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Totally umbilical, pseudo-umbilical and pointwise slant submanifolds in Kaehler manifolds

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Abstract. In this paper, pointwise slant submanifolds of a Kaehler manifold are studied. First of all, if the submanifold of a Kaehler manifold is totally umbilical or pseudo-umbilical, the conditions for this submanifold to be a pointwise slant are investigated. Then given a pointwise submanifold of a Kaehler manifold, we investigate the conditions for this submanifold to be holomorphic, totally real, slant or CR-submanifold and obtain conditions depending on the behavior of the projections defined on the submanifold and the sectional curvature. Certain conditions are also found for the submanifold to be totally geodesic.

1. Introduction

Let $\tilde{M}$ be a Kaehler manifold with complex structure $\tilde{J}$ and M a Riemannian manifold isometrically immersed in $\tilde{M}$.

- M is called holomorphic or invariant [9] (complex) if $\tilde{J}(T_pM) \subset T_pM$, for every $p \in M$, where T_pM is denotes the tangent space to M at the point p.
- M is called totally real or anti-invariant [9] if $\tilde{J}(T_pM) \subset T_pM^{\perp}$ for every $p \in M$, where $T_pM^{\perp}$ denotes the normal space to M at the point p.
- M is called a CR-submanifold [2] if there exists a differentiable distribution $D : p \to D_p \subset T_pM$ such that D is invariant with respect to $\tilde{J}$ and the complementary distribution $D^{\perp}$ is anti-invariant with respect to $\tilde{J}$.
- M is called slant [8] if for all non-zero vector X tangent to M the angle $\theta(X)$ between $\tilde{J}X$ and T_pM is a constant, i.e, it does not depend on the choice of $p \in M$ and $X \in T_pM$.

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- M is called pointwise slant [10] if, at each given point $p \in M$, the angle $\theta(X)$ between $\tilde{J}X$ and the tangent space T_pM is independent of the choice of the nonzero tangent vector $X \in T_pM$. In this case,$\theta(X)$ can be considered as a function on M, which is called the slant function of the pointwise slant submanifold. Extensions of such submanifolds, see: [1], [3], [4], [5], [11], [14]. If $\theta(X)$ is constant, i.e. it is independent of the choice of any point $p \in M$, M becomes a slant submanifold. Moreover, if $\theta(X) = 0$ it becomes a holomorphic submanifold. Furthermore, if $\theta(X) = \frac{\pi}{2}$ then M becomes anti-invariant submanifold.

Let $\tilde{M}$ be a Kaehler manifold and M be a certain type (i.e. holomorphic, totally real or CR-submanifold) submanifold of $\tilde{M}$. If M is totally umbilical or pseudo-umbilical, various characterizations are known in the literature [2], [6], [7], [13]. However, if the submanifold of a Kaehler manifold is totally umbilical or pseudo-umbilical, there are very few studies on which of these specific types the submanifold falls into and under what conditions. In the previous paper, the first author of the current paper gave conditions on which submanifolds of certain types of a Kaehler manifold belongs to if its submanifold is pseudo-umbilical [12]. In this paper, the conditions for a totally umbilical or pseudo umbilical submanifold of a Kaehler manifold to be pointwise submanifold are studied.

The paper is organized as follows. In the second section, the concepts and formulas necessary for the paper are given. In the third section, first the criterion that a pseudo-umbilical submanifold of a Kaehler manifold is pointwise slant is given. Then the condition for the pointwise slant submanifold of a totally umbilical submanifold of a Kaehler manifold is presented. In the fourth section, firstly, the necessary and sufficient conditions for reducing a pointwise slant submanifold to a slant submanifold are given. Afterwards, the conditions under which a totally umbilical pointwise slant submanifold becomes an totally real submanifold are investigated. In this section, it is also obtained that a totally umbilical and non-totally real pointwise slant submanifold is the same as being an extrinsic sphere and being a CR-submanifold. Finally, an inequality containing the sectional curvature of the submanifold and the slant function is obtained, and if this inequality is equal, geometric results are given.

In this paper, every concept such as vector field, tensor field, function and so on is considered to be differentiable. The ring $C^\infty(M)$, which is the set of differentiable functions on the manifold M, is considered as the integral domain.

2. Preliminaries

Let M be an n-manifold immersed in an almost Hermitian manifold $(\tilde{M}, \tilde{J}, \tilde{g})$ with an almost complex structure $\tilde{J}$, $\tilde{J}^2 = -I$(where I denotes the identity map) and an almost Hermitian metric $\tilde{g}$, i.e., $\tilde{g}$ is a Riemannian metric satisfying

$$\tilde{g}(\tilde{J}X, \tilde{J}Y) = \tilde{g}(X, Y) \tag{1}$$

for vector fields X and Y on $\tilde{M}$. If $\tilde{M}$ is Kaehler manifold [15], then $\tilde{J}$ is parallel, i.e.

$$(\bar{\nabla}_X \tilde{J})Y = 0. \tag{2}$$

Consider a vector U at a point p of a Kaehler manifold $\tilde{M}$. Then the pair $(U, \tilde{J}U)$ determines a plane π (since $\tilde{J}U$ is obviously orthogonal to U) element called a holomorphic sectional, whose curvature $\bar{K}$ is given by

$$\bar{K} = \frac{g(\bar{R}(U, \tilde{J}U)\tilde{J}U, U)}{(g(U, U))^2},$$

and is called the holomorphic sectional curvature with respect to U, where $\bar{R}$ is the curvature tensor field of $\tilde{M}$. If $\bar{K}$ is independent of the choice of U at each point, then $\bar{K} = c$, an absolute constant. A simply connected complete Kaehler manifold of constant sectional curvature c is called a *complex space-form*, denoted by $\tilde{M}(c)$, which can be identified with the complex projective space $P_n(c)$, the open ball D_n in C^n or C^n according as $c > 0$, $c < 0$ or $c = 0$. The curvature tensor of $\tilde{M}(c)$ is

$$\bar{R}(X, Y)Z = \frac{c}{4}[g(Y, Z)X - g(X, Z)Y + g(\tilde{J}Y, Z)\tilde{J}X - g(\tilde{J}X, Z)\tilde{J}Y + 2g(X, \tilde{J}Y)\tilde{J}Z]. \tag{3}$$

for vector fields X, Y, Z on $\tilde{M}$, [9], [15].

Let M be a submanifold of a Kaehler manifold $\tilde{M}$. For any vector X tangent to M we put

$$\tilde{J}X = PX + FX \tag{4}$$

where PX and FX are the tangential and normal components of $\tilde{J}X$, respectively. Thus P is an endomorphism of the tangent bundle TM of M and F a normal-bundle-valued 1-form on TM. From (4) and (1) we have

$$g(PX, Y) = -g(X, PY) \tag{5}$$

for vector fields for X, Y.

For a submanifold M of an almost Hermitian manifold $(\tilde{M}, \tilde{J}, \tilde{g})$, we denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and $\tilde{M}$, respectively. Then the Gauss and Weingarten formulas of M in $\tilde{M}$ are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{6}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi \tag{7}$$

where X, Y are tangent vector, ξ is a normal vector field, h is the second fundamental form, $\nabla^\perp$ the normal connection and A the shape operator of M.The second fundamental form h and the Weingarten map A are related by

$$g\left(A_\xi X, Y\right) = \tilde{g}(h(X, Y), \xi). \tag{8}$$

The Gauss equation is given by

$$\bar{R}(X, Y)Z \quad = \quad R(X, Y)Z - A_{h(Y,Z)}X + A_{h(X,Z)}Y + (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) \tag{9}$$

where $\bar{R}$ and R, respectively, denote the curvature tensor fields corresponding to the Levi-Civita connections $\bar{\nabla}$ and ∇ and $(\nabla_X h)(Y, Z)$ is defined as

$$(\nabla_X h)(Y, Z) = \nabla^\perp_h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

where X, Y, Z are vector fields on M. The mean curvature vector $\mathfrak{H}$ of M is defined by $\mathfrak{H} = \left(\frac{1}{n}\right)$ trace $h, n = \dim M$. For any vector field ξ normal to the submanifold M, we put

$$\tilde{J}\xi = t\xi + f\xi \tag{10}$$

where $t\xi$ and $f\xi$ are the tangential and the normal components of $\tilde{J}\xi$, respectively. It is easy to verify that t is an endomorphism of the normal bundle and f is a tangent-bundle-valued 1-form on the normal bundle $TM^\perp$.

Finally, in this section, the notions of totally umbilical and pseudo-umbilical submanifold are recalled. If the below condition

$$h(X, Y) = g(X, Y)\mathbf{H} \tag{11}$$

is satisfied, the submanifold M is called a totally umbilical submanifold and the vector field $\mathbf{H}$ is called the mean curvature vector field. On the other hand, if the following condition

$$g(h(X, Y), \mathbf{H}) = g(X, Y)g(\mathbf{H}, \mathbf{H}) \tag{12}$$

is satisfied, a submanifold M is called pseudo-umbilical submanifold. It is easy to see that every totally umbilical submanifold is a pseudo-umbilical submanifold, but the converse is not true.

3. Criteria for a totally umbilical or pseudo-umbilical submanifold of a Kaehler manifold to be a pointwise slant submanifold

In this section, we are going to obtain certain conditions for an arbitrary totally umbilical or pseudo-umbilical submanifold of a Kaehler manifold to be a pointwise slant submanifold.

Lemma 3.1. *Let M be a submanifold of a Kaehler manifold. Then have*

$$g((\nabla_{PX}F)Y, \mathbf{H}) = -g(Y, (\nabla_{PX}t)\mathbf{H})$$

for $X, Y \in \Gamma(TM)$, where $(\nabla_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y$ (similar expansion for t) and $\mathbf{H}$ is the mean curvature vector field.

Proof. Using (1), (6) and (10)we have

$$g((\nabla_{PX}F)Y, \mathbf{H}) = g(\bar{\nabla}_{PX}FY, \mathbf{H}) + g(\nabla_{PX}Y, t\mathbf{H})$$

for $X, Y \in \Gamma(TM)$. Since $\bar{\nabla}$ is a metric connection, using (1), (2) and (10) we obtain

$$g((\nabla_{PX}F)Y, \mathbf{H}) = -PXg(Y, t\mathbf{H}) - g(FY, \nabla^{\perp}_{PX}\mathbf{H}) + PXg(Y, t\mathbf{H}) - g(Y, \nabla_{PX}t\mathbf{H}).$$

Hence we get

$$g((\nabla_{PX}F)Y, \mathbf{H}) = g(Y, t\nabla^{\perp}_{PX}\mathbf{H}) - g(Y, \nabla_{PX}t\mathbf{H}).$$

Using again (10), we get the assertion. □

Theorem 3.2. *Let M be a pseudo-umbilical submanifold of a Kaehler manifold. Then M is a pointwise-slant submanifold if and only if*

$$\|\mathbf{H}\|^{-2}\left[(\nabla_{PX}B)\mathbf{H} - A_{f\mathbf{H}}PX\right] = \cos^2\theta X$$

for every $X \in \Gamma(TM)$, where θ is real valued function defined on the tangent bundle TM of M.

Proof. Since M is pseudo-umbilical, from (1) and (12) we derive

$$g(h(PX, PY), \mathbf{H}) = g(P^2X, Y) \tag{13}$$

for $X, Y \in \Gamma(TM)$. On the other hand, since $\tilde{M}$ is a Kaehler manifold, using (2), (4) and (6) we get

$$g(h(PX, PY), \mathbf{H}) = -g(\tilde{J}\nabla_{PX}Y, \mathbf{H}) + g(h(PX, Y), \tilde{J}\mathbf{H}) + g(\nabla^{\perp}_{PX}FY, \mathbf{H}).$$

Using again (4) we obtain

$$g(h(PX, PY), \mathbf{H}) = g((\nabla_{PX}F)Y, \mathbf{H}) + g(h(PX, Y), f\mathbf{H}).$$

Thus from (8) and Lemma 3.1 we have

$$g(h(PX, PY), \mathbf{H}) = g(A_{f\mathbf{H}}PX, Y) - g((\nabla_{PX}B)\mathbf{H}, Y) \tag{14}$$

Then proof follows from (13), (14) and [10, Lemma 1]. □

We now consider totally umbilicity case.

Theorem 3.3. *Let M be a totally-umbilical submanifold of a Kaehler manifold. Then M is a pointwise slant submanifold if and only if*

$$\|\mathbf{H}\|^{-2}(\nabla_{PX}B)\mathbf{H} = \cos^2\theta X$$

for $X \in \Gamma(TM)$ where θ is a real valued function defined on the tangent bundle TM of M.

Proof. From(1), (5) and (11), we get

$$h(PX, PY) = -g(P^2X, Y) \tag{15}$$

for $X, Y \in \Gamma(TM)$. On the other hand, using (4), (6), (7) and (10) we obtain

$$\begin{aligned} h(PX, PY) &= P\nabla_{PX}Y + F\nabla_{PX}Y + g(PX, Y)t\mathbf{H} + g(PX, Y)f\mathbf{H} + A_{FY}PX \\ &- \nabla^{\perp}_{PX}FY + \nabla_{PX}PY \end{aligned}$$

Taking the normal part of this equation, we have

$$h(PX, PY) = -(\nabla_{PX}F)Y + g(PX, Y)f\mathbf{H} \tag{16}$$

Take inner product of both sides of (16) with **H** and using (15) we derive

$$g((\nabla_{PX}F)Y, \mathbf{H}) = g(P^2X, Y)g(\mathbf{H}, \mathbf{H}) \tag{17}$$

Thus from Lemma 3.1 and (17) we get

$$g(Y, (\nabla_{PX}B)\mathbf{H}) = -g(P^2X, Y)g(\mathbf{H}, \mathbf{H})$$

Thus proof is complete □

Hence, we have the below result.

Corollary 3.4. *There do not exist non-minimal totally-umbilical pointwise slant submanifold of a Kaehler manifold with parallel t.*

4. Relations between the pointwise slant submanifold of a Kaehler manifold and some other kinds of submanifolds via notions of submanifolds

In this section, we will consider a pointwise slant submanifold and investigate certain conditions for such submanifold to be totally geodesic, extrinsic sphere, totally real submanifold, slant submanifold or CR-submanifold.

Lemma 4.1. *Let M be a pointwise slant submanifold of a Kaehler manifold $\tilde{M}$. Then we have*

$$\sin 2\theta Y(\theta)X = -\sin^2\theta\nabla_Y X - \nabla_Y tFX + A_{fFX}Y + A_{FPX}Y \tag{18}$$

and

$$\sin^2\theta h(Y, X) = -h(Y, tFX) - \nabla^{\perp}_Y fFX - \nabla^{\perp}_Y FPX \tag{19}$$

for $X, Y \in \Gamma(TM)$.

Proof. From [10, Lemma 1], we know that $-\cos^2\theta X = P^2X$ for vector field X on M. Taking the covariant derivative both sides with respect to vector field Y, and using (4), (2), (10), (6), (7) and taking the tangential parts and normal parts of the outcome, we get (18) and (19). □

Theorem 4.2. *Let M be a pointwise slant submanifold of a Kaehler manifold $\tilde{M}$. M is a slant submanifold if and only if M is either holomorphic submanifold or $\nabla_Y X = -\sec^2\theta\nabla_Y tFX$ for $X, Y \in \Gamma(TM)$.*

Proof. Let M be a pointwise slant submanifold of a Kaehler manifold $\tilde{M}$. Taking the inner product of both sides of (18) with a vector field Z on M and using (8), we derive

$$\sin 2\theta Y(\theta) g(X,Z) = -\sin^2\theta g(\nabla_Y X, Z) - g(\nabla_Y tFX, Z) + g(h(Y,Z), fFX) + g(h(Y,Z), FPX). \quad (20)$$

On the other hand, using the complex structure $\tilde{J}^2 X = -X$ for arbitrary vector field on M and applying (10) and (4), we get

$$\tilde{J}^2 X = P^2 X + FPX + tFX + fFX.$$

Taking the normal parts of the above expression, we obtain

$$FPX + fFX = 0 \quad (21)$$

If (21) is also taken into account in (20) we have

$$\sin 2\theta Y(\theta) g(X,Z) = -\sin^2\theta g(\nabla_Y X, Z) - g(\nabla_Y tFX, Z).$$

Hence we get

$$Y(\theta) g(X,Z) = g(-\frac{1}{2}\tan\theta \nabla_Y X - \sec 2\theta \nabla_Y tFX, Z).$$

Since this expression is true for any arbitrary Z and g is a Riemannian metric, we arrive at

$$Y(\theta)X = \frac{-1}{2}\tan\theta \nabla_Y X - \sec 2\theta \nabla_Y tFX.$$

Hence we have

$$Y(\theta)X = -\frac{1}{2}\tan\theta[\nabla_Y X + \sec^2\theta \nabla_Y tFX]$$

which completes proof. □

Theorem 4.3. *Let M be a pointwise slant submanifold of a Kaehler manifold $\tilde{M}$. If*

$$g(A_{\mathbf{H}} tFX, \nabla_Z A_{\mathbf{H}} X) = \cos^2\theta \|A_{\mathbf{H}} X\|^2 - \frac{1+\cos^2\theta}{2} Z(\| A_{\mathbf{H}} X \|^2) \quad (22)$$

for $X, Z \in \Gamma(TM)$. Then at least one of the following assertions are true

(i) M is a slant submanifold,
(ii) $A_{\mathbf{H}} X = 0$ for $X \in \Gamma(TM)$,
(iii) M is a holomorphic submanifold,
(iv) M is a totally real submanifold

Proof. Let M be a pointwise slant submanifold of a Kaehler manifold $\tilde{M}$. Using (21) in (19) we have $\sin^2\theta h(Y,X) = -h(Y, tFX)$ for $X, Y \in \Gamma(TM)$. Taking inner product of this expression with the mean curvature vector field and using (8) we derive

$$\sin^2\theta A_{\mathbf{H}} X = -A_{\mathbf{H}} tFX.$$

Taking the covariant derivative of both sides of this equation with respect a vector field Z on M, using the Gauss formula (6) and taking the tangential parts of the resulting equation, we get

$$\sin 2\theta Z(\theta) A_{\mathbf{H}} X + \sin^2\theta \nabla_Z A_{\mathbf{H}} X = -\nabla_Z A_{\mathbf{H}} tFX.$$

Taking the inner product of both sides of this equation with $A_{\mathbf{H}} X$, using the Gauss formula (6) and taking into account (4), we obtain

$$\sin 2\theta Z(\theta) \|A_{\mathbf{H}} X\|^2 + \sin^2\theta g(\nabla_Z A_{\mathbf{H}} X, A_{\mathbf{H}} X) = Z g(X, A_{\mathbf{H}} A_{\mathbf{H}} X) - g(PX, \tilde{J} A_{\mathbf{H}} A_{\mathbf{H}} X) + g(A_{\mathbf{H}} tFX, \nabla_Z A_{\mathbf{H}} X).$$

Since A is self-adjoint with respect to g and P is anti-symmetric, using $-\cos^2\theta X = P^2X$ and (4), we arrive at

$$\begin{aligned}\sin 2\theta Z(\theta)\,\|A_{\mathbf{H}}X\|^2 \quad + \quad & \sin^2\theta\frac{1}{2}Zg(A_{\mathbf{H}}X, A_{\mathbf{H}}X) = Zg(A_{\mathbf{H}}X, A_{\mathbf{H}}X)\\ & -\cos^2\theta g(A_{\mathbf{H}}X, A_{\mathbf{H}}X) + g(A_{\mathbf{H}}tFX, \nabla_Z A_{\mathbf{H}}X).\end{aligned}$$

Hence we have

$$\sin 2\theta Z(\theta)\,\|A_{\mathbf{H}}X\|^2 = \frac{1+\cos^2\theta}{2}Z(\|\,A_{\mathbf{H}}X\,\|^2) - \cos^2\theta\,\|\,A_{\mathbf{H}}X\,\|^2 + g(A_{\mathbf{H}}tFX, \nabla_Z A_{\mathbf{H}}X). \tag{23}$$

We now suppose that (22) is satisfied in (23), then we have $\sin 2\theta Z(\theta)\,\|A_{\mathbf{H}}X\|^2 = 0$. This implies that one of $\sin 2\theta$, $Z(\theta)$ and $\|A_{\mathbf{H}}X\|^2$ should be zero. If $\sin 2\theta = 0$, then either $\theta = 0$ or $\theta = \frac{\pi}{2}$ which gives (iii) and (iv). If $\|A_{\mathbf{H}}X\|^2 = 0$, it gives $A_{\mathbf{H}}X = 0$. Finally if $Z(\theta) = 0$, then it follows that θ is constant and M is a slant submanifold. □

Theorem 4.4. *Let M be a totally umbilical pointwise slant submanifold of a complex space form $\tilde{M}(c)$. Then either M is a totally real submanifold or $\nabla^{\perp}_X\mathbf{H} = \frac{3}{4}c\cos^2\theta(F\circ P)X$ for a vector field X on M.*

Proof. For $X \in \Gamma(TM)$, from (3), (4) and (5), we have

$$\bar{R}(X, PX)PX = \frac{c}{4}\left\{g(PX, PX)X - 3g(PX, PX)P^2X - 3g(X, P^2X)(F\circ P)X\right\}.$$

Since M is a pointwise slant submanifold we have $P^2X = -\cos^2\theta X$ and $g(PX, PX) = \cos^2\theta g(X, X)$. Thus we get

$$\bar{R}(X, PX)PX = \frac{c}{4}\left\{\cos^2\theta g(X, X)X + 3\cos^4\theta g(X, X)X + 3\cos^2\theta g(X, X)(F\circ P)X\right\}.$$

Then using Gauss equation (9) and taking tangential and normal parts we derive

$$\frac{c}{4}\left\{\cos^2\theta(1+\cos^2\theta)g(X, X)X\right\} = R(X, PX)PX - A_{h(PX,PX)}X + A_{h(X,PX)}PX \tag{24}$$

and

$$(\nabla_X h)(PX, PX) - (\nabla_{PX} h)(X, PX) = \frac{3c}{4}\cos^2\theta g(X, X)(F\circ P)X. \tag{25}$$

Since M is totally umbilical, from (25) we obtain

$$g(PX, PX)\nabla^{\perp}_X\mathbf{H} = \frac{3c}{4}\cos^2\theta g(X, X)(F\circ P)X.$$

Hence we get

$$\cos^2\theta\,\|\,X\,\|^2\,(\nabla^{\perp}_X\mathbf{H} - \frac{3c}{4}(F\circ P)X) = 0 \tag{26}$$

which shows that either $\cos^2\theta = 0$ or $\nabla^{\perp}_X\mathbf{H} = \frac{3c}{4}(F\circ P)X)$. □

From (26), we have the below result.

Theorem 4.5. *Let M be a totally umbilical non-totally real pointwise slant submanifold of a complex space form $\tilde{M}(c)$ such that the rank of P is constant. Then M is an extrinsic sphere if and only if M is a CR-submanifold.*

Proof. Proof follows from [2, Theorem 1.1, page:21] □

Finally, we examine what the character of the submanifold will be when the sectional curvature of the submanifold satisfies some constraints.

Theorem 4.6. *Let M be a totally umbilical pointwise slant submanifold of a complex space form $\tilde{M}(c)$. Then we have*

$$K(X\wedge PX)\geq \frac{c}{4}(1+3\cos^2\theta)\cos^2\theta \parallel X\parallel^4 . \tag{27}$$

The equality is satisfied if and only if either M is a totally real submanifold or M is a totally geodesic submanifold.

Proof. From (3) and (9), we obtain

$$\begin{aligned} R(X,PX)PX \quad - \quad & A_{h(PX,PX)}X + A_{h(X,PX)}PX = \frac{c}{4}\{g(PX,PX)X - g(X,PX)PX - g(PX,PX)P^2X \\ & + g(PX,P^2X)PX + 2g(X,P^2X)P^2X\}. \end{aligned}$$

Since M is totally umbilical, we have

$$h(PX,PX) = g(PX,PX)\mathbf{H}, h(X,PX) = g(X,PX)\mathbf{H} = 0.$$

Thus, from the above identities and (5), we reach the following expression

$$R(X,PX)PX - g(PX,PX)A_{\mathbf{H}}X = \frac{c}{4}\{-g(P^2X,X)X + 3g(P^2X,X)P^2X\}$$

Thus if M is pointwise slant submanifold, we derive

$$R(X,PX)PX = \cos^2\theta g(X,X)A_{\mathbf{H}}X + \frac{c}{4}\{\cos^2\theta(1+3\cos^2\theta)g(X,X)X\}$$

Using the umbilicity of M, we arrive at

$$K(X\wedge PX) = \cos^2\theta \parallel X\parallel^4 \{\frac{c}{4}(1+3\cos^2\theta) + \parallel \mathbf{H}\parallel^2\} \tag{28}$$

which proves (27). The equality case is satisfied if and only if

$$\cos^2\theta \parallel X\parallel^4 \parallel \mathbf{H}\parallel^2 = 0$$

which shows that either $\cos\theta = 0$ or $\mathbf{H} = 0$. □

From (28) we have the following result.

Theorem 4.7. *Let M be a totally umbilical pointwise slant submanifold of a complex space form $\tilde{M}(c)$. If $K(X\wedge PX) = 0$ either M is totally real submanifold or $c\leq 0$. When $c\leq 0$, if $c = 0$,then M is totally geodesic,if $c<0$, then $\parallel \mathbf{H}\parallel^2 = -\frac{c}{4}(1+3\cos^2\theta)$.*

Proof. Let M be a totally umbilical pointwise slant submanifold of a complex space form $\tilde{M}(c)$ such that $K(X\wedge PX) = 0$. Then we have $cos^2\theta = 0$ which shows that M is totally real or $\{\frac{c}{4}(1+3\cos^2\theta) + \parallel \mathbf{H}\parallel^2\} = 0$. The latter case implies that $c\leq 0$. In this case, if $c = 0$, then it follows that $\mathbf{H} = 0$. If $c<0$, then we have $\parallel \mathbf{H}\parallel^2 = -\frac{c}{4}(1+3\cos^2\theta)$. □

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