B(u,v)=\frac{(1-uvx)\sqrt{(1-x)(1-3x-x^2-x^3)}}{2(1-vx)(x(x-1)u^2v^2+(1-x)^2uv-x^2)}$$
$$+\frac{\begin{array}{l}uv(1-v)(1-uv)-uv(u^2v^3-u^2v^2+2uv^2-4v+4)x\\+(3u^3v^4-u^3v^3-u^2v^3+7u^2v^2-4uv^2+4uv+v-1)x^2\\+(-3u^3v^4-u^3v^3+u^2v^3-7u^2v^2-uv^2+3uv-2v+2)x^3\\+(u^3v^4+u^3v^3+2u^2v^3+uv^2-3uv-v+1)x^4-uv(1-v)(1-uv)x^5\end{array}}{2(1-vx)(1-v)((x-1)u^2v^2+(1-x)^2uv-x^2)(x(x-1)u^2v^2+(1-x)^2uv-x^2)}.$$

Note that the formula for $B(u,v)$ that we found satisfies (1) and (2), thus it is the solution for (1). Hence, from $F_{P;0}(x)=x+xF_{P;0}(x)+xF_{P;b_{0,1}}(x)$, we obtain

$$F_P(x)=\frac{x}{1-x}+\frac{x}{1-x}B(0,0),$$

which gives the following result (see [2]).

Theorem 3.1. *The generating function $F_{\{021\}}(x)$ is given by*

$$\frac{(1-x)^2-\sqrt{(1-x)(1-3x-x^2-x^3)}}{2x(1-x)}.$$

3.6. Pattern 110

Let $P=\{110\}$, $a_m=0102\cdots0m0m$ with $m\geq1$, $b_{m,j}=0102\cdots0(m-1)jm$ with $0\leq j\leq m02$, $c_m=a_{m-1}m$ with $m\geq2$, and $d_{m,j}=0102\cdots0mj$ with $0\leq j\leq m$ and $m\geq1$. The generating tree $\mathcal{T}(P)$ satisfies the following succession rules

$$\begin{aligned}
0&\rightsquigarrow 0,01,\\
01&\rightsquigarrow d_{1,0},d_{1,1},\\
d_{m,m}&\rightsquigarrow d_{m,m},\quad m\geq1,\\
d_{m,j}&\rightsquigarrow d_{m,j},d_{m,j+1},\ldots,d_{m,m-1},a_m,b_{m+1,j},\quad 0\leq j\leq m-1,\\
c_m&\rightsquigarrow d_{m,m-1},d_{m,m},\quad m\geq2,\\
b_{m,j}&\rightsquigarrow d_{m,j},d_{m,j+1},\ldots,d_{m,m},\quad 0\leq j\leq m-2,\\
a_m&\rightsquigarrow a_m,c_{m+1},\quad m\geq1.
\end{aligned}$$

By translating these rules to equations, we obtain

$$\begin{aligned}
F_{P;0}(x)&=x+xF_{P;0}(x)+xF_{P;01}(x),\\
F_{P;01}(x)&=x+xF_{P;d_{1,0}}(x)+xF_{P;d_{1,1}}(x),\\
F_{P;d_{m,m}}(x)&=x+xF_{P;d_{m,m}}(x),\quad m\geq1,\\
F_{P;d_{m,j}}(x)&=x+x\sum_{i=j}^{m-1}F_{P;d_{m,i}}(x)+xF_{P;a_m}(x)+xF_{P;b_{m+1,j}}(x),\quad 0\leq j\leq m-1,\\
F_{P;c_m}(x)&=x+xF_{P;d_{m,m-1}}(x)+xF_{P;d_{m,m}}(x),\quad m\geq2,\\
F_{P;b_{m,j}}(x)&=x+x\sum_{i=j}^{m}F_{P;d_{m,i}}(x),\quad 0\leq j\leq m-2,\\
F_{P;a_m}(x)&=x+xF_{P;a_m}(x)+xF_{P;c_{m+1}}(x),\quad m\geq1.
\end{aligned}$$

Define

$$A(v) = \sum_{m\geq 1} F_{P;a_m}(x)v^{m-1}, \qquad C(v) = \sum_{m\geq 1} F_{P;c_m}(x)v^{m-2},$$

$$B(u,v) = \sum_{m\geq 2}\sum_{j=0}^{m-2} F_{P;b_{m,j}}(x)v^{m-2}u^j, \qquad D(u,v) = \sum_{m\geq 1}\sum_{j=0}^{m} F_{P;d_{m,j}}(x)v^{m-2}u^j.$$

Thus, the above recurrence relations can be written as

$$\begin{aligned}
F_{P;0}(x) &= \frac{x}{1-x} + \frac{x}{1-x}F_{P;01}(x),\\
F_{P;01}(x) &= \frac{x}{1-x} + xD(0,0),\\
D(u,v) &= \frac{x}{(1-v)(1-uv)} + \frac{x}{1-u}(D(u,v) - uD(1,uv)) + \frac{x}{1-u}(A(v) - uA(uv)) + xB(u,v),\\
C(v) &= \frac{x}{(1-x)(1-v)} - \frac{x}{v}(D(0,v) - D(0,0)),\\
B(u,v) &= \frac{x}{(1-x)(1-v)(1-uv)} + \frac{x}{uv(1-u)}(D(u,v) - uD(1,uv)) - \frac{x}{uv}D(0,v).
\end{aligned}$$

By guessing, we found that the solution to this system is given by

$$\begin{aligned}
F_{P;0}(x) &= \frac{x}{1-x} + \frac{x}{1-x}F_{P;01}(x),\\
F_{P;01}(x) &= \frac{x}{1-x} + xD(0,0),\\
A(v) &= \frac{1-x-\sqrt{1-2x-3x^2}}{2x(1-x)(1-v)},\\
B(u,v) &= \frac{((x-1)u^2v^2 + (x^2-3x+2)uv + x^3 - (x-1)^2)\sqrt{1-2x-3x^2}}{2x^2(1-x)(1-v)(1-uv)(u^2v^2+(x-1)uv+x^2)}\\
&\quad + \frac{(x-1)(2x^2+x-1)u^2v^2 + (2x^4-3x^3+5x-2)uv + 2x^5 - x^4 + x^2 - 3x + 1}{2x^2(1-x)(1-v)(1-uv)(u^2v^2+(x-1)uv+x^2)},\\
C(v) &= \frac{1-x-2x^2-\sqrt{1-2x-3x^2}}{2x^2(1-v)},\\
D(u,v) &= \frac{(uv+x-1)\sqrt{1-2x-3x^2}}{2x(1-x)(1-v)(u^2v^2+(x-1)uv+x^2)} + \frac{(x-1)uvx - x^2 - 2x + 1}{2x(1-x)(1-v)(u^2v^2+(x-1)uv+x^2)}.
\end{aligned}$$

Hence, we can state the following result.

Theorem 3.2. *The generating function* $F_{\{110\}}(x)$ *is*

$$\frac{1-x-\sqrt{1-2x-3x^2}}{2x(1-x)}.$$

As we have seen, the use of generating trees and the subsequent resolution of the corresponding system of equations yields an explicit formula for the generating function $F_{\{\tau\}}(x)$ for each $\tau \in \{000, 001, 010, 011, 012, 021, 100, 101, 102, 1$ However, for the remaining cases $\tau \in \{120, 201, 210\}$, we were unable to determine the succession rules for the generating tree $\mathcal{T}(\{\tau\})$ or to solve the associated system of equations.

4. Pairs of patterns of length three

The number of pairs of patterns of length three is $\binom{13}{2} = 78$. By computing the sequences $|\mathcal{DS}_n(P)|_{n=1}^{11}$ for each pair P, we identify 23 candidate classes. In this section, we prove Theorem 1.1(2), namely, that there are exactly 23 D-Wilf-equivalence classes. Our method consists of deriving an explicit formula for the generating function $F_P(x)$ in all but two cases, namely $\{120, 201\}$ and $\{120, 210\}$; see Table 1. Note that we omit the derivation of the generating function $F_P(x)$ whenever the enumerations are simple or similar to other classes.

No.	P	$F_P(x)$
1	$\{000, 001\}$, $\{000, 010\}$, $\{000, 011\}$	$x + 2x^2 + 2x^3 + x^4$
2	$\{001, 010\}$, $\{001, 011\}$, $\{010, 011\}$	$x + \frac{2x^2}{1-x}$
3	$\{000, 012\}$, $\{000, 102\}$	$x + 2x^2 + 3x^3 + 3x^4$
4	$\{000, 101\}$	$x + 2x62 + 3x^3 + 3x^4 + 2x^5 + x^6$
5	$\{000, 110\}$, $\{001, 100\}$, $\{001, 110\}$	$x + 2x^2 + \frac{3x^3}{1-x}$
6	$\{000, 021\}$	$x + 2x^2 + 3x^3 + 4x^4 + 3x^5 + 2x^6$
7	$\{011, 100\}$	$x + 2x62 + 3x^3 + \frac{4x^4}{1-x}$
8	$\{000, 100\}$, $\{000, 120\}$, $\{000, 201\}$, $\{000, 210\}$	$\frac{x(1+2x+2x^2+2x^3+x^4+x^5)}{1-x^2}$
9	$\{001, \tau\}$ with $\tau \in \{012, 021, 101, 102, 120, 201, 210\}$ $\{010, \tau\}$ with $\tau \in \{012, 021, 100, 101, 102, 110, 120, 201, 210\}$ $\{011, \tau\}$ with $\tau \in \{012, 102\}$	$\frac{x}{(1-x)^2}$
10	$\{011, 120\}$	$\frac{x(1+x^3)}{(1-x)^2}$
11	$\{011, \tau\}$ with $\tau \in \{021, 101, 110, 201, 210\}$	$\frac{x(1+x)}{1-x-x^2}$
12	$\{012, 100\}$, $\{012, 101\}$, $\{012, 110\}$, $\{100, 101\}$, $\{100, 102\}$, $\{101, 102\}$, $\{102, 110\}$	$\frac{x(1-x+x^2)}{(1-x)^3}$
13	$\{100, 110\}$, $\{101, 110\}$	$\frac{x}{(1-x)(1-x-x^2)}$
14	$\{021, 100\}$, $\{101, 120\}$	$\frac{x(1-2x+2x^2)}{(1-x)^4}$
15	$\{021, 110\}$	$\frac{x(1-2x+x^2+x^3)}{(1-x-x^2)(1-x)^3}$, see Example 2.2
16	$\{012, \tau\}$ with $\tau \in \{021, 102, 120, 201, 210\}$ $\{021, \tau\}$ with $\tau \in \{101, 102\}$ $\{100, \tau\}$ with $\tau \in \{120, 201, 210\}$ $\{101, \tau\}$ with $\tau \in \{201, 210\}$ $\{102, \tau\}$ with $\tau \in \{120, 201, 210\}$ $\{110, 120\}$	$\frac{x}{1-2x}$
17	$\{021, 201\} \overset{d}{\sim} \{021\}$, $\{021, 210\} \overset{d}{\sim} \{021\}$	Theorem 3.1
18	$\{201, 210\}$	Theorem 4.1
19	$\{110, 201\}$	Theorem 4.2
20	$\{110, 210\}$	Theorem 4.3
21	$\{021, 120\}$	Theorem 4.4
22	$\{120, 201\}$	Open
23	$\{120, 210\}$	Open

Table 1: Pairs of patterns of length three

Theorem 4.1. *Let $P = \{201, 210\}$. Then the generating function $F_P(x)$ is*

$$\frac{xC(x)}{1-x} = \frac{1-\sqrt{1-4x}}{2(1-x)}.$$

Proof. The generating tree $\mathcal{T}(P)$ satisfies the following succession rules

$$\begin{aligned}
0 &\rightsquigarrow 0, 01,\\
01 &\rightsquigarrow 010, 01,\\
010 &\rightsquigarrow 010, a_{0,1,1}, a_{0,1,2},\\
b_{m,j} &\rightsquigarrow b_{m,j}, a_{m,j,j}, a_{m,j,j+1}, \dots, a_{m,j,m+2},\\
a_{m,i,j} &\rightsquigarrow (b_{m+1,j})^{j+1-i}, a_{m,i,j}, a_{m,i,j+1}, \dots, a_{m,i,m+2},
\end{aligned}$$

where $a_{m,i,j} = (01)^m 0i0j$ with $1 \le i \le j \le m+2$ and $i \le m+1$, and $b_{m,j} = (01)^m j0$ with $1 \le j \le m+1$. By translating these rules to equations, we obtain

$$\begin{aligned}
F_{P;0}(x) &= x + xF_{P;0}(x) + xF_{P;01}(x),\\
F_{P;01}(x) &= x + xF_{P;010}(x) + xF_{P;01}(x),\\
F_{P;010}(x) &= x + xF_{P;010}(x) + xF_{P;a_{0,1,1}}(x) + xF_{P;a_{0,1,2}}(x),\\
F_{P;a_{m,i,j}}(x) &= x + (j+1-i)xF_{P;b_{m+1,j}}(x) + x\sum_{k=j}^{m+2} F_{P;a_{m,i,k}}(x),\\
F_{P;b_{m,j}}(x) &= x + xF_{P;b_{m,j}}(x) + x\sum_{k=j}^{m+2} F_{P;a_{m,j,k}}(x).
\end{aligned}$$

Define

$$\begin{aligned}
A(u,v,w) &= \sum_{m\ge 0}\sum_{i=1}^{m+1}\sum_{j=i}^{m+2} F_{P;a_{m,i,j}}(x) w^m v^{m+1-j} u^{j-i},\\
B(u,v) &= \sum_{m\ge 0}\sum_{j=1}^{m+1} F_{P;b_{m,j}}(x) v^m u^{m+1-j}.
\end{aligned}$$

Thus, the above recurrence relations can be written as

$$F_{P;0}(x) = \frac{x}{(1-x)^2} + \frac{x^3}{(1-x)^3} + \frac{x^3}{(1-x)^3}\left(A(0,0,0) + \frac{\partial}{\partial u}A(0,0,0)\right), \tag{4}$$

$$\begin{aligned}
A(u,v,w) &= \frac{x(1+u-uvw)}{(1-w)(1-vw)(1-uvw)}\\
&\quad + \frac{x}{vw}\frac{\partial}{\partial u}\left(\frac{uB(v,w) - u^2vB(1/u,uvw)}{1-uv} - uB(0,w)\right)\\
&\quad + \frac{x}{1-u}(A(1,v,w) - uA(u,v,w)),
\end{aligned} \tag{5}$$

$$B(u,v) = \frac{x}{(1-v)(1-uv)} + x * B(u,v) + xA(1,u,v). \tag{6}$$

To solve this system of equations, we assume the following:

$$B(0,0) = xC^3(x), \tag{7}$$

$$B(u,v) = \frac{1-uv}{1-v}B(1,uv), \tag{8}$$

$$A(u,v,w) = \frac{1-vw}{1-w}A(u,1,vw). \tag{9}$$

Now, we solve the system (4)–(9) and then we check if our solution satisfies only the original system (4)–(6). By (5)–(9), we obtain

$$\begin{aligned}&vw(ux-u+1)(x-1)(1-vw)^2(1-uvw)^2A(u,1,vw)\\&\quad=x(1-vw)^2(vw(1-uvw)^2(x-1)+ux-x)A(1,1,vw)\\&\quad+x^2(1-x)(1-u)(1-vw)(1-uvw)^2C^3\\&\quad-x(uv^2w^2(1-x)(uvw-u-2)+vw(1-x)(1+u)+x)(1-u).\end{aligned}\tag{10}$$

By taking $u=1/(1-x)$, we obtain an explicit formula for $A(1,1,vw)$. Then by (10), we obtain an explicit formula for $A(u,1,vw)$. Therefore, by (5) and (6), we have

$$\begin{aligned}A(u,v,w)&=A_1(u,v,w)C^3(x)+A_2(u,v,w),\\B(u,v,w)&=\frac{x(1-x-uv)(x^2(1-x-uv)C^3(x)-uv)}{(1-v)(1-uv)(u^2v^2+2uvx-uv+x^2)(1-x)^2},\end{aligned}$$

where

$$\begin{aligned}A_1(u,v,w)&=-(2ux^2-x^2-2x+1)(x-1)x^2\\&\quad+x^2v(u^2x^3-u^2x^2+ux^3-6ux^2+6ux-x^2-2u+4x-2)w\\&\quad-v^2(x-1)x^2(u^2x^2-2u^2x+u^2-4ux+4u+1)w^2\\&\quad-2uv^3(x-1)x^2(ux-u-1)w^3-(x-1)x^2u^2v^4w^4,\\A_2(u,v,w)&=-(x-1)x^2(ux-1)\\&\quad+xv(u^2x^3-u^2x^2+ux^3-4ux^2+3ux-x^2-u+3x-1)w\\&\quad-v^2(x-1)x(u^2x^2-2u^2x+u^2-3ux+3u+1)w^2\\&\quad-2uv^3(x-1)x(ux-u-1)w^3-(x-1)xv^4u^2w^4.\end{aligned}$$

Note that the expressions of $A(u,v,w)$ and $B(u,v)$ that we obtained are satisfied (5)–(9). Hence, we obtained an explicit solution for (5)–(6). Hence, by (4), we complete the proof. □

Note that Theorem 4.1 shows that the number of descent sequences of length n that avoid $\{201,210\}$ is given by $\sum_{j=0}^{n-1}\frac{1}{j+1}\binom{2j}{j}$, for all $n\geq 1$. Thus, it will be interesting to find a direct combinatorial proof for this fact.

Theorem 4.2. *Let $P=\{110,201\}$. Then the generating function $F_P(x)$ is*

$$\frac{(x^2+x-1)}{(x-1)(x^3-x^2-2x+1)}.$$

Proof. The generating tree $\mathcal{T}(P)$ satisfies the following succession rules

$$\begin{aligned}&0\rightsquigarrow 0,01, && 01\rightsquigarrow d_1,c_1,\\&e_m\rightsquigarrow d_m,c_m, && d_m\rightsquigarrow d_m,b_m,a_{m+1},\\&c_m\rightsquigarrow c_m, && b_m\rightsquigarrow b_m,e_{m+1},\\&a_m\rightsquigarrow d_m,d_m,c_m,\end{aligned}$$

where $a_m=0102\cdots 0m$, $b_m=a_m0m$, $c_m=a_mm$, $d_m=a_m0$, and $e_m=a_{m-1}0(m-1)m$. So, as in Example 2.2, we find the generating function $F_P(x)=F_{P;0}(x)$. □

Theorem 4.3. *Let $P=\{110,210\}$. Then the generating function $F_P(x)$ is given by*

$$\frac{1-x-\sqrt{1-2x-3x^2}}{2x(1-x)}.$$

Proof. The generating tree $\mathcal{T}(P)$ satisfies the following succession rules

$$\begin{aligned} 0 &\rightsquigarrow 0, 01, \\ 01 &\rightsquigarrow a_{1,0}, a_{1,1}, \\ b_m &\rightsquigarrow a_{m,m-1}, a_{m,m}, \\ a_{m,j} &\rightsquigarrow a_{m,j}, a_{m,j+1}, \ldots, a_{m,m-1}, c_m, d_{m+1,j}, \quad 0 \le j \le m-1, \\ a_{m,m} &\rightsquigarrow a_{m,m}, \\ c_m &\rightsquigarrow c_m, b_{m+1}, \\ d_{m,j} &\rightsquigarrow a_{m,j}, a_{m,j+1}, \ldots, a_{m,m}, \quad 0 \le j \le m-2, \end{aligned}$$

where $a_{m,j} = 0102\cdots 0mj$, $b_m = 0102\cdots 0(m-1)0(m-1)m$, $c_m = 0102\cdots 0m0m$, and $d_{m,j} = 0102\cdots 0(m-1)jm$. By translating the succession rules to equations and then solving for $F_{P;0}(x)$, $F_{P;01}(x)$, $F_{P;a_{m,j}}(x)$, $F_{P;b_m}(x)$, $F_{P;c_m}(x)$, and $F_{P;d_{m,j}}(x)$, we obtain

$$F_{P;0}(x) = f, \qquad F_{P;01}(x) = \frac{(1-x)f - x}{x},$$
$$F_{P;a_{m,j}}(x) = \frac{(1-x)^{m-j} f^{m-j+1}}{x^{m-j}}, \qquad F_{P;b_m}(x) = \frac{(1-x)f}{x} + 1,$$
$$F_{P;d_{m,j}}(x) = \frac{x}{1-x} + x\sum_{i=j}^{m-1} \frac{(1-x)^{m-i} f^{m-i+1}}{x^{m-i}}, \qquad F_{P;c_m}(x) = f,$$

where $f = \frac{1-x-\sqrt{1-2x-3x^2}}{2x(1-x)}$, which completes the proof. □

Theorem 4.4. *Let $P = \{021, 120\}$. Then the generating function $F_P(x)$ is*

$$\frac{x(1-x)}{1-3x+2x^2-x^3}.$$

Proof. The generating tree $\mathcal{T}(P)$ satisfies the following succession rules

$$\begin{aligned} 0 &\rightsquigarrow 0, 01, \\ 01 &\rightsquigarrow b_1, 01, \\ b_m &\rightsquigarrow b_m, a_{m,1}, a_{m,2}, \ldots, a_{m,m+1}, \\ a_{m,1} &\rightsquigarrow b_{m+1}, a_{m,1}, a_{m,2}, \ldots, a_{m,m+1}, \\ a_{m,j} &\rightsquigarrow a_{m,j}, a_{m,j+1}, \ldots, a_{m,m+1}, \end{aligned}$$

where $a_{m,j} = (01)^m 0j$ with $1 \le j \le m+1$, $b_m = a_m 0$. By translating the succession rules to equations and then solving for $F_{P;0}(x)$, $F_{P;01}(x)$, $F_{P;a_{m,j}}(x)$, and $F_{P;b_m}(x)$, we obtain

$$F_{P;0}(x) = \frac{x(1-x)}{1-3x+2x^2-x^3}, \qquad F_{P;01}(x) = \frac{x(1-x+x^2)}{1-3x+2x^2-x^3},$$
$$F_{P;a_{m,1}}(x) = \frac{x(1-x+x^2)}{(1-3x+2x^2-x^3)(1-x)^m}, \qquad F_{P;b_m}(x) = \frac{x}{(1-3x+2x^2-x^3)(1-x)^{m-1}},$$
$$F_{P;a_{m,j}}(x) = \frac{x}{(1-x)^{m+2-j}},$$

for all $m \ge 1$ and $2 \le j \le m+1$. This completes the proof. □

5. Larger sets of patterns of length three

One direction to extend the results of the previous section is to study the D-Wilf-equivalences among sets of k patterns of length three with $k \geq 3$. Here, we first compute the number of descent sequences of length $n = 1, 2, \ldots, 11$ that avoid a set P of k patterns of length three. Then, by similar methods, we examine each class P individually by constructing the generating tree $\mathcal{T}(P)$, when necessary, and solving the corresponding system of equations for $F_P(x)$. This leads to the following result.

Theorem 5.1. *Let d_k be the number of D-Wilf-equivalence classes among sets of k patterns of length three. Then $d_3 = 28$, $d_4 = 28$, $d_5 = 26$, $d_6 = 23$, $d_7 = 20$, $d_8 = 17$, $d_9 = 13$, $d_{10} = 11$, $d_{11} = 7$, $d_{12} = 3$, and $d_{13} = 1$.*

6. Patterns of length four

Another direction is to study the D-Wilf-equivalences among patterns of length four. Note that there are 75 different patterns of length four. By computing the number of descent sequences of length $n = 1, 2, \ldots, 13$ that avoid a given pattern of length four, we can conjecture the following D-Wilf-equivalences: $0100 \overset{d}{\sim} 0101 \overset{d}{\sim} 0102$, $0122 \overset{d}{\sim} 1022$, $0123 \overset{d}{\sim} 1023$, $0132 \overset{d}{\sim} 1032$, and $1002 \overset{d}{\sim} 1012$, which we next establish.

Lemma 6.1. *Let 01τ be any pattern of length $m + 2 \geq 3$ such that each letter in τ is greater than 1. Then $01\tau \overset{d}{\sim} 10\tau$.*

Proof. Let π be any descent sequence that contains the pattern 01τ. Then there exists a subsequence $\pi_i\pi_j\pi_{k_1}\cdots\pi_{k_m}$ with $i < j < k_1 < \cdots < k_m$, such that $\pi_i < \pi_j < \min_{1\leq a\leq m}\pi_{k_a}$ and the subsequence $\pi_{k_1}\cdots\pi_{k_m}$ is order-isomorphic to τ. Since $\pi_{k_1} > \pi_j$, there must be a descent in the segment $\pi_j\pi_{j+1}\cdots\pi_{k_1-1}$, say $\pi_r\pi_{r+1}$, with $\pi_{k_1} > \pi_r > \pi_{r+1}$. It follows that the subsequence $\pi_r\pi_{r+1}\pi_{k_1}\cdots\pi_{k_m}$ is order-isomorphic to 10τ, which implies that π contains the pattern 10τ.

On the other hand, if a descent sequence $\pi = \pi_1\pi_2\cdots\pi_n$ contains the pattern 10τ, then π also contains 01τ, since $\pi_1 = 0$. □

By the above lemma, it follows immediately that $012 \overset{d}{\sim} 102$ (as we showed before), $0122 \overset{d}{\sim} 1022$, $0123 \overset{d}{\sim} 1023$, and $0132 \overset{d}{\sim} 1032$. Thus, it remains to show that $0100 \overset{d}{\sim} 0101 \overset{d}{\sim} 0102$ and $1002 \overset{d}{\sim} 1012$.

The generating trees $\mathcal{T}(\{0100\})$, $\mathcal{T}(\{0101\})$, and $\mathcal{T}(\{0102\})$ have a root 0 and satisfies the succession rule $0 \rightsquigarrow 0, 0$. Thus, $0100 \overset{d}{\sim} 0101 \overset{d}{\sim} 0102$ and $F_{\{0100\}}(x) = F_{\{0101\}}(x) = F_{\{0102\}}(x) = \frac{x}{1-2x}$.

Theorem 6.2. *The number of D-Wilf-equivalence classes among patterns of length four is 69.*

Proof. By the above discussion, it remains to show that $1002 \overset{d}{\sim} 1012$. We establish this by finding the generating functions $F_{\{1002\}}(x)$ and $F_{\{1012\}}(x)$.

Let us first determine the generating function $F_{\{1012\}}(x)$. Any nonempty descent sequence that avoids 1012 can be written as either 0π or 01π. Thus, $F_{\{1012\}}(x) = x + xF_{\{1012\}}(x) + f_{01}$. Any descent sequence of the form 01π that avoids 1012 can be written as either 01, $010\pi'$, or $011\pi''$, which implies that $f_{01} = x^2 + f_{010} + xf_{01}$. Now, consider the descent sequences of the form 010π that avoid 1012. There are two cases:

(1) π contains a letter 1, say $\pi_j = 1$ with j minimal. In this case, $\pi_i \in \{0, 1\}$ for all $i > j$.

(2) π does not contain any occurrence of the letter 1.

In Case (1), we map 010π to $0\pi'_1\cdots\pi'_{j-1}$, where

$$\pi'_i = \begin{cases} 0 & \text{if } \pi_i = 0, \\ \pi_i - 1 & \text{if } \pi_i \geq 2. \end{cases}$$

In Case (2), we map 010π to $0\pi'_1\pi'_2\cdots\pi'_n$, where π'_i is given by the above definition. This transformation ensures that $0\pi'_1\cdots\pi'_{j-1}$ or $0\pi'_1\pi'_2\cdots\pi'_n$ is a descent sequence that avoids 1012. Hence,

$$f_{010} = g + \frac{x}{1-2x}g, \quad \text{where } g = x^2F_{\{1012\}}(x).$$

Solving this system of equations, we find

$$F_{\{1012\}}(x) = \frac{x(1-2x)}{(1-x)(1-3x+x^2)}.$$

Similarly, we determine the generating function $F_{\{1002\}}(x)$. Any nonempty descent sequence that avoids 1002 can be written as either 0π or 01π, leading to $F_{\{1002\}}(x) = x + xF_{\{1002\}}(x) + f'_{01}$. Again, any descent sequence of the form 01π that avoids 1002 can be written as either 01, $010\pi'$, or $011\pi''$, so $f'_{01} = x^2 + f'_{010} + xf'_{01}$. Now consider descent sequences of the form 010π that avoid 1002. There are two cases:

(1) π contains a letter 0, say $\pi_j = 0$ with j minimal. In this case, $\pi_i \in \{0,1\}$ for all $i > j$.

(2) π does not contain any occurrence of the letter 0.

In case (1), we map 010π to $0(\pi_1-1)\cdots(\pi_{j-1}-1)$, which is a descent sequence that avoids 1002. In Case (2), we map 010π to $0(\pi_1-1)\cdots(\pi_n-1)$, which is a descent sequence that avoids 1002. Therefore,

$$f'_{010} = g' + \frac{x}{1-2x}g', \quad \text{where } g' = x^2F_{\{1002\}}(x).$$

Solving this system of equations yields

$$F_{\{1002\}}(x) = \frac{x(1-2x)}{(1-x)(1-3x+x^2)}.$$

Hence, $1002 \overset{d}{\sim} 1012$.

There are 75 patterns of length four. By computer data, we see that all the cases are trivial, but only 5 Wilf classes are not trivial, one class has 3 four-letter patterns, and 4 classes each have 2 four-letter patterns, so the number of D-Wilf equivalence classes is $75 - 3 - 4 \times 2 + 5 = 69$, as claimed. □

7. Further Directions

The results of this paper suggest several directions for future work. One direction is to extend Theorem 1.1 to D-Wilf-equivalence classes of sets of k patterns of length 3. Another direction is to study natural statistics on pattern-avoiding descent sequences. A further direction is to study vincular patterns in descent sequences, following the work of Lin and Yan [10] (also, see [11]) on inversion sequences.

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