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New integral inequalities for functions with concave powers of second derivatives

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Abstract. In this study, we derived several new integral inequalities for functions whose second derivatives in absolute value, raised to the power $q > 1$, are concave. Furthermore, new identities involving fractional operators for twice differentiable functions are obtained.

1. Introduction and preliminaries

The aim of this paper is to establish new inequalities related to concave functions. To achieve this, we will make use of the Hermite-Hadamard inequality for a continuous concave function $f : [a, b] \to \mathbb{R}$:

$$f\left(\frac{a+b}{2}\right) \geq \frac{1}{b-a}\int_a^b f(x)dx \geq \frac{f(a)+f(b)}{2}. \tag{1}$$

A function $f : [a, b] \to \mathbb{R}$ is defined as concave if, for all $x, y \in [a, b]$ and $t \in [0, 1]$, the inequality $f(tx+(1-t)y) \geq tf(x)+(1-t)f(y)$ holds. Geometrically, this means that the graph of f on the interval $[a, b]$ lies entirely above or on the chord that connects the endpoints $(a, f(a))$ and $(b, f(b))$.

We proceed by listing the necessary results.

Proposition 1.1 ([10, 15]). *If $f : I \to \mathbb{R}$ is a concave function and $g : \mathbb{R} \to \mathbb{R}$ is a non-decreasing concave function, then its composition $g \circ f$ is concave.*

Theorem 1.2 (The Favard inequality, [6]). *Let f be a non-negative concave function on $[a, b] \subset \mathbb{R}$. If $q > 1$, then*

$$\frac{2^q}{q+1}\left(\frac{1}{b-a}\int_a^b f(x)dx\right)^q \geq \frac{1}{b-a}\int_a^b f^q(x)dx.$$

If $0 < q < 1$, then the inequality reverses.

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Theorem 1.3 (The Young inequality, [9, 10]). *If $x, y \geq 0$ and $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

The equality holds if and only if $x^p = y^q$.

Theorem 1.4 (The Rogers-Hölder inequality, [9]). *Let $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$. Let $f, g : [a, b] \to \mathbb{R}$ be such that $|f|^p$ and $|g^q|$ are integrable functions on $[a, b]$. Then*

$$\int_a^b \left|f(x)g(x)\right| dx \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}}.$$

The equality holds if and only if $A\left|f(x)\right|^p = B\left|g(x)\right|^q$ almost everywhere, where A and B are constants.

Favard's inequality complements a well known consequence of the Rogers-Hölders inequality:

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \left(\frac{1}{b-a}\int_a^b f(x)^p dx\right)^{\frac{1}{p}}.$$

We follow with a simple result of the Rogers-Hölder inequality.

Theorem 1.5 (Power mean integral inequality, [9]). *Let $f, g : [a, b] \to \mathbb{R}$ be such that $|f|$ and $|f||g|^q$ are integrable functions on $[a, b]$, with $q > 1$. Then*

$$\int_a^b \left|f(x)g(x)\right| dx \leq \left(\int_a^b |f(x)| dx\right)^{1-\frac{1}{q}} \left(\int_a^b |f(x)||g(x)|^q dx\right)^{\frac{1}{q}}.$$

Theorem 1.6 (The Chebyshev inequality, [8]). *Let $f, g : [a, b] \to \mathbb{R}$ be two integrable functions, monotonic in the same sense. Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right).$$

If f and g are monotonic in the opposite sense, then the inequality reverses.

Theorem 1.7 (The Jensen inequality, [9]). *Let $f : [a, b] \to \mathbb{R}$ be an integrable function and let $p : [a, b] \to \mathbb{R}$ be a non-negative function. If Φ is a concave function given on an interval I such that $f([a, b]) \subseteq I$, then*

$$\Phi\left(\frac{1}{\int_a^b p(x)dx}\int_a^b p(x)f(x)\, dx\right) \geq \frac{1}{\int_a^b p(x)dx}\int_a^b p(x)\Phi(f(x))\, dx.$$

Lemma 1.8 ([2]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I°, $a, b \in I$ with $a < b$. If f'' is integrable on $[a, b]$, then*

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{(b-a)^2}{2}\int_0^1 t(1-t)f''(ta + (1-t)b)dt.$$

Recall, the beta function is the function of two complex variables defined by Euler's integral of the first kind

$$B(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt, \qquad \Re(z), \Re(w) > 0.$$

It is related to the gamma function with

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \qquad z,w \notin \mathbb{Z}_0^- = \{0,-1,-2,\ldots\}.$$

Furthermore, by $L^p[a,b]$, $1 \leq p < \infty$, we denote the space of all Lebesgue measurable functions f for which $\int_a^b |f(t)|^p dt < \infty$, where

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt \right)^{\frac{1}{p}}.$$

The paper is structured as follows: after these preliminary results and definitions, in Section 2 we estimate the quantity

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right|,$$

which represents the deviation between the arithmetic mean of the function values at the endpoints and the mean value of the function over the interval. This expression characterizes the error involved in approximating the integral of a function using the trapezoidal rule. To achieve this, we employ a range of classical inequalities and establish several new bounds of Hermite-Hadamard type for functions whose second derivatives, in absolute value raised to a power $q > 1$, are concave on the interval of integration.

Section 3 is dedicated to deriving equalities involving fractional operators. In recent years, fractional calculus, dealing with derivatives and integrals of non-integer order, has gained attention due to its ability to model memory and hereditary effects in materials and processes more accurately than classical calculus.

The research presented herein is continuation and refinement of the results given in [1]-[5], [11]-[14], [16].

2. The results

Theorem 2.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}^+$ be a twice differentiable function on I°, and let $a, b \in I$ with $0 < a < b$, such that $f'' \in L^1[a,b]$. If $\left|f''\right|$ is concave on $[a,b]$, then*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{12} \left| f''\left(\frac{a+b}{2} \right) \right|. \tag{2}$$

Proof. Applying Lemma 1.8, the triangle inequality for integrals, and Jensen's inequality from Theorem 1.7, we obtain

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ &\leq \frac{(b-a)^2}{2} \int_0^1 t(1-t)|f''(ta+(1-t)b)|dt \\ &\leq \frac{(b-a)^2}{2} \left(\int_0^1 t(1-t)dt \right) \left| f''\left(\frac{\int_0^1 t(1-t)(ta+(1-t)b)dt}{\int_0^1 t(1-t)dt} \right) \right|. \end{aligned}$$

The result now follows from the evaluation of the above integrals. □

Remark 2.2. *Theorems 6 and 7 in [1] provide two estimates for the error $\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right|$, namely:*

$$\frac{(b-a)^2}{4} \left| f''\left(\frac{a+2b}{3} \right) \right|$$

and

$$\frac{(b-a)^2}{12}\left[2\left|f''\left(\frac{3a+5b}{8}\right)\right|+\left|f''\left(\frac{3a+b}{4}\right)\right|\right].$$

A direct comparison shows that inequality (2) *provides a better approximation.*

Theorem 2.3. *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, and let $f: I\subseteq\mathbb{R}\to\mathbb{R}^+$ be a twice differentiable function on I°. Suppose $a,b\in I$ with $0<a<b$, such that $f''\in L^q[a,b]$. Then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^{2-\frac{1}{q}}}{2}\frac{\Gamma^{\frac{2}{p}}(p+1)}{\Gamma^{\frac{1}{p}}(2p+2)}\left(\int_a^b \left|f''(x)\right|^q dx\right)^{\frac{1}{q}}. \tag{3}$$

Proof. Since $q>1$, we have $L^q[a,b]\subset L^1[a,b]$; hence, $f''\in L^q[a,b]$ implies $f''\in L^1[a,b]$. Consequently, using Lemma 1.8 together with the triangle inequality for integrals and the Rogers-Hölder inequality from Theorem 1.4, we establish the resulting inequality. □

Corollary 2.4. *Under the assumptions of Theorem 2.3, suppose $\left|f''\right|^q$ is concave on $[a,b]$. Then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{(1+q)^{\frac{1}{q}}}\frac{\Gamma^{\frac{2}{p}}(p+1)}{\Gamma^{\frac{1}{p}}(2p+2)}\int_a^b \left|f''(x)\right| dx. \tag{4}$$

Proof. We consider two functions, $g(x)=|f''(x)|^q$ which is assumed to be concave, and $h(x)=x^{\frac{1}{q}}$, which is concave and increasing on $\mathbb{R}^+$ for $q>1$. Then, by Proposition 1.1, the composition $h\circ g=|f''|$ is also concave. Further, by applying Favard's inequality from Theorem 1.2 on a concave function $\left|f''\right|$, we obtain

$$\left(\frac{1}{b-a}\int_a^b \left|f''(x)\right|^q dx\right)^{\frac{1}{q}} \le \frac{2}{(1+q)^{\frac{1}{q}}}\left(\frac{1}{(b-a)}\int_a^b \left|f''(x)\right| dx\right). \tag{5}$$

The result follows from (3) and (5). □

Theorem 2.5. *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, and let $f: I\subseteq\mathbb{R}\to\mathbb{R}^+$ be a twice differentiable function on I°. Suppose $a,b\in I$ with $0<a<b$, such that $f''\in L^q[a,b]$. If $\left|f''\right|^q$ is concave on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2\,\Gamma^2(p+1)}{2p\,\Gamma(2p+2)}+\frac{2^{q-1}}{q(q+1)(b-a)^{q-2}}\left(\int_a^b \left|f''(x)\right| dx\right)^q. \tag{6}$$

Proof. Given that $t(1-t)>0$ and $\left|f''(ta+(1-t)b)\right|>0$, Young's inequality from Theorem 1.3 can be applied to their product, that is

$$t(1-t)\ \left|f''(ta+(1-t)b)\right| \le \frac{1}{p}t^p(t-1)^p+\frac{1}{q}\left|f''(ta+(1-t)b)\right|^q.$$

Integration over $[0,1]$ and application of Favard's inequality completes the proof. □

Theorem 2.6. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}^+$ be a twice differentiable function on I° and let $q>1$. Suppose $a,b\in I$ with $0<a<b$, such that $f''\in L^q[a,b]$. If $\left|f''\right|^q$ is increasing on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{6(q+1)^{\frac{1}{q}}}\int_a^b \left|f''(x)\right| dx. \tag{7}$$

Proof. First we apply Theorem 1.5 on Lemma 1.8 to obtain

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\leq \frac{(b-a)^2}{2}\frac{1}{6^{1-\frac{1}{q}}}\left(\frac{1}{(b-a)^3}\int_a^b (b-x)(x-a)\left|f''(x)\right|^q dx\right)^{\frac{1}{q}}. \end{aligned} \tag{8}$$

A function $|f''|^q$ is assumed to be increasing, and, if we observe a function $g(x) = (b-x)(x-a)$, a straightforward computation shows that $g''(x) = -2 < 0$ implying that g is concave downward on $[a,b]$, hence g is a decreasing function. Therefore, the conditions are met to apply Chebyshev's integral inequality (Theorem 1.6):

$$\frac{1}{b-a}\int_a^b (x-a)(b-x)\left|f''(x)\right|^q dx \leq \frac{(b-a)^2}{6}\left(\frac{1}{b-a}\int_a^b \left|f''(x)\right|^q dx\right). \tag{9}$$

Now from (8) and (9) follows

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{12}\left(\frac{1}{b-a}\int_a^b \left|f''(x)\right|^q dx\right)^{\frac{1}{q}}.$$

If we apply Favard's inequality (5) to the inequality above, we obtain (7). □

Remark 2.7. *As $q \longrightarrow 1^+$, the value of $\frac{1}{6(q+1)^{\frac{1}{q}}}$ approaches $\frac{1}{12}$, leading to the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{12}\int_a^b \left|f''(x)\right| dx.$$

Theorem 2.8. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}^+$ be a twice differentiable function on I° and let $q > 1$. Suppose $a, b \in I$ with $0 < a < b$, such that $f \in L^q[a,b]$. If f^q is concave on $[a,b]$, then*

$$\left(\int_a^{\frac{a+b}{2}} f^q(x)dx\right)^{\frac{1}{q}} + \left(\int_{\frac{a+b}{2}}^b f^q(x)dx\right)^{\frac{1}{q}} \leq \frac{2^{2-\frac{1}{q}}}{(b-a)^{1-\frac{1}{q}}(q+1)^{\frac{1}{q}}}\int_a^b f(x)dx. \tag{10}$$

Proof. Note that

$$\frac{a+b}{2}-a = b-\frac{a+b}{2} = \frac{b-a}{2}.$$

By applying Favard's inequality from Theorem 1.2 to the concave function f^q, we derive

$$\begin{aligned} \left(\int_a^{\frac{a+b}{2}} f^q(x)dx\right)^{\frac{1}{q}} &= \left(\frac{b-a}{2}\right)^{\frac{1}{q}}\left(\frac{1}{\frac{a+b}{2}-a}\int_a^{\frac{a+b}{2}} f^q(x)dx\right)^{\frac{1}{q}} \\ &\leq \frac{2^{2-\frac{1}{q}}}{(b-a)^{1-\frac{1}{q}}(q+1)^{\frac{1}{q}}}\int_a^{\frac{a+b}{2}} f(x)dx, \\ \left(\int_{\frac{a+b}{2}}^b f^q(x)dx\right)^{\frac{1}{q}} &= \left(\frac{b-a}{2}\right)^{\frac{1}{q}}\left(\frac{1}{b-\frac{a+b}{2}}\int_{\frac{a+b}{2}}^b f^q(x)dx\right)^{\frac{1}{q}} \\ &\leq \frac{2^{2-\frac{1}{q}}}{(b-a)^{1-\frac{1}{q}}(q+1)^{\frac{1}{q}}}\int_{\frac{a+b}{2}}^b f(x)dx. \end{aligned}$$

The result now follows by adding above inequalities. □

3. Fractional identities for twice differentiable functions

Fractional calculus is a generalization of classical calculus that extends the concepts of differentiation and integration to non-integer (fractional) orders.

A fundamental operator in this field is the Riemann–Liouville integral. The **left-sided** Riemann-Liouville fractional integral of order $\alpha > 0$, denoted by J_{a+}^{α}, and the **right-sided** counterpart J_{b-}^{α}, are defined as in [7] for functions $f \in L^1[a,b]$ by

$$\begin{aligned} J_{a^+}^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x \in (a,b], \\ J_{b_-}^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x \in [a,b). \end{aligned}$$

The fractional version of Hermite-Hadamard's inequality, involving Riemann-Liouville fractional integrals, along with an associated identity related to its right-hand side, are given in [16].

Theorem 3.1 ([16]). *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L^1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \le \frac{f(a)+f(b)}{2}.$$

Lemma 3.2 ([16]). *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L^1[a,b]$, then the following equality for fractional integral holds*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b_-}^{\alpha} f(a)\right] = \frac{b-a}{2} \int_0^1 \left((1-t)^{\alpha} - t^{\alpha}\right) f'(ta+(1-t)b)dt. \tag{11}$$

To derive new fractional identities for twice differentiable functions, we first examine several properties of the Riemann-Liouville integral. We begin by observing that integration by parts implies

$$J_{a^+}^{\alpha+1} f'(x) = -\frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)} f(a) + J_{a^+}^{\alpha} f(x), \tag{12}$$

$$J_{b_-}^{\alpha+1} f'(x) = \frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)} f(b) - J_{b_-}^{\alpha} f(x). \tag{13}$$

Next, with appropriate substitutions we have

$$\begin{aligned} J_{a^+}^{\alpha} f(b) &= \frac{(b-a)^{\alpha}}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt \\ &= \frac{(b-a)^{\alpha}}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f((1-t)a+tb)dt, \end{aligned} \tag{14}$$

$$\begin{aligned} J_{b_-}^{\alpha} f(a) &= \frac{(b-a)^{\alpha}}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(ta+(1-t)b)dt \\ &= \frac{(b-a)^{\alpha}}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} f((1-t)a+tb)dt. \end{aligned} \tag{15}$$

Remark 3.3. *Using* (14) *and* (15) *we have*

$$\int_0^1 \left((1-t)^{\alpha} - t^{\alpha}\right) f'(ta+(1-t)b)dt = \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} \left[J_{b_-}^{\alpha+1} f'(a) - J_{a^+}^{\alpha+1} f'(b)\right],$$

from which we obtain an alternative form of equation (11)

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b_-}^{\alpha} f(a)\right] = \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{b_-}^{\alpha+1} f'(a) - J_{a^+}^{\alpha+1} f'(b)\right]. \tag{16}$$

Let us prove two similar result for twice differentiable functions.

Lemma 3.4. *Let $\alpha > 0$ and let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I°, $a, b \in I$ with $a < b$. If $f'' \in L^1[a,b]$, then*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b_-}^\alpha f(a)\right] \\ &= \frac{b-a}{2(\alpha+1)}\left[f'(b)-f'(a)\right] - \frac{(b-a)^2}{2(\alpha+1)}\int_0^1 \left((1-t)^{\alpha+1} + t^{\alpha+1}\right) f''(ta+(1-t)b)dt. \end{aligned} \tag{17}$$

Proof. Let

$$I = \int_0^1 \left((1-t)^{\alpha+1} + t^{\alpha+1}\right) f''(ta+(1-t)b)dt.$$

The proof is completed once integration by parts, with $u = (1-t)^{\alpha+1} + t^{\alpha+1}$ and $dv = f''(ta+(1-t)b)dt$, is applied to obtain

$$I = \frac{f'(b)-f'(a)}{b-a} - \frac{\alpha+1}{b-a}\left[\int_0^1 \left((1-t)^\alpha - t^\alpha\right) f'(ta+(1-t)b)dt\right].$$

□

Remark 3.5. *Again we use* (14) *and* (15)*, to obtain*

$$\int_0^1 \left((1-t)^{\alpha+1} + t^{\alpha+1}\right) f''(ta+(1-t)b)dt = \frac{\Gamma(\alpha+2)}{(b-a)^{\alpha+2}}\left[J_{b_-}^{\alpha+2} f''(a) + J_{a^+}^{\alpha+2} f''(b)\right].$$

An alternative representation of equation (17) *is*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b_-}^\alpha f(a)\right] \\ &= \frac{b-a}{2(\alpha+1)}\left[f'(b)-f'(a)\right] - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^{\alpha+2} f''(b) + J_{b_-}^{\alpha+2} f''(a)\right]. \end{aligned} \tag{18}$$

Furthermore, from (16) *and* (18) *we can obtain*

$$\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)}\left[f'(b)-f'(a)\right] = J_{a^+}^{\alpha+2} f''(b) + J_{b_-}^{\alpha+2} f''(a) - J_{a^+}^{\alpha+1} f'(b) + J_{b_-}^{\alpha+1} f'(a).$$

Lemma 3.6. *Let $\alpha > 0$ and let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I°, $a, b \in I$ with $a < b$. If $f'' \in L^1[a,b]$, then*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b_-}^\alpha f(a)\right] \\ &= \frac{\Gamma(\alpha+1)}{2(\alpha+1)(b-a)^{\alpha-1}}\left[J_{b_-}^\alpha f'(a) - J_{a^+}^\alpha f'(b)\right] \\ &\quad + \frac{(b-a)^2}{2(\alpha+1)}\left[\int_0^1 t^\alpha(1-t) f''(ta+(1-t)tb)dt + \int_0^1 t^\alpha(1-t) f''((1-t)ta+tb)dt\right]. \end{aligned} \tag{19}$$

Proof. First we set

$$I = \int_0^1 t^\alpha(1-t) f''(ta+(1-t)b)dt$$

and use integration by parts to obtain

$$\begin{aligned} I &= \frac{1}{b-a}\int_0^1 \left(\alpha t^{\alpha-1} - (\alpha+1)t^{\alpha}\right) f'(ta+(1-t)b)dt \\ &= \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J_{a^+}^{\alpha} f'(b) - \frac{\Gamma(\alpha+2)}{(b-a)^{\alpha+2}} J_{a^+}^{\alpha+1} f'(b). \end{aligned}$$

Now from (12) follows

$$I = \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J_{a^+}^{\alpha} f'(b) + \frac{\alpha+1}{(b-a)^2} f(a) - \frac{\Gamma(\alpha+2)}{(b-a)^{\alpha+2}} J_{a^+}^{\alpha} f(b). \tag{20}$$

Similarly, for

$$J = \int_0^1 t^{\alpha}(1-t) f''((1-t)a+tb)dt$$

with integration by parts we have

$$\begin{aligned} J &= -\frac{1}{b-a}\int_0^1 \left(\alpha t^{\alpha-1} - (\alpha+1)t^{\alpha}\right) f'((1-t)a+tb)dt \\ &= -\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J_{b_-}^{\alpha} f'(a) + \frac{\Gamma(\alpha+2)}{(b-a)^{\alpha+2}} J_{b_-}^{\alpha+1} f'(a), \end{aligned}$$

and with (13) we get

$$J = -\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J_{b_-}^{\alpha} f'(a) + \frac{\alpha+1}{(b-a)^2} f(b) - \frac{\Gamma(\alpha+2)}{(b-a)^{\alpha+2}} J_{b_-}^{\alpha} f(a). \tag{21}$$

The result now follows by adding (20) and (21). □

Remark 3.7. *Again we use* (14) *and* (15), *to obtain*

$$\begin{aligned} &\int_0^1 t^{\alpha}(1-t) f''(ta+(1-t)b)dt + \int_0^1 t^{\alpha}(1-t) f''((1-t)ta+b)dt \\ &= \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} \left[J_{b_-}^{\alpha+1} f''(a) + J_{a^+}^{\alpha+1} f''(b)\right] - \frac{\Gamma(\alpha+2)}{(b-a)^{\alpha+2}} \left[J_{b_-}^{\alpha+2} f''(a) + J_{a^+}^{\alpha+2} f''(b)\right]. \end{aligned}$$

An alternative representation of equation (19) *is*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b_-}^{\alpha} f(a)\right] \\ &= \frac{\Gamma(\alpha+1)}{2(\alpha+1)(b-a)^{\alpha-1}} \left[J_{a^+}^{\alpha+1} f''(b) + J_{b_-}^{\alpha+1} f''(a) - J_{a^+}^{\alpha} f'(b) + J_{b_-}^{\alpha} f'(a)\right] \\ &\quad - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{b_-}^{\alpha+2} f''(a) + J_{a^+}^{\alpha+2} f''(b)\right]. \end{aligned} \tag{22}$$

Furthermore, from (16) *and* (22) *we can obtain*

$$\begin{aligned} &J_{a^+}^{\alpha+2} f''(b) + J_{b_-}^{\alpha+2} f''(a) - J_{a^+}^{\alpha+1} f'(b) + J_{b_-}^{\alpha+1} f'(a) \\ &= \frac{b-a}{\alpha+1} \left[J_{a^+}^{\alpha+1} f''(b) + J_{b_-}^{\alpha+1} f''(a) - J_{a^+}^{\alpha} f'(b) + J_{b_-}^{\alpha} f'(a)\right]. \end{aligned}$$

Future research could generalize these results to wider classes of functions, particularly those involving fractional or higher-order derivatives, and lead to the development of new inequalities. As an illustration, we present the following Hermite-Hadamard type inequality.

Theorem 3.8. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}^+$ be a twice differentiable function on I°, and let $a, b \in I$ with $0 < a < b$, such that $f'' \in L^1[a,b]$. If $\left|f''\right|$ is concave on $[a,b]$, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b) + J^\alpha_{b_-}f(a)\right]\right| \\
\leq\ & \frac{\Gamma(\alpha+1)}{2(\alpha+1)(b-a)^{\alpha-1}}\left|J^\alpha_{b_-}f'(a) - J^\alpha_{a^+}f'(b)\right| + \frac{(b-a)^2}{(\alpha+1)^2(\alpha+2)}\left|f''\left(\frac{a+b}{2}\right)\right|. \qquad (23)
\end{aligned}$$

Proof. By the triangle inequality for integrals and Jensen's inequality from Theorem 1.7 on a concave function $|f''|$, we obtain

$$\begin{aligned}
&\left|\int_0^1 t^\alpha(1-t)f''(ta+(1-t)tb)dt\right| \\
\leq\ & \left(\int_0^1 t^\alpha(1-t)dt\right)\left|f''\left(\frac{\int_0^1 t^\alpha(1-t)(ta+(1-t)b)dt}{\int_0^1 t^\alpha(1-t)dt}\right)\right| \\
=\ & \frac{1}{(\alpha+1)(\alpha+2)}\left|f''\left(\frac{(1+\alpha)a+2b}{\alpha+3}\right)\right|.
\end{aligned}$$

Analogously,

$$\left|\int_0^1 t^\alpha(1-t)f''((1-t)a+tb)dt\right| \leq \frac{1}{(\alpha+1)(\alpha+2)}\left|f''\left(\frac{2a+(1+\alpha)b}{\alpha+3}\right)\right|.$$

Hence, from Lemma 3.6 follows

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b) + J^\alpha_{b_-}f(a)\right]\right| \\
\leq\ & \frac{\Gamma(\alpha+1)}{2(\alpha+1)(b-a)^{\alpha-1}}\left|J^\alpha_{b_-}f'(a) - J^\alpha_{a^+}f'(b)\right| \\
&+ \frac{(b-a)^2}{2(\alpha+1)}\int_0^1 t^\alpha(1-t)\left|f''(ta+(1-t)tb)\right|dt \\
&+ \frac{(b-a)^2}{2(\alpha+1)}\int_0^1 t^\alpha(1-t)\left|f''((1-t)ta+tb)\right|dt \\
\leq\ & \frac{\Gamma(\alpha+1)}{2(\alpha+1)(b-a)^{\alpha-1}}\left|J^\alpha_{b_-}f'(a) - J^\alpha_{a^+}f'(b)\right| \\
&+ \frac{(b-a)^2}{2(\alpha+1)^2(\alpha+2)}\left(\left|f''\left(\frac{(1+\alpha)a+2b}{\alpha+3}\right)\right| + \left|f''\left(\frac{2a+(1+\alpha)b}{\alpha+3}\right)\right|\right).
\end{aligned}$$

The concavity of $|f''|$ implies

$$\begin{aligned}
&\frac{1}{2}\left|f''\left(\frac{(1+\alpha)a+2b}{\alpha+3}\right)\right| + \frac{1}{2}\left|f''\left(\frac{2a+(1+\alpha)b}{\alpha+3}\right)\right| \\
\leq\ & \left|f''\left(\frac{1}{2}\frac{(1+\alpha)a+2b}{\alpha+3} + \frac{1}{2}\frac{2a+(1+\alpha)b}{\alpha+3}\right)\right| \\
=\ & \left|f''\left(\frac{a+b}{2}\right)\right|,
\end{aligned}$$

from which inequality (23) follows. □

4. Conclusion

This research enhances the theoretical framework for numerical integration and error analysis. By employing classical inequalities and deriving new integral estimates of Hermite–Hadamard type, we have established upper bounds for the deviation between the arithmetic mean of the function values at the endpoints and the mean value of the function over the interval.

In addition, fractional identities for twice differentiable functions are established, providing a basis for future research to explore applications in the context of probabilistic and operator inequalities.

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