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Characterizations of a space-time admitting a generalized $\varphi(Ric)$-vector

Hülya Bağdatli Yilmaz[a], Uday Chand De[b], Krishnendu De[c,*]

[a]*Marmara University, Faculty of Sciences, Department of Mathematics, Istanbul, Turkey*
[b]*Department of Pure Mathematics,University of Calcutta, 35 Ballygaunge Circular Road, Kolkata -700019, West Bengal, India*
[c]*Department of Mathematics, Kabi Sukanta Mahavidyalaya, The University of Burdwan, Bhadreswar, P.O.-Angus, Hooghly, Pin 712221, West Bengal, India*

Abstract. This study focuses on a generalized $\varphi(Ric)$-vector in space-times. We illustrate that a conformally flat space-time, which possesses a generalized $\varphi(Ric)$-vector that is a unit time-like vector, behaves as a perfect fluid and exhibits quasi-constant curvature under a certain condition. Additionally, we establish that this space-time can be classified as both a generalized Robertson-Walker space-time and a Robertson-Walker space-time. Furthermore, we demonstrate that this space-time represents the quintessence phase. Then, we adopt the flat Friedmann-Robertson-Walker metric to derive Hubble, jerk, deceleration and snap parameters and we observe that the null, dominant and weak energy conditions are fulfilled, but the strong energy condition has failed. Finally, we scrutinize the space-time with the generalized $\varphi(Ric)$-vector in cases it admits Ricci solitons and generalized Ricci solitons.

1. Introduction

Einstein spaces represent a distinct category of cosmological models characterized by the proportional relationship between the Ricci tensor $\mathfrak{R}_{ij}$ and the metric tensor $\mathfrak{g}_{ij}$. In these spaces, it is also possible to express concircular vectors using the equation $\nabla_i \phi_j = \varrho g_{ij}$, ϱ is a function on the manifold. In [21] the authors investigated a class of vectors determined by the curvature in Riemann spaces which are modifications of concircular vectors, and initiated the $\varphi(Ric)$-vectors characterized as:

$$\nabla_j \varphi_i = \alpha \mathfrak{R}_{ij}, \tag{1}$$

in which α is a constant. In [22], the existence of $\varphi(Ric)$-vectors associated with Ricci flow on conformally flat spaces is examined. It is shown that such vectors exist only in specific subprojective spaces introduced by Kagan. In [13], the study also focuses on $\varphi(Ric)$-vectors within the context of warped product manifolds. In [31], the concept of $\varphi(Ric)$-vectors was enhanced by introducing generalized $\varphi(Ric)$-vectors, wherein

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* Corresponding author: Krishnendu De
Email addresses: hbagdatli@marmara.edu.tr (Hülya Bağdatli Yilmaz), uc_de@yahoo.com (Uday Chand De), krishnendu.de@outlook.in (Krishnendu De)
ORCID iDs: https://orcid.org/0000-0003-2568-7577 (Hülya Bağdatli Yilmaz), https://orcid.org/0000-0002-8990-4609 (Uday Chand De), https://orcid.org/0000-0001-6520-4520 (Krishnendu De)

α is a smooth function. Furthermore, some special pseudo-Riemannian spaces are investigated, namely equidistant and Kähler spaces, which admit generalized $\varphi(Ric)$-vectors.

Throughout this paper, we consider the four-dimensional Lorentzian manifold N admitting a globally time-like vector known as space-time.

A space-time N is called a perfect fluid (PF) space-time if its non-zero Ricci tensor $\mathfrak{R}_{ij}$, obeys

$$\mathfrak{R}_{ij} = \gamma \mathfrak{g}_{ij} + \beta\varphi_i\varphi_j, \tag{2}$$

here γ, β are scalar functions and φ is a unit time-like vector, that is, $\varphi_i\varphi^i = -1$ ([5], [6]). The topic of space-time is widely studied in ([12], [17], [18], [27], [33] etc...).

A generalized Robertson-Walker (GRW) space-time is a Lorentzian manifold of dimension n demonstrated by the metric:

$$ds^2 = -dt^2 + a^2(t)\tilde{\mathfrak{g}}_{ij}(x)dx^i dx^j,$$

in which $\tilde{\mathfrak{g}}_{ij}(x)$ serves as the metric tensor of the Riemannian submanifold, $a > 0$ represents a smooth warping function or scale factor. The study of large-scale cosmology depends on this structure, which is not just a small addition but rather a significant and global improvement of Robertson-Walker (RW) space-time. Both GRW and RW space-times are essential to cosmology because they show how a homogenous, isotropic world evolves over time. Numerous in-depth studies have clearly established the significance of these space-times (see, [1], [2], [5], [9], [10], [16], [23], [30], etc...).

In geometry, spaces with constant curvature—such as Lobachevsky space, Euclidean space, and the sphere—have a specific place. It has new uses in the field of relativity. Anti-De Sitter space-time, De Sitter space-time and Minkowski space-time are determined by space-times with constant curvature. Additionally, it is known that a four-dimensional space-time with constant curvature permits a maximum 10-parameter group of isometries, making it a homogeneous space.

A space-time with constant curvature can be generalized to a space-time with quasi-constant curvature. Ganchev and Mihova examined the geometric characteristics of a Riemannian manifold with quasi-constant curvature [14]. In [11], it is illustrated that a space-time with a quasi-constant curvature is a PF space-time and is consistent with the universe's current state. The authors also give several examples of a space-time with quasi-constant curvature. Mantica and Molinari established that any RW space-time has a quasi-constant curvature [23]. Furthermore, the influence of quasi-constant curvature in modified gravity has been illustrated in [11]. Additionally, a number of researchers have studied quasi-constant curvature in various structures.

The theory of general relativity describes how matter is distributed within space-time using the energy-momentum tensor (EMT) denoted as $\mathcal{T}_{ij}$. In general relativity, the EMT is one of the source of the gravitational field. The properties of a PF can be accurately characterized by its rest frame mass density and isotropic pressure. Thus, $\mathcal{T}_{ij}$ in a PF space-time can be expressed as:

$$\mathcal{T}_{ij} = p\mathfrak{g}_{ij} + (p + \sigma)\varphi_i\varphi_j, \tag{3}$$

where p represents the isotropic pressure and σ denotes the energy density. If the stress tensor σ and pressure p which are connected by an equation of state (EOS) expressed as $p = p(\sigma)$, the corresponding space-time is termed isentropic. There are specific cases that categorize the space-time: $\sigma = 3p$ represents the radiation era, $p = \sigma$ yields a stiff matter fluid, and $p + \sigma = 0$ corresponds to the dark matter era [4]. Additionally, the relationship $p = \omega\sigma$ introduces the EOS parameter ω. This parameter plays a crucial role in defining the evolution of energy density and the expansion of the universe. The universe is considered to be in an accelerating phase if $\omega < -\frac{1}{3}$, in a quintessence phase if $-1 < \omega < 0$, and in a phantom era if $\omega < -1$.

A nonlinear system, as used in science and mathematics, is one in which the change in the input is not proportionate to the change in the output. Since most systems are nonlinear by nature, biologists, engineers, mathematicians, physicists and many other sciences are interested in nonlinear issues.

In mathematics, a nonlinear system of equations describes the behaviour of a nonlinear system. Nonlinear systems are frequently approximated by linear equations because nonlinear dynamical equations are challenging to solve. Einstein's field equations (EFEs), which are very nonlinear equations, are typically used in theory of relativity. The EFEs, omitting the cosmological constant, are represented as:

$$\mathfrak{R}_{ij} - \frac{\mathfrak{R}}{2}\mathfrak{g}_{ij} = \kappa\mathcal{T}_{ij}, \tag{4}$$

where κ indicates the gravitational constant and $\mathfrak{R}$ is the scalar curvature, $\mathfrak{R} = \mathfrak{R}_{ij}\mathfrak{g}^{ij}$. Several researchers obtained solutions of EFEs under certain restrictions. In this paper, we intend to acquire the solution of EFEs with the help of generalized $\varphi(Ric)$-vectors.

The (0,4) Weyl conformal curvature tensor C_{hijk} on a semi-Riemannian manifold of dimension 4 is demonstrated as [32]

$$C_{hijk} = \mathfrak{R}_{hijk} - \frac{1}{2}\left\{\mathfrak{R}_{hk}\mathfrak{g}_{ij} - \mathfrak{R}_{hj}\mathfrak{g}_{ik} + \mathfrak{R}_{ij}\mathfrak{g}_{hk} - \mathfrak{R}_{ik}\mathfrak{g}_{hj}\right\} + \frac{\mathfrak{R}}{6}\left\{\mathfrak{g}_{hk}\mathfrak{g}_{ij} - \mathfrak{g}_{hj}\mathfrak{g}_{ik}\right\}, \tag{5}$$

$\mathfrak{R}^i_{jkl}$ denotes the curvature tensor of $(1,3)$ type and $\mathfrak{R}_{hjkl} = \mathfrak{g}_{ih}\mathfrak{R}^i_{jkl}$.

2. Conformally flat space-times with a generalized $\varphi(Ric)$-vector

Suppose that the space-time with a generalized $\varphi(Ric)$-vector is conformally flat and φ_i is a unit time-like vector, that is, $\varphi_i\varphi^i = -1$.

Let us write the integrability conditions of (1) for generalized $\varphi(Ric)$-vectors. Then, we have

$$\varphi_p\mathfrak{R}^p_{ijk} = \alpha\left(\nabla_k\mathfrak{R}_{ij} - \nabla_j\mathfrak{R}_{ik}\right) + \alpha_k\mathfrak{R}_{ij} - \alpha_j\mathfrak{R}_{ik}. \tag{6}$$

Substituting $\nabla_p\mathfrak{R}^p_{ijk} = \nabla_k\mathfrak{R}_{ij} - \nabla_j\mathfrak{R}_{ik}$ into (6) yields

$$\varphi_p\mathfrak{R}^p_{ijk} = \alpha\nabla_p\mathfrak{R}^p_{ijk} + \alpha_k\mathfrak{R}_{ij} - \alpha_j\mathfrak{R}_{ik}. \tag{7}$$

Multiplying (7) by $\mathfrak{g}^{ij}$ and then using $\nabla_p\mathfrak{R}^p_k = \frac{1}{2}\nabla_k\mathfrak{R}$, one can obtain

$$\varphi_p\mathfrak{R}^p_k = \frac{\alpha}{2}\nabla_k\mathfrak{R} + \alpha_k\mathfrak{R} - \alpha_j\mathfrak{R}^j_k, \tag{8}$$

where $\mathfrak{R}^j_k = \mathfrak{R}_{ik}\mathfrak{g}^{ij}$.

Since the space-time is conformally flat, from (5), we achieve

$$\mathfrak{R}_{hijk} = \frac{1}{2}\left\{\mathfrak{R}_{hk}\mathfrak{g}_{ij} - \mathfrak{R}_{hj}\mathfrak{g}_{ik} + \mathfrak{R}_{ij}\mathfrak{g}_{hk} - \mathfrak{R}_{ik}\mathfrak{g}_{hj}\right\} - \frac{\mathfrak{R}}{6}\left\{\mathfrak{g}_{hk}\mathfrak{g}_{ij} - \mathfrak{g}_{hj}\mathfrak{g}_{ik}\right\}. \tag{9}$$

Contracting (9) by $\mathfrak{g}^{hp}$ gives

$$\mathfrak{R}^p_{ijk} = \frac{1}{2}\left\{\mathfrak{R}^p_k\mathfrak{g}_{ij} - \mathfrak{R}^p_j\mathfrak{g}_{ik} + \mathfrak{R}_{ij}\delta^p_k - \mathfrak{R}_{ik}\delta^p_j\right\} - \frac{\mathfrak{R}}{6}\left\{\delta^p_k\mathfrak{g}_{ij} - \delta^p_j\mathfrak{g}_{ik}\right\}. \tag{10}$$

Multiplying (10) with φ_p, we obtain

$$\varphi_p\mathfrak{R}^p_{ijk} = \frac{1}{2}\left\{\varphi_p\mathfrak{R}^p_k\mathfrak{g}_{ij} - \varphi_p\mathfrak{R}^p_j\mathfrak{g}_{ik} + \varphi_k\mathfrak{R}_{ij} - \varphi_j\mathfrak{R}_{ik}\right\} - \frac{\mathfrak{R}}{6}\left\{\varphi_k\mathfrak{g}_{ij} - \varphi_j\mathfrak{g}_{ik}\right\}. \tag{11}$$

Differentiating (10), we have

$$\nabla_p\mathfrak{R}^p_{ijk} = \frac{1}{2}\left\{\nabla_p\mathfrak{R}^p_k\mathfrak{g}_{ij} - \nabla_p\mathfrak{R}^p_j\mathfrak{g}_{ik} + \nabla_p\mathfrak{R}_{ij}\delta^p_k - \nabla_p\mathfrak{R}_{ik}\delta^p_j\right\} - \frac{1}{6}\left\{\nabla_k\mathfrak{R}\mathfrak{g}_{ij} - \nabla_j\mathfrak{R}\mathfrak{g}_{ik}\right\}. \tag{12}$$

Also, executing the covariant differentiation of $\varphi_i\varphi^i = -1$, we obtain

$$(\nabla_j\varphi_i)\varphi^i + \varphi_i(\nabla_j\varphi^i) = 0.$$

Substituting (1) into the above equation yields

$$2\alpha\varphi^i\mathfrak{R}_{ij} = 0. \tag{13}$$

Due to $\alpha \neq 0$, (13) means that

$$\varphi^i\mathfrak{R}_{ij} = 0. \tag{14}$$

Substituting (11) and (12) into (7) and considering (14), we reveal that

$$\frac{1}{2}\left\{\varphi_k\mathfrak{R}_{ij} - \varphi_j\mathfrak{R}_{ik}\right\} - \frac{\mathfrak{R}}{6}\left\{\varphi_k\mathfrak{g}_{ij} - \varphi_j\mathfrak{g}_{ik}\right\}$$

$$= \frac{\alpha}{2}\left\{\frac{1}{2}(\nabla_k\mathfrak{R}\mathfrak{g}_{ij} - \nabla_j\mathfrak{R}\mathfrak{g}_{ik}) + \nabla_k\mathfrak{R}_{ij} - \nabla_j\mathfrak{R}_{ik}\right\} - \frac{\alpha}{6}\left\{\nabla_k\mathfrak{R}\mathfrak{g}_{ij} - \nabla_j\mathfrak{R}\mathfrak{g}_{ik}\right\} + \alpha_k\mathfrak{R}_{ij} - \alpha_j\mathfrak{R}_{ik}. \tag{15}$$

Moreover, from $\mathfrak{C}_{hijk} = 0$, it follows that $\nabla_h\mathfrak{C}^h_{ijk} = 0$. Thus, it is well known that

$$\nabla_k\mathfrak{R}_{ij} - \nabla_j\mathfrak{R}_{ik} = \frac{1}{6}\left\{\nabla_k\mathfrak{R}\mathfrak{g}_{ij} - \nabla_j\mathfrak{R}\mathfrak{g}_{ik}\right\} \tag{16}$$

Using (15) and (16), we can get

$$\varphi_k\mathfrak{R}_{ij} - \varphi_j\mathfrak{R}_{ik} - \frac{\mathfrak{R}}{3}\left\{\varphi_k\mathfrak{g}_{ij} - \varphi_j\mathfrak{g}_{ik}\right\} = \frac{\alpha}{3}\left\{\nabla_k\mathfrak{R}\mathfrak{g}_{ij} - \nabla_j\mathfrak{R}\mathfrak{g}_{ik}\right\} + 2(\alpha_k\mathfrak{R}_{ij} - \alpha_j\mathfrak{R}_{ik}). \tag{17}$$

Moreover, from (8) and (14), it follows that

$$\alpha_k\mathfrak{R} - \alpha_j\mathfrak{R}^j_k = -\frac{\alpha}{2}\nabla_k\mathfrak{R}. \tag{18}$$

Multiplying (17) by φ^k and then using (14), we can get

$$-\mathfrak{R}_{ij} + \frac{\mathfrak{R}}{3}\left\{\mathfrak{g}_{ij} + \varphi_i\varphi_j\right\} = \frac{\alpha}{3}\left\{\varphi^k\nabla_k\mathfrak{R}\mathfrak{g}_{ij} - \varphi_i\nabla_j\mathfrak{R}\right\} + 2\alpha_k\varphi^k\mathfrak{R}_{ij}. \tag{19}$$

Again, multiplying (19) by φ^i gives

$$\nabla_j\mathfrak{R} = -\varphi_j\varphi^k\nabla_k\mathfrak{R}, \tag{20}$$

where (14) was used.

Putting (20) in (19), it can be found that

$$\mathfrak{R}_{ij} = \frac{(\mathfrak{R} - \alpha\varphi^k\nabla_k\mathfrak{R})}{3(1 + 2\alpha_k\varphi^k)}(\mathfrak{g}_{ij} + \varphi_i\varphi_j), \tag{21}$$

for $\alpha_k\varphi^k \neq -\frac{1}{2}$.

Hence, we can write (20) as follows

$$\mathfrak{R}_{ij} = f(\mathfrak{g}_{ij} + \varphi_i\varphi_j) \tag{22}$$

here

$$f = \frac{\mathfrak{R} - \alpha\varphi^k\nabla_k\mathfrak{R}}{3(1 + 2\alpha_k\varphi^k)}, \tag{23}$$

f is scalar, and

$$\mathfrak{R} = 3f \tag{24}$$

Therefore we can state:

Theorem 2.1. *A conformally flat space-time that admits a generalized $\varphi(Ric)$-vector, which is a unit time-like vector, is PF under the condition $\alpha_k\varphi^k \neq -\frac{1}{2}$.*

Example 2.2. *Example of PF space-time: The quark-gluon is the closest known material to a PF. In general relativity, PFs are used to simulate idealistic matter distributions, such the inside of a star or an isotropic cosmos. In the latter scenario, the evolution of the cosmos may be described by the FLRW equation using the equation of state of the PF.*

Since $C = 0$, we have $div\, C = 0$. Additionally, in [24], it is illustrated that a PF space-time with $div\, C = 0$ qualifies as a GRW space-time. Therefore, considering Theorem 2.1, it follows that the space-time under consideration is a GRW space-time.

Based on the above discussion, we write:

Theorem 2.3. *A conformally flat space-time that admits a generalized $\varphi(Ric)$-vector which is a unit time-like vector, is classified as a GRW space-time, provided $\alpha_k\varphi^k \neq -\frac{1}{2}$.*

Based on [3], since a conformally flat GRW space-time becomes a RW space-time, considering Theorem 2.3, we reach the following result:

Theorem 2.4. *A conformally flat space-time that admits a generalized $\varphi(Ric)$-vector is a RW space-time, provided $\alpha_k\varphi^k \neq -\frac{1}{2}$.*

Substituting (22) and (24) into (9), we obtain

$$\mathfrak{R}_{hjkl} = \frac{f}{2}\left\{\varphi_h\varphi_k\mathfrak{g}_{ij} - \varphi_h\varphi_j\mathfrak{g}_{ik} + \varphi_i\varphi_j\mathfrak{g}_{hk} - \varphi_i\varphi_k\mathfrak{g}_{hj}\right\} + \frac{f}{2}\left\{\mathfrak{g}_{hk}\mathfrak{g}_{ij} - \mathfrak{g}_{hj}\mathfrak{g}_{ik}\right\}. \tag{25}$$

Recalling the definition of quasi-constant curvature in [7], (25) implies that such a space-time is of quasi-constant curvature.

Therefore, we can state that:

Corollary 2.5. *A conformally flat space-time that admits a generalized $\varphi(Ric)$-vector is of quasi-constant curvature, provided $\alpha_k\varphi^k \neq -\frac{1}{2}$.*

Remark 2.6. *According to [11], a space-time of quasi-constant curvature represents a RW space-time.*

Combining (3) and (4) give

$$\mathfrak{R}_{ij} = \kappa(p+\sigma)\varphi_i\varphi_j + (\kappa p + \frac{\mathfrak{R}}{2})\mathfrak{g}_{ij}. \tag{26}$$

Multiplying (26) by φ_i and then, using (14), one can be found that

$$\sigma = \frac{\mathfrak{R}}{2\kappa}. \tag{27}$$

Contracting (26) by $\mathfrak{g}^{ij}$, it becomes the form

$$p = -\frac{\mathfrak{R}}{6\kappa}. \tag{28}$$

Considering (27) and (28), we achieve

$$p = -\frac{\sigma}{3}. \tag{29}$$

According to equation (29), we can conclude that this space-time is isentropic, meaning that it describes radiation and characterizes the early universe. Moreover, since the EOS parameter ω is equal to $-\frac{1}{3}$, this space-time represents a quintessence phase [4].

Thus, we have:

Theorem 2.7. *A conformally flat space-time that admits a generalized $\varphi(Ric)$-vector which is a unit time-like vector, represents a quintessence phase, provided $\alpha_k\varphi^k \neq -\frac{1}{2}$.*

Example 2.8. *A scalar field gradually rolling down a potential, analogous to inflationary cosmology, is a typical illustration of quintessence. Quintessence is a negative pressure component of the cosmic fluid that varies over time and is spatially in homogenous. Quintessence acts as a link between the basic idea of nature, string theory or other, and the observable structure of the universe.*

From (27) and (28), we infer $p + \sigma = \frac{\Re}{2\kappa} \neq 0$. In [15], it is demonstrated that a four-dimensional PF space-time, where $p + \sigma \neq 0$, is characterized as a Yang pure space if and only if it is a RW space-time.

Thus, we have from Theorem 2.4

Corollary 2.9. *A space-time admitting a generalized $\varphi(Ric)$-vector is a Yang pure space.*

Example 2.10. *In the context of Yang-Mills theory, "Yang's pure space equations" refer to a generalization of Einstein's gravitational equations derived from gauge theory and an example of a Yang-Mills solution is the vacuum solution on the light cone.*

3. Flat Friedmann-Robertson-Walker metric in Conformally flat space-times with a generalized $\varphi(Ric)$-vector

The space-time metric in the homogeneous and isotropic Friedmann-Robertson-Walker (FRW) model takes its subsequent form:

$$ds^2 = -dt^2 + a^2(t)[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)], \tag{30}$$

r is the co-moving radial coordinates, and $a(t)$ is the scale factor that characterizes how the distance between any two world lines varies with cosmic time t. A few observations on the scale factor $a(t)$ in the FRW model, as provided by (30), are crucial. Since its development throughout time guarantees the universe's expansion, it plays an essential role in the study of cosmic dynamics. The Friedmann equation, which provides cosmic dynamics, is solved to obtain it.

The non zero components of the Ricci tensor are

$$\Re_{00} = -3\frac{\ddot{a}}{a};\ \Re_{11} = \frac{a\ddot{a} + 2\ddot{a}^2 + 2k}{1-kr^2};$$

$$\Re_{22} = r^2(a\ddot{a} + 2\ddot{a}^2 + 2k);\ \Re_{33} = r^2(a\ddot{a} + 2\ddot{a}^2 + 2k)sin^2\theta,$$

and the scalar curvature is then

$$\Re = 6[\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{k}{a^2}]. \tag{31}$$

From here and elsewhere, ordinary derivative with regard to t is shown by an overhead dot.

According to the EFEs with the cosmological constant,

$$\Re_{km} - \frac{\Re}{2}g_{km} = T_{km} - \Lambda g_{km}. \tag{32}$$

Using equations (3) and (22) in equation (32), one can acquire

$$\Re = 2(\sigma + \Lambda). \tag{33}$$

Again from equation (32), we obtain

$$\Re + 3p - \sigma - 4\Lambda = 0. \tag{34}$$

From equations (33) and (34), we find

$$\sigma = \frac{\Re}{2} - \Lambda; \; p = \Lambda - \frac{\Re}{6}. \tag{35}$$

Now for flat FRW metric ($k = 0$)

$$-3\frac{\ddot{a}}{a} = -(\sigma + \Lambda). \tag{36}$$

$$\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 = \sigma + \Lambda. \tag{37}$$

Combining equations (36) and (37), we have

$$\sigma = 3(\frac{\dot{a}}{a})^2 - \Lambda. \tag{38}$$

Hence, from equations (31), (33) and (38), we get

$$\dot{H} = -H^2, \tag{39}$$

where the Hubble parameter is given by $H = \frac{\dot{a}}{a}$.
When (39) is integrated with respect to t, the result is

$$H(t) = \frac{1}{t + c_1}, \tag{40}$$

c_1 being the integrating constant.
We may quickly get the value of the scale factor $a(t)$ using equation (40) as follows:

$$a(t) = c_2(t + c_1), \tag{41}$$

where c_2 is an integration constant. The selection of cosmic time's origin has an impact on this integration constant.

Theorem 3.1. *If a conformally flat space-time with a generalized $\varphi(Ric)$-vector admits a flat FRW metric, then Hubble parameter is given by equation* (40) *and the scale factor is expressed by equation* (41).

The deceleration, snap and jerk parameters are written by

$$q = -\frac{1}{H^2}\frac{\ddot{a}}{a}, \qquad s = \frac{1}{H^4}\frac{\ddddot{a}}{a} \;\text{ and }\; \qquad j = \frac{1}{H^3}\frac{\dddot{a}}{a}, \tag{42}$$

respectively.
Using equation (41) in the above expressions we readily acquire $q = j = s = 0$.

Remark 3.2. *If a conformally flat space-time with a generalized $\varphi(Ric)$-vector admits a flat FRW metric, then deceleration, jerk, and snap parameters vanish.*

Energy conditions, which are often derived from the well-known Raychaudhuri equation, are among the best instruments for analyzing the self-stability of cosmological models [29]. They also assist us in describing the geometrical behavior of the space-time curve. They are characterized as:

Null Energy Condition(NEC)	if and only if	$\sigma + p \geq 0$,
Dominant Energy Condition(DEC)	if and only if	$\sigma \geq 0$ and $\sigma - \lvert p\rvert \geq 0$,
Strong Energy Condition(SEC)	if and only if	$\sigma + 3p \geq 0$ and $\sigma + p \geq 0$,
Weak Energy Condition(WEC)	if and only if	$\sigma \geq 0$ and $\sigma + p \geq 0$.

The computations for pressure p and energy density σ may be written as:

$$\sigma = \frac{3}{8\pi G(t+c_1)^2} \tag{43}$$

and

$$p = \frac{2-3(t+c_1)^2}{8\pi G(t+c_1)^4}. \tag{44}$$

Figure 1 displays the profiles of all energy conditions using equations (43) and (44).

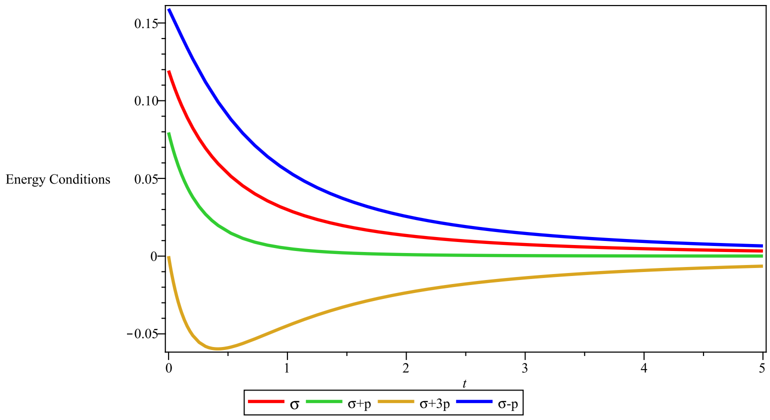

Fig. 1: Energy conditions with regard to time t

From Fig. 1, it has been noted that although the others energy conditions are satisfied, the SEC is violated.

4. Ricci solitons

Ricci flow as a solution to obtain a canonical metric on a differentiable manifold is described by Hamilton. This manifold satisfies the evolution equation

$$\frac{\partial}{\partial t}\mathfrak{g}_{ij}(t) = -2\mathfrak{R}_{ij}.$$

It is known as Ricci flow. Moreover, when the metric of the manifold meets the specified relation, it exhibits certain properties that contribute to its overall characteristics

$$\frac{1}{2}\mathfrak{L}_{\varphi}\mathfrak{g}_{ij} + \mathfrak{R}_{ij} = \lambda\mathfrak{g}_{ij}. \tag{45}$$

A Ricci soliton is a mathematical structure described by the pair (φ, λ), where φ is a differentiable vector on the manifold and $\mathfrak{L}$ represents the Lie derivative operator ([19], [20]). The parameter λ is a real constant. The classification of a Ricci soliton as expanding, shrinking, or steady depends on the value of λ: it is considered expanding if λ is negative, shrinking if λ is positive, and steady if λ is zero [19].

Recalling the equation below

$$\mathfrak{L}_{\varphi}\mathfrak{g}_{ij} = \nabla_i\varphi_j + \nabla_j\varphi_i, \tag{46}$$

and then, substituting (46) into (45), it can be found that:

$$\nabla_i\varphi_j + \nabla_j\varphi_i + 2\mathfrak{R}_{ij} = 2\lambda\mathfrak{g}_{ij}. \tag{47}$$

Combining (1) and (47) yields

$$(\alpha + 1)\Re_{ij} = \lambda \mathfrak{g}_{ij}. \tag{48}$$

Multiplying (48) by φ^i and then using (14), we obtain

$$\lambda \varphi_j = 0.$$

Due to $\varphi_j \neq 0$, it must be that

$$\lambda = 0. \tag{49}$$

Therefore, (48) and (49) means that for $\alpha \neq -1$, it is Ricci flat.

Thus, we can express

Theorem 4.1. *Let a space-time obey a generalized $\varphi(Ric)$-vector. Then the Ricci soliton (φ, λ) is steady, and it is Ricci flat, provided $\alpha \neq -1$.*

By Theorem 4.1, we get $\Re_{ij} = 0$. Then, from the definition of Ricci soliton, it follows that the potential vector is a homothetic vector. It is known that [25] if V is a non-null homothetic vector, it is shear-free and has constant expansion given by 3λ. Hence, from (49), the expansion becomes zero.

Based on the discussion above, we can establish the following theorem.

Theorem 4.2. *In a space-time with a generalized $\varphi(Ric)$-vector which is a unit time-like vector, if it admits a Ricci soliton (φ, λ), $\alpha \neq -1$, then φ_i is homothetic, shear-free, and the expansion reaches zero.*

A modified class of Ricci soliton evolution (45) was developed by Pigola et al. [28], substituting a variable function Φ for the soliton constant λ. This new formulation is referred to as an almost Ricci soliton, denoted as ARS.

If we consider an ARS, we find that the potential vector is a conformal vector. This leads us to the conclusion that for an ARS, $\Re_{ij} = 0$, indicating that the potential vector is indeed a conformal vector.

Since $\Re_{ij} = 0$, we can apply the EFEs to conclude that $\mathcal{T}_{ij} = 0$. Therefore, this space-time is classified as a vacuum. Furthermore, Collinson-French [8] demonstrated that a non-flat vacuum space-time possessing a proper conformal vector is classified as Petrov type N and exhibits parallel rays of a plane-fronted gravitational wave.

Therefore, we deduce the following result:

Theorem 4.3. *If a space-time with a generalized $\varphi(Ric)$-vector which is a unit time-like vector, admits an ARS, $\alpha \neq -1$, then it is of Petrov type N and exhibits parallel rays of a plane-fronted gravitational wave.*

A class of solitons known as the generalized Ricci soliton equations is described in [26] and are given by

$$\frac{1}{2}\mathfrak{L}_{\varphi}\mathfrak{g}_{ij} + \eta\varphi_i\varphi_j = \vartheta\Re_{ij} + \lambda\mathfrak{g}_{ij}, \tag{50}$$

here η, ϑ and λ stand for arbitrary real constants. A semi-Riemannian manifold is called a generalized Ricci soliton (GRS) if it admits the differentiable vector φ satisfying (50) [26].

Let us now put (46) in (50). Then, we get

$$\nabla_i\varphi_j + \nabla_j\varphi_i + 2\eta\varphi_i\varphi_j = 2\vartheta\Re_{ij} + 2\lambda\mathfrak{g}_{ij}. \tag{51}$$

Taking into account (1), (51) reduces to the following one:

$$(\eta - \vartheta)\Re_{ij} + \eta\varphi_i\varphi_j = \lambda\mathfrak{g}_{ij}. \tag{52}$$

Multiplying (52) by $\varphi^i\varphi^j$ and then using (14) and $\varphi_i\varphi^i = -1$, we get

$$\eta = -\lambda. \tag{53}$$

Combining (52) and (53), we acquire

$$\Re_{ij} = \frac{\lambda}{\eta - \vartheta}\left\{g_{ij} + \varphi_i\varphi_j\right\}, \tag{54}$$

for $\eta \neq \vartheta$.

Therefore, we can conclude that

Theorem 4.4. *If a space-time with a generalized $\varphi(Ric)$-vector which is a unit time-like vector, admits a GRS, then this space-time is a PF space-time for $\eta \neq \vartheta$.*

Conclusions

The idea of generalized $\varphi(Ric)$-vectors represents an advancement of the existing framework of $\varphi(Ric)$-vectors within the context of semi-Riemannian spaces. This enhancement is acquired by removing the previous fulfillment that the coefficient of proportionality be constant, thereby allowing for a non-constant coefficient.

In this study, we first investigate a conformally flat space-time that admits a generalized $\varphi(Ric)$-vector, which is a unit time-like vector. We conclude that, under a certain condition, such a space-time exhibits quasi-constant curvature and becomes perfect fluid. These spaces hold significance in differential geometry and relativity theory. We conclude that in the current space-time with flat FRW metric, the SEC profile exhibits negative behavior, but the NEC is always fulfilled. WEC is satisfied and DEC also verified for this set up which is also consistent with present observational studies that reveal the universe is expanding. Then, we classify Ricci solitons, which became even more fascinating after G. Perelman applied it to solve the Poincaré conjecture, in space-time with a generalized $\varphi(Ric)$-vector.

5. Declarations

5.1. *Funding*

NA.

5.2. *Code availability*

NA.

5.3. *Availability of data*

NA.

5.4. *Conflicts of interest*

The authors have no conflicts to disclose.

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