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Weak and strong laws of large numbers for asymptotic negatively associated random variables

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Abstract. For a sequence of asymptotic negatively associated and identically distributed random variables, we show a maximal weak law of large numbers and two general strong laws of large numbers in which the coefficient of sum and the weight are both general functions. As a consequence, we obtain the Marcinkiewicz strong law of large numbers. Finally, we get a Chung type strong law of large numbers for asymptotic negatively associated random variables.

1. Introduction

As we all know, the limiting behavior of dependent random variables plays a very important role in probability limit theory and mathematical statistics. As a type of dependent random variables, asymptotic negatively associated (ANA, for short) random variables provide insights and tools for studying limit theorems, moment inequalities, and stochastic processes. Consequently, studying the limiting behaviors for sequences of ANA random variables remains a hot topic in probability theory.

Now let us recall some concepts of dependence structures. The first one is well known as the concept of negatively associated (NA, for short) random variables, which was introduced by Joag-Dev and Proschan (1983) as follows.

Definition 1.1. A finite family $\{X_i, 1 \le i \le n\}$ of random variables is said to be NA if for every pair of disjoint subsets A and B of $\{1, 2, ..., n\}$ and any real coordinatewise nondecreasing functions f_1 on $\mathbb{R}^A$ and f_2 on $\mathbb{R}^B$,

$$Cov(f_1(X_i, i \in A), f_2(X_j, j \in B)) \le 0,$$

whenever the covariance above exists. An infinite family of random variables is NA if every finite subfamily is NA.

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Another important concept of dependent random variables is ρ^*-mixing, which was introduced by Bradley (1992) as follows.

Definition 1.2. A sequence $\{X_k, k \geq 1\}$ of random variables is called ρ^*-mixing, if

$$\rho^*(s) = \sup\{\rho(S,T); S,T \subset \mathbb{N}, dist(S,T) \geq s\} \to 0$$

as $s \to \infty$, where

$$\rho(S,T) = \sup\left\{ \frac{|Cov(X,Y)|}{\sqrt{Var(X)Var(Y)}} : X \in L_2(\sigma(X_k, k \in S)), Y \in L_2(\sigma(X_k, k \in T)) \right\}.$$

Based on the concepts of NA and ρ^*-mixing, Zhang and Wang (1999) introduced the following concept of ANA random variables.

Definition 1.3. A sequence $\{X_k, k \geq 1\}$ of random variables is called ANA (*or* $\rho^- - mixing$), if

$$\rho^-(s) = \sup\{\rho^-(S,T) : S,T \subset \mathbb{N}, dist(S,T) \geq s\} \to 0$$

as $s \to \infty$, where

$$\rho^-(S,T) = 0 \vee \left\{ \frac{Cov(f_1(X_i, i \in S), f_2(X_j, j \in T))}{\sqrt{Var(f_1(X_i, i \in S))Var(f_2(X_j, j \in T))}} : f_1, f_2 \in \mathbb{C} \right\}$$

and $\mathbb{C}$ is the set of nondecreasing functions.

An array $\{X_{n,i}, 1 \leq i \leq n, n \geq 1\}$ of random variables is said to be rowwise ANA if for every $n \geq 1$, $\{X_{n,i}, 1 \leq i \leq n\}$ are ANA.

It is easy to see that $\rho^-(s) \leq \rho^*(s)$ and ANA implies NA if and only if $\rho^-(1) = 0$. Therefore, ANA random variables include ρ^*-mixing random variables and NA random variables as special cases. Consequently, the study of the limit properties for ANA random variables is of great interest. Since the concept of ANA random variables was introduced by Zhang and Wang (1999), many interesting results have been established. For instance, Ko (2014) studied the Hajek-Renyi inequality and the strong law of large numbers (SLLN, for short) for ANA random variables; Huang et al. (2016a) established some complete convergence, complete moment convergence, and mean convergence results for arrays of rowwise ANA random variables; Huang et al. (2016b) investigated complete convergence and complete moment convergence for weighted sums of arrays of rowwise ANA random variables; Huang (2018) investigated the complete convergence for weighted sums of ANA random variables with different distributions and obtained some equivalent conditions of complete convergence theorem for weighted sums and partial sums of ANA random variables; Chen et al. (2019) studied the complete convergence and complete moment convergence for weighted sums of ANA random variables and presented several sufficient conditions of the complete convergence and complete moment convergence for weighted sums of ANA random variables; Wang and Wang (2020) presented the Khintchine-Kolmogorov type convergence theorem, three series theorem and the Kolmogorov SLLN for partial sums of ANA random vectors in Hilbert spaces; Deng et al. (2021) presented the SLLN for weighted sums of m-ANA random variables; Wu et al. (2021) established a general result on complete moment convergence and the Marcinkiewicz-Zygmund-type SLLN for weighted sums of m-ANA random variables, which improve and extend some existing ones; Meng and Wu (2024) studied the SLLN of linear processes with random coefficients generated by ANA sequences; Ko (2023) extended the corresponding ones for random variables to H-valued random vectors and presented weighted version of SLLN and complete integral convergence result, and so on.

Recently, Thành (2024) proved the following theorem.

Theorem 1.1. Let $\{u_n, n \geq 1\}$ and $\{v_n, n \geq 1\}$ be two sequences of integers such that $u_n < v_n$ for all $n \geq 1$ and $\lim_{n\to\infty}(v_n - u_n) = \infty$. Let $1 \leq p < 2$ and $\{X_{n,i}, u_n \leq i \leq v_n, n \geq 1\}$ be an array of random variables such that for each $n \geq 1$. Suppose that there exists $M_n \geq 1$ which may depend on n such that for all $a > 0$, the collection $\{X_{n,i}, u_n \leq i \leq v_n\}$ satisfies

$$E\left(\sum_{i=u_n}^{v_n}\left(X_{n,i}^{(a)} - EX_{n,i}^{(a)}\right)\right)^2 \leq M_n \sum_{i=u_n}^{v_n} E\left(X_{n,i}^{(a)}\right)^2, \tag{1.1}$$

where

$$X_{n,i}^{(a)} = -aI(X_{n,i} < -a) + X_{n,i}I(|X_{n,i}| \leq a) + aI(X_{n,i} > a).$$

Let $\{b_n, n \geq 1\}$ be a sequence of positive constants. If

$$\sup_{n\geq 1} \frac{M_n}{b_n^p} \sum_{i=u_n}^{v_n} E|X_{n,i}|^p < \infty, \tag{1.2}$$

and

$$\lim_{n\to\infty} M_n \sum_{i=u_n}^{v_n} P(|X_{n,i}| > \varepsilon b_n) = 0 \text{ for all } \varepsilon > 0, \tag{1.3}$$

then

$$\frac{1}{b_n}\sum_{i=u_n}^{v_n}(X_{n,i} - EY_{n,i}) \xrightarrow{\mathbb{P}} 0, \text{ as } n \to \infty, \tag{1.4}$$

where

$$Y_{n,i} = -b_nI(X_{n,i} < -b_n) + X_{n,i}I(|X_{n,i}| \leq b_n) + b_nI(X_{n,i} > b_n), u_n \leq i \leq v_n, n \geq 1.$$

Meng and Lin (2010) showed two general SLLN in which the coefficient of sum and the weight are both general functions for the sequence of $\tilde{\rho}$-mixing and identically distributed random variables.

Hypothesis A. Let $b(x), g(x)$ be real positive functions defined on the same domain $[h, \infty)$, $\varphi(x) = b(x)g(x)$, $(0 \leq h \leq 1$, $b(x)$ or $g(x)$ may not be well defined at the point h, but if so, $\lim_{x\to h+0} b(x)g(x)$ exists, and let $\varphi(h)$ be equal to the limit at this point) and the following conditions are satisfied:

(1) $b(x)$ is increasing on its domain, and $\lim_{x\to\infty} b(x) = \infty$;

(2) $\varphi(x)$ is strictly increasing on $[h, \infty)$, $\lim_{x\to\infty} \varphi(x) = \infty$, and its range is $[0, \infty)$;

(3) there exist constants $a, c \in \mathbb{R}$ such that for every $t \in \mathbb{R}$,

$$t^2 \int_{\varphi^{-1}(|t|)}^{\infty} \frac{dx}{\varphi^2(x)} \leq a\varphi^{-1}(|t|) + c.$$

Theorem 1.2. Let $b(x), g(x), \varphi(x)$ be functions satisfying the conditions of Hypothesis A, and let $\{X_n, n \geq 1\}$ be a sequence of $\tilde{\rho}$-mixing and identically distributed random variables. Set

$$L_n = EX_nI(|X_n| < \varphi(n)).$$

If $E[\varphi^{-1}(|X_1|)] < \infty$, then

$$\frac{1}{b(n)}\sum_{k=1}^{n}\frac{X_k - L_k}{g(k)} \to 0 \text{ a.s., as } n \to \infty.$$

Theorem 1.3. Let $b(x), g(x), \varphi(x)$ be functions satisfying the conditions of Hypothesis A, and let $\varphi(x)$ satisfy the following conditions:

(i) If $\int_r^\infty dx/\varphi(x)$ is finite, then $\int_r^\infty dx/\varphi(x) \le lr/\varphi(r)(r \ge 1, l$ is a constant);

(ii) If $\int_r^\infty dx/\varphi(x)$ doesn't exist or is infinite, then $x/\varphi(x)$ is nondecreasing and $\int_1^t dx/\varphi(x) \le mt/\varphi(t)(r \ge 1, t \ge 1, m$ is a positive constant).

Suppose that $\{X_n, n \ge 1\}$ is a sequence of $\tilde{\rho}$-mixing and identically distributed random variables. If $E[\varphi^{-1}(|X_1|)] < \infty$, and $EX_1 = 0$ when (ii) holds, then

$$\frac{1}{b(n)} \sum_{k=1}^{n} \frac{X_k}{g(k)} \to 0 \ a.s., \text{ as } n \to \infty.$$

Inspired by Theorems 1.1-1.3, we aim to generalize these results to the case of ANA random variables. The main results of the paper include a maximal weak law of large numbers and two general SLLN for ANA random variables. However, by far, general SLLN in which the coefficient of sum and the weight are both general functions for ANA sequence has not been obtained yet. In this paper we study such case and show three general results, and as a consequence we obtain the Marcinkiewicz SLLN. Finally, we get a Chung type SLLN for ANA random variables.

The rest of the paper is organized as follows. In Section 2, we present some lemmas, which will be applied to prove the main results of the paper. Section 3 establishes the main results and provides the proofs.

Throughout the paper, $I(A)$ defines the indicator function of the event A. The symbol C denotes a positive universal constant which is not necessarily the same in each appearance.

2. Preliminary lemmas

In this section, we will provide some important lemmas, which will be applied to prove the main results in this paper.

The first lemma is about the Rosenthal-type maximum inequality and Marcinkiewicz-Zygmund type maximum inequality for ANA random variables. The first inequality was established by Wang and Lu (2006) and the second one can be obtained by the first one and the same method used in Theorem 2.1 of Chen et al. (2014).

Lemma 2.1. Let $\{X_n, n \ge 1\}$ be a sequence of ANA random variables, $EX_n = 0, E|X_n|^q < \infty$ for some $q > 1$ and for every $n \ge 1$. Then there exists a positive constant C depending only on q and $\rho^-(.)$ such that

$$E\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j} X_i\Big|\Big)^q \le C\Big\{\sum_{i=1}^{n} E|X_i|^q + \Big(\sum_{i=1}^{n} EX_i^2\Big)^{q/2}\Big\}, \text{ for } q \ge 2, \tag{2.1}$$

$$E\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j} X_i\Big|^q\Big) \le C\sum_{i=1}^{n} E|X_i|^q, \text{ for } 1 < q < 2. \tag{2.2}$$

By Lemma 2.1 and the Markov's inequality we can get the Kolmogorov inequality for ANA random variables.

Lemma 2.2. Suppose that $\{X_n, n \ge 1\}$ is a sequence of ANA random variables with $EX_n = 0$ and $EX_n^2 < \infty$. Then for any $\varepsilon > 0$, there exists a positive constant C such that

$$P\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^{k} X_i\Big| > \varepsilon\Big) \le \frac{C}{\varepsilon^2}\sum_{i=1}^{n} EX_i^2.$$

By Lemma 2.2 and the subsequence method, we can get the following result which is similar to that for the independent random variables.

Lemma 2.3. Let $\{X_n, n \geq 1\}$ be a sequence of ANA random variables. If

$$\sum_{n=1}^{\infty} Var(X_n) < \infty, \tag{2.3}$$

then $\sum_{n=1}^{\infty}(X_n - EX_n)$ converges *a.s.*

Proof of Lemma 2.3: Without loss of generality, we set $EX_n = 0$, $S_k := \sum_{i=1}^{k} X_i$. For each $m > n$, by Lemma 2.2 and (2.3) we have

$$0 \leq P(|S_m - S_n| \geq \varepsilon) \leq \frac{C}{\varepsilon^2} \sum_{i=n+1}^{m} EX_i^2 \xrightarrow{m,n\to\infty} 0,$$

then for any $\varepsilon > 0$,

$$\lim_{m,n\to\infty} P(|S_m - S_n| \geq \varepsilon) = 0,$$

and thus $\{S_n\}$ is a Cauchy convergence sequence under probabilistic convergence. By virtue of completeness, there exists a random variable S such that $S_n \xrightarrow{\mathbb{P}} S$. Thus, there exists a subsequence $\{n_k\}$ such that as $k \to \infty$,

$$S_{n_k} \xrightarrow{a.s.} S. \tag{2.4}$$

Also due to

$$\begin{aligned} & \sum_{k=1}^{\infty} P\left(\max_{n_{k-1}<n\leq n_k} |S_n - S_{n_{k-1}}| \geq \varepsilon\right) \\ \leq & \sum_{k=1}^{\infty} \left[\frac{C}{\varepsilon^2}\left(\sum_{i=n_{k-1}+1}^{n_k} EX_i^2\right)\right] \\ = & \frac{C}{\varepsilon^2} \sum_{k=1}^{\infty} EX_k^2 < \infty, \end{aligned}$$

which together with Borel-Cantelli lemma yields that

$$\max_{n_{k-1}<n\leq n_k} |S_n - S_{n_{k-1}}| \xrightarrow{a.s.} 0, \text{ as } k \to \infty. \tag{2.5}$$

By (2.4), (2.5) and subsequence Lemma, we obtain

$$S_n \xrightarrow{a.s.} S, \text{ as } n \to \infty,$$

which implies that

$$\sum_{n=1}^{\infty} X_n \text{ converges } a.s.$$

That is to say,

$$\sum_{n=1}^{\infty}(X_n - EX_n) \text{ converges } a.s.$$

The proof is completed. □

We can get the next lemma by Zhang and Wang (1999).

Lemma 2.4. Let $\{X_n, n \geq 1\}$ be a sequence of ANA random variables. If $f_n(\cdot), n \geq 1$ are all nondecreasing (or nonincreasing), then $\{f_n(X_n), n \geq 1\}$ is still a sequence of ANA random variables.

By Lemmas 2.3 and 2.4, we can easily get the following three series lemma.

Lemma 2.5. Let $\{X_n, n \geq 1\}$ be a sequence of ANA random variables, and $X_n^c = -\varphi(n)I(X_n < -\varphi(n)) + X_nI(|X_n| \leq \varphi(n)) + \varphi(n)I(X_n > \varphi(n))$, where $\varphi(n)$ is a real positive function, which is strictly increasing on the domain. If the following conditions are satisfied:

$$\sum_{n=1}^{\infty} P(|X_n| > \varphi(n)) < \infty, \tag{2.6}$$

$$\sum_{n=1}^{\infty} EX_n^c \text{ converges}, \tag{2.7}$$

$$\sum_{n=1}^{\infty} Var(X_n^c) < \infty, \tag{2.8}$$

then $\sum_{n=1}^{\infty} X_n$ converges *a.s.*

Proof of Lemma 2.5: By Lemma 2.4, we have $\{X_n^c, n \geq 1\}$ is a sequence of ANA random variables. By (2.8) and Lemma 2.3, we have

$$\sum_{n=1}^{\infty} (X_n^c - EX_n^c) \text{ converges } a.s., \tag{2.9}$$

which together with (2.7) derives that

$$\sum_{n=1}^{\infty} X_n^c \text{ converges } a.s. \tag{2.10}$$

Moreover, by (2.6) we have

$$\sum_{n=1}^{\infty} P(X_n \neq X_n^c) = \sum_{n=1}^{\infty} P(|X_n| > \varphi(n)) < \infty.$$

Then by Borel-Cantelli lemma, we have

$$P(\{X_n \neq X_n^c\}, i.o.) = 0. \tag{2.11}$$

By (2.10) and (2.11), we have

$$\sum_{n=1}^{\infty} X_n \text{ converges } a.s.$$

The proof is completed. □

3. Main results and their proofs

Our main results and proofs are presented as follows.

Theorem 3.1. Let $1 \leq p < 2$ and $\{X_{n,i}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise ANA random variables, $\{b(n), n \geq 1\}$ be a sequence of positive constants. If

$$\sup_{n \geq 1} \frac{1}{b^p(n)} \sum_{i=1}^{n} E|X_{n,i}|^p < \infty, \tag{3.1}$$

and

$$\lim_{n\to\infty}\sum_{i=1}^{n}P(|X_{n,i}|>\varepsilon b(n))=0 \text{ for all } \varepsilon>0, \tag{3.2}$$

then

$$\frac{1}{b(n)}\max_{1\le j\le n}\left|\sum_{i=1}^{j}(X_{n,i}-EY_{n,i})\right| \xrightarrow{\mathbb{P}} 0, \text{ as } n\to\infty, \tag{3.3}$$

where

$Y_{n,i}=-b(n)I(X_{n,i}<-b(n))+X_{n,i}I(|X_{n,i}|\le b(n))+b(n)I(X_{n,i}>b(n)), 1\le i\le n, n\ge 1.$

Proof of Theorem 3.1: Let $\varepsilon_1>0$ be arbitrary but fixed. Then

$$\begin{aligned}
& P\left(\frac{1}{b(n)}\max_{1\le j\le n}\left|\sum_{i=1}^{j}(X_{n,i}-EY_{n,i})\right|>\varepsilon_1\right)\\
= \;& P\Bigg(\left\{\frac{1}{b(n)}\max_{1\le j\le n}\left|\sum_{i=1}^{j}(X_{n,i}-EY_{n,i})\right|>\varepsilon_1, |X_{n,i}|\le b(n), \forall\, 1\le i\le n\right\}\\
& \bigcup\left\{\frac{1}{b(n)}\max_{1\le j\le n}\left|\sum_{i=1}^{j}(X_{n,i}-EY_{n,i})\right|>\varepsilon_1, |X_{n,i}|>b(n), \exists\, 1\le i\le n\right\}\Bigg)\\
\le \;& P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}(Y_{n,i}-EY_{n,i})\right|>\varepsilon_1 b(n)\right)+P\Bigg(\bigcup_{i=1}^{n}(|X_{n,i}|>b(n))\Bigg)\\
\le \;& \sum_{i=1}^{n}P(|X_{n,i}|>b(n))+P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}(Y_{n,i}-EY_{n,i})\right|>\varepsilon_1 b(n)\right):=I_1+I_2.
\end{aligned}$$

It follows from (3.2) that $\lim_{n\to\infty}I_1=0$. It thus remains to prove $\lim_{n\to\infty}I_2=0$. By using Markov's inequality and Lemma 2.1, we have

$$\begin{aligned}
I_2 \;\le\;& \frac{1}{\varepsilon_1^2 b^2(n)}E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}(Y_{n,i}-EY_{n,i})\right|\right)^2\\
\le\;& \frac{C}{\varepsilon_1^2 b^2(n)}\sum_{i=1}^{n}EY_{n,i}^2\\
=\;& \frac{C}{\varepsilon_1^2 b^2(n)}\sum_{i=1}^{n}\left(EX_{n,i}^2 I(|X_{n,i}|\le b(n))+b^2(n)P(|X_{n,i}|>b(n))\right)\\
=\;& \frac{C}{\varepsilon_1^2}\sum_{i=1}^{n}\frac{1}{b^2(n)}\int_0^{b^2(n)}P\left(|X_{n,i}|>u^{1/2}\right)du.
\end{aligned} \tag{3.4}$$

Let $0<\varepsilon<1/2$ be arbitrary. By using Markov's inequality, (3.1) and (3.4), we have

$$\begin{aligned}
I_2 \;\le\;& \frac{C}{\varepsilon_1^2}\sum_{i=1}^{n}\frac{1}{b^2(n)}\int_0^{\varepsilon^2 b^2(n)}P\left(|X_{n,i}|>u^{1/2}\right)du\\
& +\frac{C}{\varepsilon_1^2}\sum_{i=1}^{n}\frac{1}{b^2(n)}\int_{\varepsilon^2 b^2(n)}^{b^2(n)}P\left(|X_{n,i}|>u^{1/2}\right)du
\end{aligned}$$

$$\leq \frac{C}{\varepsilon_1^2}\sum_{i=1}^{n}\frac{E|X_{n,i}|^p}{b^2(n)}\int_0^{\varepsilon^2 b^2(n)}\frac{1}{u^{p/2}}du$$
$$+\frac{C}{\varepsilon_1^2}\sum_{i=1}^{n}\frac{1}{b^2(n)}\int_{\varepsilon^2 b^2(n)}^{b^2(n)}P(|X_{n,i}|>\varepsilon b(n))du$$
$$\leq \frac{C}{\varepsilon_1^2}\sum_{i=1}^{n}\frac{E|X_{n,i}|^p}{b^p(n)}\varepsilon^{2-p}+\frac{C}{\varepsilon_1^2}\sum_{i=1}^{n}P(|X_{n,i}|>\varepsilon b(n))$$
$$\leq C\varepsilon^{2-p}+\frac{C}{\varepsilon_1^2}\sum_{i=1}^{n}P(|X_{n,i}|>\varepsilon b(n)). \tag{3.5}$$

Since $0<\varepsilon<1/2$ is arbitrary and $1\leq p<2$, it follows by (3.2) and (3.5) that $\lim_{n\to\infty} I_2=0$. The proof is completed. □

Theorem 3.2. Let $b(x), g(x), \varphi(x)$ be functions satisfying the conditions of Hypothesis A, and let $\{X_n, n\geq 1\}$ be a sequence of ANA and identically distributed random variables. Set

$$Y_n=-\varphi(n)I(X_n<-\varphi(n))+X_nI(|X_n|\leq\varphi(n))+\varphi(n)I(X_n>\varphi(n)).$$

If $E[\varphi^{-1}(|X_1|)]<\infty$, then

$$\frac{1}{b(n)}\sum_{k=1}^{n}\frac{X_k-EY_k}{g(k)}\to 0\ a.s., \text{ as } n\to\infty.$$

Now we present two examples to illustrate Theorem 3.2.

Example 3.1. Consider the Cauchy random variable introduced by Adler (2007) as follows: A random variable X is said to have an asymmetrical Cauchy distribution with a slight twist, if its probability density function is

$$f(x)=\begin{cases}\frac{p}{\pi(1+x^2)}, & \text{if } x\geq 0,\\ \frac{q}{\pi(1+x^2)}, & \text{if } x<0,\end{cases}$$

where $p,q\geq 0$ with $p+q=2$. It is the usual Cauchy distribution if $p=q=1$. It is easy to check that $E|X|=\infty$ but $E|X|\log^{-2}|X|<\infty$. Take $g(x)=x\log^{2-\beta}x$, $b(x)=\log^{\beta}x$ and thus $\varphi(x)=x\log^2 x$, where $\beta>0$. It follows from Theorem 3.2 that for a sequence $\{X_n, n\geq 1\}$ of random variables identically distributed as X,

$$\frac{1}{\log^{\beta} n}\sum_{k=1}^{n}\frac{X_k-EY_k}{k\log^{2-\beta}k}\to 0\ a.s., \text{ as } n\to\infty.$$

Moreover, Adler (2007) proved that $\sum_{n=1}^{\infty}P(|X|>\varphi(n))<\infty$ and $\frac{1}{\log^{\beta} n}\sum_{k=1}^{n}\frac{EXI(|X|\leq\varphi(k))}{k\log^{2-\beta}k}\to\frac{p-q}{\pi\beta}$. Therefore, if the distribution is asymmetrical, i.e., $p\neq q$, then we have

$$\frac{1}{\log^{\beta} n}\sum_{k=1}^{n}\frac{X_k}{k\log^{2-\beta}k}\to\frac{p-q}{\pi\beta}\ a.s., \text{ as } n\to\infty,$$

which has been obtained by Adler (2007); and if $p=q$, we still have

$$\frac{1}{\log^{\beta} n}\sum_{k=1}^{n}\frac{X_k}{k\log^{2-\beta}k}\to 0\ a.s., \text{ as } n\to\infty,$$

since $EXI(|X|\leq\varphi(k))=0$ for any fixed $1\leq k\leq n, n\geq 1$.

The following example shows the generation of a Cauchy sequence that satisfies the ANA structure.

Example 3.2. Let $U = (U_1, \cdots, U_n)$ obey the multivariate normal distribution $N_n(\mathbf{0}, \mathbf{\Sigma})$, where $\mathbf{\Sigma} = (\sigma_{ij})_{n\times n}$ is the covariance matrix such that $\sigma_{kk} = 1$ and $\sigma_{kj} \le 0$ for $k \ne j, 1 \le k, j \le n$. Let $\{W_1, \cdots, W_n\}$ be a sequence of independent $\chi^2(1)$ random variables which is also independent of U. Now let $X_k = U_k/\sqrt{W_k}$. It is easy to see that X_k follows the standard Cauchy distribution for each $1 \le k \le n$ but the sequence $\{X_1, \cdots, X_n\}$ is dependent. It is not known whether it is ANA since the covariance is not defined. However, from the proof of Theorem 3.2, one can see that it suffices to show that $\{Y_1, \cdots, Y_n\}$ is ANA. It is evident that

$$\mathrm{Cov}(Y_k, Y_j) = E\Big[\mathrm{Cov}[(Y_k, Y_j)|W]\Big] \le 0.$$

Therefore, Theorem 3.2 also holds true under this situation.
Proof of Theorem 3.2: Note that

$$\sum_{n=1}^{\infty} P\Big(|X_n| > \varphi(n)\Big) = \sum_{n=1}^{\infty} P\Big(\varphi^{-1}(|X_1|) > n\Big) \le E\left[\varphi^{-1}(|X_1|)\right] < \infty.$$

We have

$$\sum_{n=1}^{\infty} P(X_n \ne Y_n) = \sum_{n=1}^{\infty} P\Big(|X_n| > \varphi(n)\Big) < \infty.$$

Set

$$A_n = \{X_n \ne Y_n\},\ A = \limsup_{n\to\infty} A_n.$$

By the Borel-Cantelli lemma, we obtain

$$P(A) = 0,\ P(\overline{A}) = 1.$$

Note that

$$\sum_{n=1}^{\infty} \frac{EY_n^2}{\varphi^2(n)} = \sum_{n=1}^{\infty} \frac{EX_n^2 I\{|X_n| \le \varphi(n)\}}{\varphi^2(n)} + \sum_{n=1}^{\infty} \frac{E\varphi^2(n) I\{|X_n| > \varphi(n)\}}{\varphi^2(n)} := I_1 + I_2.$$

For I_2, we have

$$I_2 = \sum_{n=1}^{\infty} P(|X_n| > \varphi(n)) \le E[\varphi^{-1}(|X_1|)] < \infty.$$

Next, we deal with I_1. For every $\omega \in \overline{A}$, there exists an $m_0 \in \mathbb{N}$, such that $\varphi^{-1}(|X_1|) < m_0$, and $\varphi^{-1}(|X_1|) \ge m_0 - 1$. So for every $\omega \in \overline{A}$, we get

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{X_1^2 I\{|X_1| \le \varphi(n)\}}{\varphi^2(n)} &= \sum_{n=1}^{m_0} \frac{X_1^2 I\{|X_1| \le \varphi(n)\}}{\varphi^2(n)} + \sum_{n=m_0+1}^{\infty} \frac{X_1^2 I\{|X_1| \le \varphi(n)\}}{\varphi^2(n)} \\
&< 2 + \sum_{n=m_0+1}^{\infty} \frac{X_1^2}{\varphi^2(n)} \le 2 + X_1^2 \int_{m_0}^{\infty} \frac{dx}{\varphi^2(x)} \\
&\le 2 + X_1^2 \int_{\varphi^{-1}(|X_1|)}^{\infty} \frac{dx}{\varphi^2(x)} \le 2 + a\varphi^{-1}(|X_1|) + c.
\end{aligned}$$

Then we have

$$I_1 = \sum_{n=1}^{\infty} \frac{EX_1^2 I\{|X_1| \le \varphi(n)\}}{\varphi^2(n)} = E\sum_{n=1}^{\infty} \frac{X_1^2 I\{|X_1| \le \varphi(n)\}}{\varphi^2(n)} \le 2 + aE[\varphi^{-1}(|X_1|)] + c < \infty.$$

Consequently, we obtain

$$\sum_{n=1}^{\infty} \frac{Var(Y_n)}{\varphi^2(n)} < \infty.$$

Since $\{Y_n/\varphi(n), n \geq 1\}$ is also a sequence of ANA random variables, it follows by Lemma 2.3 that

$$\sum_{n=1}^{\infty} \frac{Y_n - EY_n}{\varphi(n)} \text{ converges } a.s.$$

By $0 < b(n) \uparrow \infty$ and the Kronecker lemma, we have

$$\frac{1}{b(n)} \sum_{k=1}^{n} \frac{Y_k - EY_k}{g(k)} \to 0 \; a.s., \text{ as } n \to \infty.$$

□

Remark 3.1. Theorem 3.2 gives a class of SLLN for the sequence of ANA and identically distributed random variables. If we take different $b(x)$, $g(x)$ and $\varphi(x)$ that satisfy the conditions of Theorem 3.2, we can get some SLLN in specific forms. For example, if we take $b(x) = x^{\frac{1}{p}} (0 < p < 2), g(x) = 1, \varphi(x) = b(x)g(x), x \in [0, \infty)$, then we can get the Marcinkiewicz type SLLN $n^{-\frac{1}{p}} \sum_{k=1}^{n} (X_k - EY_k) \to 0 \; a.s.$, as $n \to \infty$. If we take $b(x) = \log x, g(x) = x^{\alpha} (\alpha > 1/2), \varphi(x) = b(x)g(x), x \in [1, \infty)$, then we can get logarithmic SLLN $\frac{1}{\log n} \sum_{k=1}^{n} \frac{X_k - EY_k}{k^{\alpha}} \to 0 \; a.s.$, as $n \to \infty$.

Theorem 3.3. Let $b(x), g(x), \varphi(x)$ be functions satisfying the conditions of Hypothesis A, and let $\varphi(x)$ satisfy the following conditions:

(i) If $\int_r^{\infty} dx/\varphi(x)$ is finite, then $\int_r^{\infty} dx/\varphi(x) \leq lr/\varphi(r) (r \geq 1, l$ is a constant);

(ii) If $\int_r^{\infty} dx/\varphi(x)$ doesn't exist or is infinite, then $x/\varphi(x)$ is nondecreasing and $\int_1^{t} dx/\varphi(x) \leq mt/\varphi(t) (r \geq 1, t \geq 1, m$ is a constant).

Suppose that $\{X_n, n \geq 1\}$ is a sequence of ANA and identically distributed random variables. If $E[\varphi^{-1}(|X_1|)] < \infty$, and $EX_1 = 0$ when (ii) holds, then

$$\frac{1}{b(n)} \sum_{k=1}^{n} \frac{X_k}{g(k)} \to 0 \; a.s., \text{ as } n \to \infty.$$

Proof of Theorem 3.3: In the proof of Theorem 3.2, we have proved

$$\sum_{n=1}^{\infty} P(|X_n| > \varphi(n)) < \infty,$$

and

$$\sum_{n=1}^{\infty} \frac{Var(Y_n)}{\varphi^2(n)} < \infty,$$

where

$$Y_n = -\varphi(n)I(X_n < -\varphi(n)) + X_n I(|X_n| \leq \varphi(n)) + \varphi(n)I(X_n > \varphi(n)).$$

Noting that $\{Y_n/\varphi(n), n \geq 1\}$ is a sequence of ANA random variables, by Lemma 2.5 in order to prove $\frac{1}{b(n)} \sum_{k=1}^{n} \frac{X_k}{g(k)} \to 0 \; a.s.$, as $n \to \infty$, it is only needed to prove $\sum_{n=1}^{\infty} EY_n/\varphi(n)$ converges. Now we shall prove

$\sum\limits_{n=1}^{\infty} EY_n/\varphi(n)$ converges both in case (*i*) and case (*ii*).

Suppose (*i*) holds. We have

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\frac{EY_n}{\varphi(n)}\right| &\leq \sum_{n=1}^{\infty}\frac{1}{\varphi(n)}E\Big|X_1 I\{|X_1|\leq\varphi(n)\}\Big| \\ &\quad +\sum_{n=1}^{\infty}\frac{1}{\varphi(n)}E\Big|-\varphi(n)I\{X_n<-\varphi(n)\}+\varphi(n)I\{X_n>\varphi(n)\}\Big| := I_3+I_4.\end{aligned}$$

For I_4, we obtain

$$\begin{aligned}I_4 &\leq \sum_{n=1}^{\infty}\frac{1}{\varphi(n)}E\Big|\varphi(n)I\{|X_n|>\varphi(n)\}\Big| = \sum_{n=1}^{\infty}P\{|X_n|>\varphi(n)\} \\ &\leq \sum_{n=1}^{\infty}P\Big(\varphi^{-1}(|X_1|)>n\Big)\leq E[\varphi^{-1}(|X_1|)]<\infty.\end{aligned}$$

For I_3, we have

$$\begin{aligned}I_3 &\leq \sum_{n=1}^{\infty}\frac{1}{\varphi(n)}\sum_{j=1}^{n}E|X_1|I\{j-1<\varphi^{-1}(|X_1|)\leq j\} \\ &\leq \sum_{n=1}^{\infty}\frac{1}{\varphi(n)}\sum_{j=1}^{n}\varphi(j)P(j-1<\varphi^{-1}(|X_1|)\leq j) \\ &= \sum_{j=1}^{\infty}\varphi(j)P(j-1<\varphi^{-1}(|X_1|)\leq j)\sum_{n=j}^{\infty}\frac{1}{\varphi(n)} \\ &\leq \sum_{j=1}^{\infty}\varphi(j)P(j-1<\varphi^{-1}(|X_1|)\leq j)\Big[\frac{1}{\varphi(j)}+\int_j^{\infty}\frac{dx}{\varphi(x)}\Big] \\ &\leq \sum_{j=1}^{\infty}\varphi(j)P(j-1<\varphi^{-1}(|X_1|)\leq j)\Big[\frac{1}{\varphi(j)}+\frac{lj}{\varphi(j)}\Big] \\ &\leq (l+1)\sum_{n=1}^{\infty}jP(j-1<\varphi^{-1}(|X_1|)\leq j) \\ &= (l+1)\Big(1+\sum_{n=1}^{\infty}P(\varphi^{-1}(|X_1|)>n)\Big) \\ &\leq (l+1)(1+E[\varphi^{-1}(|X_1|)])<\infty.\end{aligned}$$

Consequently, we have

$$\sum_{n=1}^{\infty}\frac{EY_n}{\varphi(n)} \text{ converges.}$$

Suppose (*ii*) holds. Noting that $EX_1=0$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\frac{EY_n}{\varphi(n)}\right| &= \sum_{n=1}^{\infty}\frac{1}{\varphi(n)}\Big|EX_n-E\Big(-\varphi(n)I(X_n<-\varphi(n))+X_nI(|X_n|\leq\varphi(n))+\varphi(n)I(X_n>\varphi(n))\Big)\Big| \\ &\leq \sum_{n=1}^{\infty}\frac{1}{\varphi(n)}E|X_1|I(|X_1|>\varphi(n))+\sum_{n=1}^{\infty}\frac{1}{\varphi(n)}E\varphi(n)I(|X_n|>\varphi(n)) := I_5+I_6.\end{aligned}$$

For I_6, we have

$$I_6 = \sum_{n=1}^{\infty} P(|X_n| > \varphi(n)) \leq \sum_{n=1}^{\infty} P\Big(\varphi^{-1}(|X_1|) > n\Big) \leq E[\varphi^{-1}(|X_1|)] < \infty.$$

For I_5, we obtain

$$\begin{aligned}
I_5 &= \sum_{n=1}^{\infty} \frac{1}{\varphi(n)} \sum_{j=n}^{\infty} E|X_1|I\{j < \varphi^{-1}(|X_1|) \leq j+1\} \\
&\leq \sum_{n=1}^{\infty} \frac{1}{\varphi(n)} \sum_{j=n}^{\infty} \varphi(j+1)P(j < \varphi^{-1}(|X_1|) \leq j+1) \\
&= \sum_{j=1}^{\infty} \varphi(j+1)P(j < \varphi^{-1}(|X_1|) \leq j+1) \sum_{n=1}^{j} \frac{1}{\varphi(n)} \\
&\leq \sum_{j=1}^{\infty} \varphi(j+1)P(j < \varphi^{-1}(|X_1|) \leq j+1)\Big[\frac{1}{\varphi(1)} + \int_1^{j+1} \frac{dx}{\varphi(x)}\Big] \\
&\leq \sum_{j=1}^{\infty} \varphi(j+1)P(j < \varphi^{-1}(|X_1|) \leq j+1)\Big[\frac{1}{\varphi(1)} + \frac{m(j+1)}{\varphi(j+1)}\Big] \\
&\leq (m+1) \sum_{j=1}^{\infty} (j+1)P(j < \varphi^{-1}(|X_1|) \leq j+1) \\
&\leq (m+1) \sum_{j=1}^{\infty} jP(j-1 < \varphi^{-1}(|X_1|) \leq j) \\
&= (m+1)\Big(1 + \sum_{n=1}^{\infty} P(\varphi^{-1}(|X_1|) > n)\Big) \\
&\leq (m+1)(1 + E[\varphi^{-1}(|X_1|)]) < \infty.
\end{aligned}$$

Consequently, we obtain that

$$\sum_{n=1}^{\infty} \frac{EY_n}{\varphi(n)} \text{ converges.}$$

Thus, $\sum_{n=1}^{\infty} EY_n/\varphi(n)$ converges both in case (i) and case (ii). So Theorem 3.3 is proved. □

By Theorem 3.2 and Theorem 3.3, we can get the Marcinkiewicz SLLN for ANA random variables.

Corollary 3.1. Suppose that $\{X_n, n \geq 1\}$ is a sequence of ANA and identically distributed random variables. If $E|X_1|^p < \infty$ for some $0 < p < 2$, then for some finite constant a,

$$\frac{1}{n^{1/p}} \sum_{k=1}^{n} (X_k - a) \to 0 \ a.s., \ \text{as } n \to \infty.$$

If $0 < p < 1$, then a can be taken arbitrary real number. If $1 \leq p < 2$, then $a = EX_1$.

Proof of Corollary 3.1: If $0 < p < 1$, then for arbitrary real number a,

$$n^{-1/p} \sum_{k=1}^{n} a \to 0 \text{ as } n \to \infty.$$

Thus in order to prove

$$n^{-1/p}\sum_{k=1}^{n}(X_k-a)\to 0\ a.s.,\ \text{as } n\to\infty,$$

it is enough to show that

$$n^{-1/p}\sum_{k=1}^{n}X_k\to 0\ a.s.,\ \text{as } n\to\infty.$$

If $1<p<2, a=EX_1$, without loss of generality, we put $EX_1=0$. So, for $0<p<2, p\neq 1$, we only need to show that

$$n^{-1/p}\sum_{k=1}^{n}X_k\to 0\ a.s.,\ \text{as } n\to\infty,$$

which is true by taking

$$\begin{aligned}&b(x)=x^{1/p},\ 0<p<2,\ p\neq 1,\ x\in[0,\infty);\\ &g(x)=1,\ x\in[0,\infty);\\ &\varphi(x)=b(x)g(x),\ x\in[0,\infty)\end{aligned}$$

in Theorem 3.3. For $p=1, b(x), g(x)$ and $\varphi(x)$ satisfy the conditions of Theorem 3.2, so we obtain

$$n^{-1}\sum_{k=1}^{n}(X_k-EY_k)\to 0\ a.s.,\ \text{as } n\to\infty,$$

where $Y_k=-\varphi(k)I(X_k<-\varphi(k))+X_kI(|X_k|\leq\varphi(k))+\varphi(k)I(X_k>\varphi(k))$. And noting that $EY_k\to EX_1$, we have $n^{-1}\sum_{k=1}^{n}X_k\to EX_1\ a.s.$, as $n\to\infty$. So Corollary 3.1 is now proved. □

Theorem 3.4. Suppose that $\{X_n, n\geq 1\}$ is a sequence of ANA random variables with $EX_n=0$ for all n, and $\{c_n, n\geq 1\}$ is a sequence of positive real numbers with $c_n\uparrow\infty$. Let $\psi_n:\mathbb{R}_+\to\mathbb{R}_+$ be Borel functions and let $\alpha_n\geq 1, \beta_n\leq 2, C_n>0, D_n>0\ (n\geq 1)$ be constants satisfying

$$\nu\leq\mu\ \Rightarrow\ C_n\frac{\mu^{\alpha_n}}{\nu^{\alpha_n}}\leq\frac{\psi_n(\mu)}{\psi_n(\nu)}\leq D_n\frac{\mu^{\beta_n}}{\nu^{\beta_n}}. \tag{3.6}$$

If

$$\sum_{n=1}^{\infty}A_n\frac{E\psi_n(|X_n|)}{\psi_n(c_n)}<\infty, \tag{3.7}$$

where $A_n=\max\{1/C_n, D_n\}$, then $\sum_{n=1}^{\infty}X_n/c_n$ converges $a.s.$, and in consequence, $c_n^{-1}\sum_{i=1}^{n}X_i\to 0\ a.s.$, as $n\to\infty$.

If we take $\alpha_n=1, \beta_n=2, C_n=D_n=1$, then we can get the Chung type SLLN.

Corollary 3.2. Suppose that $\{X_n, n\geq 1\}$ is a sequence of ANA random variables with $EX_n=0$ for all n, and $\{c_n, n\geq 1\}$ is a sequence of positive real numbers with $c_n\uparrow\infty$. Let $\psi_n:\mathbb{R}\to\mathbb{R}$ be positive, even and continuous functions such that $\psi_n(x)/x$ and $x^2/\psi_n(x)$ are nondecreasing for $x>0$. If $\sum_{n=1}^{\infty}E\psi_n(X_n)/\psi_n(c_n)<\infty$, then $c_n^{-1}\sum_{i=1}^{n}X_i\to 0\ a.s.$, as $n\to\infty$.

Proof of Theorem 3.4: Put $Z_n = -c_n I(X_n < -c_n) + X_n I(|X_n| \le c_n) + c_n I(X_n > c_n), n \ge 1$. It follows from (3.6) that on the set $\{x : |x| \le c_n\}$ we have

$$\frac{|x|^2}{c_n^2} \le \frac{|x|^{\beta_n}}{c_n^{\beta_n}} \le A_n \frac{\psi_n(|x|)}{\psi_n(c_n)}.$$

Thus, we obtain

$$\sum_{n=1}^{\infty} E\left(\frac{Z_n^2}{c_n^2}\right) = \sum_{n=1}^{\infty} \int_{\{|x| \le c_n\}} \frac{x^2}{c_n^2} dF_{X_n}(x) + \sum_{n=1}^{\infty} P(|X_n| > c_n) := I_7 + I_8.$$

For I_7, we obtain

$$I_7 \le \sum_{n=1}^{\infty} A_n \int_{\{|x| \le c_n\}} \frac{\psi_n(|x|)}{\psi_n(c_n)} dF_{X_n}(x) \le \sum_{n=1}^{\infty} A_n \frac{E\psi_n(|X_n|)}{\psi_n(c_n)} < \infty. \tag{3.8}$$

Next, we deal with I_8. On the set $\{x : |x| > c_n\}$, we have

$$\frac{|x|}{c_n} \le \frac{|x|^{\alpha_n}}{c_n^{\alpha_n}} \le A_n \frac{\psi_n(|x|)}{\psi_n(c_n)}.$$

By Markov's inequality, we get

$$I_8 \le \sum_{n=1}^{\infty} \frac{E|X_n| I(|X_n| > c_n)}{c_n} \le \sum_{n=1}^{\infty} A_n \frac{E\psi_n(|X_n|)}{\psi_n(c_n)} < \infty. \tag{3.9}$$

Consequently, by (3.8) and (3.9), we have

$$\sum_{n=1}^{\infty} E\left(\frac{Z_n^2}{c_n^2}\right) < \infty.$$

Noting that $EX_n = 0$, we get

$$\sum_{n=1}^{\infty} \frac{|EZ_n|}{c_n} \le \sum_{n=1}^{\infty} \frac{1}{c_n} \left| \int_{\{|x| > c_n\}} x dF_{X_n}(x) \right| + \sum_{n=1}^{\infty} P(|X_n| > c_n) := I_9 + I_{10}.$$

For I_9, we have

$$I_9 \le \sum_{n=1}^{\infty} A_n \int_{\{|x| > c_n\}} \frac{\psi_n(|x|)}{\psi_n(c_n)} dF_{X_n}(x) \le \sum_{n=1}^{\infty} A_n \frac{E\psi_n(|X_n|)}{\psi_n(c_n)} < \infty. \tag{3.10}$$

Using the same method as (3.9), we obtain

$$I_{10} < \infty. \tag{3.11}$$

Consequently, by (3.10) and (3.11), we have

$$\sum_{n=1}^{\infty} \frac{|EZ_n|}{c_n} < \infty.$$

Finally, we have

$$\sum_{n=1}^{\infty} P\left(\left|\frac{X_n}{c_n}\right| > 1\right) = \sum_{n=1}^{\infty} \int_{\{|x| > c_n\}} dF_{X_n}(x) \le \sum_{n=1}^{\infty} A_n \int_{\{|x| > c_n\}} \frac{\psi_n(|x|)}{\psi_n(c_n)} dF_{X_n}(x)$$

$$\leq \sum_{n=1}^{\infty} A_n \frac{E\psi_n(|X_n|)}{\psi_n(c_n)} < \infty.$$

Theorem 3.4 now follows from Lemma 2.5. □

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