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Riemannian manifolds satisfying certain conditions on pseudo-quasi-conformal curvature tensor

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Abstract. In this paper, we explore various properties of the Pseudo-Quasi-Conformal Curvature Tensor, denoted as $\tilde{V}$, on Riemannian manifolds, with a particular focus on generalized quasi-Einstein manifolds as defined by Chaki. Initially, we examine a Pseudo-Quasi-Conformally Ricci semisymmetric generalized quasi-Einstein manifold. Subsequently, we investigate the Pseudo-Quasi-Conformal flatness of this manifold. Additionally, we provide a non-trivial example of a generalized quasi-Einstein manifold to demonstrate its existence.

1. Introduction

In 2005, A.A. Shaikh and S.K. Jana [30] defined and studied a tensor field $\tilde{V}$ on a Riemannian manifold (M^n, g) of dimension n(> 2) which includes the Projective curvature, Quasi-conformal curvature, Conformal Curvature and Concircular curvature. This tensor field $\tilde{V}$ is known as Pseudo-Quasi-Conformal Curvature Tensor and it is defined as

$$\begin{aligned}\tilde{V}(X,Y)Z =&(p+d)R(X,Y)Z + (q - \frac{d}{n-1})[S(Y,Z)X - S(X,Z)Y] + q[g(Y,Z)QX - g(X,Z)QY] \\ &- \frac{r}{n(n-1)}\{p + 2(n-1)q\}[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{1}$$

where $X, Y, Z \in \zeta(M^n)$, and p, q, d are real numbers such that $p^2 + q^2 + d^2 > 0$. R denotes the curvature tensor, S represents the Ricci tensor, and r is the scalar curvature.
In particular, if
(i) $p = q = 0, d = 1 \Longrightarrow$ Projectively Curvature,
(ii) $p \neq 0, q \neq 0, d = 0 \Longrightarrow$ Quasi-Conformal Curvature,
(iii) $p = 1, q = -\frac{1}{n-2}, d = 0 \Longrightarrow$ Conformal Curvature,
(iv) $p = 1, q = d = 0 \Longrightarrow$ Concircular Curvature.
In 2005, Shaikh and Jana [30] introduced and studied a tensor field, called pseudo-quasi-conformal curvature

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tensor $\tilde{V}$ on a Riemannian manifold with dimension 3 or higher.The pseudo-quasi-conformal curvature tensor has recently been studied by several authors in various contexts, including Prakasha et al. [31], and many others.

It is important to note that the pseudo-quasi-conformal curvature tensor exhibits the following symmetry properties.:
(i) $\tilde{V}(X, Y, Z, W) = -\tilde{V}(Y, X, Z, W)$,
(ii) $\tilde{V}(X, Y, Z, W) \neq \pm\tilde{V}(X, Y, W, Z)$
for all $X, Y, Z, W \in \zeta(M)$.

Let $[e_i]$ represent a set of orthonormal vectors in the tangent space at every point on the manifold, where $1 \leq i \leq n$. Now, from (1), we have

$$\sum_{i=1}^{n} \tilde{V}(X, Y, e_i, e_i) = \sum_{i=1}^{n} \tilde{V}(e_i, e_i, Z, W,) = 0, \tag{2}$$

$$\sum_{i=1}^{n} \tilde{V}(e_i, Y, Z, e_i) = [p + (n-2)][S(Y, Z) - \frac{r}{n}g(Y, Z)] = [p + (n-2)]V(Y, Z) \tag{3}$$

and

$$\begin{aligned} \sum_{i=1}^{n} \tilde{V}(X, e_i, e_i, W) &= [p + \frac{n}{n-1}d + (n-2)][S(X, W) - \frac{r}{n}g(X, W)] \\ &= [p + \frac{n}{n-1}d + (n-2)]V(X, W). \end{aligned} \tag{4}$$

Also,

$$\sum_{i=1}^{n} V(e_i, e_i) = 0. \tag{5}$$

If $p = q = 0$ and $d = 1$ in (1), then the Pseudo-Quasi-Conformal Curvature Tensor takes the form

$$\tilde{V}(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[S(Y, Z)X - S(X, Z)Y] = P(X, Y)Z, \tag{6}$$

where P represents the projective curvature tensor., [18]. Therefore, the projective curvature tensor is a special case of the Pseudo-Quasi-Conformal Curvature Tensor.
The projective curvature tensor plays a significant role in differential geometry. In a Riemannian manifold M, if there is a one-to-one correspondence between each coordinate neighborhood of M and a domain in Euclidean space, such that every geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is referred to as being locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if it has constant curvature. Therefore, the projective curvature tensor measures how much a Riemannian manifold deviates from having constant curvature. For recent advancements on the projective curvature tensor, we refer to [23], [27], and [33].

In 2000, M.C. Chaki and R.K. Maity introduced the concept of quasi-Einstein manifolds as a generalization of Einstein manifolds. U.C. De and G.C. Ghosh discuss quasi-Einstein and special quasi-Einstein manifolds, and investigate several global properties of general quasi-Einstein manifolds([4],[5],[6],[7]). According to their definition, a Riemannian manifold $(M^n, g)(n > 2)$ is called a quasi-Einstein manifold [2] if its Ricci tensor of type (0,2) is not identically zero and satisfies the following condition:

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) \tag{7}$$

for all $X, Y \in \zeta(M^n)$ where a and b are real-valued, non-zero scalar functions. of which $b \neq 0$ on (M^n, g) and A represents a non-zero 1-form, equivalent to the unit vector field U, that is,

$$g(X, U) = A(X), g(U, U) = 1 \tag{8}$$

A is referred to as an associated 1-form, and U is called a generator of (M^n, g). If b=0, then the manifold reduces to an Einstein manifold. Such an n-dimensional manifold is denoted by $(QE)_n$.

The notion of generalized quasi Einstein manifold has been firstly introduced by M.C. Chaki in 2001 [3] and then U.C. De and S. Mallick discussed existence and some properties of generalized quasi-Einstein manifolds([12],[13],[14]). Also U.C. De discussed certain applications to Relativity[15]. A Riemannian manifold $(M^n, g)(n > 2)$ is said to be generalized quasi Einstein manifold if its Ricci tensor of type (0,2) is not identically zero and it satisfies the following condition [3]

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) + c[A(X)B(Y) + A(Y)B(X)] \tag{9}$$

for all $X, Y \in \zeta(M^n)$ where a,b,c are real valued, non-zero scalar functions of which $b \neq 0, c \neq 0$ on (M^n, g), A and B are two non-zero 1-forms such that

$$g(X, U) = A(X), g(X, V) = B(X), g(U, V) = 0, g(U, U) = g(V, V) = 1 \tag{10}$$

That is. U and V are orthonormal vector fields corresponding to the 1-forms A and B, respectively. Similarly, a,b and c are referred to as associated scalars, A and B as associated 1-forms and U and V as generators of manifold. Such an n-dimensional manifold has been denoted by $G(QE)_n$. If c=0, then the manifold reduces to a quasi-Einstein manifold and if b = c = 0, the manifold reduces to an Einstein manifold. Additionally, an operator Q defined by g(QX,Y) = S(X,Y) is called the Ricci operator. Contracting (9) over X and Y, the scalar curvature function of this manifold is given by

$$r = an + b. \tag{11}$$

In view of the equations (9) and (10), in a generalized quasi-Einstein manifold, we have

$$S(Y, U) = (a + b)A(Y) + cB(Y), S(Y, V) = aB(Y) + cA(Y) \tag{12}$$

Let R denote the Riemannian curvature tensor of M. The k-nullity distribution N(k) of a Riemannian manifold M is defined as

$$N(k) : p \longrightarrow N_p(k) = \{Z \in \zeta_p(M) : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y], \forall X, Y \in \zeta(M)\}, \tag{13}$$

where k is some smooth function [24]. In a quasi-Einstein manifold M, if the generator U belongs to a k-nullity distribution N(k), then M is referred to as an N(k)-quasi-Einstein manifold. According to C. Ozgur and M.M. Tripathi [20], in an n-dimensional N(k)-quasi-Einstein manifold, the function k is equal to $\frac{a+b}{n-1}$.

Many authors have investigated the pseudo-quasi-conformal curvature tensor on K-Contact manifolds and P-Sasakian manifolds, deriving conditions under which these manifolds are Einstein, η-Einstein, or pseudo-conformally flat ([1], [26]). Additionally, J.P. Jaiswal and R.H. Ojha [17] have studied weakly pseudo-quasi-conformally symmetric and Ricci-symmetric manifolds, examining the behavior of the scalar curvature in these spaces. S. Kishor and A. Singh [32] have presented results on Ricci Solitons on Para-Sasakian Manifolds Satisfying Pseudo-Symmetry Curvature Condition. On the other hand, in [16], the authors explored certain properties of generalized quasi-Einstein manifolds that satisfy specific Ricci conditions on various curvature tensors, including conformal, concircular, and projective curvature tensors. Building upon these studies, this paper investigates the Pseudo-Quasi-Conformal Curvature Tensor under specific conditions on certain Riemannian manifolds, with a focus on generalized quasi-Einstein manifolds. This paper is structured as follows: First, we demonstrate that any Pseudo-Quasi-Conformally Ricci Semisymmetric generalized quasi-Einstein manifold (i.e. it satisfies the condition $\tilde{V}$.S =0) is an N(k)-quasi-Einstein manifold introduced in [**?**]. Next, we investigate the pseudo-quasi-conformal flatness of this manifold and establish several results related to it. Additionally, we provide a non-trivial example of a generalized quasi-Einstein manifold to demonstrate its existence.

2. Pseudo-Quasi Conformally Ricci Semisymmetric $G(QE)_n$

For a (0,k)-tensor T, where k ≥ 1, a (0,k+2)-tensor R.T is defined by

$$\begin{aligned}(R.T)(X_1, X_2, X_3, ..., X_k; X, Y) &= (R(X, Y).T)(X_1, X_2, ..., X_k)\\ &= -T(R(X, Y)X_1, ..., X_k) - ... - T(X_1, ..., R(X, Y)X_k),\end{aligned} \tag{14}$$

where R denotes the Riemannian curvature tensor.
Let A be a symmetric (0,2) - tensor and T a (0,k)-tensor. Then the tensor Q(A,T) is called a Tachibana tensor [9] of A and T and it is given by

$$\begin{aligned}&Q(A, T)(X_1, X_2, ..., X_k; X, Y) = ((X \wedge_A Y).T)(X_1, X_2, ..., X_k)\\ &= -T((X \wedge_A Y)X_1, ..., X_k) - ... - T(X_1, ..., (X \wedge_A Y)X_k).\end{aligned} \tag{15}$$

For symmetric (0,2)-tensor E and F, their Kulkarni-Nomizu product $E \wedge F$ is defined by

$$\begin{aligned}(E \wedge F)(X_1, X_2, X_3, X_4) = E(X_1, X_4)F(X_2, X_3) + E(X_2, X_3)F(X_1, X_4)\\ -E(X_1, X_3)F(X_2, X_4) - E(X_2, X_4)F(X_1, X_3)\end{aligned} \tag{16}$$

for all $X_1, X_2, X_3, X_4 \in \zeta(M)$

Definition 2.1. *([10]) An n-dimensional Riemannian manifold (M^n, g) is called Ricci-pseudosymmetric if and only if the tensor R.S and Q(g,S) are linearly dependent. That means, the equation*

$$(R(X, Y).S)(Z, W) = L_S Q(g, S)(Z, W; X, Y) \tag{17}$$

holds on $U_S = \{x \in M : S \neq \frac{r}{n} g \text{ at } x\}$ and L_S is a certain function on U_S.

Analogously, we can give the following definition:

Definition 2.2. *An n-dimensional Riemannian manifold (M^n, g) is called pseudo-quasi conformally Ricci semisymmetric if the pseudo-quasi conformal curvature tensor satisfies the condition $\tilde{V}.S = 0$.
That is, in a pseudo-quasi conformally Ricci semisymmetric manifold, the condition*

$$(\tilde{V}(X, Y).S)(Z, W) = -S(\tilde{V}(X, Y)Z, W) - S(Z, \tilde{V}(X, Y)W) = 0; \forall X, Y, Z \in \zeta(M^n) \tag{18}$$

*holds.
In this section, we consider pseudo-quasi conformally Ricci semisymmetric $G(QE)_n$. The main purpose of this section is to prove the following theorem.*

Theorem 2.3. *Every pseudo-quasi conformally Ricci semisymmetric $G(QE)_n$ is an N(k)-quasi Einstein manifold.*

Proof. Combining (9) and (18), in a pseudo-quasi conformally Ricci semisymmetric $G(QE)_n$ we get

$$\begin{aligned}&a[\tilde{V}(X, Y, Z, W) + \tilde{V}(X, Y, W, Z)] + b[A(\tilde{V}(X, Y)Z)A(W) + A(Z)A(\tilde{V}(X, Y)W)]\\ &+ c[A(\tilde{V}(X, Y)Z)B(W) + A(W)B(\tilde{V}(X, Y)Z)\\ &+ A(Z)B(\tilde{V}(X, Y)W) + A(\tilde{V}(X, Y)W)B(Z)] = 0,\end{aligned} \tag{19}$$

where $g(\tilde{V}(X, Y)Z, W) = \tilde{V}(X, Y, Z, W)$. In view of (1),(19) yields

$$\begin{aligned}&-\frac{ad}{n-1}[S(Y, Z)g(X, W) - S(X, Z)g(Y, W) + S(Y, W)g(X, Z) - S(X, W)g(Y, Z)]\\ &+ b[\tilde{V}(X, Y, Z, U)A(W) + A(Z)\tilde{V}(X, Y, W, U)] + c[\tilde{V}(X, Y, Z, U)B(W)\\ &+ A(W)\tilde{V}(X, Y, Z, V) + A(Z)\tilde{V}(X, Y, W, V) + \tilde{V}(X, Y, W, U)B(Z)] = 0.\end{aligned} \tag{20}$$

Putting Z=U and W = V in (20) and using (10), we get

$$\begin{aligned}&-\frac{ad}{n-1}[S(Y,U)B(X)-S(X,U)B(Y)+S(Y,V)A(X)-S(X,V)A(Y)]+b\tilde{V}(X,Y,V,U)\\&+c[\tilde{V}(X,Y,U,U)+\tilde{V}(X,Y,V,V)]=0\end{aligned} \tag{21}$$

By virtue of (1), we have the following three relations:

$$\begin{aligned}\tilde{V}(X,Y,V,U)=&(p+d)R(X,Y,V,U)+(q-\frac{d}{n-1})[S(Y,V)A(X)-S(X,V)A(Y)]\\&+q[S(X,U)B(Y)-S(Y,U)B(X)]-\frac{r}{n(n-1)}\{p+2(n-1)q\}[A(X)B(Y)-A(Y)B(X)],\end{aligned} \tag{22}$$

$$\tilde{V}(X,Y,U,U)=-\frac{d}{n-1}[S(Y,U)A(X)-S(X,U)A(Y)] \tag{23}$$

$$\tilde{V}(X,Y,V,V)=-\frac{d}{n-1}[S(Y,V)B(X)-S(X,V)B(Y)]. \tag{24}$$

By using (22), (23) and (24) in (21) (as b is different than zero) with the help of (12), in a $G(QE)_n$ satisying the condition $\tilde{V}.S=0$, we obtain the following relation:

$$R(X,Y,U,V)=\frac{1}{(p+d)}[\frac{r}{n(n-1)}\{p+2(n-1)q\}-(2a+b)q][A(Y)B(X)-A(X)B(Y)] \tag{25}$$

Contracting (20) over X and W and using the equation (4), we get

$$\begin{aligned}&-\frac{ad}{(n-1)}[nS(Y,Z)-rg(Y,Z)]+b[\tilde{V}(U,Y,Z,U)\\&-A(Z)(p+\frac{n}{n-1}d+(n-2)q)\{S(Y,U)-\frac{r}{n}A(Y)\}]\\&+c[\tilde{V}(V,Y,Z,U)+\tilde{V}(U,Y,Z,V)\\&-A(Z)(p+\frac{n}{n-1}d+(n-2)q)\{S(Y,V)-\frac{r}{n}B(Y)\}\\&-B(Z)(p+\frac{n}{n-1}d+(n-2)q)\{S(Y,U)-\frac{r}{n}A(Y)\}]=0\end{aligned} \tag{26}$$

Putting Z = U in (26), we get

$$\begin{aligned}&-\frac{ad}{(n-1)}[nS(Y,U)-rA(Y)]+b[\tilde{V}(U,Y,U,U)\\&-(p+\frac{n}{n-1}d+(n-2)q)\{S(Y,U)-\frac{r}{n}A(Y)\}]\\&+c[\tilde{V}(V,Y,U,U)+\tilde{V}(U,Y,U,V)\\&-(p+\frac{n}{n-1}d+(n-2)q)\{S(Y,V)-\frac{r}{n}B(Y)\}]=0\end{aligned} \tag{27}$$

By using (10),(12),(22),(23) and (25) in (27), we obtain

$$\begin{aligned}&[\frac{ad}{n-1}(r-n(a+b))-b(p+\frac{n}{n-1}d+(n-2)q)(a+b-\frac{r}{n})\\&+\frac{dc^2}{(n-1)}-c^2(p+\frac{n}{n-1}d+(n-2)q)]A(Y)\\&+[-\frac{adnc}{(n-1)}-\frac{bdc}{(n-1)}+b(p+\frac{n}{n-1}d+(n-2)q)c\\&-c[\frac{r}{n(n-1)}\{p+2(n-1)q\}-(2a+b)q]\\&-c[(q-\frac{d}{n-1})(a+b)+aq-\frac{r}{n(n-1)}\{p+2(n-1)q\}]\\&-c[(p+\frac{n}{n-1}d+(n-2)q)(a-\frac{r}{n})]]B(Y)=0\end{aligned} \tag{28}$$

Putting Y = U in (28), we get

$$\frac{ad}{n-1}(r-n(a+b))-b(p+\frac{n}{n-1}d+(n-2)q)(a+b-\frac{r}{n})+\frac{dc^2}{(n-1)}-c^2(p+\frac{n}{n-1}d+(n-2)q)=0. \tag{29}$$

Putting Y=V in (28), we get

$$\begin{aligned}&[-\frac{adnc}{(n-1)}-\frac{bdc}{(n-1)}+b(p+\frac{n}{n-1}d+(n-2)q)c-c[\frac{r}{n(n-1)}\{p+2(n-1)q\}-(2a+b)q]\\&-c[(q-\frac{d}{n-1})(a+b)+aq-\frac{r}{n(n-1)}\{p+2(n-1)q\}]-c[(p+\frac{n}{n-1}d+(n-2)q)(a-\frac{r}{n})]]=0\end{aligned} \tag{30}$$

From (30), we have

$$c[-ad+(b-a+\frac{r}{n})(p+\frac{n}{n-1}d+(n-2)q)]=0 \tag{31}$$

Then by (31), we have either c =0 or $[-ad+(b-a+\frac{r}{n})(p+\frac{n}{n-1}d+(n-2)q)]=0$. If c=0, then by (29), we get $b=\frac{r}{n}-a$ or $p+\frac{n}{n-1}d+(n-2)q+\frac{adn}{b(n-1)}=0$ and so If $a=\frac{r}{n}$ then as both of b and c are zero, the manifold reduces to an Einstein manifold, but this is a contradiction. Thus $b\neq 0$ or $a\neq\frac{r}{n}$ and so $p+\frac{n}{n-1}d+(n-2)q+\frac{adn}{b(n-1)}=0$ On the other hand, if $c\neq 0$, then again we obtain $[-ad+(b-a+\frac{r}{n})(p+\frac{n}{n-1}d+(n-2)q)]=0$. Using this in (29), we get $c=0$ if $a+b=\frac{r}{n}$.
That is, we conclude that

$$\text{Either}\quad c=0\quad\text{for}\quad a+b=\frac{r}{n}\quad\text{and}\quad p+\frac{n}{n-1}d+(n-2)q+\frac{adn}{b(n-1)}=0. \tag{32}$$

Since c = 0, the manifold reduces to a quasi Einstein manifold and from (32) and (25), we have

$$R(X,Y)U=\frac{1}{(p+d)}[\frac{a+b}{(n-1)}\{p+2(n-1)q\}-(2a+b)q][A(Y)X-A(X)Y], \tag{33}$$

$$R(X,Y)U=\frac{1}{(p+d)}[\frac{p(a+b)+b(n-1)q}{n-1}][A(Y)X-A(X)Y], \tag{34}$$

which means that the generator U belongs to the some k-nullity distribution, $k=\frac{1}{(p+d)}[\frac{p(a+b)+b(n-1)q}{n-1}]$. Hence the manifold under this consideration is an N(k)-quasi-Einstein manifold. Hence, the proof is completed. Additionally, contracting (34), we get S(Y,U) = $\frac{p(a+b)+b(n-1)q}{p+d}$A(Y); i.e., QY = $\frac{p(a+b)+b(n-1)q}{p+d}$Y, which means that $\frac{p(a+b)+b(n-1)q}{p+d}$ is an eigenvalue of the Ricci operator Q. Hence we can state:

Theorem 2.4. *In a pseudo-quasi conformally Ricci semisymmetric $G(QE)_n$, $\frac{p(a+b)+b(n-1)q}{p+d}$ is an eigenvalue of the Ricci operator Q.*

Remark 2.5. *By using the equations, (14),(15) and (16), the following relations holds:*

$$(\tilde{V}.S) = (p+d)(R.S) - \frac{r}{n(n-1)}[p+2(n-1)q]Q(g,S). \tag{35}$$

Thus, if in a Riemannian manifold, the condition $\tilde{V}.S = 0$, then by (35), as $(p+d) \neq 0$ we get

$$(R.S) = \frac{r}{(p+d)n(n-1)}[p+2(n-1)q]Q(g,S). \tag{36}$$

That is, the manifold reduces to a Ricci-pseudosymmetric manifold.

Thus, from above remark and Theorem (2.3), the following corollary obtained in [16] can be stated by a simple way:

Corollary 2.6. *([16]) Every non-Einstein Ricci-pseudosymmetric $G(QE)_n$ is an N(k) - quasi Einstein manifold.*

3. Pseudo-Quasi-Conformally Flat and Pseudo-Quasi-Conformally Conservative Riemannian Manifolds

In [23], projective curvature tensors of a non-symmetric affine connection space are expressed as functions of the affine connection coefficients and Weyl projective tensor of the corresponding associated affine connection space. Moreover, projective flatness of non-symmetric affine connection spaces were analysed. Accordingly, in this section, we determine some properties of pseudo-quasi-conformal curvature tensor and then we investigate n-dimensional(n¿2) pseudo-quasi-conformally flat manifold (M^n, g). In this case, the tensor $\tilde{V}$ vanishes. Then by (1), we have

$$\begin{aligned}(p+d)R(X,Y,Z,W) &+ (q - \frac{d}{n-1})[S(Y,Z)g(X,W) - S(X,Z)g(Y,W)]\\ &+q[g(Y,Z)g(QX,W) - g(X,Z)g(QY,W)]\\ = \frac{r}{n(n-1)}&\{p+2(n-1)q\}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\end{aligned} \tag{37}$$

Contracting (37) over X and W, we obtain

$$(p+(n-2)q)[S(Y,Z) - \frac{r}{n}g(Y,Z)] = 0. \tag{38}$$

Then we get

$$p+(n-2)q = 0 \qquad or \qquad S(Y,Z) = \frac{r}{n}g(Y,Z). \tag{39}$$

If $S(Y,Z) = \frac{r}{n}g(Y,Z)$, then the manifold under consideration becomes an Einstein manifold. Next, if $p+(n-2)q = 0$ holds, then using this relation in (1) we obtain $\tilde{V} = dP$, if $p = 0$, where P denotes the projective curvature tensor given in (6). Thus, in this case in a pseudo -quasi-conformally flat manifold, as $d \neq 0$ and $p = 0$, the projective curvature tensor vanishes and so this manifold is of constant curvature, i.e. this manifold is again Einstein. Hence, every pseudo-quasi-conformally flat manifold is an Einstein manifold if $p = 0$.

Now, let (M^n, g), $(n > 2)$ be a pseudo-quasi-confromally conservative manifold so $div\tilde{V} = 0$. Taking covariant derivative of (1), we obtain

$$\begin{aligned}(\nabla_W\tilde{V})(X,Y)Z &= (p+d)(\nabla_W R)(X,Y)Z + (q - \frac{d}{n-1})[(\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y]\\ -\frac{dr(W)}{n(n-1)}&\{p+2(n-1)q\}[g(Y,Z)X - g(X,Z)Y] + q[(\nabla_W g)(Y,Z)QX - (\nabla_W g)(X,Z)QY].\end{aligned} \tag{40}$$

Contracting (40) over W, we get

$$
\begin{aligned}
(div\tilde{V})(X,Y)Z =&(p+d)(divR)(X,Y)Z + (q - \frac{d}{n-1})[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] \\
&- \frac{\{p+2(n-1)q\}}{n(n-1)}[g(Y,Z)dr(X) - g(X,Z)dr(Y)].
\end{aligned} \tag{41}
$$

It is known that

$$
(divR)(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z). \tag{42}
$$

Combining (41) and (42), the divergence of the pseudo -quasi-conformal curvature tensor can be expressed as

$$
\begin{aligned}
(div\tilde{V})(X,Y)Z =&(p+d)[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] + (q - \frac{d}{n-1})[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] \\
&- \frac{\{p+2(n-1)q\}}{n(n-1)}[g(Y,Z)dr(X) - g(X,Z)dr(Y)].
\end{aligned} \tag{43}
$$

Thus, in a pseudo-quasi-conformally conservative manifold (M^n, g), $(n > 2)$, the following relations holds:

$$
\begin{aligned}
&(p+q+\frac{(n-2)}{(n-1)}d)[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X.Z)] \\
&= \frac{\{p+2(n-1)q\}}{n(n-1)}[g(Y,Z)dr(X) - g(X,Z)dr(Y)]
\end{aligned} \tag{44}
$$

Contracting (44) over Y and Z and using contracted second Bianchi identity (as $n > 2$), we get for all $X \in \zeta(M)$, $[\frac{(n-2)}{2n}p - \frac{(3n-1)}{2n}q + \frac{(n-2)}{2(n-1)}d]dr(X) = 0$ and so either $[\frac{(n-2)}{2n}p - \frac{(3n-1)}{2n}q + \frac{(n-2)}{2(n-1)}d] = 0$ or the manifold has constant scalar curvature. Moreover, by $[\frac{(n-2)}{2n}p - \frac{(3n-1)}{2n}q + \frac{(n-2)}{2(n-1)}d] = 0$, we get $p = q = d = 0$ but $p^2 + q^2 + d^2 > 0$ gives a contradiction, this implies $[\frac{(n-2)}{2n}p - \frac{(3n-1)}{2n}q + \frac{(n-2)}{2(n-1)}d] \neq 0$ So we have $dr(X) = 0$, then the scalar curvature is constant and so from (44), we obtain

$$
(p+q+\frac{(n-2)}{(n-1)}d)[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X.Z)] = 0 \tag{45}
$$

Thus, if $(p+q+\frac{(n-2)}{(n-1)}d) \neq 0$, then by (45) the Ricci tensor of this manifold satisfies the condition

$$
(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z), \tag{46}
$$

which means that the manifold has the Codazzi type Ricci tensor,[11]. Hence, firstly we can state the following theorem.

Theorem 3.1. *Every pseudo-quasi-conformally flat manifold is Einstein if $p = 0$. Let $(M^n, g), (n > 2)$ be a pseudo -quasi- conformally conservative manifold. Then it has constant scalar curvature. Moreover, if (M^n, g) is of constant scalar curvature and the scalars p, q and d appeared in $\tilde{V}$ satisfies the relation $(p+q+\frac{(n-2)}{(n-1)}d) \neq 0$, then the Ricci tensor of this manifold is of Codazzi type.*

Now, we consider $(M^n, g), (n > 2)$ be a pseudo-quasi conformally conservative generalized quasi-Einstein manifols whose associated scalars are constants. In this case, the scalar curvature $r = an + b$ is constant. Now, we also assume that $(p+q+\frac{(n-2)}{(n-1)}d) \neq 0$. Then, by Theorem (3.1), the Ricci tensor of this manifold is of Codazzi type. Thus, taking covariant derivative of the Ricci tensor of $G(QE)_n$, we get

$$
\begin{aligned}
(\nabla_Z S)(X,Y) = b[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)] \\
+c[(\nabla_Z A)(X)B(Y) + A(X)(\nabla_Z B)(Y) + (\nabla_Z A)(Y)B(X) + A(Y)(\nabla_Z B)(X)].
\end{aligned} \tag{47}
$$

Combining, the equations (47) and (46), we obtain

$$\begin{aligned} & b[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y) - (\nabla_X A)(Z)A(Y) - A(Z)(\nabla_X A)(Y)] \\ & +c[(\nabla_Z A)(X)B(Y) + A(X)(\nabla_Z B)(Y) + (\nabla_Z A)(Y)B(X) + A(Y)(\nabla_Z B)(X) \\ & -(\nabla_X A)(Z)B(Y) - A(Z)(\nabla_X B)(Y) - (\nabla_X A)(Y)B(Z) - A(Y)(\nabla_X B)(Z)] = 0. \end{aligned} \tag{48}$$

Putting Y = U in (48), we get

$$b[(\nabla_Z A)(X) - (\nabla_X)(Z)] + c[A(X)(\nabla_Z B)(U) + (\nabla_Z B)(X) - A(Z)(\nabla_X B)(U) - (\nabla_X B)(Z)] = 0 \tag{49}$$

and putting X = U in (49), we get

$$-b(\nabla_U A)(Z) + c[2(\nabla_Z B)(U) - A(Z)(\nabla_U B)(U) - (\nabla_U B)(Z)] = 0 \tag{50}$$

Next, putting Z = V in (50), we get

$$-b(\nabla_U A)(V) + 2c(\nabla_V B)(U) = 0. \tag{51}$$

On the other hand, putting Y = Z = V and X = U in (48), we obtain

$$b(\nabla_V A)(V) + 2c(\nabla_U B)(U) = 0. \tag{52}$$

Since g(U,V) = 0, from the properties of connection ∇, we have

$$g(\nabla_U U, V) + g(U, \nabla_U V) = 0 \ \ and \ \ g(\nabla_V U, V) + g(U, \nabla_V V) = 0 \tag{53}$$

Thus, by using (53) in (51) and (52) we obtain the following system of equations:

$$-bg(\nabla_U U, V) + 2cg(\nabla_V V, U) = 0, \qquad -bg(\nabla_V V, U) - 2cg(\nabla_U U, V) = 0 \tag{54}$$

Hence from the above system, we obtain $c[(g(\nabla_U U, V))^2 + (g(\nabla_V V, U))^2] = 0$ and so we have either c = 0 or $(g(\nabla_U U, V))^2 + (g(\nabla_V V, U))^2 = 0$. If c = 0 , then from the system (54), we obtain $g(\nabla_U U, V) = g(\nabla_V V, U) = 0$. (Otherwise, b = 0 and this means that the manifold reduces to an Einstein manifold.) On the other hand, if $(g(\nabla_U U, V))^2 + (g(\nabla_V V, U))^2 = 0$, then again we obtain $g(\nabla_U U, V) = g(\nabla_V V, U) = 0$. Thus, in each case, the generators of the manifold satisfy the following relations:

$$g(\nabla_U U, V) = g(\nabla_V V, U) = 0. \tag{55}$$

Now, putting X = V in (49) and using (55), we get

$$b[(\nabla_Z A)(V) - (\nabla_V A)(Z)] - c(\nabla_V B)(Z) = 0. \tag{56}$$

Similarly, putting Y = V and Z = U in (48) and again using (55), we get

$$-b(\nabla_X A)(V) + c(\nabla_U A)(X) = 0. \tag{57}$$

Also, contracting (47) over X and Z, we obtain

$$b[div(A)A(Y) + (\nabla_U A)(Y)] + c[div(A)B(Y) + div(B)A(Y) + (\nabla_U B)(Y) + (\nabla_V A)(Y)] = 0. \tag{58}$$

Putting Y = U in (58) and using (55) we obtain

$$bdiv(A) + cdiv(B) = 0. \tag{59}$$

Furthermore, putting Y = V in (58) and by virtue of (55), we obtain cdiv(A) = 0 so we have either c = 0 or div(A) = 0. If c = 0, then by (59), (in this case we may take $b \neq 0$) again we get div(A) = 0. Thus, in each case the 1-form A is divergence - free. Moreover, in this case from (59), we have either c = 0 or div(B) = 0.

Case I: Firstly, we assume that $c \neq 0$ and so div(B) = 0. Since div(A) = 0 always holds, the equation (58) reduces to the following form:

$$b(\nabla_U A)(Y) + c[(\nabla_U B)(Y) + (\nabla_V A)(Y)] = 0. \tag{60}$$

Then, summing the equations (50) and (60), using (55) and $c \neq 0$, we get $(\nabla_V A)(Y) = 2(\nabla_Y A)(V)$. In view of the last equation, (56) yields

$$b(\nabla_Z A)(V) + c(\nabla_V B)(Z) = 0. \tag{61}$$

Thus, it follows from (57) and (61) and as $c \neq 0$, we obtain $\nabla_U U + \nabla_V V = 0$.
Case II: In this case, we assume c = 0. Then, by (49)(as $b \neq 0$),

$$(\nabla_Z A)(X) = (\nabla_X A)(Z). \tag{62}$$

Putting X = U in (48) and using the equation (61) and c = 0, we obtain for all $X, Z \in \zeta(M)$,

$$(\nabla_Z A)(X) = g(\nabla_Z U, X) = 0. \tag{63}$$

Thus, the generator U is a parallel vector field and so for all $X \in \zeta(M)$, we have $\nabla_X U = 0$. Also, putting X = U, we get $\nabla_U U = 0$, i.e., the integral curves of U are geodesics. Moreover, from (47) and (63), the Ricci tensor satisfies the condition $\nabla S = 0$, i.e., this manifold becomes Ricci symmetric. Additionally, taking second covariant derivative of (63) and using Ricci identity, we get for all $X, Y, Z \in \zeta(M)$, R(U,X,Y,Z)=0.
Thus, contracting the last equation over X and Y and as c = 0, we get S(U,Z) = (a+b)A(Z) = 0 and so we obtain $a + b = 0$.
As a consequence, we can state the following theorem.

Theorem 3.2. *Let (M^n, g),$(n > 2)$ be a pseudo - quasi-conformally conservative $G(QE)_n$ whose associated scalars are constants and the scalars p, q and d appeared in $\tilde{V}$ satisfies the relation $(p + q + \frac{(n-2)}{(n-1)}d) \neq 0$. Then, the followings hold:*
(1) The generator V is orthogonal to $\nabla_U U$ and the generator U is orthogonal to $\nabla_V V$.
(2) The 1-form A is divergence - free.
(3) If the 1-form B has non-zero divergence,(or $\nabla_U U + \nabla_V V \neq 0$), then
(i) (M^n, g) is a $(QE)_n$ in which sum of the associated scalar functions is zero.
(ii) the main generator U is a parallel vector field.
(iii) the integral curves of the generator vector field U are geodesics.
(iv) (M^n, g) is Ricci symmetric.

Remark 3.3. *([25]) Let M be a complete $n \geq 2$ dimensional Riemannian manifold admitting a special concircular vector field ρ satisfying*

$$\nabla_\mu \nabla_{\lambda\rho} = (-l_p + m)g_{\mu\lambda}, \tag{64}$$

for some scalars μ and λ. Then, if $l = m = 0$, then M is the direct $M^* \times l$ of an (n-1)-dimensional complete Riemannian manifold M^* with straight line l.

This remark is a special case of De Rham's decomposition theorem, [8]. Thus, in view of the above remark, the following corollary is obtained:

Corollary 3.4. *Let (M^n, g), $(n > 2)$ be a pseudo-quasi conformally conservative $G(QE)_n$ whose associated scalars are constants and the scalars p, q and d appeared in $\tilde{V}$ satisfies the relation $(p + q + \frac{(n-2)}{(n-1)}d) \neq 0$. If the main generator of this manifold is gradient and $divB \neq 0$, then this manifold is the direct product of the form $M^* \times l$, where M^* is an (n-1)-dimensional complete Riemannian manifold and l is the straight line.*

4. Existence of a 4-Dimensional Generalized Quasi Einstein Manifold

In general relativity and cosmology, the purpose of studying various types of semi-Riemannian manifolds is to represent the different phases in the evolution of the universe.The evolution of the universe to its present state can be divided into three phases: *The initial phase* is just after the Big-Bang when the effects of both viscosity and heat flux were quite pronounced.In *the intermediate phase,* the effect of viscosity was no longer significant but the heat flux was still not negligible. *The final phase* extends to the present state of the universe when both the effects of viscosity and the heat flux have become negligible and the matter content of the universe may be assumed to be a perfect fluid[19].

Thus, the importance of quasi-Einstein and generalized quasi-Einstein manifolds lies in the fact that these semi-Riemannian manifolds represent the second and the third phase respectively in the evolution of the universe. Additionally, a semi-Riemannian $G(QE)_4$ is related to the study of general relativistic fluid space-time admitting heat flux,[22]. Because of all these reasons, in this section, we prove the existence of a generalized quasi-Einstein manifold with non-zero and non-constant scalar curvature by constructing a non trivial example.

Now, we shall consider a Riemannian metric g on the 4-dimensional real number space R^4 by

$$ds^2 = g_{ij}dx^i dx^j = (e^{2x^4})(dx^1)^2 + (x^4)^4[(dx^2)^2 + (dx^3)^2] + (dx^4)^2, \tag{65}$$

where $-1-\sqrt{7} < x^4 < 1-\sqrt{3}$ or $-1+\sqrt{7} < x^4 < 1+\sqrt{3}$ and x^1, x^2, x^3, x^4 are the standard coordinates of R^4. Then the only non vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are

$$\Gamma^1_{14} = 1, \Gamma^4_{11} = -e^{2x^4}, \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{x^4}, \Gamma^4_{22} = \Gamma^4_{33} = -2(x^4)^3, \tag{66}$$

$$R_{1221} = R_{1331} = 2e^{2x^4}(x^4)^3, R_{1441} = e^{2(x^4)}, \tag{67}$$

$$R_{2332} = 4(x^4)^6, R_{2442} = R_{3443} = 2(x^4)^2, \tag{68}$$

$$R_{11} = \frac{4e^{2(x^4)}}{x^4} + e^{2(x^4)}, R_{22} = R_{33} = 2(x^4)^2[(x^4) + 3], R_{44} = 1 + \frac{4}{(x^4)^2} \tag{69}$$

and the components which can be obtained from these by symmetry properties. Also it can be shown that the scalar curvature is

$$r = \frac{16}{(x^4)^2} + \frac{8}{x^4} + 2, \tag{70}$$

which is non zero and non constant.

Let us now define associated scalar functions as

$$a = \frac{6}{(x^4)^2 + \frac{2}{(x^4)}}, b = 2 - \frac{8}{(x^4)^2}, c = (\frac{4}{(x^4)^2} - 1)tan(2\lambda) \tag{71}$$

and the 1-forms

$$A_i(x) = \begin{cases} e^{(x^4)}sin(\lambda) & if \;\; i = 1 \\ 0 & if \;\; i = 2,3 \\ cos(\lambda) & if \;\; i = 4 \end{cases} \tag{72}$$

and

$$B_i(x) = \begin{cases} e^{(x^4)}cos(\lambda) & if \;\; i = 1 \\ 0 & if \;\; i = 2,3 \\ -sin(\lambda) & if \;\; i = 4 \end{cases} \tag{73}$$

Here λ is some non-zero function of (x^4) satisfying the conditions
$sin^2(\lambda) = \frac{(x^4)^2+2(x^4)-6}{4(x^4)-4}$ and $cos^2(\lambda) = \frac{(x^4)^2-2(x^4)-2}{-4(x^4)+4}$
Then, we can show that
1. $R_{11} = ag_{11} + bA_1A_1 + 2cA_1B_1$,
2. $R_{22} = ag_{22} + bA_2A_2 + 2cA_2B_2$,
3. $R_{33} = ag_{33} + bA_3A_3 + 2cA_3B_3$,
4. $R_{44} = ag_{44} + bA_4A_4 + 2cA_4B_4$.
Since all the cases other than (1)-(4) are trivial, we obtain

$$R_{ij} = ag_{ij} + bA_iA_j + c(A_iB_j + A_jB_i), \quad for \quad i, j = 1, 2, 3, 4. \tag{74}$$

Moreover, we find

$$g^{ij}A_iA_j = 1, \quad g^{ij}B_iB_j = 1, \quad and \quad g^{ij}A_iB_j = 0 \tag{75}$$

and so

$$r = 4a + b = \frac{16}{(x^4)^2} + \frac{8}{(x^4)} + 2 \tag{76}$$

Therefore, this proves that the manifold under consideration is a generalized quasi Einstein manifold with non-zero and non-constant scalar curvature.
Hence we can state:

Theorem 4.1. *Let* $M^4 = \{(x^1, x^2, x^3, x^4) \in R^4 : x^4 \in (-1 - \sqrt{7}, 1 - \sqrt{3}) \cup (-1 + \sqrt{7}, 1 + \sqrt{3})\}$*be an open subset of* R^4 *endowed with the Riemannian metric given by*
$ds^2 = g_{ij}dx^idx^j = (e^{2x^4})(dx^1)^2 + (x^4)^4[(dx^2)^2 + (dx^3)^2] + (dx^4)^2$,
where x^1, x^2, x^3, x^4 *are the standard coordinates of* R^4*. Then* (M^4, g) *is a generalized quasi Einstein manifold with non zero and non constant scalar curvature* $r = \frac{16}{(x^4)^2} + \frac{8}{(x^4)} + 2$.

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