



Radii results of convex combinations and convolutions of generalized univalent harmonic mappings

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Abstract. In this article, we introduce the generalized harmonic Koebe function and find its coefficient bounds. We obtain radii of fully starlikeness and convexity for convex combinations two generalized univalent harmonic mappings. We also obtain radii of fully starlikeness and convexity for convolutions of two generalized univalent harmonic mappings.

1. Introduction

Let \mathcal{H} represent the class of complex-valued functions $f = u + iv$ that are harmonic on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, where u and v are real-valued harmonic functions in \mathbb{D} . A function $f \in \mathcal{H}$ can be expressed as $f = h + \bar{g}$, where h and g are analytic in \mathbb{D} , called the analytic and co-analytic parts of f , respectively.

The Jacobian of $f = h + \bar{g}$ is given by

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2.$$

A classical result of Lewy [11] showed that each harmonic function $f = h + \bar{g} \in \mathcal{H}$ is locally univalent and sense-preserving if and only if $J_f(z) > 0$ in \mathbb{D} , which is equivalent to the existence of an analytic function $w_f(z) = g'(z)/h'(z)$ in \mathbb{D} such that $|w_f(z)| < 1$. As usual, w_f is called the dilatation of f . For more details, we refer the reader to [1, 4] (see also [16, 17]).

The familiar class of univalent harmonic functions $f = h + \bar{g} \in \mathcal{H}$ with normalized conditions $h(0) = g(0) = 0$ and $h'(0) = 1$ is denoted by $\mathcal{S}_{\mathcal{H}}$. For a mapping $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}$, h and g are of the forms

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \quad (|b_1| < 1; z \in \mathbb{D}). \quad (1)$$

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The subclass of functions $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}$ that satisfy the additional condition $g'(0) = 0$ is denoted by $\mathcal{S}_{\mathcal{H}}^0$. Moreover, the subclasses of starlike, convex, and close-to-convex functions f in $\mathcal{S}_{\mathcal{H}}$ ($\mathcal{S}_{\mathcal{H}}^0$) are, respectively, denoted by $\mathcal{S}_{\mathcal{H}}^*$ ($\mathcal{S}_{\mathcal{H}}^{*,0}$), $\mathcal{K}_{\mathcal{H}}$ ($\mathcal{K}_{\mathcal{H}}^0$) and $\mathcal{C}_{\mathcal{H}}$ ($\mathcal{C}_{\mathcal{H}}^0$).

The convolution or Hadamard product of two analytic functions $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ is denoted by $f * g$ and is defined by $(f * g)(z) := \sum_{n=1}^{\infty} a_n b_n z^n$. The convolution of two harmonic functions $f = h + \bar{g}$ and $F = H + \bar{G}$ is defined by $(f * F)(z) := (h * H)(z) + \overline{(g * G)(z)}$.

A domain $\Omega \subset \mathbb{C}$ is said to be convex in the direction $\varphi \in [0, \pi)$, if for all $t \in \mathbb{C}$, the set $\Omega \cap \{t + re^{i\varphi} : r \in \mathbb{R}\}$ is connected or empty. A function is said to be convex in the direction φ if it maps \mathbb{D} univalently onto a domain convex in the direction φ .

In 1984, Clunie-Sheil-Small [3] introduced a method, known as shearing technique, to construct a univalent harmonic mapping from a related conformal mapping. By this method, a generalized result can be presented as follows:

Lemma 1.1. *A locally univalent harmonic function $f = h + \bar{g}$ in \mathbb{D} is a univalent harmonic mapping of \mathbb{D} onto a domain convex in the direction φ if and only if $h - e^{2i\varphi}g$ is a univalent analytic mapping of \mathbb{D} onto a domain convex in the direction φ .*

By shearing the Koebe function

$$k(z) = h(z) - g(z) = \frac{z}{(1-z)^2}$$

horizontally with dilatation $w(z) = z$, Clunie-Sheil-Small [3] introduced the harmonic Koebe function $k_0 = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}^0$ such that

$$k_0(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} + \overline{\frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}}. \tag{2}$$

They conjectured that the Taylor coefficients of the series of h and g satisfy the inequalities

$$|a_n| \leq \frac{1}{6}(2n+1)(n+1) \text{ and } |b_n| \leq \frac{1}{6}(2n-1)(n-1)$$

for all $n \geq 2$.

By shearing the right half-plane mapping

$$s(z) = h(z) + g(z) = \frac{z}{1-z}$$

vertically with dilatation $w(z) = -z$, Clunie-Sheil-Small [3] also considered the harmonic half-plane mapping $s_0 = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}^0$ such that

$$s_0(z) = \frac{z - \frac{1}{2}z^2}{(1-z)^2} - \overline{\frac{\frac{1}{2}z^2}{(1-z)^2}}. \tag{3}$$

They proved that the Taylor coefficients of the series of h and g satisfy the inequalities

$$|a_n| \leq \frac{n+1}{2} \text{ and } |b_n| \leq \frac{n-1}{2}$$

for all $n \geq 2$.

Convexity and starlikeness are hereditary properties for conformal mappings. To be more precise, if a function f is convex or starlike with respect to the origin in a domain \mathbb{D} , then it maps \mathbb{D} onto a convex domain or a starlike domain with respect to the origin, respectively. However, hereditary properties do not extend to harmonic mappings. The failure of hereditary properties has motivated the introduction of the

concepts of fully convex and fully starlike functions, which were first proposed by Chuaqui-Duren-Osgood [2].

A harmonic mapping f in \mathbb{D} is said to be fully convex of order α , denoted by $\mathcal{FK}_{\mathcal{H}}$, if it maps every circle $|z| = r < 1$ in a one-to-one manner onto a convex curve that satisfies the condition

$$\frac{\partial}{\partial \theta} \left(\arg \left(\frac{\partial}{\partial \theta} f(re^{i\theta}) \right) \right) > \alpha \quad (0 < \theta < 2\pi; 0 < r < 1).$$

A harmonic mapping f in \mathbb{D} with $f(0) = 0$ is said to be fully starlike of order α , denoted by $\mathcal{FS}_{\mathcal{H}}^*$, if it maps every circle $|z| = r < 1$ in a one-to-one manner onto a curve bounded by a domain starlike with respect to the origin that satisfies the inequality

$$\frac{\partial}{\partial \theta} \left(\arg f(re^{i\theta}) \right) > \alpha \quad (0 < \theta < 2\pi; 0 < r < 1).$$

Clearly, fully convex mappings are fully starlike, but the converse is not true. According to the Radó-Kneser-Choquet theorem [8, Section 3.1], a fully convex harmonic mapping is guaranteed to be univalent in \mathbb{D} . However, a fully starlike mapping does not have to be univalent.

Construction of univalent harmonic mappings is an important task in the context of univalent harmonic mappings. One of useful way to construct new examples of univalent harmonic mappings is taking convex combinations of two one-to-one harmonic mappings for getting a new one-to-one harmonic mappings. Convolution is another approach to construct univalent harmonic functions. It also requires certain conditions in order to guarantee that the resulting functions are univalent. In [15], MacGregor showed that the convex combination $tf + (1 - t)g$ ($t \in [0, 1]$) of two univalent analytic functions need not to be univalent, even if f and g are convex analytic functions. Since then, the inquiry for sufficient conditions on convex combinations of univalent analytic functions or univalent harmonic mappings to be univalent has emerged as a fascinating area of research. In [7], Dorff and Rolf applied to another way of constructing a univalent harmonic mapping by taking two suitable harmonic mappings F_1 and F_2 with the same dilatation, whose convex combination is univalent and convex in the direction of the imaginary axis. Wang-Liu-Li [23] derived several sufficient conditions on harmonic univalent functions F_1 and F_2 so that their convex combinations are univalent and convex in the direction of the real axis. Moreover, the convolutions of two harmonic functions convex in one direction have studied by Dorff [5] and Dorff-Nowak-Woloszkiewicz [6]. More recent results on the convex combinations and convolutions of harmonic mappings can also be found in [9, 12, 13].

Liu and Ponnusamy [12] introduced and studied the generalized right half-plane mapping s_a for generalized harmonic mappings $f_a = h_a + \bar{g}_a \in \mathcal{S}_{\mathcal{H}}^a$, where $-1 < a < 1$. By shearing the conformal mapping

$$h_a(z) + g_a(z) = \frac{(1 + a)z}{1 - z}$$

vertically with dilatation $w(z) = (a - z)/(1 - az)$, where $-1 < a < 1$, the authors considered the harmonic right half-plane mapping $s_a = h_a + \bar{g}_a \in \mathcal{S}_{\mathcal{H}}^a$ as

$$s_a(z) = \frac{z - \frac{1}{2}(1 + a)z^2}{(1 - z)^2} + \overline{\frac{az - \frac{1}{2}(1 + a)z^2}{(1 - z)^2}}. \tag{4}$$

They proved that the Taylor coefficients of the series of h_a and g_a satisfy the inequalities

$$|a_n| \leq \frac{(1 + a) + n(1 - a)}{2} \quad \text{and} \quad |b_n| \leq \frac{(1 + a) - n(1 - a)}{2}$$

for all $n \geq 2$. For $a = 0$, we get the harmonic right half-plane mapping given of the form (3), and its coefficient estimates (see also [18–20]).

In this article, we construct the generalized harmonic Koebe function k_a for generalized harmonic mappings $f_a = h_a + \bar{g}_a \in \mathcal{S}_{\mathcal{H}}^a$, where $-1 < a < 1$. By shearing the function

$$h_a(z) - g_a(z) = \frac{(1-a)z}{(1-z)^2}$$

with dilatation $w(z) = (a+z)/(1+az)$, where $-1 < a < 1$, we get the generalized harmonic Koebe function $k_a = h_a + \bar{g}_a \in \mathcal{S}_{\mathcal{H}}^a$ as

$$k_a(z) = \frac{z + \frac{a-1}{2}z^2 + \frac{1+a}{6}z^3}{(1-z)^3} + \frac{\overline{az + \frac{1-a}{2}z^2 + \frac{1+a}{6}z^3}}{(1-z)^3}. \tag{5}$$

Taylor coefficients of the series of h_a and g_a satisfy the inequalities

$$|a_n| \leq \frac{2(1+a)n^2 + 3(1-a)n + (1+a)}{6} \quad \text{and} \quad |b_n| \leq \frac{2(1+a)n^2 + 3(a-1)n + (1+a)}{6}$$

for all $n \geq 2$. For $a = 0$, we get the harmonic Koebe function given of the form (2), and its coefficient estimates.

In the paper, we find the radii of fully starlikeness and fully convexity for convex combinations and convolutions of generalized harmonic mappings $f_a = h_a + \bar{g}_a \in \mathcal{S}_{\mathcal{H}}^a$, where h_a and g_a are of the form (1). Let two generalized harmonic mappings be of the form $f_{a,j} = h_{a,j} + \bar{g}_{a,j} \in \mathcal{S}_{\mathcal{H}}^a$ ($j = 1, 2$) such that their convex combinations are shown by

$$F_a = t f_{a,1} + (1-t) f_{a,2}, \quad (t \in [0, 1])$$

and their convolutions are shown by

$$G_a = f_{a,1} * f_{a,2}.$$

In Section 2, we derive radii of convex combinations of generalized harmonic Koebe and generalized harmonic right half-plane mappings. In Section 3, we derive radii of convolutions of two generalized harmonic Koebe mappings. We observe that the main theorems are generalization of several radius results appeared in literature.

2. Radii of convex combinations of generalized harmonic mappings

In this section, we obtain the radii of fully starlikeness and radii of fully convexity for convex combinations of two generalized harmonic mappings. The following elementary identities and lemmas are required to prove the main results.

Lemma 2.1. *We have*

- (i) $\sum_{n=2}^{\infty} n r^{n-1} = \frac{r(2-r)}{(1-r)^2}$;
- (ii) $\sum_{n=2}^{\infty} n^2 r^{n-1} = \frac{r(4-3r+r^2)}{(1-r)^3}$;
- (iii) $\sum_{n=2}^{\infty} n^3 r^{n-1} = \frac{r(8-5r+4r^2-r^3)}{(1-r)^4}$;
- (iv) $\sum_{n=2}^{\infty} n^4 r^{n-1} = \frac{r(16+r+11r^2-5r^3+r^4)}{(1-r)^5}$;
- (v) $\sum_{n=2}^{\infty} n^5 r^{n-1} = \frac{r(32+51r+46r^2-14r^3+6r^4-r^5)}{(1-r)^6}$;
- (vi) $\sum_{n=2}^{\infty} n^6 r^{n-1} = \frac{r(64+281r+337r^2+22r^3+22r^4-7r^5+r^6)}{(1-r)^7}$.

Lemma 2.2. ([22]) *Let $f = h + \bar{g} \in \mathcal{H}$, where h and g are of the form (1) with $|b_1| < 1$.*

(a) *If*

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha} |b_n| \leq 1, \tag{6}$$

then f is fully starlike in \mathbb{D} .

(b) If

$$\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n(n+\alpha)}{1-\alpha} |b_n| \leq 1, \tag{7}$$

then f is fully convex in \mathbb{D} .

Theorem 2.3. Let $f_{a,j} = h_{a,j} + \bar{g}_{a,j} \in \mathcal{S}_{\mathcal{H}}^a$ be given by (1) with

$$h_{a,j}(z) = z + \sum_{n=2}^{\infty} a_{j,n} z^n, \quad g_{a,j}(z) = \sum_{n=1}^{\infty} b_{j,n} z^n \quad (z \in \mathbb{D}, j = 1, 2).$$

Also, suppose that the coefficients satisfy the conditions

$$|a_{1,n}| \leq \frac{2(1+a)n^2 + 3(1-a)n + (1+a)}{6} \quad \text{and} \quad |b_{1,n}| \leq \frac{2(1+a)n^2 + 3(a-1)n + (1+a)}{6} \tag{8}$$

and

$$|a_{2,n}| \leq \frac{(1+a) + n(1-a)}{2} \quad \text{and} \quad |b_{2,n}| \leq \frac{(1+a) - n(1-a)}{2} \tag{9}$$

for all $n \geq 2$. Then for $F_a = t f_{a,1} + (1-t) f_{a,2} \in \mathcal{S}_{\mathcal{H}}^a$ ($t \in [0, 1]$),

i) the fully starlikeness of order α is in the disk $|z| \leq r_s$, where $r_s = r_s(a, \alpha, t)$ is the unique real root of the equation

$$\begin{aligned} &(\alpha(2at - a - 2) + 2at - a + 2)r^4 \\ &+ (-2\alpha(2at - a - 2) + \alpha(-3at + a + t + 3) - 8at + 4a - 8)r^3 \\ &+ (\alpha(2at - a - 2) - 2\alpha(-3at + a + t + 3) + 12at - 7a - \alpha - 2t + 13)r^2 \\ &+ (\alpha(-3at + a + t + 3) - 2(5at - 2a + t + 4) + 2\alpha)r + 1 - \alpha = 0 \end{aligned} \tag{10}$$

in the interval $(0, 1)$.

ii) the univalent radius is r_u , where r_u is the unique real root of the equation

$$(2at - a + 2)r^4 + (-8at + 4a - 8)r^3 + (12at - 7a - 2t + 13)r^2 - 2(5at - 2a + t + 4)r + 1 = 0 \tag{11}$$

in the interval $(0, 1)$.

Moreover, all the results are sharp.

Proof. From $F_a = t f_{a,1} + (1-t) f_{a,2}$, we easily get

$$\begin{aligned} F_a &= t f_{a,1}(z) + (1-t) f_{a,2}(z) \\ &= z + \sum_{n=2}^{\infty} [t a_{1,n} + (1-t) a_{2,n}] z^n + \overline{\sum_{n=1}^{\infty} [t b_{1,n} + (1-t) b_{2,n}] z^n}. \end{aligned}$$

For $r \in (0, 1)$, it is sufficient to show that F_r is fully starlike, where

$$\begin{aligned} F_r(z) &= r^{-1} F_a(rz) \\ &= z + \sum_{n=2}^{\infty} [t a_{1,n} + (1-t) a_{2,n}] r^{n-1} z^n + \overline{\sum_{n=1}^{\infty} [t b_{1,n} + (1-t) b_{2,n}] r^{n-1} z^n}. \end{aligned} \tag{12}$$

Consider the sum

$$S = \sum_{n=2}^{\infty} \frac{n - \alpha}{1 - \alpha} |ta_{1_n} + (1 - t)a_{2_n}|r^{n-1} + \sum_{n=2}^{\infty} \frac{n + \alpha}{1 - \alpha} |tb_{1_n} + (1 - t)b_{2_n}|r^{n-1},$$

by Lemma 2.2(a), it suffices to show that $S \leq 1$ for $|z| < r_s$. Using the coefficient inequalities given by (8), (9) and Lemma 2.1, we see that

$$\begin{aligned} S &\leq \sum_{n=2}^{\infty} \frac{n - \alpha}{1 - \alpha} \left[t \left(\frac{2(1 + a)n^2 + 3(1 - a)n + (1 + a)}{6} \right) + (1 - t) \left(\frac{(1 + a) + n(1 - a)}{2} \right) \right] r^{n-1} \\ &\quad + \sum_{n=2}^{\infty} \frac{n + \alpha}{1 - \alpha} \left[t \left(\frac{2(1 + a)n^2 + 3(a - 1)n + (1 + a)}{6} \right) + (1 - t) \left(-\frac{(1 + a) - n(1 - a)}{2} \right) \right] r^{n-1} \\ &\leq \frac{1}{3(1 - \alpha)} \left[(2at + 2t) \sum_{n=2}^{\infty} n^3 r^{n-1} + (3 - 3a - 3t + 3at) \sum_{n=2}^{\infty} n^2 r^{n-1} \right. \\ &\quad \left. + (3aat + at - 3at + t) \sum_{n=2}^{\infty} nr^{n-1} + (-3\alpha - 3a\alpha + 3at + 3aat) \sum_{n=2}^{\infty} r^{n-1} \right] \\ &= \frac{1}{3(1 - \alpha)(1 - r)^4} \left[-3(a\alpha(2t - 1) + a(2t - 1) - \alpha + 1)r^4 \right. \\ &\quad - 3(-3aat - 2a\alpha(2t - 1) + a\alpha + a(4 - 8t) + at + 3\alpha - 4)r^3 \\ &\quad - 3(6aat + a\alpha(2t - 1) - 2a\alpha + a(12t - 7) - 2at - 3\alpha - 2t + 7)r^2 \\ &\quad \left. - 3(-3aat + a\alpha - 10at + 4a + at + \alpha - 2t - 4)r \right]. \end{aligned}$$

Due to $S \leq 1$, then we get

$$\begin{aligned} &(\alpha(2at - a - 2) + 2at - a + 2)r^4 \\ &+ (-2\alpha(2at - a - 2) + \alpha(-3at + a + t + 3) - 8at + 4a - 8)r^3 \\ &+ (\alpha(2at - a - 2) - 2\alpha(-3at + a + t + 3) + 12at - 7a - \alpha - 2t + 13)r^2 \\ &+ (\alpha(-3at + a + t + 3) - 2(5at - 2a + t + 4) + 2\alpha)r + 1 - \alpha \geq 0. \end{aligned}$$

By Lemma 2.2(a), we find that F_r is fully starlike for all $0 < r \leq r_s$, where $r_s = r_s(a, \alpha, t)$ is the real root of the equation given by (10) in the interval $(0, 1)$. Furthermore, for $-1 < a < 1$ and $t \in (0, 1]$ the roots of (10) in $(0, 1)$ are decreasing with respect to the values $\alpha \in [0, 1)$, as a result $r_s(a, \alpha, t) \leq r_s(a, 0, t)$. Hence, taking $\alpha = 0$, equation (10) reduces to (11). Then by Lemma 2.2(a), we observe that F_a is harmonic univalent in $|z| \leq r_u$, where $r_u = r_s(a, 0, t)$.

Next, to prove the sharpness of the statement of the theorem, we consider the function

$$f_0(z) = tf_{a,1}(z) + (1 - t)f_{a,2}(z),$$

where

$$f_{a,1}(z) = z - \sum_{n=2}^{\infty} \frac{2(1 + a)n^2 + 3(1 - a)n + (1 + a)}{6} z^n + \overline{\sum_{n=2}^{\infty} \frac{2(1 + a)n^2 + 3(a - 1)n + (1 + a)}{6} z^n},$$

and

$$f_{a,2}(z) = z - \sum_{n=2}^{\infty} \frac{(1 + a) + n(1 - a)}{2} z^n + \overline{\sum_{n=2}^{\infty} -\frac{(1 + a) - n(1 - a)}{2} z^n}.$$

Then, we get

$$\begin{aligned} f_0(z) &= t f_{a,1}(z) + (1-t) f_{a,2}(z) \\ &= z - \sum_{n=2}^{\infty} \left(t \sum_{n=2}^{\infty} \frac{2(1+a)n^2 + 3(1-a)n + (1+a)}{6} + (1-t) \frac{(1+a) + n(1-a)}{2} \right) z^n \\ &\quad + \sum_{n=2}^{\infty} \left(t \frac{2(1+a)n^2 + 3(a-1)n + (1+a)}{6} + (1-t) \left(-\frac{(1+a) - n(1-a)}{2} \right) \right) z^n \\ &= h_0(z) + \overline{g_0(z)}. \end{aligned}$$

Routine computations give us

$$\begin{aligned} [h'_0(z) - g'_0(z)]|_{r=r_u} &= \frac{1}{(1-r)^4} \times \left((2at - a + 2)r^4 + (-8at + 4a - 8)r^3 \right. \\ &\quad \left. + (12at - 7a - 2t + 13)r^2 + (-10at + 4a - 2t - 8)r + 1 \right). \end{aligned}$$

Hence, due to (11), we observe that

$$J_{f_0} = [h'_0(z) + g'_0(z)][h'_0(z) - g'_0(z)]|_{r=r_u} = 0.$$

Therefore, in view of Lewy’s theorem, the function f_0 is not univalent in $|z| < r$ if $r > r_u$. This shows that r_u is sharp.

Regarding starlikeness, we observe that

$$\begin{aligned} &\left. \frac{\partial}{\partial \theta} \left(\arg \left(f_0(re^{i\theta}) \right) \right) \right|_{\theta=0} \\ &= \frac{r h'_0(r) - r g'_0(r)}{h_0(r) + g_0(r)} \\ &= \frac{(2at - a + 2)r^4 + (-8at + 4a - 8)r^3 + (12at - 7a - 2t + 13)r^2 + (-10at + 4a - 2t - 8)r + 1}{(1-r)^2(-a(2t-1) - 2)r^2 - (-3at + a + t + 3)r + 1}. \end{aligned}$$

Also, from equation (10) we arrive at

$$\alpha = \frac{(2at - a + 2)r^4 + (-8at + 4a - 8)r^3 + (12at - 7a - 2t + 13)r^2 - 2(5at - 2a + t + 4)r + 1}{(1-r)^2(-a(2t-1) - 2)r^2 - (-3at + a + t + 3)r + 1}.$$

It follows from the last two equations, we get

$$\left. \frac{\partial}{\partial \theta} \left(\arg \left(f_0(re^{i\theta}) \right) \right) \right|_{\theta=0, r=r_s(\alpha)} = \alpha.$$

This shows that r_s is the best possible. \square

Special cases in Theorem 2.3 gives [14, Theorem 2.1, Eq. (2.3) and (2.4)].

Remark 2.4. For $F_a = t f_{a,1} + (1-t) f_{a,2} \in \mathcal{S}_{\mathcal{H}}^a$, ($t \in [0, 1]$):

a) If $a = 0$, then the radius of fully starlikeness of order α is $|z| < r_0$, where r_0 is the real root of

$$(2 - 2\alpha)r^4 + (\alpha t + 7\alpha - 8)r^3 + (-2\alpha t - 9\alpha - 2t + 13)r^2 + (\alpha t + 5\alpha - 2t - 8)r + 1 - \alpha = 0$$

in $(0, 1)$. Noting that for fixed $t \in (0, 1]$, the root of this equation in $(0, 1)$ is unique.

b) If $a = 0$, $\alpha = 0$, then the univalent radius is r_1 , where r_1 is the unique real root of the equation

$$2r^4 - 8r^3 - 2r^2 t + 13r^2 - 2rt - 8r + 1 = 0$$

in $(0, 1)$.

Remark 2.5. For $a = 0, \alpha = 0$ and $t = 1$ in (10), we get the equation

$$2r^4 - 8r^3 + 11r^2 - 10r + 1 = 0,$$

whose real root in $(0, 1)$ is

$$r_2 = \frac{1}{2} \left(2 + \frac{1}{\sqrt{2}} - \sqrt{4\sqrt{2} + \frac{1}{2}} \right) \approx 0.112903.$$

Observe that Theorem 2.3 gives the radius obtained by Kalaj-Ponnusamy-Vuorinen [10]. The value is the radius of fully starlikeness for mappings in $\mathcal{S}_{\mathcal{H}}^{*,0}$. The root of this equation in $(0, 1)$ is unique. Also, $r_2 \approx 0.112903$ cannot be replaced by a larger number because the function cannot be univalent in $|z| < r$ if $r > r_2$.

Similarly, we get the following result.

Theorem 2.6. Under the hypothesis of Theorem 2.3, let $F_a = tf_{a,1} + (1 - t)f_{a,2} \in \mathcal{S}_{\mathcal{H}}^a$ ($t \in [0, 1]$). Then F_a is fully convex of order α in the disk $|z| < r_c$, where $r_c = r_c(a, \alpha, t)$ is the unique real root of the equation

$$\begin{aligned} &(-\alpha(2at - a - 2) - 2at + a - 2)r^5 \\ &- (-2\alpha(2at - a - 2) + \alpha(-6at + 3a + 6) - 5(2at - a + 2))r^4 \\ &- (\alpha(2at - a - 2) + \alpha(6at - 2a - 2t - 5) - 2\alpha(-6at + 3a + 6) + 20at - 9a + 2t + 19)r^3 \\ &- (-2\alpha(6at - 2a - 2t - 5) + \alpha(-6at + 3a + 6) - 16at + 13a + \alpha + 10t - 23)r^2 \\ &- (\alpha(6at - 2a - 2t - 5) + 20at - 8a - 2\alpha + 4t + 13)r + 1 - \alpha = 0 \end{aligned} \tag{13}$$

in the interval $(0, 1)$.

Proof. Let $r \in (0, 1)$, it is sufficient to show that F_r is fully convex, where F_r is defined by (12). Consider the sum

$$K = \sum_{n=2}^{\infty} \frac{n(n - \alpha)}{1 - \alpha} |ta_{1_n} + (1 - t)a_{2_n}|r^{n-1} + \sum_{n=2}^{\infty} \frac{n(n + \alpha)}{1 - \alpha} |tb_{1_n} + (1 - t)b_{2_n}|r^{n-1}.$$

According to Lemma 2.2(b), it is enough to show that $K \leq 1$. Considering the assumptions (8) and (9), we get

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{n(n - \alpha)}{1 - \alpha} \left[t \left(\frac{2(1 + a)n^2 + 3(1 - a)n + (1 + a)}{6} \right) + (1 - t) \left(\frac{(1 + a) + n(1 - a)}{2} \right) \right] r^{n-1} \\ &+ \sum_{n=2}^{\infty} \frac{n(n + \alpha)}{1 - \alpha} \left[t \left(\frac{2(1 + a)n^2 + 3(a - 1)n + (1 + a)}{6} \right) + (1 - t) \left(-\frac{(1 + a) - n(1 - a)}{2} \right) \right] r^{n-1} \\ &\leq 1, \end{aligned}$$

and using Lemma 2.1, the last inequality reduces to

$$\begin{aligned} &(-\alpha(2at - a - 2) - 2at + a - 2)r^5 \\ &- (-2\alpha(2at - a - 2) + \alpha(-6at + 3a + 6) - 5(2at - a + 2))r^4 \\ &- (\alpha(2at - a - 2) + \alpha(6at - 2a - 2t - 5) - 2\alpha(-6at + 3a + 6) + 20at - 9a + 2t + 19)r^3 \\ &- (-2\alpha(6at - 2a - 2t - 5) + \alpha(-6at + 3a + 6) - 16at + 13a + \alpha + 10t - 23)r^2 \\ &- (\alpha(6at - 2a - 2t - 5) + 20at - 8a - 2\alpha + 4t + 13)r + 1 - \alpha \geq 0. \end{aligned}$$

By Lemma 2.2(b), we find that F_r is fully convex for all $0 < r \leq r_c$, where $r_c = r_c(a, \alpha, t)$ is the real root of the equation given by (13) in the interval $(0, 1)$.

To prove the sharpness, we consider the function

$$f_0(z) = t f_{a,1}(z) + (1 - t) f_{a,2}(z),$$

where

$$f_{a,1}(z) = z - \sum_{n=2}^{\infty} \frac{2(1+a)n^2 + 3(1-a)n + (1+a)}{6} z^n - \overline{\sum_{n=2}^{\infty} \frac{2(1+a)n^2 + 3(a-1)n + (1+a)}{6} z^n},$$

and

$$f_{a,2}(z) = z - \sum_{n=2}^{\infty} \frac{(1+a) + n(1-a)}{2} z^n - \overline{\sum_{n=2}^{\infty} -\frac{(1+a) - n(1-a)}{2} z^n}.$$

Then, we get

$$\begin{aligned} f_0(z) &= t f_{a,1}(z) + (1 - t) f_{a,2}(z) \\ &= z - \sum_{n=2}^{\infty} \left(t \sum_{n=2}^{\infty} \frac{2(1+a)n^2 + 3(1-a)n + (1+a)}{6} + (1-t) \frac{(1+a) + n(1-a)}{2} \right) z^n \\ &\quad - \overline{\sum_{n=2}^{\infty} \left(t \frac{2(1+a)n^2 + 3(a-1)n + (1+a)}{6} + (1-t) \left(-\frac{(1+a) - n(1-a)}{2} \right) \right) z^n} \\ &= h_0(z) + \overline{g_0(z)}. \end{aligned}$$

By calculations, we arrive at

$$\begin{aligned} \left. \frac{\partial}{\partial \theta} \left(\arg \left(\frac{\partial}{\partial \theta} f_0(re^{i\theta}) \right) \right) \right|_{\theta=0} &= \frac{h'_0(r) + g'_0(r) + r(h''_0(r) + g''_0(r))}{h'_0(r) - g'_0(r)} \\ &= \frac{(2at - a + 2)r^5 + (-10at + 5a - 10)r^4 + (20at - 9a + 2t + 19)r^3}{(1-r)^2(-a(2t-1) - 2)r^3 - (a(3-6t) + 6)r^2 - (a(6t-2) - 2t - 5)r - 1} \\ &\quad + \frac{(-16at + 13a + 10t - 23)r^2 + (20at - 8a + 4t + 13)r - 1}{(1-r)^2(-a(2t-1) - 2)r^3 - (a(3-6t) + 6)r^2 - (a(6t-2) - 2t - 5)r - 1}. \end{aligned}$$

Moreover, using (13) we obtain

$$\alpha = \frac{(2at - a + 2)r^5 + (-10at + 5a - 10)r^4 + (20at - 9a + 2t + 19)r^3}{(1-r)^2(-a(2t-1) - 2)r^3 - (a(3-6t) + 6)r^2 - (a(6t-2) - 2t - 5)r - 1} + \frac{(-16at + 13a + 10t - 23)r^2 + (20at - 8a + 4t + 13)r - 1}{(1-r)^2(-a(2t-1) - 2)r^3 - (a(3-6t) + 6)r^2 - (a(6t-2) - 2t - 5)r - 1}.$$

Thus, we observe that

$$\left. \frac{\partial}{\partial \theta} \left(\arg \left(\frac{\partial}{\partial \theta} f_0(re^{i\theta}) \right) \right) \right|_{\theta=0, r=r_c(\alpha)} = \alpha.$$

This shows that the bound given by (13) is sharp.

□

For special values in Theorem 2.6, we get [14, Theorem 2.2, Eq. (2.8)].

Remark 2.7. For $F_a = t f_{a,1} + (1 - t) f_{a,2} \in \mathcal{S}_{\mathcal{H}}^a$, ($t \in [0, 1]$):

a) If $a = 0$, then the radius of fully convexity of order α is $|z| < r_3$, where r_3 is the real root of

$$(2\alpha - 2)r^5 + (10 - 10\alpha)r^4 + (2\alpha t + 19\alpha - 2t - 19)r^3 + (-4\alpha t - 17\alpha - 10t + 23)r^2 + (2\alpha t + 7\alpha - 4t - 13)r + 1 - \alpha = 0$$

in $(0, 1)$. Noting that for fixed $t \in (0, 1]$, the root of this equation in $(0, 1)$ is unique.

b) If $a = 0, \alpha = 0$, then the univalent radius is r_4 , where r_4 is the unique real root of the equation,

$$2r^5 - 10r^4 + 2r^3t + 19r^3 + 10r^2t - 23r^2 + 4rt + 13r - 1 = 0$$

in $(0, 1)$.

Remark 2.8. For $a = 0, \alpha = 0$ and $t = 1$ in (13), we get the equation

$$2r^5 - 10r^4 + 21r^3 - 13r^2 + 17r - 1 = 0,$$

whose real root in $(0, 1)$ is $r_5 \approx 0.0614$. The value is the radius of fully convexity for mappings in $\mathcal{S}_{\mathcal{H}}^{*,0}$ studied in [21, Theorem 3.3].

3. Radii of convolutions of generalized harmonic mappings

The convolution of two generalized harmonic functions

$$f_{a,1}(z) = z + \sum_{n=2}^{\infty} a_{1n} z^n + \sum_{n=1}^{\infty} \overline{b_{1n} z^n} \quad \text{and} \quad f_{a,2}(z) = z + \sum_{n=2}^{\infty} a_{2n} z^n + \sum_{n=1}^{\infty} \overline{b_{2n} z^n}$$

is defined by

$$G_a(z) = (f_{a,1} * f_{a,2})(z) = z + \sum_{n=2}^{\infty} \underbrace{a_{1n} a_{2n}}_{a_n} z^n + \sum_{n=1}^{\infty} \underbrace{\overline{b_{1n} b_{2n}}}_{b_n} z^n.$$

In this part, we get radii of convolutions of two generalized harmonic Koebe mappings.

Theorem 3.1. Let $f_{a,j} = h_{a,j} + \bar{g}_{a,j} \in \mathcal{S}_{\mathcal{H}}^a$ be given by (1) and let the coefficients of the series satisfy the conditions

$$|a_n| \leq \frac{1}{36} (2(1+a)n^2 + 3(1-a)n + (1+a))^2, \quad n \geq 2 \tag{14}$$

and

$$|b_n| \leq \frac{1}{36} (2(1+a)n^2 + 3(a-1)n + (1+a))^2, \quad n \geq 1 \tag{15}$$

Then for $G_a = f_{a,1} * f_{a,2} \in \mathcal{S}_{\mathcal{H}}^a$,

i) the fully starlikeness of order α is in the disk $|z| \leq r_s$, where $r_s = r_s(a, \alpha)$ is the unique real root of the equation

$$\begin{aligned} & (3(a^2 - 2)\alpha + 3(a^2 + 2))r^6 \\ & + (-18(a^2 - 2)\alpha - 18(a^2 + 2))r^5 \\ & + (27(a^2 - 2)\alpha + 3(5a^2 - 11)\alpha + 42a^2 + 87)r^4 \\ & + (-12(a^2 - 2)\alpha - 6(5a^2 - 11)\alpha + 3(10 - 6a^2)\alpha - 3(27a^2 + 10a + 47))r^3 \\ & + (3(5a^2 - 11)\alpha - 6(10 - 6a^2)\alpha + 13a^2 - 100a - 3\alpha + 58)r^2 \\ & + (3(10 - 6a^2)\alpha - 3(13a^2 + 10a + 19) + 6\alpha)r + 3(1 - \alpha) = 0 \end{aligned} \tag{16}$$

in the interval $(0, 1)$.

ii) the univalent radius is r_u , where r_u is the unique real root of the equation

$$3(a^2 + 2)r^6 - 18(a^2 + 2)r^5 + (42a^2 + 87)r^4 - 3(27a^2 + 10a + 47)r^3 + (13a^2 - 100a + 58)r^2 - 3(13a^2 + 10a + 19)r + 3 = 0 \tag{17}$$

in the interval $(0, 1)$.

Moreover, all the results are sharp.

Proof. For $r \in (0, 1)$, it is sufficient to show that G_r is fully starlike, where

$$G_r(z) = r^{-1}G_a(rz) = z + \sum_{n=2}^{\infty} a_n r^{n-1} z^n + \overline{\sum_{n=1}^{\infty} b_n r^{n-1} z^n}. \tag{18}$$

Consider the sum

$$R = \sum_{n=2}^{\infty} \frac{n - \alpha}{1 - \alpha} |a_n| r^{n-1} + \sum_{n=1}^{\infty} \frac{n + \alpha}{1 - \alpha} |b_n| r^{n-1}.$$

In view of Lemma 2.2(a) it is enough to show that $R \leq 1$. Considering the assumptions (14) and (15), we get

$$\sum_{n=2}^{\infty} \frac{n - \alpha}{36(1 - \alpha)} (2(1 + a)n^2 + 3(1 - a)n + (1 + a))^2 r^{n-1} + \sum_{n=1}^{\infty} \frac{n + \alpha}{36(1 - \alpha)} (2(1 + a)n^2 + 3(a - 1)n + (1 + a))^2 r^{n-1} \leq 1,$$

and using Lemma 2.1, we obtain

$$\begin{aligned} & (3(a^2 - 2)\alpha + 3(a^2 + 2))r^6 \\ & + (-18(a^2 - 2)\alpha - 18(a^2 + 2))r^5 \\ & + (27(a^2 - 2)\alpha + 3(5a^2 - 11)\alpha + 42a^2 + 87)r^4 \\ & + (-12(a^2 - 2)\alpha - 6(5a^2 - 11)\alpha + 3(10 - 6a^2)\alpha - 3(27a^2 + 10a + 47))r^3 \\ & + (3(5a^2 - 11)\alpha - 6(10 - 6a^2)\alpha + 13a^2 - 100a - 3\alpha + 58)r^2 \\ & + (3(10 - 6a^2)\alpha - 3(13a^2 + 10a + 19) + 6\alpha)r + 3(1 - \alpha) \geq 0. \end{aligned}$$

By Lemma 2.2(a), we find that G_r is fully starlike for all $0 < r \leq r_s$, where $r_s = r_s(a, \alpha)$ is the real root of the equation given by (16) in the interval $(0, 1)$. Furthermore, taking $\alpha = 0$ equation (16) reduces to (17). Then by Lemma 2.2(a), we know that G_a is harmonic univalent in $|z| \leq r_u$, where $r_u = r_s(a, 0)$.

To prove the sharpness, we consider the function

$$h_0(z) = 2z - \sum_{n=1}^{\infty} \frac{1}{36} (2(1 + a)n^2 + 3(1 - a)n + (1 + a))^2 z^n,$$

and

$$g_0(z) = \sum_{n=2}^{\infty} \frac{1}{36} (2(1 + a)n^2 + 3(1 - a)n + (1 + a))^2 z^n.$$

Then, we get

$$[h'_0(z) - g'_0(z)]|_{r=r_u} = \frac{1}{3(1-r)^6} \times \left(3(a^2 + 2)r^6 - 18(a^2 + 2)r^5 + (42a^2 + 87)r^4 - 3(27a^2 + 10a + 47)r^3 + (13a^2 - 100a + 58)r^2 - 3(13a^2 + 10a + 19)r + 3 \right).$$

Hence, due to (17), we observe that

$$J_{f_0} = [h'_0(z) + g'_0(z)][h'_0(z) - g'_0(z)]|_{r=r_u} = 0.$$

Therefore, in view of Lewy’s theorem, the function f_0 is not univalent in $|z| < r$ if $r > r_u$. This shows that r_u is sharp.

To get starlikeness, we observe that

$$\frac{\partial}{\partial \theta} \left(\arg(f_0(re^{i\theta})) \right) \Big|_{\theta=0} = \frac{rh'_0(r) - rg'_0(r)}{h_0(r) + g_0(r)}$$

$$\begin{aligned} & 3(a^2 + 2)r^6 - 18(a^2 + 2)r^5 + (42a^2 + 87)r^4 - 3(27a^2 + 10a + 47)r^3 \\ & \quad + (13a^2 - 100a + 58)r^2 - 3(13a^2 + 10a + 19)r + 3 \\ = & \frac{\hspace{10em}}{-3(r-1)^2((a^2-2)r^4 - 4(a^2-2)r^3 + (5a^2-11)r^2 + (10-6a^2)r - 1)}. \end{aligned}$$

From equation (16), we also get

$$\alpha = \frac{3(a^2 + 2)r^6 - 18(a^2 + 2)r^5 + (42a^2 + 87)r^4 - 3(27a^2 + 10a + 47)r^3 + (13a^2 - 100a + 58)r^2 - 3(13a^2 + 10a + 19)r + 3}{-3(r-1)^2((a^2-2)r^4 - 4(a^2-2)r^3 + (5a^2-11)r^2 + (10-6a^2)r - 1)}.$$

It follows that

$$\frac{\partial}{\partial \theta} \left(\arg(f_0(re^{i\theta})) \right) \Big|_{\theta=0, r=r_s(\alpha)} = \alpha.$$

This shows that r_s is the best possible. \square

For special values in Theorem 3.1, we get [14, Theorem 3.1, Eq. (3.2) and (3.3)].

Remark 3.2. For $G_a = f_{a,1} * f_{a,2} \in \mathcal{S}_{\mathcal{H}}^a$

a) If $a = 0$, then the radius of fully starlikeness of order α is $|z| < r_6$, where r_6 is the real root of

$$(6 - 6\alpha)r^6 + (36\alpha - 36)r^5 + (87 - 87\alpha)r^4 + (120\alpha - 141)r^3 + (58 - 96\alpha)r^2 + (36\alpha - 57)r - 3\alpha + 3 = 0$$

in $(0, 1)$.

b) If $a = 0, \alpha = 0$, the univalent radius is $r_7 \approx 0.0553$, where r_7 is the unique real root of the equation

$$6r^6 - 36r^5 + 87r^4 - 141r^3 + 58r^2 - 57r + 3 = 0$$

in $(0, 1)$.

Theorem 3.3. Under the hypothesis of Theorem 3.1, let $G_a = f_{a,1} * f_{a,2} \in \mathcal{S}_{\mathcal{H}}^a$. Then G_a is fully convex of order α in the disk $|z| < r_c$, where $r_c = r_c(a, \alpha)$ is the unique real root of the equation

$$\begin{aligned} & \left((2 - a^2)\alpha - a^2 - 2 \right) r^7 \\ & + \left(7(a^2 - 2)\alpha + 7(a^2 + 2) \right) r^6 \\ & + \left(-11(a^2 - 2)\alpha + (21 - 11a^2)\alpha - 22a^2 - 43 \right) r^5 \\ & + \left(5(a^2 - 2)\alpha + (3a^2 - 13)\alpha + 2(11a^2 - 21)\alpha + 16a^2 - 20a + 51 \right) r^4 \\ & + \left(2(13 - 3a^2)\alpha + (21 - 11a^2)\alpha + (17 - 12a^2)\alpha - 5(19a^2 + 28a + 26) \right) r^3 \\ & + \left((3a^2 - 13)\alpha + 2(12a^2 - 17)\alpha - 39a^2 - 140a - \alpha - 18 \right) r^2 \\ & + \left((17 - 12a^2)\alpha - 26a^2 - 20a + 2\alpha - 33 \right) r + 1 - \alpha = 0 \end{aligned} \tag{19}$$

in the interval $(0, 1)$. The result is sharp.

Proof. The proof is similar to the proof of Theorem 3.1, hence it is omitted. For sharpness we take the functions

$$h_0(z) = 2z - \sum_{n=1}^{\infty} \frac{1}{36} (2(1+a)n^2 + 3(1-a)n + (1+a))^2 z^n,$$

and

$$g_0(z) = - \sum_{n=2}^{\infty} \frac{1}{36} (2(1+a)n^2 + 3(1-a)n + (1+a))^2 z^n.$$

□

For special values in Theorem 3.3, we get [14, Theorem 3.1, Eq. (3.7)].

Remark 3.4. For $G_a = f_{a,1} * f_{a,2} \in \mathcal{S}_{\mathcal{H}}^a$,

a) If $a = 0$, the radius of fully convexity of order α is $|z| < r_8$, where r_8 is the real root of

$$\begin{aligned} & (2\alpha - 2)r^7 + (14 - 14\alpha)r^6 + (43\alpha - 43)r^5 + (51 - 65\alpha)r^4 \\ & + (64\alpha - 130)r^3 + (-48\alpha - 18)r^2 + (19\alpha - 33)r + 1 - \alpha = 0 \end{aligned}$$

in $(0, 1)$.

b) If $a = 0, \alpha = 0$, the univalent radius is $r_9 \approx 0.0297$, where r_9 is the unique real root of the equation

$$-2r^7 + 14r^6 - 43r^5 + 51r^4 - 130r^3 - 18r^2 - 33r + 1 = 0$$

in $(0, 1)$.

4. Conclusions

In this paper, we introduced the generalized harmonic Koebe function and get its coefficient estimates. By using the concept of convex combination, we obtained radii of fully starlikeness and radii of fully convexity for convex combinations of generalized harmonic Koebe and generalized harmonic right half-plane mappings. Moreover, we get radii of fully starlikeness and convexity for convolutions of harmonic mappings by taking two generalized harmonic Koebe mappings.

To introduce the generalized harmonic Koebe function, and to get radius results by using several methods are focal points of this paper. The significance of the paper is due to the novelty of radius results in the context of harmonic mappings.

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