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New generalizations of fractional Iyengar-type inequalities involving multi-point quadratures and L_p norms

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Abstract. In this paper, we establish several new generalizations of Iyengar-type integral inequalities within the setting of Riemann-Liouville fractional calculus. By employing a fractional version of the Montgomery identity, we derive sharp estimates for the deviation between the fractional integral of a function and its discrete weighted averages at n arbitrary nodes. We extend these results to functions whose derivatives belong to the $L_r[a, b]$ spaces by utilizing Hölder's inequality. The established inequalities provide a unified setting that recovers the classical Iyengar-type results as special cases when the fractional order $\alpha = 1$. Moreover, the influence of the node distribution on the associated error bounds is examined, and several corollaries corresponding to midpoint and trapezoidal-type rules are derived.

1. Introduction

The study of integral inequalities plays a fundamental role in many areas of mathematical analysis, particularly in numerical integration, approximation theory, and the qualitative analysis of differential equations. Such inequalities provide effective tools for estimating the deviation between integral means and discrete averages of functions.

One of the most classical and influential results in this direction was established by Iyengar in 1938 [6], who derived a sharp bound for the difference between the integral mean of a function and the arithmetic mean of its endpoint values.

Theorem 1.1 ([6]). *Let Ω be a differentiable function on (σ, ρ) such that $|\Omega'(\xi)| \leq M$ for all $\xi \in [\sigma, \rho]$. Then the following inequality holds:*

$$\left| \frac{1}{\rho - \sigma} \int_{\sigma}^{\rho} \Omega(\xi)\, d\xi - \frac{\Omega(\sigma) + \Omega(\rho)}{2} \right| \leq \frac{M(\rho - \sigma)}{4} - \frac{(\Omega(\rho) - \Omega(\sigma))^2}{4M(\rho - \sigma)}. \tag{1}$$

Over the past decades, inequality (1) has attracted considerable attention. Milovanovic and Pecaric [7] provided early considerations on its applications, while Qi [8] explored further generalizations for

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general integrals. By incorporating higher-order derivatives, Agarwal et al. [1] and Elezovic and Pecaric [3] obtained improved Iyengar-type inequalities, particularly focusing on estimates of error in trapezoidal rules. For smoother classes of functions, the following result is well-known:

Theorem 1.2 ([1, 3]). *Let $\Omega \in C^2[\sigma,\rho]$ and assume that $|\Omega''(\xi)| \leq M$ for all $\xi \in [\sigma,\rho]$. Define the functional*

$$I = \frac{1}{\rho-\sigma}\int_\sigma^\rho \Omega(\xi)\,d\xi - \frac{\Omega(\sigma)+\Omega(\rho)}{2} + \frac{\rho-\sigma}{8}\left(\Omega'(\rho)-\Omega'(\sigma)\right). \tag{2}$$

Then $|I| \leq \frac{M(\rho-\sigma)^2}{24}$.

Further refinements were contributed by Cheng [2] and Franjic et al. [4]. A significant advancement in the discrete sampling of such inequalities was made by Huy [5], who investigated Iyengar-type estimates involving quadratures in n knots, providing a general setting for multi-point discrete averages.

In recent years, there has been a growing interest in extending these classical results to the setting of fractional calculus, as documented in the foundational work of Samko et al. [9]. Recently, Sarikaya [10] established generalized Iyengar-type inequalities, while Sarikaya et al. [11] extended these concepts to conformable fractional integrals.

1.1. Riemann–Liouville Fractional Integrals

Among the various fractional operators, the Riemann-Liouville fractional integral plays a central role (see, Samko et al. [9]) .

Definition 1.3. *Let $\Omega \in L_1[\sigma,\rho]$ and let $\alpha > 0$. The left- and right-sided Riemann–Liouville fractional integrals of order α are defined, respectively, by*

$$J^\alpha_{\sigma+}\Omega(\xi) = \frac{1}{\Gamma(\alpha)}\int_\sigma^\xi (\xi-\tau)^{\alpha-1}\Omega(\tau)\,d\tau, \quad \xi > \sigma,$$

and

$$J^\alpha_{\rho-}\Omega(\xi) = \frac{1}{\Gamma(\alpha)}\int_\xi^\rho (\tau-\xi)^{\alpha-1}\Omega(\tau)\,d\tau, \quad \xi < \rho,$$

where $\Gamma(\alpha)$ denotes the Gamma function.

The main motivation of this paper is to unify the multi-knot approach of Huy [5] with the fractional framework of Sarikaya [10]. By employing a fractional Montgomery identity, we derive new bounds for the deviation between the fractional integral $J^\alpha_{\sigma+}\Omega(\rho)$ and a weighted point functional $\frac{1}{n}\sum_{k=1}^n \Omega\left(\sigma+(\rho-\sigma)\xi_k\right)$. Our results cover both bounded derivatives and L_p spaces, providing a comprehensive fractional extension of Iyengar-type inequalities.

2. A Fractional Montgomery Identity and an Iyengar-Type Inequality

Before presenting the main result, we introduce a Montgomery-type identity adapted to the Riemann-Liouville fractional integral.

Lemma 2.1. *Let $\Omega : [\sigma,\rho] \to \mathbb{R}$ be a differentiable function on (σ,ρ) such that $\Omega' \in L_1[\sigma,\rho]$. Then, for any $\xi \in [\sigma,\rho]$ and $\alpha > 0$, the following identity holds:*

$$\Omega(\xi) = \frac{\Gamma(\alpha+1)}{(\rho-\sigma)^\alpha}J^\alpha_{\sigma+}\Omega(\rho) + \int_\sigma^\rho K_\alpha(\xi,\tau)\,\Omega'(\tau)\,d\tau, \tag{3}$$

where $J^{\alpha}_{\sigma+}$ denotes the Riemann-Liouville fractional integral and the kernel $K_{\alpha}(\xi,\tau)$ is defined by

$$K_{\alpha}(\xi,\tau)=\begin{cases}\dfrac{(\rho-\sigma)^{\alpha}-(\rho-\tau)^{\alpha}}{(\rho-\sigma)^{\alpha}}, & \sigma\le\tau<\xi,\\ -\dfrac{(\rho-\tau)^{\alpha}}{(\rho-\sigma)^{\alpha}}, & \xi\le\tau\le\rho.\end{cases} \tag{4}$$

Proof. By the definition of the Riemann-Liouville fractional integral, we have

$$J^{\alpha}_{\sigma+}\Omega(\rho)=\frac{1}{\Gamma(\alpha)}\int_{\sigma}^{\rho}(\rho-\tau)^{\alpha-1}\Omega(\tau)\,d\tau.$$

We apply integration by parts to the integral $\int_{\sigma}^{\rho}(\rho-\tau)^{\alpha-1}\Omega(\tau)\,d\tau$ by setting

$$u=\Omega(\tau),\ \ du=\Omega'(\tau)\,d\tau,$$

and

$$dv=(\rho-\tau)^{\alpha-1}d\tau,\quad v=-\frac{(\rho-\tau)^{\alpha}}{\alpha}.$$

Hence,

$$\begin{aligned}\int_{\sigma}^{\rho}(\rho-\tau)^{\alpha-1}\Omega(\tau)\,d\tau&=\left[-\frac{(\rho-\tau)^{\alpha}}{\alpha}\Omega(\tau)\right]_{\sigma}^{\rho}+\frac{1}{\alpha}\int_{\sigma}^{\rho}(\rho-\tau)^{\alpha}\Omega'(\tau)\,d\tau\\&=\frac{(\rho-\sigma)^{\alpha}}{\alpha}\Omega(\sigma)+\frac{1}{\alpha}\int_{\sigma}^{\rho}(\rho-\tau)^{\alpha}\Omega'(\tau)\,d\tau.\end{aligned}$$

Multiplying both sides by $\frac{1}{\Gamma(\alpha)}$ and using the identity $\alpha\Gamma(\alpha)=\Gamma(\alpha+1)$, we obtain

$$J^{\alpha}_{\sigma+}\Omega(\rho)=\frac{(\rho-\sigma)^{\alpha}}{\Gamma(\alpha+1)}\Omega(\sigma)+\frac{1}{\Gamma(\alpha+1)}\int_{\sigma}^{\rho}(\rho-\tau)^{\alpha}\Omega'(\tau)\,d\tau.$$

Solving for $\Omega(\sigma)$ yields

$$\Omega(\sigma)=\frac{\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha}}J^{\alpha}_{\sigma+}\Omega(\rho)-\frac{1}{(\rho-\sigma)^{\alpha}}\int_{\sigma}^{\rho}(\rho-\tau)^{\alpha}\Omega'(\tau)\,d\tau. \tag{5}$$

By the Fundamental Theorem of Calculus,

$$\Omega(\xi)=\Omega(\sigma)+\int_{\sigma}^{\xi}\Omega'(\tau)\,d\tau.$$

Substituting (5) into this relation, we obtain

$$\begin{aligned}\Omega(\xi)&=\frac{\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha}}J^{\alpha}_{\sigma+}\Omega(\rho)-\frac{1}{(\rho-\sigma)^{\alpha}}\int_{\sigma}^{\rho}(\rho-\tau)^{\alpha}\Omega'(\tau)\,d\tau+\int_{\sigma}^{\xi}\Omega'(\tau)\,d\tau\\&=\frac{\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha}}J^{\alpha}_{\sigma+}\Omega(\rho)+\int_{\sigma}^{\xi}\left(1-\frac{(\rho-\tau)^{\alpha}}{(\rho-\sigma)^{\alpha}}\right)\Omega'(\tau)\,d\tau-\int_{\xi}^{\rho}\frac{(\rho-\tau)^{\alpha}}{(\rho-\sigma)^{\alpha}}\Omega'(\tau)\,d\tau.\end{aligned}$$

Observing that

$$1-\frac{(\rho-\tau)^{\alpha}}{(\rho-\sigma)^{\alpha}}=\frac{(\rho-\sigma)^{\alpha}-(\rho-\tau)^{\alpha}}{(\rho-\sigma)^{\alpha}},$$

we conclude that

$$\Omega(\xi) = \frac{\Gamma(\alpha+1)}{(\rho-\sigma)^\alpha} J^\alpha_{\sigma+}\Omega(\rho) + \int_\sigma^\rho K_\alpha(\xi,\tau)\,\Omega'(\tau)\,d\tau,$$

where $K_\alpha(\xi,\tau)$ is given by (4). This completes the proof. □

Theorem 2.2. *Let $\Omega : [\sigma,\rho] \to \mathbb{R}$ be a differentiable function on (σ,ρ) such that $|\Omega'(\xi)| \le M$ for all $\xi \in [\sigma,\rho]$. Let $\alpha > 0$ and let $J^\alpha_{\sigma+}$ denote the Riemann-Liouville fractional integral. Assume that the nodes $\{\xi_k\}_{k=1}^n \subset [0,1]$ satisfy*

$$\sum_{k=1}^n \xi_k = \frac{n}{2}.$$

Then the following fractional Iyengar-type inequality holds:

$$\left|\frac{\Gamma(\alpha+1)}{(\rho-\sigma)^\alpha} J^\alpha_{\sigma+}\Omega(\rho) - \frac{1}{n}\sum_{k=1}^n \Omega\left(\sigma + (\rho-\sigma)\xi_k\right)\right| \le \frac{M(\rho-\sigma)}{n}\sum_{k=1}^n \left(\xi_k + \frac{2(1-\xi_k)^{\alpha+1}-1}{\alpha+1}\right). \tag{6}$$

Proof. Applying Lemma 2.1 at the points $\sigma_k = \sigma + (\rho-\sigma)\xi_k$, we obtain

$$\Omega(\sigma_k) = \frac{\Gamma(\alpha+1)}{(\rho-\sigma)^\alpha} J^\alpha_{\sigma+}\Omega(\rho) + \int_\sigma^\rho K_\alpha(\sigma_k,\tau)\,\Omega'(\tau)\,d\tau.$$

Taking the arithmetic mean over $k = 1,\dots,n$ identities, we obtain

$$\frac{1}{n}\sum_{k=1}^n \Omega(\sigma_k) - \frac{\Gamma(\alpha+1)}{(\rho-\sigma)^\alpha} J^\alpha_{\sigma+}\Omega(\rho) = \frac{1}{n}\sum_{k=1}^n \int_\sigma^\rho K_\alpha(\sigma_k,\tau)\,\Omega'(\tau)\,d\tau.$$

Applying absolute values and using the triangle inequality yields

$$\left|\frac{1}{n}\sum_{k=1}^n \Omega(\sigma_k) - \frac{\Gamma(\alpha+1)}{(\rho-\sigma)^\alpha} J^\alpha_{\sigma+}\Omega(\rho)\right| \le \frac{1}{n}\sum_{k=1}^n \int_\sigma^\rho |K_\alpha(\sigma_k,\tau)|\,|\Omega'(\tau)|\,d\tau.$$

Using the bound $|\Omega'(\tau)| \le M$ for all $\tau \in [\sigma,\rho]$, we obtain

$$\left|\frac{1}{n}\sum_{k=1}^n \Omega(\sigma_k) - \frac{\Gamma(\alpha+1)}{(\rho-\sigma)^\alpha} J^\alpha_{\sigma+}\Omega(\rho)\right| \le \frac{M}{n}\sum_{k=1}^n \int_\sigma^\rho |K_\alpha(\sigma_k,\tau)|\,d\tau.$$

We now compute the integral of the absolute value of the kernel K_α for a fixed index k. From (4), we have

$$\int_\sigma^\rho |K_\alpha(\sigma_k,\tau)|\,d\tau = \int_\sigma^{\sigma_k} \left|\frac{(\rho-\tau)^\alpha - (\rho-\sigma)^\alpha}{(\rho-\sigma)^\alpha}\right| d\tau + \int_{\sigma_k}^\rho \frac{(\rho-\tau)^\alpha}{(\rho-\sigma)^\alpha}\,d\tau.$$

Since $(\rho-\sigma)^\alpha \ge (\rho-\tau)^\alpha$ for $\tau \in [\sigma,\sigma_k]$, the first integrand reduces to $\frac{(\rho-\sigma)^\alpha-(\rho-\tau)^\alpha}{(\rho-\sigma)^\alpha}$. Hence,

$$\int_\sigma^{\sigma_k} \left(1 - \frac{(\rho-\tau)^\alpha}{(\rho-\sigma)^\alpha}\right) d\tau = [\tau]_\sigma^{\sigma_k} + \left[\frac{(\rho-\tau)^{\alpha+1}}{(\alpha+1)(\rho-\sigma)^\alpha}\right]_\sigma^{\sigma_k}$$

$$= (\rho-\sigma)\xi_k + \frac{(\rho-\sigma)(1-\xi_k)^{\alpha+1} - (\rho-\sigma)}{\alpha+1}.$$

For the second integral, we obtain

$$\int_{\sigma_k}^{\rho} \frac{(\rho-\tau)^\alpha}{(\rho-\sigma)^\alpha}\, d\tau = \left[-\frac{(\rho-\tau)^{\alpha+1}}{(\alpha+1)(\rho-\sigma)^\alpha}\right]_{\sigma_k}^{\rho} = \frac{(\rho-\sigma)(1-\xi_k)^{\alpha+1}}{\alpha+1}.$$

Combining both parts yields

$$\int_{\sigma}^{\rho} |K_\alpha(\sigma_k,\tau)|\, d\tau = (\rho-\sigma)\left(\xi_k + \frac{2(1-\xi_k)^{\alpha+1}-1}{\alpha+1}\right).$$

Substituting this expression into the above inequality and summing over $k = 1,\ldots,n$ immediately yields the estimate (6). □

We now extend the fractional Iyengar-type inequality to the case where the derivative belongs to the space $L_r[\sigma,\rho]$ for some $r > 1$.

Theorem 2.3. *Let $\Omega : [\sigma,\rho] \to \mathbb{R}$ be a differentiable function on (σ,ρ) such that $\Omega' \in L_r[\sigma,\rho]$ for some $r > 1$. Let $\alpha > 0$ and let $p = \frac{r}{r-1}$ be the conjugate exponent. Then the following inequality holds:*

$$\left|\frac{\Gamma(\alpha+1)}{(\rho-\sigma)^\alpha} J^\alpha_{\sigma+}\Omega(\rho) - \frac{1}{n}\sum_{k=1}^{n} \Omega\left(\sigma + (\rho-\sigma)\xi_k\right)\right| \le \frac{\|\Omega'\|_r(\rho-\sigma)^{1/p}}{n} \sum_{k=1}^{n} \Psi(\alpha,p;\xi_k), \tag{7}$$

where

$$\Psi(\alpha,p;\xi_k) = \left(\int_0^{\xi_k} |1-(1-u)^\alpha|^p\, du + \frac{(1-\xi_k)^{p\alpha+1}}{p\alpha+1}\right)^{1/p}. \tag{8}$$

Proof. Applying the fractional Montgomery identity (Lemma 2.1) at the points $\sigma_k = \sigma + (\rho-\sigma)\xi_k$ and taking the arithmetic mean, we obtain

$$\frac{1}{n}\sum_{k=1}^{n} \Omega\left(\sigma + (\rho-\sigma)\xi_k\right) - \frac{\Gamma(\alpha+1)}{(\rho-\sigma)^\alpha} J^\alpha_{\sigma+}\Omega(\rho) = \frac{1}{n}\sum_{k=1}^{n} \int_{\sigma}^{\rho} K_\alpha(\sigma_k,\tau)\, \Omega'(\tau)\, d\tau.$$

Taking absolute values and applying the triangle inequality yields

$$\left|\frac{1}{n}\sum_{k=1}^{n} \Omega\left(\sigma + (\rho-\sigma)\xi_k\right) - \frac{\Gamma(\alpha+1)}{(\rho-\sigma)^\alpha} J^\alpha_{\sigma+}\Omega(\rho)\right| \le \frac{1}{n}\sum_{k=1}^{n} \int_{\sigma}^{\rho} |K_\alpha(\sigma_k,\tau)|\, |\Omega'(\tau)|\, d\tau.$$

Applying Hölder's inequality to each integral term, we obtain

$$\int_{\sigma}^{\rho} |K_\alpha(\sigma_k,\tau)|\, |\Omega'(\tau)|\, d\tau \le \left(\int_{\sigma}^{\rho} |K_\alpha(\sigma_k,\tau)|^p d\tau\right)^{1/p} \|\Omega'\|_r.$$

Using the definition of the kernel K_α and performing the change of variables $\tau = \sigma + (\rho-\sigma)u$, we compute

$$\int_{\sigma}^{\rho} |K_\alpha(\sigma_k,\tau)|^p d\tau = \int_{\sigma}^{\sigma_k} \left|\frac{(\rho-\tau)^\alpha - (\rho-\sigma)^\alpha}{(\rho-\sigma)^\alpha}\right|^p d\tau + \int_{\sigma_k}^{\rho} \left|\frac{(\rho-\tau)^\alpha}{(\rho-\sigma)^\alpha}\right|^p d\tau$$
$$= (\rho-\sigma)\int_0^{\xi_k} |1-(1-u)^\alpha|^p\, du + (\rho-\sigma)\int_{\xi_k}^{1} (1-u)^{p\alpha} du.$$

The second integral is evaluated explicitly as

$$\int_{\xi_k}^{1}(1-u)^{p\alpha}du=\left[-\frac{(1-u)^{p\alpha+1}}{p\alpha+1}\right]_{\xi_k}^{1}=\frac{(1-\xi_k)^{p\alpha+1}}{p\alpha+1}.$$

Consequently,

$$\left(\int_{\sigma}^{\rho}|K_\alpha(\sigma_k,\tau)|^p d\tau\right)^{1/p}=(\rho-\sigma)^{1/p}\left(\int_0^{\xi_k}|1-(1-u)^\alpha|^p\,du+\frac{(1-\xi_k)^{p\alpha+1}}{p\alpha+1}\right)^{1/p}.$$

Summing over $k=1,\dots,n$ and multiplying by $\frac{\|\Omega'\|_r}{n}$ completes the proof of (7). □

3. Special Cases and Corollaries

In this section, we explore several special cases of Theorem 2.2 and Theorem 2.3 by fixing the fractional order α and the distribution of the nodes ξ_k.

Corollary 3.1. *Setting $\alpha=1$ and $n=2$ with nodes $\xi_1=0$ and $\xi_2=1$ in Theorem 2.2, we recover the classical trapezoid inequality:*

$$\left|\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\Omega(\xi)\,d\xi-\frac{\Omega(\sigma)+\Omega(\rho)}{2}\right|\le\frac{M(\rho-\sigma)}{4}.$$

Proof. Setting $\alpha=1$, we observe that the expression inside the summation in (6) reduces to

$$\xi_k+\frac{2(1-\xi_k)^2-1}{2}=\frac{\xi_k^2+(1-\xi_k)^2}{2}.$$

For $n=2$ with $\xi_1=0$ and $\xi_2=1$, we obtain

$$\frac{1}{2}(1)+\frac{1}{2}(1)=1.$$

Hence, the right-hand side of (6) becomes $\frac{M(\rho-\sigma)}{2}\cdot\frac{1}{2}=\frac{M(\rho-\sigma)}{4}$, which completes the proof. □

Corollary 3.2. *Let $n=1$ and $\xi_1=\frac{1}{2}$. Then, for any $\alpha>0$, the following fractional midpoint-type inequality holds:*

$$\left|\frac{\Gamma(\alpha+1)}{(\rho-\sigma)^\alpha}J_{\sigma+}^{\alpha}\Omega(\rho)-\Omega\Big(\frac{\sigma+\rho}{2}\Big)\right|\le M(\rho-\sigma)\left(\frac{1}{2}+\frac{2^{-\alpha}-1}{\alpha+1}\right).$$

Remark 3.3. *For $\alpha=1$, Corollary 3.2 reduces to the classical midpoint-type Iyengar inequality. Indeed, substituting $\alpha=1$ into the right-hand side yields*

$$\frac{1}{2}+\frac{2^{-1}-1}{2}=\frac{1}{2}-\frac{1}{4}=\frac{1}{4}.$$

Consequently, we obtain

$$\left|\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\Omega(\xi)\,d\xi-\Omega\Big(\frac{\sigma+\rho}{2}\Big)\right|\le\frac{M(\rho-\sigma)}{4},$$

which coincides with the well-known midpoint estimate associated with Iyengar's inequality. This specific estimate relates to the error analysis of the midpoint rule in numerical integration as discussed by Elezovic and Pecaric [3] and further explored in the context of higher-order derivatives by Agarwal et al. [1].

Corollary 3.4. *Let $\alpha > 0$ and $p > 1$, and assume that Ω satisfies the hypotheses of Theorem 2.3. Setting $n = 1$ and $\xi_1 = \frac{1}{2}$, the following midpoint-type fractional inequality holds:*

$$\left|\frac{\Gamma(\alpha+1)}{(\rho-\sigma)^\alpha} J^\alpha_{\sigma+}\Omega(\rho) - \Omega\Big(\frac{\sigma+\rho}{2}\Big)\right| \tag{9}$$

$$\leq \quad \|\Omega'\|_r(\rho-\sigma)^{1/p}\left[\frac{1}{\alpha}\, B_{1-(1/2)^\alpha}\Big(\frac{1}{\alpha}, p+1\Big) + \frac{(1/2)^{p\alpha+1}}{p\alpha+1}\right]^{1/p},$$

where $B_\xi(\cdot,\cdot)$ denotes the incomplete Beta function defined by

$$B_\xi(\sigma,\rho) = \int_0^\xi \tau^{\sigma-1}(1-\tau)^{\rho-1}\, d\tau.$$

Proof. Starting from Theorem 2.3 and choosing $n = 1$ with $\xi_1 = \frac{1}{2}$, we obtain

$$\left|\frac{\Gamma(\alpha+1)}{(\rho-\sigma)^\alpha} J^\alpha_{\sigma+}\Omega(\rho) - \Omega\Big(\frac{\sigma+\rho}{2}\Big)\right| \leq \|\Omega'\|_r(\rho-\sigma)^{1/p}\,(I_1+I_2)^{1/p},$$

where

$$I_1 = \int_0^{1/2} [1-(1-u)^\alpha]^p\, du,$$

and

$$I_2 = \int_{1/2}^1 (1-u)^{p\alpha}\, du = \frac{(1/2)^{p\alpha+1}}{p\alpha+1}.$$

To compute I_1, we use the substitution $\tau = 1-(1-u)^\alpha$, which yields

$$I_1 = \frac{1}{\alpha}\int_0^{1-(1/2)^\alpha} \tau^p(1-\tau)^{\frac{1}{\alpha}-1}\, d\tau = \frac{1}{\alpha}\, B_{1-(1/2)^\alpha}\Big(\frac{1}{\alpha}, p+1\Big).$$

Combining the estimates for I_1 and I_2 gives (9), which completes the proof. □

Remark 3.5. *If we set $\alpha = 1$ in Corollary 3.4, then the Riemann-Liouville fractional integral reduces to the classical integral, i.e.,*

$$J^1_{\sigma+}\Omega(\rho) = \int_\sigma^\rho \Omega(\tau)\, d\tau.$$

Moreover, the incomplete Beta function term simplifies as

$$\frac{1}{\alpha}\, B_{1-(1/2)^\alpha}\Big(\frac{1}{\alpha}, p+1\Big) = B_{1/2}(1,p+1) = \int_0^{1/2}(1-\tau)^p\, d\tau = \frac{1-(1/2)^{p+1}}{p+1}.$$

Consequently, inequality (9) becomes

$$\left|\frac{1}{\rho-\sigma}\int_\sigma^\rho \Omega(\tau)\, d\tau - \Omega\Big(\frac{\sigma+\rho}{2}\Big)\right| \leq \|\Omega'\|_r(\rho-\sigma)^{1/p}\left[\frac{1-(1/2)^{p+1}}{p+1} + \frac{(1/2)^{p+1}}{p+1}\right]^{1/p},$$

which yields

$$\left|\frac{1}{\rho-\sigma}\int_\sigma^\rho \Omega(\tau)\, d\tau - \Omega\Big(\frac{\sigma+\rho}{2}\Big)\right| \leq \|\Omega'\|_r\frac{(\rho-\sigma)^{1/p}}{(p+1)^{1/p}}.$$

This shows that Corollary 3.4 recovers the classical midpoint-type inequality as a special case when $\alpha = 1$. This approach generalizes the estimates provided by Huy [5] for the n-knot case and Sarikaya [10] for generalized fractional operators.

4. Conclusion

In this study, we have successfully extended the classical Iyengar integral inequality to the domain of fractional calculus by incorporating multi-point quadrature functional $\frac{1}{n}\sum_{k=1}^{n}\Omega\left(\sigma+(\rho-\sigma)\xi_k\right)$. The cornerstone of our approach was the derivation and application of a fractional Montgomery identity, which ensured mathematical consistency between the fractional integral operator and the fundamental theorem of calculus.

Our main results, Theorems 2.2 and 2.3, provide unified and sharp error estimates for functions whose derivatives are either essentially bounded or belong to the Lebesgue space L_r. The analysis shows that the fractional order α plays a crucial role in shaping the associated error kernel, leading to a natural asymmetry that does not appear in the classical integer-order setting. Moreover, the application of Hölder's inequality enables the extension of Iyengar-type bounds to functions with possibly large or singular derivatives, as long as their L_r-norms are finite. It is also demonstrated that all the obtained inequalities recover the well-known classical Iyengar-type results available in the literature when $\alpha = 1$ and suitable choices of the nodes are made.

The findings of this paper provide a robust theoretical basis for error estimation in numerical fractional integration. Future research could focus on extending these results to other types of fractional operators, such as the Caputo–Fabrizio or Atangana-Baleanu derivatives, and exploring their applications in the stability analysis of fractional differential equations.

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