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Almost Ricci solitons on co-Kähler manifolds

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Abstract. This paper investigates the geometrical structure of co-Kähler manifolds admitting semi-symmetric metric ξ-connection ($\mathcal{SSM}$ ξ-connection). We derive several fundamental identities involving the curvature tensor, scalar curvature, and Ricci operator that characterize the geometry of such manifolds. In particular, we identify the criteria for the manifold to be η-Einstein and examine the behavior of Ricci soliton structures in such framework. We prove that when a co-Kähler manifold with $\mathcal{SSM}$ ξ-connection admit a Ricci soliton, then the soliton is necessarily expanding. Furthermore, we reveal that an η-Einstein co-Kähler manifold with such a connection admitting Ricci soliton structure is Einstein. Additionally, we investigate three-dimensional co-Kähler manifolds admitting Ricci soliton with a $\mathcal{SSM}$ ξ-connection, and conclude with an examination of co-Kähler manifolds equipped with such a connection admitting gradient Ricci almost solitons.

1. Introduction

Friedmann and Schouten [16] first proposed the concept of a semi-symmetric linear connection on a differentiable manifold in 1924. They considered a linear connection $\bar{\nabla}$ on an n-dimensional semi-Riemannian manifold $\mathcal{M}$, distinct from the Levi-Civita connection ∇ corresponding to the metric g. Such a connection is termed as symmetric if its torsion tensor denoted by $\bar{T}$ is expressed as

$$\bar{T}(\zeta_1, \zeta_2) = \bar{\nabla}_{\zeta_1}\zeta_2 - \bar{\nabla}_{\zeta_2}\zeta_1 - [\zeta_1, \zeta_2],$$

vanishes identically on the manifold $\mathcal{M}$. Or else, it is a non-symmetric connection. On a manifold $\mathcal{M}$, a linear connection $\bar{\nabla}$ is referred to as semi-symmetric if

$$\bar{T}(\zeta_1, \zeta_2) = \eta(\zeta_2)\zeta_1 - \eta(\zeta_1)\zeta_2, \tag{1}$$

$\forall\ \zeta_1, \zeta_2 \in \chi(\mathcal{M})$, where η is a 1-form corresponding to the vector field ξ satisfying

$$\eta(\zeta_1) = g(\zeta_1, \xi). \tag{2}$$

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In 1932, the metric connection $\bar{\nabla}$ was first proposed by Hayden [21] on a Riemannian manifold, which is commonly known by his name as *Hayden connection*. Later on, Pak [32] showed that this was a $\mathcal{SSM}$ connection. The linear connection $\bar{\nabla}$ on $\mathcal{M}$ is regarded as a *metric* if $\bar{\nabla}g = 0$, or else it is termed as a *non-metric*. The study on $\mathcal{SSM}$ connection was further developed by Yano [44]. Subsequently, this concept was studied in different perspectives by numerous geometers including Sharfuddin and Hussain [34], Yano and Imai [45], De and De [14], De and Biswas [13], Tripathi [37], Chaubey and Kumar [9], Pahan and Bhattacharyya [31], Kazan and Kazan [25], Yadav et al. [42], Dogru [15] and several others. Building upon these foundational works, Chaubey et al. [10] extended the study of $\mathcal{SSM}$ ξ-connection and its properties on Riemannian manifold. This framework was further examined in the realm of three-dimensional Riemannian manifolds admitting Yamabe soliton [8]. Additionally, Chaubey et al. [12] explored analogous structures on Lorentzian manifolds. Meanwhile, Özgür [30] obtained some properties of Ricci solitons with $\mathcal{SSM}$ connection. Recently, Li et al. [27] investigated $\mathcal{SSM}$ connection on Lorentzian manifolds admitting Ricci almost solitons, uncovering significant geometric and physical implications. Bulut and İnselöz [6] investigated the curvature tensor, the Ricci tensor and scalar curvature tensor with respect to the generalized symmetric metric connection on para-Sasaki-like manifolds. Furthermore, building upon the aforementioned works, Li et al. [28] investigated the properties of the Bach and Cotton tensors on a class of Lorentzian manifolds equipped with a $\mathcal{SSM}$ ω-connection. They demonstrated that if the Cotton tensor is parallel on such a manifold, then it is necessarily quasi-Einstein and Bach-flat. Their study also emphasized the crucial role of Bach and Cotton tensors in understanding the geometry of spacetime and its interaction with matter and energy.

Ricci solitons, introduced by Hamilton [18], regarded as a natural generalization of Einstein metrics, have garnered significant attention in the field of differential geometry, especially in the study of almost contact Riemannian manifolds, over the past decade. A semi-Riemannian manifold $\mathcal{M}$ is said to be Ricci soliton, if $\exists$ a smooth vector field V on $\mathcal{M}$ such that

$$S + \frac{1}{2}\pounds_V g = \lambda g, \tag{3}$$

where $\pounds$ denotes the Lie derivative, S indicates the Ricci tensor and λ stands for a real number i.e. $\lambda = \frac{divV + r}{n}$, provided $n = dim\mathcal{M}$ and r symbolizes the scalar curvature of $\mathcal{M}$. The nature of Ricci solitons depends on λ, i.e. it expands for $\lambda < 0$, shrinks for $\lambda > 0$, and is steady when $\lambda = 0$. If $\exists$ a potential function h satisfying $V = gradh$, then the Ricci soliton is said to be a *gradient Ricci soliton* by which the equation (3) transforms to

$$Hess_h + S = \lambda g, \tag{4}$$

where $Hess_h$ indicates the Hessian of h. Pigola et al. [33] extended the study of Ricci soliton by permitting the soliton constant λ to vary smoothly on $\mathcal{M}$, thereby transforming equations (3) and (4) into the fundamental formulations of Ricci almost and gradient Ricci almost solitons, respectively. The geometric behavior of curvature, characterized by the Ricci tensor, is intrinsically linked to the geometry of the potential function through the structure of a (gradient) Ricci almost soliton structure. Therefore, Ivey [22], Hamilton [19], Sharma [35] contributed to certain results of solitons in contact Riemannian manifolds admitting Ricci flow. Batat et al. [3] studied Ricci solitons on Lorentzian manifolds endowed with large isometry groups. The structure equations for Ricci almost soliton structure was discussed by Barros and Ribeiro [2]. Venkatesha et al. [38] worked on $*$-Ricci solitons in Kenmotsu manifolds and substantiated that the soliton constant λ is zero, and in dimension three, such manifolds have negative constant sectional curvature -1. They also showed that the manifold is Einstein when the potential vector field is collinear with the Reeb vector field ξ, and in such case, the soliton vector field coincides with ξ. Ganguly and Bhattacharyya [17] worked on almost co-Kähler manifolds admitting conformal Ricci soliton and studied the nature of soliton by determining λ, the soliton constant. Chaubey and Suh [11] characterized the Ricci soliton structure on Lorentzian manifolds having $\mathcal{SSM}$ ξ-connection considering $V = \xi$. Dogru [15] obtained certain properties of η-Ricci Bourguignon solitons on Riemannian manifolds wherein the potential vector field is torse-forming with respect to a $\mathcal{SSM}$ and semi-symmetric non-metric connection. Biswas and De [4] characterized almost co-Kähler and co-Kähler 3-manifolds admitting gradient ρ-Einstein solitons. They obtained constant scalar curvature with respect to co-Kähler 3-manifold and also showed that, the manifold is either flat or the gradient of the potential function is collinear with the Reeb vector field ξ. Recently, Li et al. [27] investigated Ricci almost and gradient Ricci almost solitons on a class of Lorentzian manifolds equipped with $\mathcal{SSM}$ connection. one of their main results proved that if a quasi-Einstein Lorentzian manifold with $\mathcal{SSM}$ connection admits a Ricci soliton, then the manifold is Einstein. They had also given some geometrical and physical motivations for future research study. For further insight and motivation of our paper, we suggest [1, 5, 7, 20, 23, 24, 26, 36], [39]-[42].

Although substantial work has been done on Ricci solitons and their variants across Riemannian, Lorentzian, Kenmotsu, Sasakian and almost contact metric manifolds, the interaction between Ricci (almost) soliton structures and $\mathcal{SSM}$ ξ-connection within the framework of co-Kähler manifolds has not been adequately addressed in the literature. Most existing studies focus either on Ricci solitons and its variants on co-Kähler manifolds without incorporating $\mathcal{SSM}$ ξ-connection.

Inspired by the works mentioned above and the identified gap, this paper investigates the geometrical behavior of Ricci soliton and gradient Ricci almost soliton structures on co-Kähler manifolds admitting $\mathcal{SSM}$ ξ-connection, and establishes several associated geometric properties. The sequence of the paper is outlined below: section 2 devotes on co-Kähler manifolds admitting $\mathcal{SSM}$ ξ-connection. Section 3 focuses on the analysis of geometric properties of Ricci solitons on co-Kähler manifolds admitting $\mathcal{SSM}$ ξ-connection. Section 4 explores the study of co-Kähler manifolds with $\mathcal{SSM}$ ξ-connection admitting gradient Ricci almost solitons. The final section, draws some concluding remarks based on the obtained results.

2. Co-Kähler Manifolds Admitting $\mathcal{SSM}$ ξ-Connection

In differential geometry, co-Kähler manifolds forms a significant class of *almost contact metric manifolds* ($\mathcal{ACMM}$) which exhibits rich geometric structures and has close analogies with Kähler geometry. These manifolds naturally arise in the study of $\mathcal{ACMM}$ with parallel structure tensors and play a significant role in various geometries including contact geometry, complex geometry, and certain models in theoretical physics.

A smooth manifold $\mathcal{M}$ of dimension $(2n+1)$ is called as an *almost contact structure* (ϕ, ξ, η) if there exist a $(1, 1)$ tensor field ϕ, a vector field ξ and a global 1-form η on $\mathcal{M}$ satisfying the following conditions:

$$\phi^2 = -I + \eta \otimes \xi \quad \& \quad \eta(\xi) = 1, \tag{5}$$

here I denotes the identity endomorphism on $\mathcal{M}$. A manifold with such a structure is known as an *almost contact manifold* ($\mathcal{ACM}$) [5], represented by the quadruple $(\mathcal{M}^{2n+1}, \phi, \xi, \eta)$. The vector field ξ is referred to be the characteristic or the Reeb vector field. From equation (5), we deduce that the below mentioned conditions hold in an almost contact structure:

$$\phi(\xi) = 0 \quad \& \quad \eta \circ \phi = 0.$$

Moreover, an $\mathcal{ACM}$ $(\mathcal{M}^{2n+1}, \phi, \xi, \eta)$ is said to admit a compatible *almost contact metric structure* if $\exists$ a Riemannian metric g satisfying

$$g(\phi\zeta_1, \phi\zeta_2) = g(\zeta_1, \zeta_2) - \eta(\zeta_1)\eta(\zeta_2),$$

$\forall\ \zeta_1, \zeta_2 \in T\mathcal{M}$, where $T\mathcal{M}$ denotes the tangent bundle of $\mathcal{M}$. In this case, the quadruple (ϕ, ξ, η, g) defines an almost contact metric structure, and the manifold $(\mathcal{M}^{2n+1}, \phi, \xi, \eta, g)$ is called an $\mathcal{ACMM}$.

The fundamental 2-form Φ on a manifold $\mathcal{M}$ is defined as

$$\Phi(\zeta_1, \zeta_2) = g(\zeta_1, \phi\zeta_2),$$

for any vector fields ζ_1, ζ_2. An $\mathcal{ACMM}$ is called as an *almost co-Kähler manifold*, if both η and Φ are closed. i.e. $d\eta = 0$ and $d\Phi = 0$.

An $\mathcal{ACMM}$ $(\mathcal{M}^{2n+1}, \phi, \xi, \eta, g)$ is said to be *normal*, if the almost complex structure J defined on $\mathcal{M} \times \mathbb{R}$ by [5]

$$J\Big(\zeta_1, f\frac{d}{dt}\Big) = \Big(\phi\zeta_1 - f\xi, \eta(\zeta_1)\frac{d}{dt}\Big),$$

is integrable, where f is a real valued function on $\mathcal{M} \times \mathbb{R}$. Moreover, if an $\mathcal{ACM}$ $(\mathcal{M}^{2n+1}, \phi, \xi, \eta)$ is normal, then it is referred as a *co-Kähler manifold*. Additionally, an $\mathcal{ACMM}$ $(\mathcal{M}^{2n+1}, \phi, \xi, \eta, g)$ is co-Kähler if and only if it holds the condition that $\nabla\phi = 0$, or equivalently, $\nabla\Phi = 0$, where ∇ is the Levi-Civita connection associated with the metric g.

This connection ∇ then induces a linear connection $\bar{\nabla}$ on $\mathcal{M}$, which is defined by

$$\bar{\nabla}_{\zeta_1}\zeta_2 = \nabla_{\zeta_1}\zeta_2 + \eta(\zeta_2)\zeta_1 - g(\zeta_1, \zeta_2)\xi, \quad \forall \ \zeta_1, \zeta_2 \in \Gamma(T\mathcal{M}). \tag{6}$$

The investigation on $\mathcal{ACMM}$ was undertaken by Mishra et al. [29] and Chaubey et al. [9], where they characterized the condition $\bar{\nabla}\xi = 0$ and yielded a number of significant geometrical interpretations. Chaubey et al. [10, 11] later expanded this concept to Riemannian and Lorentzian manifolds, introducing what is known as the $\mathcal{SSM}$ ξ-connection.

Consider $\bar{\nabla}\xi = 0$, hence by equation (6), we have

$$\nabla_{\zeta_1}\xi = -\zeta_1 + \eta(\zeta_1)\xi, \tag{7}$$

where ξ is a characteristic vector field, i.e. $g(\xi, \xi) = \eta(\xi) = 1$. Using this together with equations (2) and (6), we get

$$(\bar{\nabla}_{\zeta_1}\eta)(\zeta_2) = (\nabla_{\zeta_1}\eta)(\zeta_2) + g(\zeta_1, \zeta_2) - \eta(\zeta_1)\eta(\zeta_2).$$

Since $(\bar{\nabla}_{\zeta_1}\eta)\zeta_2 = 0$, from the preceding equation we infer

$$(\nabla_{\zeta_1}\eta)(\zeta_2) = \eta(\zeta_1)\eta(\zeta_2) - g(\zeta_1, \zeta_2). \tag{8}$$

To proceed toward the main results, we first prove the following essential lemmas.

Lemma 2.1. *A co-Kähler manifold $\mathcal{M}^{2n+1}$ admitting $\mathcal{SSM}$ ξ-connection $\bar{\nabla}$ satisfies*

$$R(\zeta_1, \zeta_2)\xi = \eta(\zeta_1)\zeta_2 - \eta(\zeta_2)\zeta_1, \tag{9}$$

$$R(\xi, \zeta_1)\zeta_2 = \eta(\zeta_2)\zeta_1 - g(\zeta_1, \zeta_2)\xi, \tag{10}$$

$$g(R(\zeta_1, \zeta_2)\zeta_3, \xi) = \eta(\zeta_2)g(\zeta_1, \zeta_3) - \eta(\zeta_1)g(\zeta_2, \zeta_3), \tag{11}$$

for $\zeta_1, \zeta_2, \zeta_3 \in \Gamma(T\mathcal{M})$.

Proof. Consider a co-Kähler manifold $\mathcal{M}^{2n+1}$. The Riemannian curvature tensor denoted by R on $\mathcal{M}$ with $\mathcal{SSM}$ connection ∇ is expressed as

$$R(\zeta_1, \zeta_2)\zeta_3 = \nabla_{\zeta_1}\nabla_{\zeta_2}\zeta_3 - \nabla_{\zeta_2}\nabla_{\zeta_1}\zeta_3 - \nabla_{[\zeta_1,\zeta_2]}\zeta_3.$$

In light of equations (2), (7) and (8), the preceding equation becomes

$$R(\zeta_1, \zeta_2)\xi = \nabla_{\zeta_1}[\eta(\zeta_2)\xi - \eta(\xi)\zeta_2] - \nabla_{\zeta_2}[\eta(\zeta_1)\xi - \eta(\xi)\zeta_1] - \eta([\zeta_1, \zeta_2])\xi + \eta(\xi)[\zeta_1, \zeta_2],$$

which on straightforward calculations gives

$$R(\zeta_1, \zeta_2)\xi = \eta(\zeta_1)\zeta_2 - \eta(\zeta_2)\zeta_1.$$

Therefore, with the help of the above equation, we can easily prove (10) and (11). □

These curvature identities characterize the interaction between the characteristic vector field ξ and the curvature tensor under the $\mathcal{SSM}$ ξ-connection. Building on these, now we examine how the Ricci operator behaves in this geometric setting.

Lemma 2.2. *A co-Kähler manifold $\mathcal{M}^{2n+1}$ admitting $\mathcal{SSM}$ ξ-connection $\bar{\nabla}$ satisfies*

$$(\nabla_{\zeta_1}Q)\xi = Q\zeta_1 + 2n\zeta_1, \tag{12}$$

$$(\nabla_{\xi}Q)\zeta_1 = 2Q\zeta_1 + 4n\zeta_1, \tag{13}$$

here Q represents the Ricci operator defined by $S(\zeta_1, \zeta_2) = g(Q\zeta_1, \zeta_2)$.

Proof. Contraction of ζ_2 in equation (9) gives

$$S(\zeta_1, \xi) = -2n\eta(\zeta_1), \tag{14}$$

which provides $Q\xi = -2n\xi$. Differentiating this with respect to ζ_1 and using equation (7) leads to equation (12).

Now, differentiating equation (9) along ζ_3 and using (8) yields

$$(\nabla_{\zeta_3} R)(\zeta_1, \zeta_2)\xi = R(\zeta_1, \zeta_2)\zeta_3 + g(\zeta_2, \zeta_3)\zeta_1 - g(\zeta_1, \zeta_3)\zeta_2.$$

Considering $\{E_i\}_{i=1}^{2n+1}$, a local orthonormal basis on $\mathcal{M}$. Setting $\zeta_1 = \zeta_3 = E_i$ and then summing over i, in the above equation leads to

$$\sum_{i=1}^{2n+1} g((\nabla_{E_i} R)(E_i, \zeta_2)\xi, \zeta_3) = -S(\zeta_2, \zeta_3) - 2ng(\zeta_2, \zeta_3). \tag{15}$$

Now, applying the second Bianchi identity leads to

$$\sum_{i=1}^{2n+1} g((\nabla_{E_i} R)(\zeta_3, \xi)\zeta_2, E_i) = g((\nabla_{\zeta_3} Q)\xi, \zeta_2) - g((\nabla_{\xi} Q)\zeta_3, \zeta_2).$$

Equation (15) along with the above equation yields

$$g((\nabla_\xi Q)\zeta_3, \zeta_2) = 2[S(\zeta_2, \zeta_3) + 2ng(\zeta_2, \zeta_3)],$$

which by the definition of the Ricci operator Q provides equation (13). □

Utilising these results, we now derive a relation involving the scalar curvature r, leading to the following lemma.

Lemma 2.3. *A co-Kähler manifold $\mathcal{M}^{2n+1}$ admitting $\mathcal{SSM}$ ξ-connection $\bar{\nabla}$ satisfies*

$$\xi(r) = 2[2n(2n+1) + r].$$

Proof. Applying g-trace on equation (12), we obtain the required identity. □

This leads naturally to the succeeding lemma, where we further explore the directional behavior of the scalar curvature through its gradient.

Lemma 2.4. *A co-Kähler manifold $\mathcal{M}^{2n+1}$ of scalar curvature r admitting $\mathcal{SSM}$ ξ-connection $\bar{\nabla}$ satisfies*

$$gradr = \xi(r)\xi. \tag{16}$$

Proof. From Lemma 2.3, we can have $\pounds_\xi r = 2[2n(2n+1) + r]$. Applying the exterior derivative d in view that $\pounds_\xi$ commutes with d in this equation, implies $\pounds_\xi dr = 2dr$, which is expressible in terms of gradient operator as $\pounds_\xi gradr = 2gradr$. Employing equation (7) yields

$$\nabla_\xi gradr = gradr + \xi(r)\xi. \tag{17}$$

As, it is known that $\xi(r) = g(\xi, gradr) = 2[2n(2n+1) + r]$. Differentiating this with respect to ζ_1 and utilising equation (7) gives

$$g(\xi, \nabla_{\zeta_1} gradr) = 3\zeta_1(r) - \eta(\zeta_1)\xi(r).$$

The metric g satisfies $g(\nabla_{\zeta_1} gradv, \zeta_2) = g(\nabla_{\zeta_2} gradv, \zeta_1)$, for a smooth function v. Hence, the aforementioned relation becomes

$$\nabla_\xi gradr = 3gradr - \xi(r)\xi.$$

Using this in (17), we obtain the required result. □

If a co-Kähler manifold $\mathcal{M}^{2n+1}$ with non-vanishing Ricci tensor S satisfy

$$S = \alpha g + \beta \eta \otimes \eta, \tag{18}$$

where α and β are smooth functions with η, a non-zero one-form, then it is said to be η-Einstein. In particular, if β is zero and α is some constant, then $\mathcal{M}$ is Einstein.

Lemma 2.5. *A co-Kähler manifold $\mathcal{M}^{2n+1}$ admitting a $\mathcal{SSM}$ ξ-connection $\bar{\nabla}$ is η-Einstein if and only if the Ricci tensor satisfy*

$$S = \Big(\frac{r}{2n} + 1\Big)g - \Big(\frac{r}{2n} + (2n+1)\Big)\eta \otimes \eta. \tag{19}$$

Proof. As it is well known that $\mathcal{M}$ is η-Einstein, taking the g-trace on (18) gives the scalar curvature r in the form

$$r = (2n+1)\alpha + \beta. \tag{20}$$

Further, by incorporating equation (14) in equation (18), we observe that $\alpha + \beta = -2n$. Solving this equation together with the previous equation yields $\alpha = (\frac{r}{2n} + 1)$ and $\beta = -(\frac{r}{2n} + (2n+1))$. Substitution of these values in equation (18) gives equation (19). □

3. Co-Kähler Manifolds with $\mathcal{SSM}$ ξ-Connection admitting Ricci Solitons

The current section explores the geometric characteristics of Ricci solitons on co-Kähler manifolds equipped with a $\mathcal{SSM}$ ξ-connection, establishing the subsequent findings.

Lemma 3.1. *If a co-Kähler manifold $\mathcal{M}$ admitting a $\mathcal{SSM}$ ξ-connection $\bar{\nabla}$ has a Ricci soliton structure (g, V), then the soliton expands and the Ricci tensor satisfies*

$$(\pounds_V S)(\zeta_1, \xi) = 0. \tag{21}$$

Proof. Upon applying the covariant derivative along ζ_3 to equation (3), we get

$$(\nabla_{\zeta_3} \pounds_V g)(\zeta_1, \zeta_2) = -2(\nabla_{\zeta_3} S)(\zeta_1, \zeta_2). \tag{22}$$

From the commutation formula [43], the symmetric nature of $\pounds_V \nabla$ leads to

$$(\pounds_V \nabla_{\zeta_1} g - \nabla_{\zeta_1} \pounds_V g - \nabla_{[V,\zeta_1]} g)(\zeta_2, \zeta_3) = -g((\pounds_V \nabla)(\zeta_1, \zeta_2), \zeta_3) - g((\pounds_V \nabla)(\zeta_1, \zeta_3), \zeta_2),$$

and by further operations, we achieve

$$2g((\pounds_V \nabla)(\zeta_1, \zeta_2), \zeta_3) = (\nabla_{\zeta_1} \pounds_V g)(\zeta_2, \zeta_3) + (\nabla_{\zeta_2} \pounds_V g)(\zeta_3, \zeta_1) - (\nabla_{\zeta_3} \pounds_V g)(\zeta_1, \zeta_2). \tag{23}$$

Utilising equation (22) in equation (23), we acquire

$$g((\pounds_V \nabla)(\zeta_1, \zeta_2), \zeta_3) = (\nabla_{\zeta_3} S)(\zeta_1, \zeta_2) - (\nabla_{\zeta_1} S)(\zeta_2, \zeta_3) - (\nabla_{\zeta_2} S)(\zeta_3, \zeta_1).$$

Replacing ζ_2 by ξ in the above relation yields

$$(\pounds_V \nabla)(\zeta_1, \xi) = -2[Q\zeta_1 + 2n\zeta_1]. \tag{24}$$

On differentiation of equation (24) with respect to ζ_2 and utilising (7), we arrive at

$$(\nabla_{\zeta_2} \pounds_V \nabla)(\zeta_1, \xi) = \pounds_V \nabla(\zeta_1, \zeta_2) - 2(\nabla_{\zeta_2} Q)\zeta_1 + 2\eta(\zeta_2)Q\zeta_1 + 4n\eta(\zeta_2)\zeta_1.$$

Employing this in the below identity [43]

$$(\pounds_V R)(\zeta_1,\zeta_2)\zeta_3 = (\nabla_{\zeta_1}\pounds_V\nabla)(\zeta_2,\zeta_3) - (\nabla_{\zeta_2}\pounds_V\nabla)(\zeta_1,\zeta_3),$$

by replacing ζ_3 as ξ, we infer

$$(\pounds_V R)(\zeta_1,\zeta_2)\xi = 4n[\eta(\zeta_1)\zeta_2 - \eta(\zeta_2)\zeta_1] + 2[\eta(\zeta_1)Q\zeta_2 - \eta(\zeta_2)Q\zeta_1 + (\nabla_{\zeta_2}Q)\zeta_1 - (\nabla_{\zeta_1}Q)\zeta_2]. \tag{25}$$

Substituting $\zeta_2 = \xi$ in (25), also utilising the expressions (12) and (13), we obtain $(\pounds_V R)(\zeta_1,\xi)\xi = 0$. Meanwhile, applying Lie differentiation on $R(\zeta_1,\xi)\xi = \eta(\zeta_1)\xi - \zeta_1$ (which is obtained from equation (9)) yields

$$(\pounds_V R)(\zeta_1,\xi)\xi - 2\eta(\pounds_V\xi)\zeta_1 + g(\pounds_V\xi,\zeta_1)\xi - (\pounds_V\eta)(\zeta_1)\xi = 0, \tag{26}$$

we know that $(\pounds_V R)(\zeta_1,\xi)\xi = 0$, leads to

$$-2\eta(\pounds_V\xi)\zeta_1 + g(\pounds_V\xi,\zeta_1)\xi = (\pounds_V\eta)(\zeta_1)\xi. \tag{27}$$

The soliton equation (3) along with equation (14) transforms to

$$\pounds_V g(\zeta_1,\xi) = 2[\lambda + 2n]\eta(\zeta_1). \tag{28}$$

Lie differentiating $\eta(\zeta_1) = g(\zeta_1,\xi)$ & $g(\xi,\xi) = 1$ along V and using (28) gives

$$\begin{aligned}(\pounds_V\eta)\zeta_1 - g(\zeta_1,\pounds_V\xi) &= 2[\lambda + 2n]\eta(\zeta_1),\\ \eta(\pounds_V\xi) &= -(\lambda + 2n).\end{aligned}$$

Utilising the above expressions in equation (26), we observe that $[\lambda+2n][\eta(\zeta_1)\xi-\zeta_1] = 0$, which on applying g-trace gives $\lambda = -2n$, that indicates the soliton expands. Also, taking g-trace of equation (25) presents

$$(\pounds_V S)(\zeta_2,\xi) = -\xi(r)\eta(\zeta_2) + \zeta_2(r).$$

This equation with Lemma 2.4 implies $(\pounds_V S)(\zeta_1,\xi) = 0$. □

With the aid of the preceding lemmas, we now proceed to the main geometric result of this section, stated and proved below.

Theorem 3.2. *If an η-Einstein co-Kähler manifold $\mathcal{M}$ admitting $\mathcal{SSM}$ ξ-connection $\bar{\nabla}$ has a Ricci soliton structure (g, V), then the manifold $\mathcal{M}$ is Einstein.*

Proof. Taking the Lie differentiation of equation (14) along V and utilising equation (28) infers

$$(\pounds_V S)(\zeta_1,\xi) + S(\zeta_1,\pounds_V\xi) = -2n[2(\lambda + 2n)\eta(\zeta_1) + g(\zeta_1,\pounds_V\xi)].$$

In view of equation (21) and (19), $\lambda = -2n$ and $\eta(\pounds_V\xi) = 0$ in the preceding equation, we get

$$[r + 2n(2n + 1)]g(\zeta_1,\pounds_V\xi) = 0. \tag{29}$$

In case, if $r \neq -2n(2n+1)$ on some open subset $\mathcal{U} \subset \mathcal{M}$, then $\pounds_V\xi = 0 = \pounds_V\eta$ on $\mathcal{U}$. Considering the following well known formula [43],

$$(\pounds_V\nabla)(\zeta_1,\zeta_2) = \pounds_V\nabla_{\zeta_1}\zeta_2 - \nabla_{\zeta_1}\pounds_V\zeta_2 - \nabla_{[V,\zeta_1]}\zeta_2.$$

Substituting ζ_2 by ξ and utilising $\pounds_V\xi = 0 = \pounds_V\eta$ and equation (7), we identify

$$(\pounds_V\nabla)(\zeta_1,\xi) = 0.$$

Comparing this equation with equation (24), we infer $Q\zeta_1 = -2n\zeta_1$. Taking g-trace of this implies $r = -2n(2n + 1)$ on $\mathcal{U}$. Therefore, this is a conflict on $\mathcal{U}$. Hence, equation (29) infers $r = -2n(2n + 1)$. Thus, in light of equation (19), we can say that $\mathcal{M}$ is Einstein. □

As a result, the study of Ricci soliton structure on co-Kähler manifold $\mathcal{M}^3$ admitting $\mathcal{SSM}$ ξ-connection gains particular interest because of Theorem 3.2. We now proceed to prove the forthcoming statement.

Theorem 3.3. *If a co-Kähler Manifold $\mathcal{M}^3$ admitting a $\mathcal{SSM}$ ξ-connection $\bar{\nabla}$ has a Ricci soliton (g, V), then it is of constant negative curvature -1.*

Proof. Considering the expression of Riemannian curvature tensor in a 3 dimensional manifold

$$\begin{aligned} R(\zeta_1, \zeta_2)\zeta_3 &= g(\zeta_2, \zeta_3)Q\zeta_1 - g(\zeta_1, \zeta_3)Q\zeta_2 + g(Q\zeta_2, \zeta_3)\zeta_1 \\ &\quad - g(Q\zeta_1, \zeta_3)\zeta_2 - \frac{r}{2}[g(\zeta_2, \zeta_3)\zeta_1 - g(\zeta_1, \zeta_3)\zeta_2]. \end{aligned} \tag{30}$$

Substituting $\zeta_2 = \zeta_3 = \xi$ in equation (30) and utilising equations (9) and (14), we get

$$Q\zeta_1 = \left(\frac{r}{2} + 5\right)\zeta_1 - \left(\frac{r}{2} + 11\right)\eta(\zeta_1)\xi. \tag{31}$$

By employing the same procedure as that of Theorem 3.2 and using the relation $(\pounds_V R)(\zeta_1, \xi)\xi = 0$, we deduce that $r = -22$. Thus, by equation (31) it follows that $Q\zeta_1 = -6\zeta_1$. This result, together with equation (30) yields

$$R(\zeta_1, \zeta_2)\zeta_3 = g(\zeta_1, \zeta_3)\zeta_2 - g(\zeta_2, \zeta_3)\zeta_1,$$

which proves that $\mathcal{M}^3$ is of negative constant curvature -1. □

This result reflects the geometric rigidity in the three-dimensional case, where the Ricci soliton compels the manifold to have negative constant curvature. We now consider the case of potential vector field aligned with ξ, a natural assumption in co-Kähler geometry and in the study of self-similar solutions to the Ricci flow. This structural alignment leads to a sharper conclusion, as stated below.

Theorem 3.4. *If a co-Kähler Manifold $(\mathcal{M}, g)$ admitting a $\mathcal{SSM}$ ξ-connection $\bar{\nabla}$ has a Ricci soliton structure (g, V) with $V = \sigma\xi$, then the manifold is Einstein.*

Proof. Assuming, $V = \sigma\xi$ where $\sigma \in C^\infty$ on $\mathcal{M}$. Differentiating it with respect to ζ_1 and utilising equation (7) provides

$$\nabla_{\zeta_1} V = \zeta_1(\sigma)\xi + \sigma(-\zeta_1 + \eta(\zeta_1)\xi).$$

In light of this, the fundamental equation (3) implies

$$2S(\zeta_1, \zeta_2) + \zeta_1(\sigma)\eta(\zeta_2) + \zeta_2(\sigma)\eta(\zeta_1) = 2(\lambda + \sigma)g(\zeta_1, \zeta_2) - 2\sigma\eta(\zeta_1)\eta(\zeta_2). \tag{32}$$

Replacing $\zeta_1 = \xi$ and $\zeta_2 = \xi$ in equation (32) and referring back to equation (14), offers $\xi(\sigma) = \lambda + 2n$. Taking it into account along with equation (14) and substituting $\zeta_2 = \xi$ in equation (32) gives $\zeta_1(\sigma) = [2n + \lambda]\eta(\zeta_1)$. This along with equation (32) extracts

$$S = (\lambda + \sigma)g - (\lambda + 2n + \sigma)\eta \otimes \eta.$$

Taking g-trace on the preceding expression gives $\lambda + \sigma = \frac{r}{2n} + 1$. Putting this into the previous equation produces

$$S = \left(\frac{r}{2n} + 1\right)g - \left(\frac{r}{2n} + (2n + 1)\right)\eta \otimes \eta. \tag{33}$$

This implies that g is η-Einstein. However, it is in fact Einstein, as followed by invoking Theorem 3.2. □

4. Co-Kähler Manifolds with $\mathcal{SSM}$ ξ-Connection admitting Gradient Ricci Almost Solitons

Here, we inspect the gradient Ricci almost soliton structure on co-Kähler manifolds equipped with a $\mathcal{SSM}$ ξ-connection, and establish the succeeding result.

Theorem 4.1. *If a co-Kähler Manifold $(\mathcal{M}, g)$ admitting a $\mathcal{SSM}$ ξ-connection $\bar{\nabla}$ has a gradient Ricci almost soliton structure (g, V), then either the manifold is Einstein or its potential vector field is pointwise collinear with the Reeb vector field ξ on an open subset $\mathcal{U} \subset \mathcal{M}$.*

Proof. The gradient Ricci almost soliton structure equation (4) can be expressed as

$$\nabla_{\zeta_1} gradh = -Q\zeta_1 + \lambda\zeta_1. \tag{34}$$

Differentiating the preceding equation along ζ_2, we achieve

$$\nabla_{\zeta_2}\nabla_{\zeta_1} gradh = -(\nabla_{\zeta_2} Q)\zeta_1 - Q(\nabla_{\zeta_2}\zeta_1) + \zeta_2(\lambda)\zeta_1 + \lambda(\nabla_{\zeta_2}\zeta_1). \tag{35}$$

Using equation (34) and (35) in the definition $R(\zeta_1, \zeta_2) = [\nabla_{\zeta_1}, \nabla_{\zeta_2}] - \nabla_{[\zeta_1,\zeta_2]}$, offers

$$R(\zeta_1, \zeta_2) gradh = (\nabla_{\zeta_2} Q)\zeta_1 - (\nabla_{\zeta_1} Q)\zeta_2 + \zeta_1(\lambda)\zeta_2 - \zeta_2(\lambda)\zeta_1. \tag{36}$$

Applying the inner product on the previous expression with ξ, and invoking equations (9) and (12), offers

$$\zeta_1(h)\eta(\zeta_2) - \zeta_2(h)\eta(\zeta_1) = \zeta_1(\lambda)\eta(\zeta_2) - \zeta_2(\lambda)\eta(\zeta_1).$$

Substituting ζ_2 by ξ in the preceding relation gives

$$\zeta_1(\lambda - h) = \xi(\lambda - h)\eta(\zeta_1), \tag{37}$$

thereby deducing to

$$d(\lambda - h) = \xi(\lambda - h)\eta. \tag{38}$$

Now, substituting $\zeta_1 = \xi$ in (36) and applying the inner product with ζ_3 infers

$$g(R(\xi, \zeta_2) gradh, \zeta_3) = -S(\zeta_2, \zeta_3) - 2ng(\zeta_2, \zeta_3) + \xi(\lambda)g(\zeta_2, \zeta_3) - \zeta_2(\lambda)\eta(\zeta_3).$$

Again, considering the inner product on equation (11) with *gradh* produces

$$g(R(\xi, \zeta_2) gradh, \zeta_3) = \xi(h)g(\zeta_2, \zeta_3) - \zeta_2(h)\eta(\zeta_3).$$

A comparison of the last two equations yields

$$S(\zeta_1, \zeta_2) = [\xi(\lambda - h) - 2n]g(\zeta_1, \zeta_2) - \xi(\lambda - h)\eta(\zeta_1)\eta(\zeta_2). \tag{39}$$

On contracting equation (39), we obtain

$$\xi(\lambda - h) = \frac{r}{2n} + (2n + 1). \tag{40}$$

On substitution of equation (40) into equation (39) directly yields equation (19). Furthermore, the g-trace of equation (36) gives

$$S(\zeta_1, gradh) = \frac{1}{2}\zeta_1(r) - 2n\zeta_1(\lambda).$$

The above equation along with equation (19) implies

$$\frac{1}{2}\zeta_1(r) - 2n\zeta_1(\lambda) = \left(\frac{r}{2n} + 1\right)\zeta_1(h) - \left(\frac{r}{2n} + (2n + 1)\right)\eta(\zeta_1)\xi(h). \tag{41}$$

Replacing ζ_1 by $-\zeta_1 + \eta(\zeta_1)\xi$ in equation (41), by Lemma 2.3 and equation (16), we have

$$-2n[-\zeta_1(\lambda) + \eta(\zeta_1)\xi(\lambda)] = \left(1 + \frac{r}{2n}\right)[-\zeta_1(h) + \eta(\zeta_1)\xi(h)]. \tag{42}$$

Considering equation (37) in equation (42) gives

$$\left((2n+1) + \frac{r}{2n}\right)(-gradh + \xi(h)\xi) = 0.$$

In case, if $r = -2n(2n+1)$, then equation (19) implies that $S = -2ng$, and thus $\mathcal{M}$ is Einstein. On the other hand, if $r \neq -2n(2n+1)$ on some open subset $\mathcal{U} \subset \mathcal{M}$, we obtain $gradh = \xi(h)\xi$, which concludes the proof. □

5. Conclusion

Despite significant advancements in Ricci soliton theory on various geometric structures, the study of co-Kähler manifolds with $\mathcal{SSM}$ ξ-connection admitting (gradient) Ricci almost solitons remains largely unexplored. Existing results predominantly focus on Kenmotsu, Lorentzian, Riemannian, Sasakian manifolds, leaving a gap in understanding how $\mathcal{SSM}$ ξ-connection influences the soliton structure in the co-Kähler setting. This work addresses that gap by deriving new curvature identities and soliton conditions tailored to this geometry. Some of our main results demonstrate that, under suitable conditions, such as when the manifold satisfies certain curvature constraints or when the potential vector field is collinear with the Reeb vector field, the underlying structure exhibits significant rigidity, often reducing to Einstein (Theorem 4.1), or negative constant curvature manifolds (Theorem 3.3).

These findings contribute to the broader understanding of geometric flows and Einstein-type metrics, which have implications in general relativity and theoretical physics. Moreover, the structural rigidity revealed is crucial in general relativity and string theory, where stable spacetime models with torsion are essential for consistent physical interpretations. Additionally, co-Kähler manifolds naturally arise in Hamiltonian mechanics and phase space geometry, where the inclusion of semi-symmetric connection may encode external fields or non-trivial geometric interactions relevant to string theory and gauge theories.

Overall, this work enhances the understanding of Ricci almost solitons on co-Kähler manifolds and opens avenues for further research in both pure geometry and its physical applications, particularly in areas where curvature, symmetry, and torsion play pivotal roles.

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