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Singular fractional double-phase problems involving logarithmic non-linearity with variable exponent

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Abstract. In this work, we study a fractional nonlocal problem involving a logarithmic-type nonlinearity, a singular term, and a vanishing potential. Our methodology is based on the computation of the critical groups of an approximate problem. By combining Morse theory with variational methods, we prove that the considered problem admits infinitely many solutions.

1. Introduction

Let $\mathcal{G} \subset \mathbb{R}^N$ an open bounded set, our goal is to establish the existence of infinitely numerous solutions for fractional double-phase singular problems characterized by a logarithmic non-linearity and a variable exponent, of the form :

$$(P_1)\quad \begin{cases} \Delta^{\phi,\varphi}_{\mathsf{p}(\mathtt{x},.),\mathsf{q}(\mathtt{x},.)}\vartheta(\mathtt{x}) = \dfrac{h\,(\mathtt{x},\vartheta(\mathtt{x}))}{(\vartheta(\mathtt{x}))^{\delta(\mathtt{x})}} + \alpha|\vartheta(\mathtt{x})|^{\upsilon(\mathtt{x})-2}\vartheta(\mathtt{x})\log|\vartheta(\mathtt{x})| & \text{in} \quad \mathcal{G}, \\ \qquad + \mathcal{P}(\mathtt{x})\,|\vartheta(\mathtt{x})|^{\varrho(\mathtt{x})-2}\,(\vartheta(\mathtt{x})) & \\ \vartheta > 0 & \text{in} \quad \mathcal{G}, \\ \vartheta = 0 & \text{in} \quad \mathbb{R}^N\backslash\mathcal{G}, \end{cases}$$

where, α denotes a positive parameter, $\mathsf{p},\mathsf{q} : \mathcal{G}\times\mathcal{G} \to (1,\infty)$, $\upsilon \in C(\mathcal{G},(1,\infty))$, $\varrho : \mathcal{G} \to (1,\infty)$, and $\delta : \mathcal{G} \to (0,1]$ are continuous functions satisfying the following assumptions :

$$\mathsf{p}(a-c, b-c) = \mathsf{p}(a,b), \text{ for all } (a,b,c) \in \mathcal{G}\times\mathcal{G}\times\mathcal{G}, \tag{1}$$

$$\mathsf{p}(a,b) = \mathsf{p}(b,a), \text{ for all } (a,b) \in \mathcal{G}\times\mathcal{G}, \tag{2}$$

$$1 < \varrho^- < \varrho^+ < \mathsf{q}^- < \mathsf{q}^+ < \mathsf{p}^- < \mathsf{p}^+ < +\infty, \tag{3}$$

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with $\varrho^- = \min_{a\in\mathcal{G}} \varrho(a), \varrho^+ = \max_{a\in\mathcal{G}} \varrho(a), \mathsf{q}^- = \min_{(a,b)\in\mathcal{G}\times\mathcal{G}} \mathsf{q}(a,b), \mathsf{q}^+ = \max_{(a,b)\in\mathcal{G}\times\mathcal{G}} \mathsf{q}(a,b), \mathsf{p}^- = \min_{(a,b)\in\mathcal{G}\times\mathcal{G}} \mathsf{p}(a,b), \mathsf{p}^+ = \max_{(a,b)\in\mathcal{G}\times\mathcal{G}} \mathsf{p}(a,b),$

$\mathcal{P}$ vanishing potential satisfies the following conditions:
(V) $\mathcal{P} : \mathbb{R}^N \to \mathbb{R}$ is a continuous function for which there exist $\theta_1 > 0$, and $0 < \eta_1 < 1$ such that:

$$\mathcal{P}(\mathsf{x}) > \theta_1 > 0 \text{ and } \int_{\mathbb{R}^N} \mathcal{P}(\mathsf{x})|\vartheta(\mathsf{x})|^{\varrho(\mathsf{x})}\, d\mathsf{x} \leq \eta_1 \|\vartheta\|_{\mathcal{W}},$$

for all $\mathsf{x} \in \mathbb{R}^N$, and $\vartheta \in \mathcal{W}^{\mathsf{z},\mathsf{m}(\mathsf{x}),\mathsf{p}(\mathsf{x},\mathsf{y})}$ with $\mathcal{W}^{\mathsf{z},\mathsf{m}(\mathsf{x}),\mathsf{p}(\mathsf{x},\mathsf{y})}$ is the fractional Sobolev space.
The function $h : \mathcal{G} \times \mathbb{R} \to \mathbb{R}$ denotes a Carathéodory function that fulfills the following condition:

(C_1) Let $\xi \in L^\infty(\mathcal{G})$, and $t : \mathcal{G} \to (1, +\infty)$ be a continuous function such that

$$1 < t(\mathsf{x}) < \mathsf{p}^\star_\phi(\mathsf{x}) = \frac{\mathsf{N}\mathsf{p}(\mathsf{x},\mathsf{x})}{\mathsf{N} - \phi \mathsf{p}(\mathsf{x},\mathsf{x})},$$

and

$$h(\mathsf{x},\mathsf{y}) \leq \xi(\mathsf{x})\left(1 + |\mathsf{y}|^{t(\mathsf{x})-1}\right), \quad \text{a.e. } \mathsf{x} \in \mathcal{G}, \mathsf{y} \in \mathbb{R},$$

$\Delta^{\phi,\varphi}_{\mathsf{p}(\mathsf{x},.),\mathsf{q}(\mathsf{x},.)}$ is a $(\mathsf{p}(\mathsf{x},.), \mathsf{q}(\mathsf{x},.))$ – fractional double phase operator with $\phi, \varphi \in (0,1)$, defined as:

$$\Delta^{\phi,\varphi}_{\mathsf{p}(\mathsf{x},.),\mathsf{q}(\mathsf{x},.)}\vartheta(\mathsf{x}) = 2 \lim_{\varepsilon\to 0^+} \int_{\mathcal{G}\setminus\mathcal{B}_\varepsilon(\mathsf{x})} \left[\frac{|\vartheta(\mathsf{x}) - \vartheta(\mathsf{y})|^{\mathsf{p}(\mathsf{x},\mathsf{y})-2}}{|\mathsf{x}-\mathsf{y}|^{N+\phi\mathsf{p}(\mathsf{x},\mathsf{y})}} + \frac{|\vartheta(\mathsf{x}) - \vartheta(\mathsf{y})|^{\mathsf{q}(\mathsf{x},\mathsf{y})-2}}{|\mathsf{x}-\mathsf{y}|^{N+\varphi\mathsf{q}(\mathsf{x},\mathsf{y})}} \right] (\vartheta(\mathsf{x}) - \vartheta(\mathsf{y}))d\mathsf{y}, \tag{4}$$

where $\mathcal{B}_\varepsilon(\mathsf{x})$ is the ball of $\mathcal{G}$ of center x and radius ε .

The double-phase operator began with Zhikov [19] in the mid-1980s. He used it to model composite materials that behave differently in various directions. Zhikov's key idea was a mathematical expression that combined two material responses: a softer one (with p-growth) and a stiffer one (with q-growth). This was written as:

$$\mathcal{H}(\mathsf{x}, \nabla\vartheta) = |\nabla\vartheta|^{\mathsf{p}} + \mu(\mathsf{x})|\nabla\vartheta|^{\mathsf{q}}.$$

Here, $1 < p < q$, and $\mu(\mathsf{x}) \geq 0$ tells us which phase we are in. It helps explain how materials change from being like a p-Laplacian to a q-Laplacian. This early work was important for understanding material elasticity.

In the 2010s, Baroni, Colombo, and Mingione in [6, 7] greatly improved our mathematical understanding of these operators. They developed strong analytical methods that showed how the two phases interact. They proved that solutions to double-phase problems naturally belong to the Musielak-Orlicz-Sobolev space $\mathcal{W}^{1,\mathcal{C}}(\Omega)$. This space includes functions with a finite energy, defined by:

$$\int_\Omega (|\nabla\vartheta|^{\mathsf{p}} + \mu(\mathsf{x})|\nabla\vartheta|^{\mathsf{q}})\, d\mathsf{x}\,.$$

Their research gave crucial tools for studying if solutions exist and how smooth they are. It also opened up new areas in the calculus of variations for functions that don't grow in a standard way.

More recently, Aberqi, Bahrouni, Radulescu, Ragusa and Tachikawa have extended this operator to more complex situations see [1–5, 9, 12, 13, 16]. This includes spaces that account for fractional derivatives (nonlocal effects) and where the exponents p and q can change from point to point (variable-exponent spaces). The modern form of the operator looks like (4).

These new versions combine both double-phase and variable-exponent features. They are very useful for modeling difficult problems in nonlocal elasticity and image processing, where material behavior or image details vary at small scales. The history of this operator shows how it grew from a specific tool for materials science into a powerful concept in nonlinear analysis.

The foundational work by Bahrouni and Rădulescu in [4] explored key properties of fractional Sobolev spaces with variable exponents, denoted as $\mathcal{W}^{\mathsf{z},\mathsf{q}(\mathsf{x}),\mathsf{p}(\mathsf{x},\mathsf{y})}(U)$. Here, $\mathsf{z} \in (0,1)$ and U is a Lipschitz domain. They investigated nonlocal problems given by:

$$\begin{cases} \Delta\vartheta(\mathsf{x}) + |\vartheta(\mathsf{x})|^{\mathsf{q}(\mathsf{x})-1}\vartheta(\mathsf{x}) = \lambda|\vartheta(\mathsf{x})|^{\upsilon(\mathsf{x})-1}\vartheta(\mathsf{x}) & \text{in } U, \\ \vartheta = 0 & \text{on } \partial U. \end{cases}$$

The non-local operator $\Delta\vartheta(\mathsf{x})$ in this problem was defined as:

$$\Delta\vartheta(\mathsf{x}) = 2\lim_{\epsilon\to 0^+}\int_{U\setminus\mathcal{B}_\epsilon(\mathsf{x})}\int_U \frac{|\vartheta(\mathsf{x})-\vartheta(\mathsf{y})|^{\mathsf{p}(\mathsf{x},\mathsf{y})-2}(\vartheta(\mathsf{x})-\vartheta(\mathsf{y}))}{|\mathsf{x}-\mathsf{y}|^{n+\mathsf{s}\mathsf{p}(\mathsf{x},\mathsf{y})}}\,d\mathsf{x}\,d\mathsf{y}.$$

Here, the condition $1 < \upsilon(\mathsf{x}) < \mathsf{p}^- = \min_{(\mathsf{x},\mathsf{y})\in U\times U}\mathsf{p}(\mathsf{x},\mathsf{y})$ applies.

Further work in [5] expanded this framework to include double-phase problems with logarithmic nonlinearities. These problems involve an operator $\Delta^{\phi,\varphi}_{\mathsf{p}(\mathsf{x},\cdot),\mathsf{q}(\mathsf{x},\cdot)}\vartheta(\mathsf{x})$ of the form:

$$\Delta^{\phi,\varphi}_{\mathsf{p}(\mathsf{x},\cdot),\mathsf{q}(\mathsf{x},\cdot)}\vartheta(\mathsf{x}) = \lambda|\vartheta(\mathsf{x})|^{\upsilon(\mathsf{x})-1}\vartheta(\mathsf{x}) + \mu(\mathsf{x})|\vartheta(\mathsf{x})|^{\upsilon(\mathsf{x})-2}\ln(|\vartheta(\mathsf{x})|).$$

This represented a significant step forward in understanding problems that combine multiple growth phases. These developments laid the theoretical groundwork for handling more intricate nonlinearities and complex operator structures in nonlocal problems.

Building on the previous results, we establish the existence of infinitely many solutions for a class of double-phase problems. These problems are driven by a double-phase operator that includes singular and logarithmic nonlinearities and a vanishing potential in the context of fractional Sobolev spaces with variable exponents. Our approach is to use variational methods and Morse theory to compute the critical groups of the energy functional for the approximated problem.

The paper proceeds as follows: In Section 2, we gather the main definitions and analytical properties of the generalized Lebesgue and Sobolev spaces. We also provide essential background on Morse theory. In Section 3, we propose an approximated problem (P_2). For this problem, we then calculate the critical groups at infinity and at the critical point 0 for its associated energy functional. Then we uses Morse relations to prove that the approximated problem (P_2) has infinitely many non-trivial solutions. Finally, in the last section, we will prove our main Theorem 10.

2. Definitions and preliminary results

This section lays the groundwork for the subsequent discussions by presenting essential concepts and results. We will cover fundamental properties of Sobolev spaces and generalized Lebesgue spaces, which are crucial for the analytical framework employed in this document. Additionally, we will introduce the basic notions and key properties of Morse theory, providing the necessary topological and geometrical background.

2.1. *Generalized fractional Sobolev space*

We consider the set given by:

$$C^+(\bar{\mathcal{G}}) = \{\, f \in C(\bar{\mathcal{G}}, \mathbb{R}^+) : 1 < f^- < f(x) < f^+ < +\infty \,\},$$

where $f^- = \min_{\mathsf{x}\in\bar{\mathcal{G}}} f(\mathsf{x})$, $f^+ = \max_{\mathsf{x}\in\bar{\mathcal{G}}} f(\mathsf{x})$.

Definition 1. *([10]) Let $\mathsf{q} \in C^+(\mathcal{G})$. The generalized Lebesgue space $L^{\mathsf{q(x)}}(\mathcal{G})$ is defined as :*

$$L^{\mathsf{q(x)}}(\mathcal{G}) = \left\{ \vartheta : \mathcal{G} \to \mathbb{R} \text{ is measurable and } : \exists \alpha > 0 \text{ such that } \int_{\mathcal{G}} \left| \frac{\vartheta(\mathsf{x})}{\alpha} \right|^{\mathsf{q(x)}} d\mathsf{x} < \infty \right\}.$$

We equip this space with the Luxemburg norm, given by:

$$|\vartheta|_{L^{\mathsf{q(x)}}(\mathcal{G})} = \inf \left\{ \delta > 0 : \int_{\mathcal{G}} \left| \frac{\vartheta(\mathsf{x})}{\delta} \right|^{\mathsf{q(x)}} d\mathsf{x} \leq 1 \right\}.$$

Lemma 1. *(Hölder's inequality [10]) For each $\mathsf{q} \in C^+(\mathbb{R}^N)$, the following inequality holds:*

$$\left| \int_{\mathbb{R}^N} \vartheta(\mathsf{x})\psi(\mathsf{x})\, d\mathsf{x} \right| \leq \left(\frac{1}{\mathsf{q}_1^-} + \frac{1}{\mathsf{q}_2^-} \right) |\vartheta|_{L^{\mathsf{q}_1(\mathsf{x})}(\mathbb{R}^N)} |\psi|_{L^{\mathsf{q}_2(\mathsf{x})}(\mathbb{R}^N)},$$

for every $(\vartheta, \psi) \in L^{\mathsf{q}_1(\mathsf{x})}(\mathbb{R}^N) \times L^{\mathsf{q}_2(\mathsf{x})}(\mathbb{R}^N)$, where $\frac{1}{\mathsf{q}_1(\mathsf{x})} + \frac{1}{\mathsf{q}_2(\mathsf{x})} = 1$.

We first fix the fractional exponent $\mathsf{z} \in (0, 1)$.
Let $\mathcal{G}$ an open bounded set of $\mathbb{R}^N$, $\mathsf{m}_1 \in C^+(\mathcal{G})$, and $\mathsf{p} : \bar{\mathcal{G}} \times \bar{\mathcal{G}} \to (1, \infty)$ is a continuous function that satisfies the conditions (1)-(3). The generalized fractional Sobolev space $\mathcal{W}^{\mathsf{z},\mathsf{m}_1(x),\mathsf{p}(x,y)}(\mathcal{G})$ is defined as :

$$\mathcal{W}^{\mathsf{z},\mathsf{m}_1(\mathsf{x}),\mathsf{p}(\mathsf{x},\mathsf{y})}(\mathcal{G}) = \left\{ \vartheta \in L^{\mathsf{m}_1(\mathsf{x})}(\mathcal{G}) : \frac{\vartheta(\mathsf{x}) - \vartheta(\mathsf{y})}{\xi |\mathsf{x} - \mathsf{y}|^{\mathsf{z} + \frac{N}{\mathsf{p}(\mathsf{x},\mathsf{y})}}} \in L^{\mathsf{p}(\mathsf{x},\mathsf{y})}(\mathcal{G} \times \mathcal{G}) \text{ for some } \xi > 0 \right\}.$$

Let $[\vartheta]^{\mathsf{z},\mathsf{p}(\mathsf{x},\mathsf{y})} = \inf \left\{ \xi > 0 : \int_{\mathcal{G} \times \mathcal{G}} \frac{|\vartheta(\mathsf{x}) - \vartheta(\mathsf{y})|^{\mathsf{p}(\mathsf{x},\mathsf{y})}}{\xi^{\mathsf{p}(\mathsf{x},\mathsf{y})} |\mathsf{x} - \mathsf{y}|^{N + \mathsf{z}\mathsf{p}(\mathsf{x},\mathsf{y})}} \, d\mathsf{x}\, d\mathsf{y} < 1 \right\}$ be the associated variable exponent Gagliardo seminorm. We equip the space $\mathcal{W}^{\mathsf{z},\mathsf{m}_1(x),\mathsf{p}(x,y)}(\mathcal{G})$ with the norm

$$\|\vartheta\|_{\mathcal{W}^{\mathsf{z},\mathsf{m}_1(\mathsf{x}),\mathsf{p}(\mathsf{x},\mathsf{y})}(\mathcal{G})} = [\vartheta]^{\mathsf{z},\mathsf{p}(\mathsf{x},\mathsf{y})} + |\vartheta|_{\mathsf{m}_1(\mathsf{x})},$$

where $\left(L^{\mathsf{m}_1(\mathsf{x})}(\mathcal{G}), |.|_{\mathsf{m}_1(\mathsf{x})} \right)$ is the generalized Lebesgue space.

Lemma 2. *([4]) Let $\mathcal{G} \subset \mathbb{R}^N$ a bounded Lipschitz domain, $\mathsf{p} : \mathcal{G} \times \mathcal{G} \to (1, +\infty)$ is a continuous function that satisfies conditions (1)-(3), and $\mathsf{m}_1 \in C^+(\overline{\mathcal{G}})$. Then $\mathcal{W}^{\mathsf{z},\mathsf{m}_1(\mathsf{x}),\mathsf{p}(\mathsf{x},\mathsf{y})}(\mathcal{G})$ is a Banach space that is both separable and reflexive.*

Theorem 1. *([14]) Let $\mathcal{G} \subset \mathbb{R}^N$ be a Lipschitz-bounded domain, $\mathsf{p} : \mathcal{G} \times \mathcal{G} \to (1, +\infty)$ be a continuous function satisfying conditions (1)-(3) $\mathsf{m}_1 \in C^+(\mathcal{G})$, and*

$$\mathsf{z}\mathsf{p}(\mathsf{x}, \mathsf{y}) < N, \ \mathsf{p}((\mathsf{x}, \mathsf{x})) < \mathsf{m}_1(\mathsf{x}), \ \forall (\mathsf{x}, \mathsf{y}) \in \mathcal{G}^2,$$

and $\ell : \overline{\mathcal{G}} \to (1, +\infty)$ denotes a continuous variable exponent satisfying:

$$\mathsf{p}_z^*(\mathsf{x}) = \frac{N\mathsf{p}((\mathsf{x}, \mathsf{x}))}{N - z\mathsf{p}((\mathsf{x}, \mathsf{x}))} > \ell(\mathsf{x}) \geq \ell^- = \min_{\mathsf{x} \in \overline{\mathcal{G}}} \ell(\mathsf{x}) > 1.$$

Thus, the space $\mathcal{W}^{\mathsf{z},\mathsf{m}_1(\mathsf{x}),\mathsf{p}(\mathsf{x},\mathsf{y})}(\mathcal{G})$ is continuously embedded in $L^{\ell(\mathsf{y})}(\mathcal{G})$. In other words, there exists a positive constant $\mathsf{C} = \mathsf{C}(\mathsf{N}, \mathsf{z}, \mathsf{p}, \mathsf{m}_1, \mathcal{G})$ such that

$$|\vartheta|_{L^{\ell(\mathsf{x})}(\mathcal{G})} \leq \mathsf{C} \|\vartheta\|_{\mathcal{W}^{\mathsf{z},\mathsf{m}_1(\mathsf{x}),\mathsf{p}(\mathsf{x},\mathsf{y})}(\mathcal{G})}, \text{ for all } \quad \vartheta \in \mathcal{W}^{\mathsf{z},\mathsf{m}_1(\mathsf{x}),\mathsf{p}(\mathsf{x},\mathsf{y})}(\mathcal{G}).$$

Furthermore, this embedding is compact.

2.2. Critical groups and Homology theory

In this paragraph, we provide a brief overview of the fundamental concepts and properties of Morse theory.

Definition 2. *([17]) Let $\mathcal{W}$ be a Banach space and $\mathcal{W}^*$ its topological dual. Suppose that $\Theta \in C^1(\mathcal{W})$.*

- *We say that Θ satisfies the Palais-Smale condition at the level $c \in \mathbb{R}$ if the following is true: "every sequence $\{\vartheta_n\}_{n\geq 1} \subset \mathcal{W}$ such that $\Theta(\vartheta_n) \to c$ and $\Theta'(\vartheta_n) \to 0$ in $\mathcal{W}^*$ admits a strongly convergent subsequence".*
- *We say that Θ satisfies the Cerami condition at the level $c \in \mathbb{R}$ if the following is true: "every sequence $\{\vartheta_n\}_{n\geq 1} \subset \mathcal{W}$ such that $\Theta(\vartheta_n) \to c$ and $(1 + \|\vartheta_n\|)\Theta'(\vartheta_n) \to 0$ in $\mathcal{W}^*$ admits a strongly convergent subsequence".*

Let $\mathcal{W}$ be a real Banach space, and let $\Theta \in C^1(\mathcal{W}, \mathbb{R})$ satisfy the Palais-Smale condition. For a given $c \in \mathbb{R}$, we define the following sets:

$$\Theta^c = \{\vartheta \in \mathcal{W} : \Theta(\vartheta) \leq c\},$$

and

$$K_\Theta = \left\{\vartheta \in \mathcal{W} : \Theta^{'}(\vartheta) = 0\right\}.$$

The critical groups of Θ at ϑ are given by

$$C_k(\Theta, \vartheta) = \mathcal{H}_k(\Theta^c \cap \mathcal{U}, \Theta^c \cap \mathcal{U} \backslash \{\vartheta\}),$$

where $k \in \mathbb{N}$, and $\mathcal{U}$ is a neighborhood of ϑ, such that $K_\Theta \cap \mathcal{U} = \{\vartheta\}$, and $\mathcal{H}_k$ stands for the singular relative homology with coefficient in an Abelian group G; see [17] for more details.

Definition 3. *([8]) Assume that Θ satisfies condition (C) and that all its critical values are bounded below by some $a < \inf \Theta(K)$. Then the critical groups of Θ at infinity are defined as*

$$C_k(\Theta, \infty) = \mathcal{H}_k(\mathcal{W}, \Theta^a), \quad \forall k \in \mathbb{N}.$$

Theorem 2. *([18]) Let $\mathcal{W}$ a real Banach space and $\Theta \in C^1(\mathcal{W}, \mathbb{R})$ satisfies the Palais–Smale condition and is bounded below. If there exists at least one nontrivial critical group of Θ, then Θ has at least three critical points.*

Definition 4. *([18]) Given X is a Banach space, $\Theta \in C(X, \mathbb{R})$, and 0 is an isolated critical point of Θ such that $\Theta(0) = 0$. We say that Θ has a local linking at 0 with respect to $X = Y \oplus Z$, $k = \dim Y < \infty$, if there exists $\zeta > 0$ small, such that*

$$\begin{cases} \Theta(\vartheta) \leq 0, & \vartheta \in Y; \quad \|\vartheta\| \leq \zeta; \\ \Theta(\vartheta) > 0, & \vartheta \in Z; \quad 0 < \|\vartheta\| \leq \zeta. \end{cases}$$

Theorem 3. *([18]) Let $\mathcal{W}$ be a Banach space and let $\Theta \in C(\mathcal{W}, \mathbb{R})$. If Θ has a local linking at $0 \in \mathcal{W}$ with respect to $\mathcal{W}$, then the critical group at 0 is nontrivial: $C_k(\Theta, 0) \neq 0$.*

Definition 5. *([18]) Given $\mathcal{W}$ is a Banach space, $\Theta \in C(\mathcal{W}, \mathbb{R})$, and 0 is an isolated critical point of Θ, where $\Theta(0) = 0$. Let $a, b \in \mathbb{N}$. We say that Θ possesses a local (a, b)-linking near the origin if there exist a neighborhood $\mathcal{U}$ of 0 and nonempty subsets $G_0, G \subset \mathcal{U}$, and $D \subset \mathcal{W}$ such that $0 \notin G_0 \subset G$, $G \cap D = \emptyset$, and*

1. $\Theta|_G \leq 0 < \Theta|_{\mathcal{U} \cap D \backslash \{0\}}$,
2. *The point 0 is the only critical point of Θ in the set $\Theta^0 \cap \mathcal{U}$, where $\Theta^0 = \{\vartheta \in X : \Theta(\vartheta) = 0\}$,*
3. $\dim(\mathrm{Im}(i^*)) - \dim(\mathrm{Im}(j^*)) \geq n$*, where*

$$i^* : \mathcal{H}_{a-1}(G_0) \to \mathcal{H}_{a-1}(\mathcal{W} \setminus D) \text{ and } j^* : \mathcal{H}_{a-1}(G_0) \to \mathcal{H}_{a-1}(G)$$

are the homomorphisms induced by the inclusion maps $i : G_0 \to W \setminus D$ and $j : G_0 \to G$.

Lemma 3. *(Morse's relation) ([17]) If $\mathcal{W}$ is a Banach space, $\Theta \in C^1(\mathcal{W}, \mathbb{R})$, $\mathtt{x}, \mathtt{y} \in \mathbb{R} \backslash \Theta(K_\Theta)$ with $\mathtt{x} < \mathtt{y}$, $\Theta^{-1}((\mathtt{x}, \mathtt{y}))$ contains finitely many critical points $\{\vartheta_i\}_{i=1}^n$, and Θ satisfies the Palais–Smale condition, then*

1. *For all* $k \in \mathbb{N}_0$, *we have* $\sum_{i=1}^{n} \operatorname{rank} C_k(\Theta, \vartheta_i) \geqslant \operatorname{rank} \mathcal{H}_k(\Theta^y, \Theta^x)$;
2. *If the Morse-type numbers* $\sum_{i=1}^{n} \operatorname{rank} C_k(\Theta, \vartheta_i)$ *are finite for all* $k \in \mathbb{N}_0$ *and vanish for all large* $k \in \mathbb{N}_0$, *then so do the Betti numbers* $\operatorname{rank} \mathcal{H}_k(\Theta^y, \Theta^x)$ *and we have*

$$\sum_{k\geqslant 0} \sum_{i=1}^{n} \operatorname{rank} C_k(\Theta, \vartheta_i) t^k = \sum_{k\geqslant 0} \operatorname{rank} \mathcal{H}_k(\Theta^y, \Theta^x) t^k + (1+t)Q(t) \text{ for all } t \in \mathbb{R},$$

where $Q(t)$ *is a polynomial in the real variable* $t \in \mathbb{R}$ *whose coefficients are non-negative integers.*

3. Main results

We begin this section by stating a crucial lemma that will be instrumental in proving our main results.

Lemma 4. *For every* $\lambda > 0$. *We have*

1. $\mathrm{x}^\lambda |\log(\mathrm{x})| \leq \frac{1}{\lambda \exp(1)}$, *for all* $\mathrm{x} \in (0, 1]$;
2. $\log(\mathrm{x}) \leq \frac{\mathrm{x}^\lambda}{\lambda \exp(1)}$, *for all* $\mathrm{x} > 1$.

Proof. For (1). We define the function $K : (0, 1] \to \mathbb{R}$ by $K(\mathrm{x}) = \mathrm{x}^\lambda |\log(\lambda)|$. This function is continuous on $(0, 1]$, and $\lim_{\mathrm{x}\to 0} \mathrm{x}^\lambda |\log(\mathrm{x})| = 0$. A direct computation reveals that the function K attains its maximum value at $\mathrm{x}_0 = \exp\left(\frac{-1}{\lambda}\right)$. Finally, we have $\mathrm{x}^\lambda |\log(\mathrm{x})| \leq \frac{1}{\lambda \exp(1)}$, for all $\mathrm{x} \in (0, 1]$.

For (2), we define the function as follows:

$$L(\mathrm{x}) = \log(\mathrm{x}) - \frac{1}{\lambda \exp(1)} \mathrm{x}^\lambda, \text{ for all } \mathrm{x} \in [1, \infty).$$

Taking the derivative of L and solving $L'(\mathrm{x}) = 0$ shows that the only critical point in $[1, \infty)$ is $\mathrm{x}^* = e^{1/\lambda}$. Moreover, $L'(\mathrm{x}) > 0$ for $\mathrm{x} < \mathrm{x}^*$ and $L'(\mathrm{x}) < 0$ for $\mathrm{x} > \mathrm{x}^*$. Hence L increases on $[1, \mathrm{x}^*]$ and decreases on $[\mathrm{x}^*, \infty)$, which proves that L attains its maximum value at x^*.Therefore $L(\mathrm{x}) \leq L(\mathrm{x}^*)$. □

Due to the singular term, the energy functional for our problem isn't differentiable, we consider a sequence of approximate problems.

3.1. The Approximated Problem

We suggest an approximate problem sequence as

$$(P_2) \quad \begin{cases} \Delta^{\phi,\varphi}_{\mathrm{p}(\mathrm{x},.),\mathrm{q}(x..)} \vartheta_n(\mathrm{x}) = \dfrac{\mathcal{H}_n(\mathrm{x}, \vartheta_n(\mathrm{x}))}{\left(\vartheta_n(\mathrm{x}) + \frac{1}{n}\right)^{\delta(\mathrm{x})}} + \nu(\mathrm{x}) |\vartheta_n(\mathrm{x})|^{\varrho(\mathrm{x})-2} \left(\vartheta_n(\mathrm{x}) + \dfrac{1}{n}\right) & \text{in } \mathcal{G}, \\ \qquad + \alpha \left|\vartheta_n(\mathrm{x}) + \dfrac{1}{n}\right|^{\upsilon(\mathrm{x})-2} \left(\vartheta_n(\mathrm{x}) + \dfrac{1}{n}\right) \log\left|\vartheta_n(\mathrm{x}) + \dfrac{1}{n}\right| & \\ \vartheta_n > 0 & \text{in } \mathcal{G}, \\ \vartheta_n = 0 & \text{in } \mathbb{R}^N \backslash \mathcal{G}, \end{cases}$$

where, $\mathcal{H}_n(\mathrm{x}, t) = \min(n, h(\mathrm{x}, t))$, $\mathcal{H}_n(\mathrm{x}, t) = \int_0^t \frac{\mathcal{H}_n(\mathrm{x}, s)}{(s + \frac{1}{n})^{\delta(\mathrm{x})}} ds$, and $\mathcal{H}_n : \mathcal{G} \times \mathbb{R} \to \mathbb{R}$ is a sequence of functions that verifies the following conditions:

(C_2) For some $\eta > \mathsf{p}^+$ and $\upsilon > 0$, such that for $\mathsf{x} \in \mathcal{G}$ a.e, and $|\mathsf{x}| \geq \upsilon$,

$$0 < \eta H_n(\mathsf{x}, t) \leq \frac{t h_n(\mathsf{x}, t)}{(t + \frac{1}{n})^{\delta(\mathsf{x})}}.$$

(C_3) It is satisfied

$$\lim_{t \to +\infty} \frac{h_n(\mathsf{x}, t)}{t \mathsf{p}^+} = l_1 \text{ uniformly for } \mathsf{x} \in \mathcal{G} \text{ a.e.}$$

(C_4) There exist $\eta > \varrho^-$ and $a_3 > 0$ such that

$$h_n(\mathsf{x}, t)t - \eta H_n(\mathsf{x}, t) \geq -a_3 |t|^{\mathsf{p}^-},$$

for all $\mathsf{x} \in \mathcal{G}$ and $t \in \mathbb{R}$.

Notation: To simplify expressions, we write:

$$\mathcal{W} = \mathcal{W}^{\mathsf{z}, \mathsf{m}(\mathsf{x}), \mathsf{p}(\mathsf{x},\mathsf{y})}(\mathcal{G}), \;\; \mathcal{W}_1 = \mathcal{W}^{\phi, \mathsf{m}_1(\mathsf{x}), \mathsf{p}(\mathsf{x},\mathsf{y})}(\mathcal{G}), \;\; \mathcal{W}_2 = \mathcal{W}^{\varphi, \mathsf{m}_2(\mathsf{x}), \mathsf{q}(\mathsf{x},\mathsf{y})}(\mathcal{G}).$$

Definition 6. *We say that* $\{\vartheta_n\}_{n \in \mathbb{N}}$ *is a weak solution of* (P_2) *if:*

$$\begin{aligned}
&\int_{\mathcal{G}\times\mathcal{G}} \frac{|\vartheta_n(\mathsf{x}) - \vartheta_n(\mathsf{y})|^{\mathsf{p}(\mathsf{x},\mathsf{y})-2}(\vartheta_n(\mathsf{x}) - \vartheta_n(\mathsf{y}))(\mathfrak{u}(\mathsf{x}) - \mathfrak{u}(\mathsf{y}))}{|\mathsf{x} - \mathsf{y}|^{N+\phi\mathsf{p}(\mathsf{x},\mathsf{y})}} \, d\mathsf{x}\, d\mathsf{y} \\
&+ \int_{\mathcal{G}\times\mathcal{G}} \frac{|\vartheta_n(\mathsf{x}) - \vartheta_n(\mathsf{y})|^{\mathsf{q}(\mathsf{x},\mathsf{y})-2}(\vartheta_n(\mathsf{x}) - \vartheta_n(\mathsf{y}))(\mathfrak{u}(\mathsf{x}) - \mathfrak{u}(\mathsf{y}))}{|\mathsf{x} - \mathsf{y}|^{N+\varphi\mathsf{q}(\mathsf{x},\mathsf{y})}} \, d\mathsf{x}\, d\mathsf{y} \\
&= \int_{\mathcal{G}} \left[\frac{h_n(\mathsf{x}, \vartheta_n(\mathsf{x}))}{\left(\vartheta_n(\mathsf{x}) + \frac{1}{n}\right)^{\delta(\mathsf{x})}} + \mathcal{P}(\mathsf{x}) |\vartheta_n(\mathsf{x}) + \frac{1}{n}|^{\varrho(\mathsf{x})-2} \left(\vartheta_n(\mathsf{x}) + \frac{1}{n}\right) \right] \mathfrak{u}(\mathsf{x})\, d\mathsf{x} \\
&+ \alpha \int_{\mathcal{G}} \left|\vartheta_n(\mathsf{x}) + \frac{1}{n}\right|^{\upsilon(\mathsf{x})-2} \left(\vartheta_n(\mathsf{x}) + \frac{1}{n}\right) \log \left|\vartheta_n(\mathsf{x}) + \frac{1}{n}\right| \mathfrak{u}(\mathsf{x})\, d\mathsf{x},
\end{aligned}$$

for all $\mathfrak{u} \in \mathcal{W}_1^*$*, where* $\mathcal{W}_1^*$ *is the dual space of* $\mathcal{W}_1$*.*

3.2. *Computation of critical group*

In this subsection, we consider the energy functional associated with (P_2) $\Theta : \mathcal{W}_1 \longrightarrow \mathbb{R}$ defined as:

$$\Theta(\vartheta_n) = \Theta_1(\vartheta_n) - \Theta_2(\vartheta_n) - \Theta_3(\vartheta_n) - \Theta_4(\vartheta_n),$$

where

$$\Theta_1(\vartheta_n) = \int_{\mathcal{G}\times\mathcal{G}} \left[\frac{1}{\mathsf{p}(\mathsf{x},\mathsf{y})} \frac{|\vartheta_n(\mathsf{x}) - \vartheta_n(\mathsf{y})|^{\mathsf{p}(\mathsf{x},\mathsf{y})}}{|\mathsf{x} - \mathsf{y}|^{N+\phi\mathsf{p}(\mathsf{x},\mathsf{y})}} + \frac{1}{\mathsf{q}(\mathsf{x},\mathsf{y})} \frac{|\vartheta_n(\mathsf{x}) - \vartheta_n(\mathsf{y})|^{\mathsf{q}(\mathsf{x},\mathsf{y})}}{|\mathsf{x} - \mathsf{y}|^{N+\varphi\mathsf{q}(\mathsf{x},\mathsf{y})}} \right] d\mathsf{x}\, d\mathsf{y},$$

$$\Theta_2(\vartheta_n) = \int_{\mathcal{G}} H_n(\mathsf{x}, \vartheta_n(\mathsf{x}))\, d\mathsf{x}, \text{ and } H_n(\mathsf{x}, t) = \int_0^t \frac{h_n(\mathsf{x}, s)}{(s + \frac{1}{n})^{\delta(\mathsf{x})}}\, ds \text{ is the primitive of } \frac{h_n(\mathsf{x}, s)}{(s + \frac{1}{n})^{\delta(\mathsf{x})}},$$

$$\Theta_3(\vartheta_n) = \int_{\mathcal{G}} \frac{\mathcal{P}(\mathsf{x})}{\varrho(\mathsf{x})} \left|\vartheta_n(\mathsf{x}) + \frac{1}{n}\right|^{\varrho(\mathsf{x})} d\mathsf{x},$$

$$\Theta_4(\vartheta_n) = \alpha \int_{\mathcal{G}} \frac{1}{\upsilon(\mathsf{x})} \left(\left|\vartheta_n(\mathsf{x}) + \frac{1}{n}\right|^{\upsilon(\mathsf{x})} \log \left|\vartheta_n(\mathsf{x}) + \frac{1}{n}\right| - \frac{1}{\upsilon(\mathsf{x})} \left|\vartheta_n(\mathsf{x}) + \frac{1}{n}\right|^{\upsilon(\mathsf{x})} \right) d\mathsf{x}.$$

Lemma 5. *If h satisfies the condition (C_1) and the potential $\mathcal{P}$ satisfies (V). Then $\Theta_2 + \Theta_3 \in C^1(\mathcal{W}_1, \mathbb{R})$ and*

$$\langle (\Theta_2 + \Theta_3)'(\vartheta_n), \psi_n \rangle = \int_{\mathcal{G}} \left[\frac{h_n(\mathrm{x}, \vartheta_n(\mathrm{x}))}{\left(\vartheta_n(\mathrm{x}) + \frac{1}{n}\right)\delta(\mathrm{x})} + \mathcal{P}(\mathrm{x}) \left| \vartheta_n(\mathrm{x}) + \frac{1}{n} \right| \varrho(\mathrm{x})^{-2} \left(\vartheta_n(\mathrm{x}) + \frac{1}{n} \right) \right] \psi_n(\mathrm{x})\, d\mathrm{x},$$

for all $\vartheta_n, \psi_n \in \mathcal{W}_1$.

Proof. *(i)* Θ_2 is Gateaux differentiable in $\mathcal{W}_1$.

Let ϑ_n, $\psi_n \in \mathcal{W}_1$, and $0 < t < 1$, we have

$$\begin{aligned}
\frac{1}{t}(H_n(\mathrm{x}, \vartheta_n + t\psi_n) - H_n(\mathrm{x}, \vartheta_n)) &= \frac{1}{t}\int_0^{\vartheta_n + t\psi_n} \frac{h_n(\mathrm{x}, s)}{(s + \frac{1}{n})^{\delta(\mathrm{x})}}\, ds - \frac{1}{t}\int_0^{\vartheta_n} \frac{h_n(\mathrm{x}, s)}{(s + \frac{1}{n})^{\delta(\mathrm{x})}}\, ds \\
&= \frac{1}{t}\int_{\vartheta_n}^{\vartheta_n + t\psi_n} \frac{h_n(\mathrm{x}, s)}{(s + \frac{1}{n})^{\delta(\mathrm{x})}}\, ds.
\end{aligned}$$

Applying the mean value Theorem, there is a ζ, with $0 < \zeta < 1$, such that:

$$\frac{1}{t}(H_n(\mathrm{x}, \vartheta_n + t\psi_n) - H_n(\mathrm{x}, \vartheta_n)) = \frac{h_n(\mathrm{x}, \vartheta_n + \zeta t\psi_n)}{(\vartheta_n + \zeta t\psi_n + \frac{1}{n})^{\delta(\mathrm{x})}}\psi_n.$$

Combining (C_1) with Young's inequality, we have

$$\begin{aligned}
h_n(\mathrm{x}, \vartheta_n + \zeta t\psi_n) &\le f(\mathrm{x}, \vartheta_n + \zeta t\psi_n) \\
&\le \xi\left(|\psi_n| + |\vartheta_n + \zeta t\psi_n|^{\upsilon(\mathrm{x})} |\psi_n| \right) \\
&\le \xi 2^{\upsilon^+}\left(1 + |\vartheta_n|^{\upsilon(\mathrm{x})} + |\psi_n|^{\upsilon(\mathrm{x})} \right).
\end{aligned}$$

Since $\upsilon(\mathrm{x}) \in \left(1, \mathrm{p}^*_\phi(\mathrm{x})\right)$, we have $\vartheta_n, \psi_n \in L^{\upsilon(\mathrm{x})}(\mathcal{G})$. By the Lebesgue Dominated Convergence Theorem, it follows that:

$$\begin{aligned}
\lim_{t\to 0} \frac{1}{t}(H_n(\mathrm{x}, \vartheta_n + t\psi_n) - H_n(\mathrm{x}, \vartheta_n)) &= \lim_{t\to 0} \int_{\mathcal{G}} \frac{h_n(\mathrm{x}, \vartheta_n + \zeta t\psi_n)}{\left(\vartheta_n + \zeta t\psi_n + \frac{1}{n}\right)^{\delta(\mathrm{x})}}\psi_n\, d\mathrm{x} \\
&= \int_{\mathcal{G}} \lim_{t\to 0} \frac{h_n(\mathrm{x}, \vartheta_n + \zeta t\psi_n)}{\left(\vartheta_n + \zeta t\psi_n + \frac{1}{n}\right)^{\delta(\mathrm{x})}}\psi_n\, d\mathrm{x} \\
&= \int_{\mathcal{G}} \frac{h_n(\mathrm{x}, \vartheta_n)}{\left(\vartheta_n + \frac{1}{n}\right)^{\delta(\mathrm{x})}}\psi_n\, d\mathrm{x},
\end{aligned}$$

$$\begin{aligned}
\left\langle \Theta_3'(\vartheta_n), \psi_n \right\rangle &= \lim_{t\to 0} \frac{\Theta_3(\vartheta_n + t\psi_n) - \Theta_3(\vartheta_n)}{t} \\
&= \lim_{t\to 0} \int_{\mathcal{G}} \frac{\mathcal{P}(\mathrm{x})}{t\varrho(\mathrm{x})}\left(\left|\vartheta_n + \psi_n t + \frac{1}{n}\right|^{\varrho(\mathrm{x})} - \left|\vartheta_n + \frac{1}{n}\right|^{\varrho(\mathrm{x})} \right) d\mathrm{x}.
\end{aligned} \tag{5}$$

Considering the function defined by $M : [0,1] \to \mathbb{R}$ as $M(z) = \frac{\mathcal{P}(\mathrm{x})}{\varrho(\mathrm{x})}\left|\vartheta_n + z\psi_n t + \frac{1}{n}\right|^{\varrho(\mathrm{x})}$. By the Mean Value Theorem, there exists a ε, with $0 < \varepsilon < 1$, such that:

$$M'(z)(\varepsilon) = M(1) - M(0). \tag{6}$$

Combining (5) with (6), it follows that

$$\left\langle \Theta_3' (\vartheta_n), \psi_n \right\rangle = \int_{\mathcal{G}} \mathcal{P}(\mathrm{x}) \left| \vartheta_n(\mathrm{x}) + \frac{1}{n} \right|^{\varrho(\mathrm{x})-2} \left(\vartheta_n(\mathrm{x}) + \frac{1}{n} \right) \psi_n(\mathrm{x})\, d\mathrm{x}.$$

(ii) Continuity of the Gâteaux derivatives:

Let $\{\vartheta_{n,k}\}_{k\in\mathbb{N}} \subset \mathcal{W}_1$ such that $\vartheta_{n,k} \to \vartheta_n$ strongly in $\mathcal{W}_1$ as $k \to +\infty$. Combining Hölder's inequality and condition (C_1), we obtain:

$$\begin{aligned}
\int_{\mathcal{G}} \left| \frac{h_n(\mathrm{x}, \vartheta_{n,k})}{\left(\vartheta_{n,k} + \frac{1}{n}\right)^{\delta(\mathrm{x})}} \right|^{\upsilon'(\mathrm{x})} d\mathrm{x} &\le \int_{\mathcal{G}} \left| h_n(\mathrm{x}, \vartheta_{n,k}) \right|^{\upsilon'(\mathrm{x})} d\mathrm{x} \\
&\le \int_{\mathcal{G}} \left| f(\mathrm{x}, \vartheta_{n,k}) \right|^{\upsilon'(\mathrm{x})} d\mathrm{x} \\
&\le 2^{\frac{\upsilon^+ + 1}{\upsilon^+ - 1}} \|\xi\|_\infty^{\frac{\upsilon^+ + 1}{\upsilon^+ - 1}} \int_{\mathcal{G}} \left| \vartheta_{n,k} \right|^{\upsilon(\mathrm{x})} d\mathrm{x} \\
&\le \mathsf{C}(\xi, \upsilon^+) \int_{\mathcal{G}} \left| \vartheta_{n,k} \right|^{\upsilon(\mathrm{x})} d\mathrm{x} \\
&\le \mathsf{C}(\|\xi\|_\infty, \upsilon^+) \left\| \left| \vartheta_{n,k} \right| \right\|_{L^{\frac{\mathrm{p}_{\phi^*(\mathrm{x})}}{\upsilon(\mathrm{x})}}(\mathcal{G})} \|1\|_{L^{\frac{\mathrm{p}_{\phi^*}(\mathrm{x})}{\mathrm{p}_{\phi^*(\mathrm{x})} - \upsilon(\mathrm{x})}}(\mathcal{G})}.
\end{aligned}$$

So, the sequence $\left\{ \left| \frac{h_n(\mathrm{x}, \vartheta_{n,k})}{\left(\vartheta_{n,k} + \frac{1}{n}\right)^{\delta(\mathrm{x})}} - \frac{h_n(\mathrm{x}, \vartheta_n)}{\left(\vartheta_n + \frac{1}{n}\right)^{\delta(\mathrm{x})}} \right|^{\upsilon(\mathrm{x})} \right\}_{k\in\mathbb{N}}$ is uniformly bounded and equi-integrable in $L^1(\mathcal{G})$. The Vitali convergence theorem implies that:

$$\lim_{k\to+\infty} \int_{\mathcal{G}} \left| \frac{h_n(\mathrm{x}, \vartheta_{n,k})}{\left(\vartheta_{n,k} + \frac{1}{n}\right)^{\delta(\mathrm{x})}} - \frac{h_n(\mathrm{x}, \vartheta_n)}{\left(\vartheta_n + \frac{1}{n}\right)^{\delta(\mathrm{x})}} \right|^{\upsilon'(\mathrm{x})} d\mathrm{x} = 0,$$

where $\frac{1}{\upsilon'(\mathrm{x})} + \frac{1}{\upsilon(\mathrm{x})} = 1$. Consequently, based on Theorem 1 and Hölder's inequality, we find that

$$\begin{aligned}
\left\| \Theta_2'(\vartheta_{n,k}) - \Theta_2'(\vartheta_n) \right\|_{\mathcal{W}_1^*} &= \sup_{\psi_n \in \mathcal{W}_1} \left\| \left\langle \Theta_2'(\vartheta_{n,k}) - \Theta_2'(\vartheta_n), \psi_n \right\rangle \right\|_{\mathcal{W}_1} \\
&\le \left| \left\langle \Theta_2'(\vartheta_{n,k}) - \Theta_2'(\vartheta_n), \psi_n \right\rangle \right| \\
&\le \left\| \frac{h_n(\mathrm{x}, \vartheta_{n,k})}{\left(\vartheta_{n,k} + \frac{1}{n}\right)^{\delta(\mathrm{x})}} - \frac{h_n(\mathrm{x}, \vartheta_n)}{\left(\vartheta_n + \frac{1}{n}\right)^{\delta(\mathrm{x})}} \right\|_{L^{\mathrm{q}_1'(\mathrm{x})}(\mathcal{G})} \left\| \psi_n \right\|_{L^{\mathrm{q}_1(\mathrm{x})}(\mathcal{G})} \to 0 \text{ as } k \to +\infty,
\end{aligned}$$

where $\mathcal{W}_1^*$ is the dual space of $\mathcal{W}_1$. Similarly, we prove that Θ_3' continuous in $\mathcal{W}_1$. □

Lemma 6. *Let $\mathcal{G} \subset \mathbb{R}^N$ be a Lipschitz-bounded domain, $\alpha \in \mathbb{R}^+$, and a continuous function $\upsilon : U \to (1, \infty)$. Then, we have $\Theta_4 \in C^1(\mathcal{W}_1, \mathbb{R})$, and*

$$\left\langle \Theta_4'(\vartheta_n), \psi_n \right\rangle = \alpha \int_{\mathcal{G}} \left| \vartheta_n(\mathrm{x}) + \frac{1}{n} \right|^{\upsilon(\mathrm{x})-2} \left(\vartheta_n(\mathrm{x}) + \frac{1}{n} \right) \log \left| \vartheta_n(\mathrm{x}) + \frac{1}{n} \right| \psi_n(\mathrm{x})\, d\mathrm{x},$$

for all $\vartheta_n, \psi_n \in \mathcal{W}_1$.

Proof. Let $\psi_n, \vartheta_n \in \mathcal{W}^{\mathsf{z},\mathsf{q}(\mathsf{x}),\mathsf{p}(\mathsf{x},\mathsf{y})}(\mathcal{G})$. For each $\mathsf{x} \in \mathcal{G}$, and $0 < t < 1$.
From the definition of Gâteaux differentiability, we obtain:

$$\left\langle \Theta_4'(\vartheta_n), \psi_n \right\rangle = \lim_{t\to 0} \frac{\Theta_4(\vartheta_n + t\psi_n) - \Theta_4(\vartheta_n)}{t}$$
$$= \lim_{t\to 0} \alpha \int_{\mathcal{G}} \frac{1}{\upsilon(\mathsf{x})} \frac{\left(|\vartheta_n + t\psi_n|^{\upsilon(\mathsf{x})} \log|\vartheta_n + t\psi_n| - |\vartheta_n|^{\upsilon(\mathsf{x})}\log|\vartheta_n| + \frac{1}{\upsilon(\mathsf{x})}\left(|\vartheta_n + t\psi_n|^{\upsilon(\mathsf{x})} - |\vartheta_n|^{\upsilon(\mathsf{x})}\right)\right)}{t} d\mathsf{x}.$$

We consider the function $L : [0, 1] \to \mathbb{R}$ defined as follows:

$$L(\mathsf{y}) = \frac{|\vartheta_n + \mathsf{y}t\psi_n|^{\upsilon(\mathsf{x})} \log|\vartheta_n + \mathsf{y}t\psi_n|}{\upsilon(\mathsf{x})} - \frac{|\vartheta_n + \mathsf{y}t\psi_n|^{\upsilon(\mathsf{x})}}{\upsilon^2(\mathsf{x})}.$$

Applying the Mean Value Theorem, we find some $\theta \in (0, 1)$ for which

$$L'(\mathsf{y})(\theta) = L(1) - L(0).$$

Using the Lebesgue Dominated Convergence Theorem and a direct computation, we obtain:

$$\left\langle \Theta_4'(\vartheta_n), \psi_n \right\rangle = \alpha \int_{\mathcal{G}} \left|\vartheta_n(\mathsf{x}) + \frac{1}{n}\right|^{\upsilon(\mathsf{x})-2} \left(\vartheta_n(\mathsf{x}) + \frac{1}{n}\right) \log\left|\vartheta_n(\mathsf{x}) + \frac{1}{n}\right| \psi_n(\mathsf{x})\, d\mathsf{x},$$

using the same method as appear in paper [12], we can prove that $\Theta_4 \in C^1(\mathcal{W}_1, \mathbb{R})$. □

From the Lemma 5 and Lemma 6, we have that $\Theta \in C^1(\mathcal{W}_1, \mathbb{R})$, and

$$\begin{aligned}\langle \Theta'(\vartheta_{n,k}), \psi_n \rangle = &\int_{\mathcal{G}\times\mathcal{G}} \frac{|\vartheta_{n,k}(\mathsf{x}) - \vartheta_{n,k}(\mathsf{y})|^{\mathsf{p}(\mathsf{x},\mathsf{y})-2}(\vartheta_{n,k}(\mathsf{x}) - \vartheta_{n,k}(\mathsf{y}))(\psi_n(\mathsf{x}) - \psi_n(\mathsf{y}))}{|\mathsf{x}-\mathsf{y}|^{N+\phi\mathsf{p}(\mathsf{x},\mathsf{y})}} d\mathsf{x}\, d\mathsf{y}\\
&+ \int_{\mathcal{G}\times\mathcal{G}} \frac{|\vartheta_{n,k}(\mathsf{x}) - \vartheta_{n,k}(\mathsf{y})|^{\mathsf{q}(\mathsf{x},\mathsf{y})-2}(\vartheta_{n,k}(\mathsf{x}) - \vartheta_{n,k}(\mathsf{y}))(\psi_n(\mathsf{x}) - \psi_n(\mathsf{y}))}{|\mathsf{x}-\mathsf{y}|^{N+\varphi\mathsf{q}(\mathsf{x},\mathsf{y})}} d\mathsf{x}\, d\mathsf{y}\\
&- \int_{\mathcal{G}} \left[\frac{h_n(\mathsf{x}, \vartheta_n(\mathsf{x}))}{(\vartheta_n(\mathsf{x}) + \frac{1}{n})^{\delta(\mathsf{x})}} + \mathcal{P}(\mathsf{x})|\vartheta_n(\mathsf{x}) + \frac{1}{n}|^{\varrho(\mathsf{x})-2}\left(\vartheta_n(\mathsf{x}) + \frac{1}{n}\right)\right]\psi_n(\mathsf{x})\, d\mathsf{x}\\
&+ \alpha \int_{\mathcal{G}} \left|\vartheta_n(\mathsf{x}) + \frac{1}{n}\right|^{\upsilon(\mathsf{x})-2}\left(\vartheta_n(\mathsf{x}) + \frac{1}{n}\right)\log\left|\vartheta_n(\mathsf{x}) + \frac{1}{n}\right|\psi_n(\mathsf{x})\, d\mathsf{x}.\end{aligned}$$

Theorem 4. *The functional Θ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$.*

Proof. The proof of this theorem proceeds in two steps. In first, we will demonstrate that the sequence $\{\vartheta_{n,k}\}_{k\in\mathbb{N}}$ is uniformly bounded in $\mathcal{W}_1$, followed by showing that $\vartheta_{n,k} \to \vartheta_n$ strongly in $\mathcal{W}_1$ as $k \to \infty$.

Let $\{\vartheta_{n,k}\} \subset \mathcal{W}_1$ be a sequence such that

$$\Theta(\vartheta_{n,k}) \to c \quad \text{as } k \to +\infty \quad \text{and} \quad \Theta'(\vartheta_{n,k}) \to 0 \quad \text{as } k \to +\infty \quad \text{in } \mathcal{W}_1^*. \tag{7}$$

i) By using the contradiction approach, we prove that the sequence $\{\vartheta_{n,k}\}$ is uniformly bounded in $\mathcal{W}_1$. We assume this does not hold, that is up to a subsequence still denoted by $\{\vartheta_{n,k}\}_{k\in\mathbb{N}}$ such that $\left\|\vartheta_{n,k}\right\|_{\mathcal{W}_1} \to +\infty$ as $k \to +\infty$ in $\mathcal{W}_1$. Let us $\psi_{n,k} := \frac{\vartheta_{n,k}}{\|\vartheta_{n,k}\|_{\mathcal{W}_1}}$. Clearly $\{\psi_{n,k}\}_{k\in\mathbb{N}}$ is bounded in $\mathcal{W}_1$. Since $\mathcal{W}_1$ is a reflexive Banach space, up to a subsequence still denoted by $\{\psi_{n,k}\}_{k\in\mathbb{N}}$ such that:

$$\begin{cases} \psi_{n,k} \rightharpoonup \psi_n \text{ weakly in } \mathcal{W}_1 \text{ as } k \to \infty,\\ \psi_{n,k} \to \psi_n \text{ strongly } k \to +\infty \text{ in } L^{a(\mathsf{x})}(\mathcal{G}) \text{ for all } 1 < a(\mathsf{x}) < \mathsf{p}_\phi^*(\mathsf{x}),\\ \psi_{n,k} \to \psi_n \text{ a.e in } \mathcal{G} \text{ as } k \to \infty.\end{cases}$$

Combining (7) with $\lim_{k\to+\infty}\dfrac{1}{\|\vartheta_{n,k}\|_{\mathcal{W}_1}}=0$, we have

$$\frac{\|\psi_{n,k}\|_{\mathcal{W}_1}^{\mathsf{p}^+}}{\mathsf{p}^-}+\frac{\|\vartheta_{n,k}\|_{\mathcal{W}_1}^{\mathsf{p}^+-\mathsf{q}^-}\|\psi_{n,k}\|_{\mathcal{W}_2}^{\mathsf{q}^+}}{\mathsf{q}^-}-\|\vartheta_{n,k}\|_{\mathcal{W}_1}^{-\mathsf{p}^-}\int_{\mathcal{G}}H_n(\mathrm{x},\vartheta_{n,k})\,d\mathrm{x}$$
$$-\frac{\|\vartheta_{n,k}\|_{\mathcal{W}_1}^{\varrho^--\mathsf{p}^+}}{\varrho^+}\int_{\mathcal{G}}\mathcal{P}(\mathrm{x})\left|\vartheta_n(\mathrm{x})+\frac{1}{n}\right|^{\varrho(\mathrm{x})}d\mathrm{x}\quad\longrightarrow 0\quad\text{as}\quad k\to+\infty, \tag{8}$$

and

$$\|\psi_{n,k}\|_{\mathcal{W}_1}^{\mathsf{p}^+}+\|\vartheta_{n,k}\|_{\mathcal{W}_1}^{\mathsf{q}^+-\mathsf{p}^-}\|\psi_{n,k}\|_{\mathcal{W}_2}^{\mathsf{q}^+}-\|\vartheta_{n,k}\|_{\mathcal{W}_1}^{-\mathsf{p}^-}\int_{\mathcal{G}}\frac{h_n(\mathrm{x},\vartheta_{n,k})}{\left(\vartheta_{n,k}+\frac{1}{n}\right)^{\delta(\mathrm{x})}}\vartheta_{n,k}(\mathrm{x})\,d\mathrm{x}$$
$$-\|\vartheta_{n,k}\|_{\mathcal{W}_1}^{\varrho^--\mathsf{p}^+}\int_{\mathcal{G}}\mathcal{P}(\mathrm{x})\left|\vartheta_n(\mathrm{x})+\frac{1}{n}\right|^{\varrho(\mathrm{x})}d\mathrm{x}\quad\longrightarrow 0\quad\text{as}\quad k\to+\infty. \tag{9}$$

We use (8) and (9), we have

$$\left(\frac{\eta}{\mathsf{p}^-}-1\right)\|\psi_{n,k}\|_{\mathcal{W}_1}^{\mathsf{p}^-}+\left(\frac{\eta}{\mathsf{q}^-}-1\right)\|\vartheta_{n,k}\|_{\mathcal{W}_1}^{\mathsf{q}^--\mathsf{p}^+}\|\psi_{n,k}\|_{\mathcal{W}_2}^{\mathsf{q}^+}-\left(\frac{\eta}{\varrho^-}-1\right)\|\vartheta_{n,k}\|_{\mathcal{W}_1}^{\varrho^--\mathsf{p}^+}\int_{\mathcal{G}}\mathcal{P}(\mathrm{x})\left|\vartheta_n(\mathrm{x})+\frac{1}{n}\right|^{\varrho(\mathrm{x})}d\mathrm{x}$$
$$-\eta\|\vartheta_{n,k}\|_{\mathcal{W}_1}^{-\mathsf{p}^-}\int_{\mathcal{G}}\left(H_n(\mathrm{x},\vartheta_{n,k})-\frac{h_n(\mathrm{x},\vartheta_{n,k})}{\left(\vartheta_{n,k}+\frac{1}{n}\right)^{\delta(\mathrm{x})}}\vartheta_{n,k}(\mathrm{x})\right)d\mathrm{x}\quad\longrightarrow 0\quad\text{as}\quad k\to+\infty.$$

We use (C_4), we can write

$$\begin{aligned}\left(\frac{\eta}{\mathsf{p}^-}-1\right)\|\psi_{n,k}\|_{\mathcal{W}_1}^{\mathsf{p}^-}&=\left(1-\frac{\eta}{\mathsf{q}^+}\right)\|\vartheta_{n,k}\|_{\mathcal{W}_1}^{\mathsf{q}^+-\mathsf{p}^-}\|\psi_{n,k}\|_{\mathcal{W}_2}^{\mathsf{q}^+}\\
&\quad+\eta\|\vartheta_{n,k}\|_{\mathcal{W}_1}^{-\mathsf{p}^-}\left(\int_{\mathcal{G}}h_n(\mathrm{x},\vartheta_{n,k})-\frac{h_n(\mathrm{x},\vartheta_{n,k})}{\left(\vartheta_{n,k}+\frac{1}{n}\right)^{\delta(\mathrm{x})}}\vartheta_{n,k}(\mathrm{x})\,d\mathrm{x}\right)\\
&\quad+\left(1-\frac{\eta}{\varrho^+}\right)\|\vartheta_{n,k}\|_{\mathcal{W}_1}^{\varrho^--\mathsf{p}^+}\int_{\mathcal{G}}\mathcal{P}(\mathrm{x})\left|\vartheta_n(\mathrm{x})+\frac{1}{n}\right|^{\varrho(\mathrm{x})}d\mathrm{x}+o(1)\\
&\le\left(1-\frac{\eta}{\mathsf{q}^+}\right)\|\vartheta_{n,k}\|_{\mathcal{W}_1}^{\mathsf{q}^+-\mathsf{p}^-}\|\psi_{n,k}\|_{\mathcal{W}_2}^{\mathsf{q}^+}\\
&\quad+\left(1-\frac{\eta}{\varrho^+}\right)\|\vartheta_{n,k}\|_{\mathcal{W}_1}^{\varrho^--\mathsf{p}^+}\int_{\mathcal{G}}\mathcal{P}(\mathrm{x})\left|\vartheta_n(\mathrm{x})+\frac{1}{n}\right|^{\varrho(\mathrm{x})}d\mathrm{x}\\
&\quad+a_3\|\vartheta_{n,k}\|_{\mathcal{W}_1}^{\mathsf{p}^--\mathsf{q}^-}\|\psi_n\|_{L^{\mathsf{p}^-}(\mathcal{G})}^{\mathsf{p}^-}+o(1)\\
&=o(1),\end{aligned}$$

as $k\to\infty$. This is a contradiction as $\|\psi_n\|_{\mathcal{W}_1}=1$, and hence the sequence $\{\vartheta_{n,k}\}$ is uniformly bounded in $\mathcal{W}_1$. Consequently, there exists $\vartheta_n\in\mathcal{W}_1$ such that up to a subsequence

$$\begin{cases}\vartheta_{n,k}\rightharpoonup\vartheta_n \text{ weakly in } \mathcal{W}_1 \text{ as } k\to\infty,\\ \vartheta_{n,k}\to\vartheta_n \text{ strongly in } L^{a(\mathrm{x})}(\mathcal{G}) \text{ as } k\to+\infty \text{ for all } 1<a(\mathrm{x})<\mathsf{p}_\phi^*(\mathrm{x}),\\ \vartheta_{n,k}\to\vartheta_n \text{ a.e in } \mathcal{G} \text{ as } k\to\infty.\end{cases} \tag{10}$$

From (C_1) and (ψ), we get

$$\int_{\mathcal{G}} \frac{h_n(\mathrm{x}, \vartheta_{n,k})\, \vartheta_{n,k}}{\frac{1}{n} + \vartheta_{n,k}}\, d\mathrm{x} = \int_{\mathcal{G}} \frac{h_n(\mathrm{x}, \vartheta_n)\, \vartheta_n}{\frac{1}{n} + \vartheta_n}\, d\mathrm{x} + o(1),$$

$$\int_{\mathcal{G}} H_n(\mathrm{x}, \vartheta_{n,k})\, d\mathrm{x} = \int_{\mathcal{G}} H_n(\mathrm{x}, \vartheta_n)\, d\mathrm{x} + o(1),$$

$$\int_{\mathcal{G}} \mathcal{P}(\mathrm{x}) \left|\vartheta_{n,k} + \frac{1}{n}\right|^{\varrho(\mathrm{x})} d\mathrm{x} = \int_{\mathcal{G}} \mathcal{P}(\mathrm{x}) \left|\vartheta_n + \frac{1}{n}\right|^{\varrho(\mathrm{x})} d\mathrm{x} + o(1).$$

We have also $\{\vartheta_{n,k}\}_{k\in\mathbb{N}}$ is bounded in $\mathcal{W}_2$. Since $\vartheta_{n,k} \to \vartheta_n$ in $\mathcal{G}$ a.e. as $k \to +\infty$, we have that

$$\frac{\left|\vartheta_{n,k}(\mathrm{x}) - \vartheta_{n,k}(\mathrm{y})\right|^{\mathrm{p(x,y)}-2} (\vartheta_{n,k}(\mathrm{x}) - \vartheta_{n,k}(\mathrm{y}))}{|\mathrm{x}-\mathrm{y}|^{\left(\frac{N}{\mathrm{p(x,y)}}+\phi\right)(\mathrm{p(x,y)}-1)}} \to \frac{|\vartheta_n(\mathrm{x}) - \vartheta_n(\mathrm{y})|^{\mathrm{p(x,y)}-2} (\vartheta_n(\mathrm{x}) - \vartheta_n(\mathrm{y}))}{|\mathrm{x}-\mathrm{y}|^{\left(\frac{N}{\mathrm{p(x,y)}}+\phi\right)(\mathrm{p(x,y)}-1)}}$$

$(\mathrm{x}, \mathrm{y}) \in \mathcal{G} \times \mathcal{G}$ a.e as $k \to +\infty$. Since $\{\vartheta_{n,k}\}_{k\in\mathbb{N}}$ is bounded in $\mathcal{W}_1$, there exist $c > 0$ such that

$$\int_{\mathcal{G}\times\mathcal{G}} \left| \frac{\left|\vartheta_{n,k}(\mathrm{x}) - \vartheta_{n,k}(\mathrm{y})\right|^{\mathrm{p(x,y)}-2} (\vartheta_{n,k}(\mathrm{x}) - \vartheta_{n,k}(\mathrm{y}))}{|\mathrm{x}-\mathrm{y}|^{\left(\frac{N}{\mathrm{p(x,y)}}+\phi\right)(\mathrm{p(x,y)}-1)}} \right|^{\frac{\mathrm{p(x,y)}}{\mathrm{p(x,y)}-1}} d\mathrm{x}\, d\mathrm{y} \le C\cdot$$

So, we have that

$$\frac{\left|\vartheta_{n,k}(\mathrm{x}) - \vartheta_{n,k}(\mathrm{y})\right|^{\mathrm{p(x,y)}-2} (\vartheta_{n,k}(\mathrm{x}) - \vartheta_{n,k}(\mathrm{y}))}{|\mathrm{x}-\mathrm{y}|^{\left(\frac{N}{\mathrm{p(x,y)}}+\phi\right)(\mathrm{p(x,y)}-1)}} \rightharpoonup \frac{|\vartheta_n(\mathrm{x}) - \vartheta_n(\mathrm{y})|^{\mathrm{p(x,y)}-2} (\vartheta_n(\mathrm{x}) - \vartheta_n(\mathrm{y}))}{|\mathrm{x}-\mathrm{y}|^{\left(\frac{N}{\mathrm{p(x,y)}}+\phi\right)(\mathrm{p(x,y)}-1)}}$$

weakly in $L^{\mathrm{p}'(\mathrm{x,y})}(\mathcal{G}\times\mathcal{G})$ as $k \to \infty$, where $\frac{1}{\mathrm{p}'(\mathrm{x,y})} + \frac{1}{\mathrm{p(x,y)}} = 1$. Let $\vartheta_n \in \mathcal{W}_1$, it is follows that

$$\frac{\vartheta_n(\mathrm{x}) - \vartheta_n(\mathrm{y})}{|\mathrm{x}-\mathrm{y}|^{\frac{N}{\mathrm{p(x,y)}}+\phi}} \in L^{\mathrm{p(x,y)}}(\mathcal{G}\times\mathcal{G}), \text{ and } \frac{\vartheta_n(\mathrm{x}) - \vartheta_n(\mathrm{y})}{|\mathrm{x}-\mathrm{y}|^{\frac{N}{\mathrm{q(x,y)}}+\varphi}} \in L^{\mathrm{q(x,y)}}(\mathcal{G}\times\mathcal{G}).$$

Finally, we get that

$$\int_{\mathcal{G}\times\mathcal{G}} \frac{\left|\vartheta_{n,k}(\mathrm{x}) - \vartheta_{n,k}(\mathrm{y})\right|^{\mathrm{p(x,y)}-2} (\vartheta_{n,k}(\mathrm{x}) - \vartheta_{n,k}(\mathrm{y}))}{|\mathrm{x}-\mathrm{y}|^{\left(\frac{N}{\mathrm{p(x,y)}}+\phi\right)\mathrm{p(x,y)}}} d\mathrm{x}\, d\mathrm{y}$$
$$\to \int_{\mathcal{G}\times\mathcal{G}} \frac{|\vartheta_n(\mathrm{x}) - \vartheta_n(\mathrm{y})|^{\mathrm{p(x,y)}-2} (\vartheta_n(\mathrm{x}) - \vartheta_n(\mathrm{y}))}{|\mathrm{x}-\mathrm{y}|^{\left(\frac{N}{\mathrm{p(x,y)}}+\phi\right)\mathrm{p(x,y)}}} d\mathrm{x}\, d\mathrm{y} \quad \text{as } k \to \infty,$$

and

$$\int_{\mathcal{G}\times\mathcal{G}} \frac{\left|\vartheta_{n,k}(\mathrm{x}) - \vartheta_{n,k}(\mathrm{y})\right|^{\mathrm{q(x,y)}-2} (\vartheta_{n,k}(\mathrm{x}) - \vartheta_{n,k}(\mathrm{y}))}{|\mathrm{x}-\mathrm{y}|^{\left(\frac{N}{\mathrm{p(x,y)}}+\varphi\right)\mathrm{q(x,y)}}} d\mathrm{x}\, d\mathrm{y}$$
$$\to \int_{\mathcal{G}\times\mathcal{G}} \frac{|\vartheta_n(\mathrm{x}) - \vartheta_n(\mathrm{y})|^{\mathrm{q(x,y)}-2} (\vartheta_n(\mathrm{x}) - \vartheta_n(\mathrm{y}))}{|\mathrm{x}-\mathrm{y}|^{\left(\frac{N}{\mathrm{q(x,y)}}+\varphi\right)\mathrm{q(x,y)}}} d\mathrm{x}\, d\mathrm{y} \text{ as } k \to \infty.$$

Now, we show that

$$\lim_{n\to\infty} \int_{\mathcal{G}} \left|\vartheta_{n,k}(\mathrm{x})\right|^{\upsilon(\mathrm{x})-2} \vartheta_{n,k}(\mathrm{x}) (\vartheta_{n,k}(\mathrm{x}) - \vartheta_{n,k}(\mathrm{x})) \log\left|\vartheta_{n,k}(\mathrm{x})\right| d\mathrm{x}$$
$$= \lim_{n\to\infty} \int_{\mathcal{G}} |\vartheta_n(\mathrm{x})|^{\upsilon(\mathrm{x})-2} \vartheta_n(\mathrm{x}) (\vartheta_{n,k}(\mathrm{x}) - \vartheta_n(\mathrm{x})) \log|\vartheta_n(\mathrm{x})|\, d\mathrm{x}.$$

From (10), we have

$$\lim_{n\to\infty}\left|\vartheta_{n,k}(\mathsf{x})\right|^{\upsilon(\mathsf{x})}|\log\left|\vartheta_{n,k}(\mathsf{x})\right||=|\vartheta_n(\mathsf{x})|^{\upsilon(\mathsf{x})}|\log|\vartheta_n(\mathsf{x})||\text{ a.e in }\mathcal{G}. \tag{11}$$

Let $\kappa\in(0,\mathsf{p}_z^{*-}-\upsilon^+)$. From Lemma 4, and Theorem 1, we have that

$$\begin{aligned}\int_{\mathcal{G}}\left|\vartheta_{n,k}(\mathsf{x})\right|^{\upsilon(\mathsf{x})}|\log|\vartheta_{n,k}(\mathsf{x})||\,d\mathsf{x}&=\int_{\mathcal{G}\cap\{|\vartheta_{n,k}(\mathsf{x})|\le1\}}\left|\vartheta_{n,k}(\mathsf{x})\right|^{\upsilon(\mathsf{x})}|\log|\vartheta_{n,k}(\mathsf{x})||\,d\mathsf{x}\\&\quad+\int_{\mathcal{G}\cap\{|\vartheta_{n,k}(\mathsf{x})|>1\}}\left|\vartheta_{n,k}(\mathsf{x})\right|^{\upsilon(\mathsf{x})}|\log|\vartheta_{n,k}(\mathsf{x})||\,d\mathsf{x}\\&\le\frac{|\mathcal{G}|}{\upsilon^-\exp(1)}+\frac{1}{\kappa\exp(1)}\int_{\mathcal{G}}\left|\vartheta_{n,k}(\mathsf{x})\right|^{\upsilon^++\kappa}\,d\mathsf{x}\\&\le\frac{|\mathcal{G}|}{\upsilon^-\exp(1)}+C\frac{|\mathcal{G}|C_{\upsilon^++\kappa}^{\upsilon^++\kappa}}{\kappa\exp(1)},\end{aligned}\tag{12}$$

where $C=\sup\left\|\vartheta_{n,k}\right\|^{\upsilon^++\kappa}<\infty$. Hence, the sequence $\left\{\left|\vartheta_{n,k}(\mathsf{x})\right|^{\upsilon(\mathsf{x})}|\log|\vartheta_{n,k}(\mathsf{x})||\right\}_{n\ge1}$ is equi-integral and uniformly bounded in $L^1(\mathcal{G})$, and uniformly bounded.
Using (12), (11) with Vitali's convergence theorem, we have that

$$\lim_{n\to\infty}\int_{\mathcal{G}}\left|\vartheta_{n,k}(\mathsf{x})\right|^{\upsilon(\mathsf{x})}|\log|\vartheta_{n,k}(\mathsf{x})||\,d\mathsf{x}=\int_{\mathcal{G}}|\vartheta_n(\mathsf{x})|^{\upsilon(\mathsf{x})}\left|\log|\vartheta_n(\mathsf{x})|\right|\,d\mathsf{x}. \tag{13}$$

By similar arguments, we prove that

$$\lim_{n\to\infty}\int_{\mathcal{G}}u\left|\vartheta_{n,k}(\mathsf{x})\right|^{\upsilon(\mathsf{x})-2}\vartheta_{n,k}(\mathsf{x})|\log|\vartheta_{n,k}(\mathsf{x})||d\mathsf{x}=\int_{\mathcal{G}}|\vartheta_n(\mathsf{x})|^{\upsilon(\mathsf{x})}|\log|\vartheta_n(\mathsf{x})||\,d\mathsf{x}, \tag{14}$$

and

$$\lim_{n\to\infty}\int_{\mathcal{G}}\vartheta_{n,k}(\mathsf{x})\left|\vartheta_{n,k}(\mathsf{x})\right|^{\upsilon(\mathsf{x})-2}|\log|\vartheta_n(\mathsf{x})||\vartheta_n(\mathsf{x})d\mathsf{x}=\int_{\mathcal{G}}|\vartheta_n(\mathsf{x})|^{\upsilon(\mathsf{x})}|\log|\vartheta_n(\mathsf{x})||\,d\mathsf{x}. \tag{15}$$

From (13), (14), and (15), we have that

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathcal{G}}\left|\vartheta_{n,k}(\mathsf{x})\right|^{\upsilon(\mathsf{x})-2}\vartheta_{n,k}(\mathsf{x})\left(\vartheta_{n,k}(\mathsf{x})-\vartheta_{n,k}(\mathsf{x})\right)\log\left|\vartheta_{n,k}(\mathsf{x})\right|\,d\mathsf{x}\\&\quad=\lim_{n\to\infty}\int_{\mathcal{G}}|\vartheta_n(\mathsf{x})|^{\upsilon(\mathsf{x})-2}\vartheta_n(\mathsf{x})\left(\vartheta_{n,k}(\mathsf{x})-\vartheta_n(\mathsf{x})\right)\log|\vartheta_n(\mathsf{x})|\,d\mathsf{x}.\end{aligned}$$

ii) We will show that $\vartheta_{n,k}\to\vartheta_n$ strongly in $\mathcal{W}_1$ as $k\to+\infty$.
Considering the sequence defined as $\psi_{n,k}=\vartheta_{n,k}-\vartheta_n$. Since $\vartheta_{n,k}\to\vartheta_n$ in $\mathcal{G}$ a.e and $\{\vartheta_{n,k}\}_{k\in\mathbb{N}}$ is uniformly bounded in $\mathcal{W}_1$ and $\mathcal{W}_2$. Applying the Brezis-Lieb Lemma in [11], we have that

$$\begin{aligned}&\int_{\mathcal{G}\times\mathcal{G}}\frac{\left|\vartheta_{n,k}(\mathsf{x})-\vartheta_{n,k}(\mathsf{y})\right|^{\mathsf{p}(\mathsf{x},\mathsf{y})-2}\left(\vartheta_{n,k}(\mathsf{x})-\vartheta_{n,k}(\mathsf{y})\right)}{|\mathsf{x}-\mathsf{y}|^{\left(\frac{N}{\mathsf{p}(\mathsf{x},\mathsf{y})}+\phi\right)\mathsf{p}(\mathsf{x},\mathsf{y})}}\,d\mathsf{x}\,d\mathsf{y}\\&=\int_{\mathcal{G}\times\mathcal{G}}\frac{\left|\psi_{n,k}(\mathsf{x})-\psi_{n,k}(\mathsf{y})\right|^{\mathsf{p}(\mathsf{x},\mathsf{y})-2}\left(\psi_{n,k}(\mathsf{x})-\psi_{n,k}(\mathsf{y})\right)}{|\mathsf{x}-\mathsf{y}|^{\left(\frac{N}{\mathsf{p}(\mathsf{x},\mathsf{y})}+\phi\right)\mathsf{p}(\mathsf{x},\mathsf{y})}}\,d\mathsf{x}\,d\mathsf{y}\\&+\int_{\mathcal{G}\times\mathcal{G}}\frac{|\vartheta_n(\mathsf{x})-\vartheta_n(\mathsf{y})|^{\mathsf{p}(\mathsf{x},\mathsf{y})-2}\left(\vartheta_n(\mathsf{x})-\vartheta_n(\mathsf{y})\right)}{|\mathsf{x}-\mathsf{y}|^{\left(\frac{N}{\mathsf{p}(\mathsf{x},\mathsf{y})}+\phi\right)\mathsf{p}(\mathsf{x},\mathsf{y})}}\,d\mathsf{x}\,d\mathsf{y}+o(1),\end{aligned}$$

i.e $\left\|\vartheta_{n,k}\right\|_{\mathcal{W}_1}^{\mathrm{p}^+} = \|\vartheta_n\|_{\mathcal{W}_1}^{\mathrm{p}^+} + \left\|\psi_{n,k}\right\|_{\mathcal{W}_1}^{\mathrm{p}^+} + o(1)$. Similarly, we get $\left\|\vartheta_{n,k}\right\|_{\mathcal{W}_2}^{\mathrm{q}^+} = \|\vartheta_n\|_{Y2}^{\mathrm{q}^+} + \left\|\psi_{n,k}\right\|_{\mathcal{W}_2}^{\mathrm{q}^+} + o(1)$. So, we have that

$$c + o(1) = \Theta(\vartheta_{n,k}) \leq \frac{1}{\mathrm{p}^+}\left\|\psi_{n,k}\right\|_{\mathcal{W}_1}^{\mathrm{p}^+} + \frac{1}{\mathrm{q}^+}\left\|\psi_{n,k}\right\|_{\mathcal{W}_2}^{\mathrm{q}^+} + \frac{1}{\mathrm{p}^+}\|\vartheta_n\|_{\mathcal{W}_1}^{\mathrm{p}^+} + \frac{1}{\mathrm{q}^+}\|\vartheta_n\|_{\mathcal{W}_2}^{\mathrm{q}^+} - \int_{\mathcal{G}} H_n(\mathrm{x}, \vartheta_n(\mathrm{x}))\,d\mathrm{x}$$
$$- \int_{\mathcal{G}} \mathcal{P}(\mathrm{x})\left|\vartheta_{n,k} + \frac{1}{n}\right|^{\varrho(\mathrm{x})} d\mathrm{x}.$$

Since $\Theta'(\vartheta_{n,k}) \to 0$ as $k \to +\infty$, we have

$$\lim_{k\to+\infty}\left\|\psi_{n,k}\right\|_{\mathcal{W}_1}^{\mathrm{p}^+} + \left\|\psi_{n,k}\right\|_{\mathcal{W}_2}^{\mathrm{q}^+} = \int_{\mathcal{G}} \frac{h_n(\mathrm{x}, \vartheta_n)}{\left(\vartheta_n + \frac{1}{n}\right)^{\delta(\mathrm{x})}}\vartheta_n(\mathrm{x})\,d\mathrm{x} - \|\vartheta_n\|_{\mathcal{W}_1}^{\mathrm{p}^+} - \|\vartheta_n\|_{\mathcal{W}_2}^{\mathrm{q}^+} - \int_{\mathcal{G}} \mathcal{P}(\mathrm{x})\left|\vartheta_n + \frac{1}{n}\right|^{\varrho(\mathrm{x})} d\mathrm{x}, \tag{16}$$

we combine (16) with $\Theta(\vartheta_n) = 0$, we have

$$\lim_{k\to+\infty}\left\|\psi_{n,k}\right\|_{\mathcal{W}_1}^{\mathrm{p}^+} + \left\|\psi_{n,k}\right\|_{\mathcal{W}_2}^{\mathrm{q}^+} = 0.$$

Since $\left\|\psi_{n,k}\right\|_{\mathcal{W}_1}^{\mathrm{p}^+}$ and $\left\|\psi_{n,k}\right\|_{\mathcal{W}_2}^{\mathrm{q}^+}$ are bounded sequence, we can write $\lim_{k\to+\infty}\left\|\psi_{n,k}\right\|_{\mathcal{W}_1}^{\mathrm{p}^+} = a$ and $\lim_{k\to+\infty}\left\|\psi_{n,k}\right\|_{\mathcal{W}_2}^{\mathrm{q}^+} = b$.
Given that $a, b \geq 0$ and $a + b = 0$, it follows that $a = b = 0$.
Finally $\vartheta_{n,k} \to \vartheta_{n,k}$ strongly in $\mathcal{W}_1$ as $k \to +\infty$. □

We now employ the local $(a, b)-$linking notion to compute $\dim C_k(\Theta, 0)$.

Theorem 5. *The functional Θ satisfies the local $(1, 1)-$ linking condition at the origin.*

Proof. According to (C_3) and a direct computation, we have

$$\frac{n^{\delta(\mathrm{x})} l}{2\mathrm{p}(\mathrm{x}, \mathrm{y})}|\vartheta_n(\mathrm{x})|^{\mathrm{p}^+ + 1} \leq H_n(\mathrm{x}, \vartheta_n(\mathrm{x})).$$

We define $Y = \mathbb{R}$. Evidently, Y is a one-dimensional vector subspace of $\mathcal{W}_1$. We choose $\upsilon \in (0, 1)$ such that $K_\Theta \cap \mathcal{B}_\upsilon(0) = \{0\}$, where $\mathcal{B}_\upsilon(0) = \{\vartheta_n \in \mathcal{W}_1 : \|\vartheta_n\|_{\mathcal{W}_1} < r\}$ and $K_\Theta = \{\vartheta_n \in \mathcal{W}_1 : \Theta'(\vartheta_n) = 0\}$. We consider the set $E = Y \cap \overline{\mathcal{B}_\upsilon(0)}$ for small enough $\upsilon \in (0, 1)$. As all norms are equivalent on finite-dimensional normed space.
So, by choosing $\upsilon \in (0, 1)$small enough, we obtain that

$$\|\vartheta_n\|_{\mathcal{W}_1} \leq r \Rightarrow |\vartheta_n| \leq \zeta \quad \text{for all} \quad \vartheta_n \in Y = \mathbb{R}.$$

Then for any $\vartheta_n \in Y \cap \overline{\mathcal{B}_\upsilon(0)}$, we have

$$\begin{aligned}
\Theta(\vartheta_n) &= \int_{\mathcal{G}\times\mathcal{G}} \frac{1}{\mathrm{p}(\mathrm{x}, \mathrm{y})}\frac{|\vartheta_n(\mathrm{x}) - \vartheta_n(\mathrm{y})|^{\mathrm{p}(\mathrm{x},\mathrm{y})}}{|\mathrm{x} - \mathrm{y}|^{N+\phi\mathrm{p}(\mathrm{x},\mathrm{y})}}\,d\mathrm{x}\,d\mathrm{y} + \int_{\mathcal{G}\times\mathcal{G}} \frac{1}{\mathrm{q}(\mathrm{x}, \mathrm{y})}\frac{|\vartheta_n(\mathrm{x}) - \vartheta_n(\mathrm{y})|^{\mathrm{q}(\mathrm{x},\mathrm{y})}}{|\mathrm{x} - \mathrm{y}|^{N+\varphi\mathrm{q}(\mathrm{x},\mathrm{y})}}\,d\mathrm{x}\,d\mathrm{y} \\
&- \int_{\mathcal{G}} H_n(\mathrm{x}, \vartheta_n(\mathrm{x}))d\mathrm{x} - \int_{\mathcal{G}} \frac{\mathcal{P}(\mathrm{x})}{\varrho(\mathrm{x})}|\vartheta_n(\mathrm{x}) + \frac{1}{n}|^{\varrho(\mathrm{x})}\,d\mathrm{x} \\
&- \alpha\int_{\mathcal{G}} \frac{1}{\upsilon(\mathrm{x})}\left(\left|\vartheta_n(\mathrm{x}) + \frac{1}{n}\right|^{\upsilon(\mathrm{x})}\log\left|\vartheta_n(\mathrm{x}) + \frac{1}{n}\right| - \frac{1}{\upsilon(\mathrm{x})}\left|\vartheta_n(\mathrm{x}) + \frac{1}{n}\right|^{\upsilon(\mathrm{x})}\right)d\mathrm{x} \\
&\leq \frac{1}{p-}\int_{\mathcal{G}\times\mathcal{G}} \frac{|\vartheta_n(\mathrm{x}) - \vartheta_n(\mathrm{y})|^{\mathrm{p}(\mathrm{x},\mathrm{y})}}{|\mathrm{x} - \mathrm{y}|^{N+\phi\mathrm{p}(\mathrm{x},\mathrm{y})}}\,d\mathrm{x}\,d\mathrm{y} + \frac{1}{q-}\int_{\mathcal{G}\times\mathcal{G}} \frac{|\vartheta_n(\mathrm{x}) - \vartheta_n(\mathrm{y})|^{\mathrm{q}(\mathrm{x},\mathrm{y})}}{|\mathrm{x} - \mathrm{y}|^{N+\varphi\mathrm{q}(\mathrm{x},\mathrm{y})}}\,d\mathrm{x}\,d\mathrm{y} \\
&- \frac{n^{\delta^+} l}{2(\mathrm{p}^+ + 1)}\int_{\mathcal{G}} |\vartheta_n(\mathrm{x})|^{\mathrm{p}^+ + 1}d\mathrm{x} - \frac{1}{\varrho^+}\left(\frac{1}{n}\right)^{\varrho^+}|\mathcal{G}| - \frac{\alpha}{(\upsilon^-)^2}\int_{\mathcal{G}} |\vartheta_n(\mathrm{x})|^{\upsilon(\mathrm{x})}\,d\mathrm{x} \\
&\leq 0.
\end{aligned}$$

Moreover, We define the set as follows:

$$D = \left\{ \vartheta_n \in \mathcal{W}_1 : \min(\frac{1}{\mathsf{q}^+}, \frac{1}{\mathsf{p}^+}) \|\vartheta_n\|_{\mathcal{W}_1}^{\mathsf{p}(\mathsf{x},\mathsf{y})} > \|\xi\|_\infty C(\mathcal{G}, \upsilon, N) \|\vartheta_n\|_{\mathcal{W}_1}^{l(\mathsf{x})} \frac{n^{1-\delta^-}}{1-\delta^-} + \frac{1}{\varrho^+} \eta_1 \|\vartheta_n\|_{\mathcal{W}_1} \right\},$$

where $l : \mathcal{G} \to (1, \infty)$ is the continuous function such that $l(\mathsf{x}) \leq \mathsf{p}_z^*(\mathsf{x})$, and $C(\mathcal{G}, \upsilon, N)$ is the positive constant. Using conditions (C_1) , (V), and Theorem 1, we have that for any $\vartheta_n \in D$,

$$\begin{aligned}
\Theta(\vartheta_n) &= \int_{\mathcal{G}\times\mathcal{G}} \frac{1}{\mathsf{p}(\mathsf{x},\mathsf{y})} \frac{|\vartheta_n(\mathsf{x}) - \vartheta_n(\mathsf{y})|^{\mathsf{p}(\mathsf{x},\mathsf{y})}}{|\mathsf{x}-\mathsf{y}|^{N+\phi\mathsf{p}(\mathsf{x},\mathsf{y})}} d\mathsf{x}\, d\mathsf{y} + \int_{\mathcal{G}\times\mathcal{G}} \frac{1}{\mathsf{q}(\mathsf{x},\mathsf{y})} \frac{|\vartheta_n(\mathsf{x}) - \vartheta_n(\mathsf{y})|^{\mathsf{q}(\mathsf{x},\mathsf{y})}}{|\mathsf{x}-\mathsf{y}|^{N+\varphi\mathsf{q}(\mathsf{x},\mathsf{y})}} d\mathsf{x}\, d\mathsf{y} \\
&- \int_{\mathcal{G}} H_n(\mathsf{x}, \vartheta_n(\mathsf{x})) d\mathsf{x} - \int_{\mathcal{G}} \frac{\mathcal{P}(\mathsf{x})}{\varrho(\mathsf{x})} |\vartheta_n(\mathsf{x}) + \frac{1}{n}|^{\varrho(\mathsf{x})} d\mathsf{x} \\
&- \alpha \int_{\mathcal{G}} \frac{1}{\upsilon(\mathsf{x})} \left(\left|\vartheta_n(\mathsf{x}) + \frac{1}{n}\right|^{\upsilon(\mathsf{x})} \log\left|\vartheta_n(\mathsf{x}) + \frac{1}{n}\right| - \frac{1}{\upsilon(\mathsf{x})} \left|\vartheta_n(\mathsf{x}) + \frac{1}{n}\right|^{\upsilon(\mathsf{x})} \right) d\mathsf{x} \\
&\geq \frac{1}{\mathsf{p}^+} \|\vartheta_n\|_{\mathcal{W}_1}^{\mathsf{p}(\mathsf{x},\mathsf{y})} + \frac{1}{\mathsf{q}^+} \|\vartheta_n\|_{\mathcal{W}_2}^{\mathsf{q}(\mathsf{x},\mathsf{y})} - 2\|\xi\|_\infty \|\vartheta_n\|_{\mathcal{W}_1}^{l(\mathsf{x})} - \frac{1}{\varrho^+} \eta_1 \|\vartheta_n\|_{\mathcal{W}_1} - \frac{1}{\upsilon^+} \|\vartheta_n\|_{\mathcal{W}_1} > 0.
\end{aligned}$$

Let $\mathcal{U} = \overline{\mathcal{B}_\upsilon(0)}$, $E_0 = Y \cap \partial\mathcal{B}_\upsilon(0)$, $E = Y \cap \overline{\mathcal{B}_\upsilon(0)}$, and D defined as before, the conditions $0 \notin E_0 \subset E \subset \mathcal{U} = \overline{\mathcal{B}_\upsilon(0)}$ and $E_0 \cap D = \emptyset$ lead directly to the following conclusion:

$$\Theta_{|E} \leq 0 < \Theta_{|D \cap \overline{\mathcal{B}_\upsilon(0)}}.$$

We define Z as the topological complement of Y. We have that $\mathcal{W}_1 = Y \bigoplus Z$. So, every $\vartheta_n \in \mathcal{W}_1$ can be uniquely expressed in the form:

$$\vartheta_n = \psi_n + y_n \quad \text{with} \quad \psi_n \in Y, y_n \in Z.$$

We consider the map $g : [0,1] \times \mathcal{W}_1 \backslash D \to \mathcal{W}_1 \backslash D$ defined by

$$g(t, \vartheta_n) = (1-t)\vartheta_n + t\upsilon \frac{\psi_n}{\|\psi_n\|}.$$

We have $g(0, \vartheta_n) = \vartheta_n$ and $g(1, \vartheta_n) = \upsilon \frac{\psi_n}{\|\psi_n\|} \in Y \cap \partial\mathcal{B}_\upsilon(0) = E_0$. Consequently, E_0 is a deformation retract of $\mathcal{W}_1 \backslash D$. Hence

$$i^* : \mathcal{H}_0(E_0) \to \mathcal{H}_0(\mathcal{W}_1),$$

i an isomorphism. Note that $E_0 = \{a, -a\}$ with $a \neq 0$.
Therefore, from $\dim \mathcal{H}_0(E_0) = 2$, since $\mathcal{H}_0(E_0) = \mathbb{R} \bigoplus \mathbb{R}$.
Thus $\dim \mathrm{im}(i^*) = 2$.
The set $E = Y \cap \mathcal{B}_\upsilon(0)$ is contractible (it is an interval). Using Theorem 11.5 in [1], we have that $\mathcal{H}_0(E, E_0) = 0$.
By the Remark 6.1.26 in [18], we get $\dim Im((j^*)) = 1$.
So, finally

$$\dim(Im(i^*)) - \dim Im((j^*)) = 2 - 1 = 1.$$

Since the hypotheses of definition 5 are satisfied. We conclude that Θ has a local $(1,1)-$ linking at 0 . □

Remark 1. *The critical groups of* Θ *at the origin are nontrivial for all* $k \in \mathbb{N}$, *i.e* $\forall k \in \mathbb{N}, C_k(\Theta, 0) \neq 0$.

Proof. Given that Θ has a local $(1,1)$ linking at the origin. Using proposition 2.1 in [15], we get that $\dim C_k(\Theta, 0) \geq 1$. □

We now turn to computing the group critical of Θ at infinitely.

Theorem 6. *Suppose that the condition* (C_3) *is satisfied. Then, there exists* $k \in \mathbb{N}$ *such that* $C_k(\Theta, \infty) = 0$.

Proof. Firstly, we prove that there exists a positive constant A such that Θ^a is homotopic to exists a constant $A > 0$ such that Θ^a is homotopic to $S^1 = \left\{\vartheta_n \in \mathcal{W}_1 : \|\vartheta_n\|_{\mathcal{W}_1} = 1\right\}$, for all $a < -A$. From the condition (C_3), it follows that

$$\begin{aligned}\Theta(t\vartheta_n) &= \int_{\mathcal{G}\times\mathcal{G}} \frac{t^{\mathsf{p}(x,y)}}{\mathsf{p}(x,y)} \frac{|\vartheta_n(x)-\vartheta_n(y)|^{\mathsf{p}(x,y)}}{|x-y|^{N+\phi\mathsf{p}(x,y)}}\,dx\,dy + \int_{\mathcal{G}\times\mathcal{G}} \frac{t^{\mathsf{q}(x,y)}}{\mathsf{q}(x,y)} \frac{|\vartheta_n(x)-\vartheta_n(y)|^{\mathsf{q}(x,y)}}{|x-y|^{N+\varphi\mathsf{q}(x,y)}}\,dx\,dy\\
&- \int_{\mathcal{G}} \frac{\mathcal{P}(x)}{\varrho(x)}|t|^{\varrho(x)}\left|\vartheta_n(x)+\frac{1}{n}\right|^{\varrho(x)} dx - \int_{\mathcal{G}} H_n(x, t\vartheta_n(x))\,dx\\
&- \alpha\int_{\mathcal{G}} \frac{1}{\upsilon(x)}|t|^{\upsilon(x)}\left(\left|\vartheta_n(x)+\frac{1}{n}\right|^{\upsilon(x)}\log\left|\vartheta_n(x)+\frac{1}{n}\right| - \frac{1}{\upsilon(x)}\left|\vartheta_n(x)+\frac{1}{n}\right|^{\upsilon(x)}\right) dx\\
&\le \frac{t^{\mathsf{p}^+}}{\mathsf{p}^-} + \frac{t^{\mathsf{q}^+}}{\mathsf{q}^-}\|\vartheta_n\|_{\mathcal{W}_2}^{\mathsf{q}(x,y)} - \frac{lt^{\mathsf{p}_\phi^{+*}}}{2\mathsf{p}_\phi^{+*}}\int_{\mathcal{G}} \vartheta_n(x)^{\mathsf{p}_\phi^{+*}}\,dx - \frac{\theta_1 t^{\varrho^+}}{\varrho^+}\int_{\mathcal{G}}\left|\vartheta_n(x)+\frac{1}{n}\right|^{\varrho(x)} - \frac{t^{\upsilon^+}}{(\upsilon^-)^2}\alpha\int_{\mathcal{G}}|\vartheta_n(x)|^{\upsilon(x)}\,dx,\end{aligned}$$

where $\mathsf{p}_\phi^{+*} = \max_{x\in\mathcal{G}} \mathsf{p}_\phi^*(x)$. Since $\mathsf{p}_\phi^{+*} > \mathsf{p}^+ > \mathsf{q}^+ > \varrho^+$, we have that $\Theta(t\vartheta_n) \to -\infty$ as $t \to +\infty$. Let $A \in \mathbb{R}$ there exists $t \in \mathbb{R}$ such that $\|t\vartheta_n\|_{\mathcal{W}_1} \ge A$, we have that $\Theta(t\vartheta_n) \le A$. Since $\vartheta_n \in S^1$, we have that

$$\begin{aligned}\frac{d}{dt}\Theta(t\vartheta_n) &= \int_{\mathcal{G}\times\mathcal{G}} t^{\mathsf{p}(x,y)-1}\frac{|\vartheta_n(x)-\vartheta_n(y)|^{\mathsf{p}(x,y)}}{|x-y|^{N+\phi\mathsf{p}(x,y)}}\,dx\,dy + \int_{\mathcal{G}\times\mathcal{G}} t^{\mathsf{q}(x,y)-1}\frac{|\vartheta_n(x)-\vartheta_n(y)|^{\mathsf{q}(x,y)}}{|x-y|^{N+\varphi\mathsf{q}(x,y)}}\,dx\,dy\\
&- \int_{\mathcal{G}} \vartheta_n(x)\frac{h_n(x, t\vartheta_n(x))}{\left(t\vartheta_n(x)+\frac{1}{n}\right)^{\delta(x)}}\,dx - \int_{\mathcal{G}} \mathcal{P}(x)|t|^{\varrho(x)-1}\left|\vartheta_n(x)+\frac{1}{n}\right|^{\varrho(x)-1} dx - \int_{\mathcal{G}} |t|^{\upsilon(x)}|\vartheta_n(x)|^{\upsilon(x)}\,dx\\
&\le t^{\mathsf{p}^+-1}\|\vartheta_n\|_{\mathcal{W}_1}^{\mathsf{p}(x,y)} + t^{\mathsf{q}^+-1}\|\vartheta_n\|_{\mathcal{W}_2}^{\mathsf{q}(x,y)} - \int_{\mathcal{G}} \vartheta_n(x)\frac{h_n(x, t\vartheta_n(x))}{\left(t\vartheta_n(x)+\frac{1}{n}\right)^{\delta(x)}}\,dx\\
&- t^{\varrho^+-1}\int_{\mathcal{G}} \mathcal{P}(x)\left|\vartheta_n(x)+\frac{1}{n}\right|^{\varrho(x)-1} dx - t^{\upsilon^+}\int_{\mathcal{G}} |\vartheta_n(x)|^{\upsilon(x)}\,dx\\
&\le \frac{\mathsf{p}^+}{t}\left[A + \int_{\mathcal{G}} H_n(x, t\vartheta_n(x))\,dx - \frac{\mathsf{p}^+}{t}\int_{\mathcal{G}} t\vartheta_n(x)\frac{h_n(x, t\vartheta_n(x))}{\left(t\vartheta_n(x)+\frac{1}{n}\right)^{\delta(x)}}\,dx\right]\\
&- t^{\varrho^+-1}\int_{\mathcal{G}} \mathcal{P}(x)\left|\vartheta_n(x)+\frac{1}{n}\right|^{\varrho(x)-1} dx - t^{\upsilon^+}\int_{\mathcal{G}} |\vartheta_n(x)|^{\upsilon(x)}\,dx\\
&\le \frac{\mathsf{p}^+}{t}\left[A + \left(\frac{1}{\theta} - \frac{1}{\mathsf{p}^+}\right)\int_{\mathcal{G}} t\vartheta_n(x)\frac{h_n(x, t\vartheta_n(x))}{\left(t\vartheta_n(x)+\frac{1}{n}\right)^{\delta(x)}}\,dx\right]\\
&- t^{\varrho^+-1}\int_{\mathcal{G}} \mathcal{P}(x)\left|\vartheta_n(x)+\frac{1}{n}\right|^{\varrho(x)-1} dx - t^{\upsilon^+}\int_{\mathcal{G}} |\vartheta_n(x)|^{\upsilon(x)}\,dx\\
&\le \frac{\mathsf{p}^+}{t}\left[A + C_1\left(\frac{1}{\theta} - \frac{1}{\mathsf{p}^+}\right)\right] - -t^{\varrho^+-1}\int_{\mathcal{G}} \mathcal{P}(x)\left|\vartheta_n(x)+\frac{1}{n}\right|^{\varrho(x)-1} dx - t^{\upsilon^+}\int_{\mathcal{G}} |\vartheta_n(x)|^{\upsilon(x)}\,dx\\
&< 0.\end{aligned}$$

By the Implicit Function Theorem, there exists a unique $T \in C\left(S^1, \mathbb{R}\right)$ such that for every $\vartheta_n \in S^1$,

$$\Theta(T(\vartheta_n)\vartheta_n) = A.$$

For any $\vartheta_n \neq 0$, set $\tau(\vartheta_n) = \frac{1}{\|\vartheta_n\|}T\left(\frac{\vartheta_n}{\|\vartheta_n\|}\right)$. Then $\tau \in C(\mathcal{W}_1\backslash 0, \mathbb{R})$ and for all $\vartheta_n \in \mathcal{W}_1, \Theta(\vartheta_n\tau(\vartheta_n)) = A$.

Moreover, if $\Theta(\vartheta_n) = A$, then $\tau(\vartheta_n) = 1$. We define a function $\tau_1 : \mathcal{W}_1 \to \mathbb{R}$ as

$$\tau_1(\vartheta_n) := \begin{cases} \tau(\vartheta_n), & \text{if } \Theta(\vartheta_n) \geq A, \\ 1, & \text{if } \Theta(\vartheta_n) < A. \end{cases}$$

Since $\Theta(\vartheta_n) = A$ implies that $\tau(\vartheta_n) = 1$, it follows thatt $\tau_1 \in C(\mathcal{W}_1 \backslash 0, \mathbb{R})$.
Lastly, we define $G : [0,1] \times \mathcal{W}_1 \backslash 0 \to \mathcal{W}_1 \backslash 0$ as

$$G(t, \vartheta_n) = (1-t)\vartheta_n + t\tau_1(\vartheta_n)\vartheta_n.$$

We have $G(0, \vartheta_n) = \vartheta_n, G(1, \vartheta_n) = \tau_1(\vartheta_n)\vartheta_n \in \Theta^A$, and $G(t,.)_{|\Theta^A} = id_{|\Theta^A}$ for all $t \in [0,1]$.
Therefore,

$$\Theta^A \text{ forms a strong deformation retract of } \mathcal{W}_1 \setminus \{0\}. \tag{17}$$

We introduce the radial retraction $\upsilon : \mathcal{W}_1 \to \mathbb{R}$ defined by

$$\upsilon(\vartheta_n) = \frac{\vartheta_n}{\|\vartheta_n\|} \text{ for all } \vartheta_n \in \mathcal{W}_1.$$

This map is continuous and $\upsilon_{|S^1} = id_{|S^1}$. Consequently, S^1 is a retract of $\mathcal{W}_1 \backslash 0$. Considering the map defined by

$$g(t, \vartheta_n) = (1-t)\vartheta_n + t\upsilon(\vartheta_n) \text{ for all } (t, \vartheta_n) \in [0,1] \times \mathcal{W}_1 \backslash 0.$$

Then, $g(0, \vartheta_n) = \vartheta_n, g(1, \vartheta_n) = \upsilon(\vartheta_n) \in S^1$, and $g(1,.)_{|S^1} = id_{|S^1}$. Hence, it follows that

$$S^1 \text{ is a deformation retract of } \mathcal{W}_1 \backslash 0. \tag{18}$$

Finally, by (18) and (17) it follow that Θ^a and S^1 are homotopie equivalent. We already know that the space $\mathcal{W}_1$ is an infinite dimensional Banach space. From Remark 6.1.13 in [17], it follows that the sphere unit S^1 is contractible. So, we have that

$$G_k(\mathcal{W}_1, \Theta^a) = G_k\left(\mathcal{W}_1, S^1\right) = 0 \text{ for all }, k \in \mathbb{N}.$$

Finally, we obtain that

$$C_k(\Theta, \infty) = G_k(\mathcal{W}_1, \Theta^a) = G_k\left(\mathcal{W}_1, S^1\right) = 0, \text{ for all } k \in \mathbb{N}. \tag{19}$$

□

Theorem 7. *Suppose that conditions (V), and $(C_1) - (C_4)$ are satisfied. Then, the problem (P_2) has nontrivial weak solution in $\mathcal{W}_1$.*

Proof. Given that Θ satisfies the local $(1,1)-$ linking condition near the origin, we have that $\dim C_1(\Theta, 0) \geq 1$, i, e $C_k(\Theta, 0) \neq 0$ for some $k \in \mathbb{N}$. Applying the Theorem 6.2.42 in [17], there exists $\vartheta_n \in K_\Theta$. □

Theorem 8. *Suppose that condition (V), and $(C_1) - (C_4)$ are satisfied. Then, the problem (P_2) has at least non-trivial weak solution in $\mathcal{W}_1$.*

Proof. Theorem 4 guarantees that Θ fulfills the Palais–Smale condition and is bounded below. Furthermore, the trivial solution $\vartheta_n = 0$ is homologically nontrivial and serves as a minimizer. The desired conclusion then follows directly from Theorem 2.1 in [15]. □

Our results are follows

Theorem 9. *Suppose that conditions (V), and $(C_1) - (C_4)$ are satisfied. Then, the problem (P_2) has infinitely non-trivial weak solutions in $\mathcal{W}_1$.*

Proof. We assume that our problem admits three nontrivial solutions in $\mathcal{W}_1$. That is $K_\Theta = \{0, \vartheta_n, \psi_n\}$. From Morse's relation, it follows that

$$C_n(\Theta, 0) = \begin{cases} \mathbb{R}, & k = \mathsf{m}(0), \\ 0, & \text{otherwise}, \end{cases}$$

where $\mathsf{m}(0)$ is a Morse index of 0 . We use Morse's relation, we get that

$$\begin{aligned}\sum_{k\geq 0} \operatorname{rank} C_k(\Theta, \infty) X^k + (1+X)Q(\mathtt{x}) &= \sum_{k\geq 0} \operatorname{rank} C_k(\Theta, 0) X^k + \sum_{k\geq 0} \operatorname{rank} C_k(\Theta, \vartheta_n) X^k + \sum_{k\geq 0} \operatorname{rank} C_k(\Theta, \psi_n) X^k \\ &= X^{\mathsf{m}(0)} + 2\sum_{k\geq 0} \xi_k X^k.\end{aligned}$$

From (19), it follows that

$$(1+X)Q(\mathtt{X}) = X^{\mathsf{m}(0)} + 2\sum_{k\geq 0} \xi_k X^k,$$

where $\xi_k \geq 0$ is an integer, and Q is a polynomial whose coefficients are non-negative integers. In particular, for $\mathtt{X} = 1$ we have $2a = 1 + 2\sum_{k\geq 0} \xi_k$. Since $\xi_k \in \mathbb{N}$, we have that $\sum_{k\geq 0} \xi_k = +\infty$ leads to a contradiction. Thus, there exist infinitely solutions to the problem (1). □

Theorem 10. *Suppose that conditions (V), and $(C_1) - (C_4)$ are satisfied. Then, problem (P_1) admits an infinitely weak solutions in $\mathcal{W}_1$.*

Proof. We consider the sequence $\{\vartheta_n\}_{n\in\mathbb{N}} \subset \mathcal{W}_1$ of solutions to problem (P_2). Then, we have that

$$\begin{aligned}&\int_{\mathcal{G}\times\mathcal{G}} \frac{|\vartheta_n(\mathtt{x}) - \vartheta_n(\mathtt{y})|^{\mathsf{p}(\mathtt{x},\mathtt{y})-2} (\vartheta_n(\mathtt{x}) - \vartheta_n(\mathtt{y}))(\mathfrak{u}(\mathtt{x}) - \mathfrak{u}(\mathtt{y}))}{|\mathtt{x}-\mathtt{y}|^{N+\phi\mathsf{p}(\mathtt{x},\mathtt{y})}}\, d\mathtt{x}\, d\mathtt{y} \\ &+ \int_{\mathcal{G}\times\mathcal{G}} \frac{|\vartheta_n(\mathtt{x}) - \vartheta_n(\mathtt{y})|^{\mathsf{q}(\mathtt{x},\mathtt{y})-2} (\vartheta_n(\mathtt{x}) - \vartheta_n(\mathtt{y}))(\mathfrak{u}(\mathtt{x}) - \mathfrak{u}(\mathtt{y}))}{|\mathtt{x}-\mathtt{y}|^{N+\varphi\mathsf{q}(\mathtt{x},\mathtt{y})}}\, d\mathtt{x}\, d\mathtt{y} \\ &= \int_{\mathcal{G}} \frac{h_n(\mathtt{x}, \vartheta_n(\mathtt{x}))}{\left(\vartheta_n(\mathtt{x}) + \frac{1}{n}\right)\delta(\mathtt{x})} \mathfrak{u}(\mathtt{x})\, d\mathtt{x} + \int_{\mathcal{G}} \mathcal{P}(\mathtt{x}) \left|\vartheta_n(\mathtt{x}) + \frac{1}{n}\right|^{\varrho(\mathtt{x})-2} \left(\vartheta_n(\mathtt{x}) + \frac{1}{n}\right) \mathfrak{u}(\mathtt{x})\, d\mathtt{x}, \\ &+ \alpha \int_{\mathcal{G}} \left|\vartheta_n(\mathtt{x}) + \frac{1}{n}\right|^{\upsilon(\mathtt{x})-2} \left(\vartheta_n(\mathtt{x}) + \frac{1}{n}\right) \log\left|\vartheta_n(\mathtt{x}) + \frac{1}{n}\right| \mathfrak{u}(\mathtt{x})\, d\mathtt{x},\end{aligned} \tag{20}$$

for all $\mathfrak{u} \in \mathcal{W}_1^*$.

We take $\mathfrak{u} = \vartheta_n$ in (20), we have that

$$\begin{aligned}&\int_{\mathcal{G}\times\mathcal{G}} \frac{|\vartheta_n(\mathtt{x}) - \vartheta_n(\mathtt{y})|^{\mathsf{p}(\mathtt{x},\mathtt{y})}}{|\mathtt{x}-\mathtt{y}|^{N+\phi\mathsf{p}(\mathtt{x},\mathtt{y})}}\, d\mathtt{x}\, d\mathtt{y} + \int_{\mathcal{G}\times\mathcal{G}} \frac{|\vartheta_n(\mathtt{x}) - \vartheta_n(\mathtt{y})|^{\mathsf{q}(\mathtt{x},\mathtt{y})}}{|\mathtt{x}-\mathtt{y}|^{N+\varphi\mathsf{q}(\mathtt{x},\mathtt{y})}}\, d\mathtt{x}\, d\mathtt{y} \\ &= \int_{\mathcal{G}} \frac{h_n(\mathtt{x}, \vartheta_n(\mathtt{x}))}{\left(\vartheta_n(\mathtt{x}) + \frac{1}{n}\right)^{\delta(\mathtt{x})}} \vartheta_n(\mathtt{x})\, d\mathtt{x} + \int_{\mathcal{G}} \mathcal{P}(\mathtt{x}) \left|\vartheta_n(\mathtt{x}) + \frac{1}{n}\right|^{\varrho(\mathtt{x})-2} \left(\vartheta_n(\mathtt{x}) + \frac{1}{n}\right) \vartheta_n(\mathtt{x})\, d\mathtt{x} \\ &+ \alpha \int_{\mathcal{G}} \left|\vartheta_n(\mathtt{x}) + \frac{1}{n}\right|^{\upsilon(\mathtt{x})-2} \left(\vartheta_n(\mathtt{x}) + \frac{1}{n}\right) \log\left|\vartheta_n(\mathtt{x}) + \frac{1}{n}\right| \vartheta_n(\mathtt{x})\, d\mathtt{x} \\ &\leq \int_{\mathcal{G}} \frac{h_n(\mathtt{x}, \vartheta_n(\mathtt{x}))}{\left(\vartheta_n(\mathtt{x}) + \frac{1}{n}\right)^{\delta(\mathtt{x})}} \vartheta_n(\mathtt{x})\, d\mathtt{x} + \int_{\mathcal{G}} \mathcal{P}(\mathtt{x}) \left|\vartheta_n(\mathtt{x}) + \frac{1}{n}\right|^{\varrho(\mathtt{x})} d\mathtt{x} + \alpha \int_{\mathcal{G}} \left|\vartheta_n(\mathtt{x}) + \frac{1}{n}\right|^{\upsilon(\mathtt{x})} \log\left|\vartheta_n(\mathtt{x}) + \frac{1}{n}\right| d\mathtt{x}.\end{aligned}$$

Combining (C_1) with (V), and (12) it follows that

$$\begin{aligned}
&\int_{\mathcal{G}\times\mathcal{G}} \frac{|\vartheta_n(x)-\vartheta_n(y)|^{p(x,y)}}{|x-y|^{N+\phi p(x,y)}}\,dx\,dy + \int_{\mathcal{G}\times\mathcal{G}} \frac{|\vartheta_n(x)-\vartheta_n(y)|^{q(x,y)}}{|x-y|^{N+\varphi q(x,y)}}\,dx\,dy \\
&\le \int_{\mathcal{G}} \frac{h_n(x,\vartheta_n(x))}{\left(\vartheta_n(x)+\frac{1}{n}\right)\delta(x)}\vartheta_n(x)\,dx + \int_{\mathcal{G}} \mathcal{P}(x)\left|\vartheta_n(x)+\frac{1}{n}\right|^{\varrho(x)} dx + \alpha\int_{\mathcal{G}}\left|\vartheta_n(x)+\frac{1}{n}\right|^{\upsilon(x)}\log\left|\vartheta_n(x)+\frac{1}{n}\right| dx \\
&\le \int_{\mathcal{G}} \xi(x)\left(1+|\vartheta_n(x)|^{t(x)-1)}\right)|\vartheta_n(x)|^{1-\delta(x)}\,dx + \eta_1\|\vartheta_n\|_{\mathcal{W}_1} + \frac{|\mathcal{G}|}{\upsilon^-\exp(1)} + M\frac{|\mathcal{G}|C_{\upsilon^++\kappa}^{\upsilon^++\kappa}}{\kappa\exp(1)}.
\end{aligned}$$

Since $t(x)-1 \le t(x)-\delta(x)$ for all $x \in \mathcal{G}$, we get that

$$\|\vartheta_n\|_{\mathcal{W}_1} \le \frac{\|\xi\|_\infty C\left(t(x), p(x,y), q_1(x), \delta(x), \phi, \mathcal{G}\right)}{1-\eta_1}.$$

Consequently, the sequence $\{\vartheta_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{W}_1$. Since $\mathcal{W}_1$ is a reflexive Banach space, we can extract a subsequence (still denoted $\{\vartheta_n\}$) such that $\vartheta_n \rightharpoonup \vartheta$ weakly in $\mathcal{W}_1$. By compact embeddings, $\vartheta_n \to \vartheta$ strongly in $L^{a(x)}(\mathcal{G})$ for $1 \le a(x) < p^*_{z_1}(x)$, and $\vartheta_n \to \vartheta$ almost everywhere in $\mathcal{G}$. A similar discussion as in Theorem 4 now yields that

$$\begin{aligned}
&\lim_{n\to\infty}\left[\int_{\mathcal{G}\times\mathcal{G}} \frac{|\vartheta_n(x)-\vartheta_n(y)|^{p(x,y)-2}(\vartheta_n(x)-\vartheta_n(y))(u(x)-u(y))}{|x-y|^{N+\phi p(x,y)}}\,dx\,dy\right.\\
&\left.+\int_{\mathcal{G}\times\mathcal{G}} \frac{|\vartheta_n(x)-\vartheta_n(y)|^{q(x,y)-2}(\vartheta_n(x)-\vartheta_n(y))(u(x)-u(y))}{|x-y|^{N+\varphi q(x,y)}}\,dx\,dy\right]\\
&=\int_{\mathcal{G}\times\mathcal{G}} \frac{|\vartheta(x)-\vartheta(y)|^{p(x,y)-2}(\vartheta(x)-\vartheta(y))(u(x)-u(y))}{|x-y|^{N+\phi p(x,y)}}\,dx\,dy\\
&+\int_{\mathcal{G}\times\mathcal{G}} \frac{|\vartheta(x)-\vartheta(y)|^{q(x,y)-2}(\vartheta(x)-\vartheta(y))(u(x)-u(y))}{|x-y|^{N+\varphi q(x,y)}}\,dx\,dy.
\end{aligned}$$

Since $\vartheta_n(x) > 0$, we get that

$$\left|\frac{h_n(x,\vartheta_n(x))\,u(x)}{\left(\frac{1}{n}+\vartheta_n(x)\right)^{\delta(x)}}\right| \le |h(x,\vartheta(x))u(x)|.$$

From the dominated converge theorem, it follows that

$$\lim_{n\to+\infty}\int_{\mathcal{G}} \frac{h_n(x,\vartheta_n(x))\,u(x)}{\left(\frac{1}{n}+\vartheta_n(x)\right)^{\delta(x)}}\,dx = \int_{\mathcal{G}} \frac{h(x,\vartheta(x))u(x)}{(\vartheta(x))^{\delta(x)}}\,dx.$$

Similarly, we prove that

$$\lim_{n\to\infty}\int_{\mathcal{G}} \mathcal{P}(x)\left|\vartheta_n+\frac{1}{n}\right|^{\varrho(x)-2}\left(\vartheta_n+\frac{1}{n}\right)u(x)\,dx = \int_{\mathcal{G}} \mathcal{P}(x)|\vartheta|^{\varrho(x)-2}\vartheta(x)u(x)\,dx,$$

and

$$\begin{aligned}
&\lim_{n\to+\infty}\alpha\int_{\mathcal{G}}\left|\vartheta_n(x)+\frac{1}{n}\right|^{\upsilon(x)-2}\left(\vartheta_n(x)+\frac{1}{n}\right)\log\left|\vartheta_n(x)+\frac{1}{n}\right|u(x)\,dx\\
&=\alpha\int_{\mathcal{G}}|\vartheta(x)|^{\upsilon(x)-2}(\vartheta(x))\log|\vartheta(x)|\,u(x)\,dx.
\end{aligned}$$

Finally, passing to the limit in (20), we deduce that

$$\begin{aligned}&\int_{\mathcal{G}\times\mathcal{G}} \frac{|\vartheta(x)-\vartheta(y)|^{p(x,y)-2}(\vartheta(x)-\vartheta(y))(u(x)-u(y))}{|x-y|^{N+\phi p(x,y)}}\,dx\,dy\\ &+\int_{\mathcal{G}\times\mathcal{G}} \frac{|\vartheta(x)-\vartheta(y)|^{q(x,y)-2}(\vartheta(x)-\vartheta(y))(u(x)-u(y))}{|x-y|^{N+\varphi q(x,y)}}\,dx\,dy\\ &=\int_{\mathcal{G}} \frac{h(x,\vartheta(x))u(x)}{(\vartheta(x))^{\delta(x)}}\,dx+\int_{\mathcal{G}} \mathcal{P}(x)|\vartheta|^{\varrho(x)-2}\vartheta(x)u(x)\,dx\\ &+\alpha\int_{\mathcal{G}} |\vartheta(x)|^{\nu(x)-2}\,(\vartheta(x))\log|\vartheta(x)|\,u(x)\,dx,\ \text{for all } u\in\mathcal{W}_1^*,\end{aligned}$$

This means that ϑ is a weak solution to (P_1). □

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