Filomat 40:9 (2026), 3287–3298
https://doi.org/10.2298/FIL2609287A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Characterization of the pseudospectrum of a compact operator

Anouer Ben Ali[a]

[a]*Faculty of Sciences of Sfax, Department of Mathematics, BP1171, Sfax 3000, Tunisia*

Abstract. We study the ε-pseudospectrum of compact linear operators on complex Banach spaces. Several pseudospectral notions defined via resolvent norms, bounded perturbations, and approximate eigenvectors are compared. We say that a complex Banach space X belongs to the class $\mathcal{W}$ if, for every compact operator A on X satisfying $\|I + A\| > 1$, the operator $I + A$ attains its norm. For compact operators acting on Banach spaces in $\mathcal{W}$, we show that these pseudospectral notions coincide.

1. Introduction

Let $\mathcal{B}(X)$ denote the Banach algebra of all bounded linear operators on a complex Banach space X, and let $\mathcal{K}(X)$ be the ideal of all compact operators on X. We say that X belongs to the class $\mathcal{W}$ if, for every compact operator K on X satisfying $\|I + K\| > 1$, the operator $I + K$ attains its norm. Recall that an operator $A \in \mathcal{B}(X)$ is said to be norm attaining if there exists $x \in X$ with $\|x\| = 1$ such that $\|Ax\| = \|A\|$.

Throughout this paper, for an operator $A \in \mathcal{B}(X)$, we denote by $\sigma(A)$ its spectrum and by $\sigma_p(A)$ its point spectrum. The notion of ε-pseudospectrum provides a natural generalization of the classical spectrum.

Definition 1.1. *Let $A \in \mathcal{B}(X)$ and $\varepsilon > 0$. The ε-pseudospectrum of A is defined by*

$$\sigma_\varepsilon(A) := \Sigma_\varepsilon(A) \cup L_\varepsilon(A),$$

where

$$\Sigma_\varepsilon(A) = \left\{\lambda \in \mathbb{C} : \|(A - \lambda I)^{-1}\| > \tfrac{1}{\varepsilon}\right\} \cup \sigma(A),$$

and

$$L_\varepsilon(A) = \left\{\lambda \in \mathbb{C} : \|(A - \lambda I)^{-1}\| = \tfrac{1}{\varepsilon}\right\}.$$

The concept of pseudospectrum emerged in the 1960s and 1970s through the work of numerical analysts such as Landau, Varah, Wilkinson, Demmel, and, most notably, Trefethen, in their efforts to understand the influence of rounding errors and non-normality on eigenvalue computations. The pseudospectrum generalizes the classical spectrum by identifying those complex numbers λ for which the resolvent norm $\|(A - \lambda I)^{-1}\|$ is large, or equivalently, for which λ becomes an eigenvalue of a perturbed operator $A + E$ with $\|E\| \leq \varepsilon$. In recent years, several refinements and variants of the classical pseudospectrum have been

2020 *Mathematics Subject Classification*. Primary 47A30; Secondary 47A10.
Keywords. pseudospectrum, compact operator, norm attaining operator.
Received: 05 September 2025; Revised: 29 December 2025; Accepted: 18 January 2026
Communicated by Dragan S. Djordjević
Email address: anouer.benali.math@gmail.com (Anouer Ben Ali)
ORCID iD: https://orcid.org/0000-0001-5156-6619 (Anouer Ben Ali)

introduced in order to obtain sharper spectral characterizations in Banach spaces. In particular, Ammar, Jeribi and Mahfoudhi [4] established a characterization of the condition pseudospectrum for bounded linear operators, highlighting its connections with resolvent estimates and norm-attaining properties. Subsequently, the same authors extended this framework by introducing and studying the left and right condition pseudospectra of linear operators in [5], where finer perturbation and one-sided spectral behaviors were analyzed. These contributions provide a deeper understanding of pseudospectral structures beyond the classical setting and motivate further investigations into the relationships between different pseudospectral notions, especially for compact operators and in infinite-dimensional Banach spaces.

For compact operators, whose spectra consist of isolated eigenvalues accumulating only at zero, the pseudospectrum provides valuable insight into the sensitivity of eigenvalues under arbitrarily small perturbations and helps detect spectral pollution in finite-rank approximations. The geometric interpretation of pseudospectra, obtained by plotting level sets of the resolvent norm, offers a powerful tool for analyzing operator stability and transient behavior. The influential monograph by Trefethen and Embree [16] played a central role in popularizing pseudospectral analysis. Further properties of pseudospectra in finite-dimensional spaces and Banach algebras can be found in the works of Davies [6], Varah [17], Harrabi [8], Jeribi and coauthors [2, 3, 9, 10], and Shargorodsky [12].

We now introduce two additional pseudospectral-type sets associated with an operator $A \in \mathcal{B}(X)$ and $\varepsilon > 0$.

1. Define $\mathcal{U}_\varepsilon(A) = U_\varepsilon(A) \cup LU_\varepsilon(A)$, where

$$U_\varepsilon(A) := \Big\{\lambda \in \mathbb{C} : \lambda \in \sigma(A+E) \text{ for some } E \in \mathcal{B}(X) \text{ with } \|E\| < \varepsilon\Big\},$$

and

$$LU_\varepsilon(A) := \Big\{\lambda \in \mathbb{C} : \lambda \in \sigma(A+E) \text{ for some } E \in \mathcal{B}(X) \text{ with } \|E\| = \varepsilon\Big\}.$$

2. Define $\mathcal{V}_\varepsilon(A) = V_\varepsilon(A) \cup LV_\varepsilon(A)$, where

$$V_\varepsilon(A) := \Big\{\lambda \in \mathbb{C} : \exists\, x \in X,\ \|x\| = 1,\ \|(A-\lambda I)x\| < \varepsilon\Big\},$$

and

$$LV_\varepsilon(A) := \Big\{\lambda \in \mathbb{C} : \exists\, x \in X,\ \|x\| = 1,\ \|(A-\lambda I)x\| = \varepsilon\Big\}.$$

The following lemmas describe the relationships among the sets $\sigma_\varepsilon(A)$, $\mathcal{U}_\varepsilon(A)$, $\mathcal{V}_\varepsilon(A)$ and their strict counterparts.

Lemma 1.1. [16] *Let $A \in \mathcal{B}(X)$ with $A \neq \lambda I$ for all $\lambda \in \mathbb{C}$, and let $\varepsilon > 0$. For $\lambda \notin \sigma(A)$, we have $\lambda \in \Sigma_\varepsilon(A)$ if and only if there exists $x \in X$ such that*

$$\|(A-\lambda I)x\| < \varepsilon\|x\|.$$

Lemma 1.2. [16] *Let $A \in \mathcal{B}(X)$ with $A \neq \lambda I$ for all $\lambda \in \mathbb{C}$, and let $\varepsilon > 0$. Then $\lambda \in \Sigma_\varepsilon(A)$ if and only if $\lambda \in \sigma(A+E)$ for some $E \in \mathcal{B}(X)$ with $\|E\| < \varepsilon$.*

As a consequence of Lemmas 1.1 and 1.2, we obtain

$$\Sigma_\varepsilon(A) = U_\varepsilon(A), \qquad \Sigma_\varepsilon(A) \setminus \sigma(A) = V_\varepsilon(A) \setminus \sigma(A),$$

and hence

$$U_\varepsilon(A) \setminus \sigma(A) = V_\varepsilon(A) \setminus \sigma(A),$$

for every $A \in \mathcal{B}(X)$ with $A \neq \lambda I$.

In the finite-dimensional setting, the following result establishes the equivalence of the three pseudospectral notions.

Theorem 1.1. [16] *Let $\varepsilon > 0$ and let X be a finite-dimensional complex Banach space. If $A \in \mathcal{B}(X)$ satisfies $A \neq \lambda I$ for all $\lambda \in \mathbb{C}$, then*

$$\sigma_\varepsilon(A) = \mathcal{U}_\varepsilon(A) = \mathcal{V}_\varepsilon(A).$$

The main objective of this paper is to investigate the ε-pseudospectrum of linear operators and to establish new characterizations that yield necessary and sufficient conditions for the equality

$$\sigma_\varepsilon(A) = \mathcal{U}_\varepsilon(A) = \mathcal{V}_\varepsilon(A)$$

when A is compact. Moreover, we analyze the relationships among the boundary sets $L_\varepsilon(A)$, $LU_\varepsilon(A)$, and $LV_\varepsilon(A)$ for operators in $\mathcal{K}(X)$. Our approach follows the same general methodology as in [1].

This paper is organized as follows. In Section 2, we show that if $A \in \mathcal{B}(X)$ is not a scalar multiple of the identity and the resolvent $(A - \lambda I)^{-1}$ attains its norm for all $\lambda \in L_\varepsilon(A)$, then

$$\mathcal{V}_\varepsilon(A) = \mathcal{U}_\varepsilon(A) = \sigma_\varepsilon(A), \quad \text{and} \quad L_\varepsilon(A) \subseteq LV_\varepsilon(A) \subseteq LU_\varepsilon(A).$$

In Section 3, assuming that X belongs to the class $\mathcal{W}$, we prove that

$$\sigma_\varepsilon(A) = \mathcal{U}_\varepsilon(A)$$

for every compact operator $A \in \mathcal{K}(X)$.

2. Preliminary results

This section explores the relationship between the sets $\sigma_\epsilon(A)$, $\mathcal{U}_\epsilon(A)$,$\mathcal{V}_\epsilon(A)$ and $L_\epsilon(A)$, $LU_\epsilon(A)$, $LV_\epsilon(A)$ for any bounded operator A defined on a Banach space X.

Lemma 2.1. *If $M \in \mathcal{B}(X)$ is an invertible operator and $N \in \mathcal{B}(X)$ is not invertible, then*

$$\|N - M\| \geqslant \frac{1}{\|M^{-1}\|}$$

Proof. Suppose, $\|N - M\| < \dfrac{1}{\|M^{-1}\|}$. Then

$$\|NM^{-1} - I\| = \|(N - M)M^{-1}\| \leqslant \|N - M\|\|M^{-1}\| < 1.$$

It gives NM^{-1} is invertible. Also, as M is invertible, therefore $N = NM^{-1}M$ is invertible too. Consequently, we get a contradiction with the fact that N is non-invertible. Q.E.D.

Corollary 2.1. *Let A be a compact operator defined on an infinite complex Banach space X. If $\lambda \notin \sigma(A)$ then*

$$\|(A - \lambda I)^{-1}\| \geqslant \frac{1}{|\lambda|}.$$

Proof. Let $\lambda \notin \sigma(A)$. It's well known that any compact operator defined on an infinite Banach space is non invertible, which means A is not invertible and $A - \lambda I$ is an invertible operator. Applying Lemma 2.1

$$\|A - (A - \lambda I)\| = |\lambda| \geqslant \frac{1}{\|(A - \lambda I)^{-1}\|}.$$

It implies,

$$\|(A - \lambda I)^{-1}\| \geqslant \frac{1}{|\lambda|}.$$

Q.E.D.

Using the previous Lemma as a key step, we establish the following theorem, which shows that $\mathcal{U}_\epsilon(A) \subseteq \sigma_\epsilon(A)$.

Theorem 2.1. *Let $A \in \mathcal{B}(X)$, such that $A \neq \lambda I \ \forall \lambda \in \mathbb{C}$, and $\epsilon > 0$. Then,*

$$\mathcal{U}_\epsilon(A) \subseteq \sigma_\epsilon(A).$$

Proof. Suppose $\lambda \in \mathcal{U}_\epsilon(A)$, then $\lambda \in \sigma(A+E)$ for some $E \in \mathcal{B}(X)$ such that $\|E\| \leqslant \epsilon$. In the case $\lambda \in \sigma(A)$, the inclusion is trivial as $\sigma(A) \subseteq \sigma_\epsilon(A)$.
Now, if $\lambda \notin \sigma(A)$, then $A - \lambda I$ is invertible and $A + E - \lambda I$ is non invertible operator. Thus, by Lemma 2.1

$$\|E\| = \|(E + A - \lambda I) - (A - \lambda I)\| \geqslant \frac{1}{\|(A - \lambda I)^{-1}\|}.$$

That implies,

$$\frac{1}{\|(A - \lambda I)^{-1}\|} \leqslant \|E\| \leqslant \epsilon.$$

Therefore

$$\frac{1}{\epsilon} \leqslant \|(A - \lambda I)^{-1}\|.$$

Consequently, $\lambda \in \sigma_\epsilon(A)$. Q.E.D.

The next lemma establishes the converse inclusion, under the additional assumption that the Banach space X satisfies a certain property.

Lemma 2.2. *Let X be a complex Banach space that have the following property: "For every invertible operator $A \in \mathcal{B}(X)$ there exists a non invertible operator $E \in \mathcal{B}(X)$ such that $\|A - E\| = \frac{1}{\|A^{-1}\|}$". Hence, for every $A \in \mathcal{B}(X)$, with A is not scalar multiple of the identity, and $\lambda \in \sigma_\epsilon(A)$ there exists a non invertible operator E such that:*

$$\|E\| \leqslant \epsilon \text{ and } \lambda \in \sigma(A + E).$$

Proof. Suppose $\lambda \in \sigma_\epsilon(A)$, then we discuss this two cases:
case 1 : If $\lambda \in \sigma(A)$, so putting $E = 0$.
case 2 : In case $\lambda \in \sigma_\epsilon(A) \setminus \sigma(A)$, then $\lambda I - A$ is an invertible operator. Thus, our hypothesis ensures the existence of a non-invertible operator D such that

$$\|\lambda I - A - D\| = \frac{1}{\|(\lambda I - A)^{-1}\|}.$$

Let $E = \lambda I - A - D$, then

$$\|E\| = \frac{1}{\|(\lambda I - A)^{-1}\|} \leqslant \epsilon.$$

Since $D = \lambda I - A - E$ is a non invertible operator, then $\lambda \in \sigma(A + E)$. Q.E.D.

Next, we provide examples of Banach spaces satisfying the condition stated in the previous lemma. These examples were previously cited in [11].

Example 2.1. *Let X be a compact Hausdorff space and $g \in C(X)$ be an invertible element, so there exists $a \in X$ verifying:*

$$0 < |g(a)| \leqslant |g(x)|, \ \forall x \in X$$

Then

$$|g(a)| = \frac{1}{\|g^{-1}\|}$$

Let $h \in C(X)$ defined by $h(x) = g(a) \ \ \forall x \in X$. We notice that

$$\|g - (g - h)\| = \|h\| = |g(a)| = \frac{1}{\|g^{-1}\|}$$

In addition, $(g-h)(a) = 0$, so $g-h$ is not invertible. Therefore, the Banach algebra $C(X)$ satisfies the property indicated in Lemma 2.2.

Example 2.2. *The matrix algebra $\mathbb{C}^{n\times n}$ has also the property indicated in Lemma* 2.2.

Corollary 2.2. *Let X be a complex Banach space satisfying the assumption of Lemma* 2.2 *and $A \in \mathcal{B}(X)$, such that $A \neq \lambda I \ \forall \lambda \in \mathbb{C}$, then:*

$$\sigma_\epsilon(A) = \mathcal{U}_\epsilon(A).$$

Proof. Lemma 2.2 provides the direct inclusion, and Theorem 2.1 provides the reverse inclusion. Q.E.D.

The following lemma characterizes the connection between $LV_\epsilon(A)$ and $LU_\epsilon(A)$.

Lemma 2.3. *Let $\epsilon > 0$. If $A \in \mathcal{B}(X)$, then*

$$LV_\epsilon(A) \subseteq LU_\epsilon(A).$$

Proof. Suppose $\lambda \in LV_\epsilon(A)$, so there exists $x \in X$ with $\|x\| = 1$ such that

$$\|(A - \lambda I)x\| = \epsilon.$$

Consider $y = \frac{1}{\epsilon}(A - \lambda I)x$ be a unit vector. By Hahn Banach Theorem, there exists $f \in X^*$ (dual of X) such that $\|f\| = f(x) = \|x\| = 1$. Let

$$E : X \to X, \text{ with } E(s) = -\epsilon f(s)y.$$

Obviously, E is a bounded linear operator and $\|E(s)\| = \|-\epsilon f(s)y\| \leqslant \epsilon\|f\|\|s\|\|y\| = \epsilon$ for every unit vector $s \in X$. Hence

$$\|E\| = \sup_{\|s\|=1} \|E(s)\| \leqslant \epsilon.$$

As $\|E(x)\| = \epsilon$, therefore $\|E\| = \epsilon$. Also, we notice that

$$\lambda x = Ax - \epsilon y = Ax - \epsilon y f(x) = Ax + Ex = (A + E)x.$$

thus, $\lambda \in \sigma(A + E)$. As a result, $\lambda \in LU_\epsilon(A)$. Q.E.D.

Corollary 2.3. *Let $\epsilon > 0$. If $A \in \mathcal{B}(X)$, then*

$$\mathcal{V}_\epsilon(A) \setminus \sigma(A) \subseteq \mathcal{U}_\epsilon(A) \setminus \sigma(A).$$

Proof. Applying Lemma 1.1, Lemma 1.2 and Lemma 2.3 we obtain

$$\begin{aligned}
\mathcal{V}_\epsilon(A) \setminus \sigma(A) &= V_\epsilon(A) \setminus \sigma(A) \bigcup LV_\epsilon(A) \setminus \sigma(A) \\
&= U_\epsilon(A) \setminus \sigma(A) \bigcup LV_\epsilon(A) \setminus \sigma(A) \\
&\subseteq U_\epsilon(A) \setminus \sigma(A) \bigcup LU_\epsilon(A) \setminus \sigma(A) \\
&= \mathcal{U}_\epsilon(A) \setminus \sigma(A).
\end{aligned}$$

Q.E.D.

The following example demonstrates that $\sigma(A)$ is not necessarily a subset of $\mathcal{V}_\epsilon(A)$ for some $A \in \mathcal{B}(X)$ and some $\epsilon > 0$.

Example 2.3. *Consider the Banach space $l^\infty(\mathbb{N})$ and $0 < \epsilon < 1$. Let*

$$L : l^\infty(\mathbb{N}) \to l^\infty(\mathbb{N}) \text{ such that } L(x_1, x_2, x_3, ...) = (0, x_1, x_2, ...).$$

It is well known that, $\sigma(L) = \{z \in \mathbb{C}; |z| \leqslant 1\}$ and $\|L(x)\| = \|L\| = 1$ for any unit vector $x \in l^\infty(\mathbb{N})$. As $\|L(x)\| > \epsilon$ for any unit vector x, so $0 \notin \mathcal{V}_\epsilon(L)$ but $0 \in \sigma(L) \subseteq \mathcal{U}_\epsilon(L)$.

Lemma 2.4. *Let $\epsilon > 0$ and $A \in \mathcal{B}(X)$. If $\lambda \in L_\epsilon(A)$ and $(A - \lambda I)^{-1}$ is norm attaining, then*

$$\lambda \in LV_\epsilon(A).$$

Proof. As $(A - \lambda I)^{-1}$ is norm attaining, so there exists $y \in X$ with $\|y\| = 1$ such that

$$\|(A - \lambda I)^{-1} y\| = \|(A - \lambda I)^{-1}\|.$$

Consider $z = \alpha^{-1}(A - \lambda I)^{-1} y$ be a unit vector, where $\alpha = \|(A - \lambda I)^{-1}\|$.
Using the last equation we notice that $\alpha^{-1} y = (A - \lambda I)z$ and

$$\|(A - \lambda I)z\| = \alpha^{-1} = \frac{1}{\|(A - \lambda I)^{-1}\|} = \epsilon.$$

As a result, $\lambda \in LV_\epsilon(A)$. Q.E.D.

Corollary 2.4. *Let $\epsilon > 0$ and $A \in \mathcal{B}(X)$. If $\lambda \in L_\epsilon(A)$ and $(A - \lambda I)^{-1}$ is norm attaining, then*

$$\lambda \in LU_\epsilon(A).$$

Proof. This result is obtained directly from Lemma 2.3 and Lemma 2.4. Q.E.D.

. Let $X = l^\infty(\mathbb{N})$, the following example shows that in $\mathcal{B}(X)$ there exists $A \in \mathcal{B}(X)$, $\epsilon > 0$ and $\lambda \in LU_\epsilon(A) \bigcap LV_\epsilon(A)$ such that $(A - \lambda I)^{-1}$ is norm attaining but $\lambda \notin L_\epsilon(A)$.

Example 2.4. *Consider the Banach space $X = l^\infty(\mathbb{N})$ and $\epsilon = \frac{1}{2}$. Define $A : X \to X$ by*

$$A(x_1, x_2, x_3, .., x_n, ..) = (\frac{x_1}{2}, \frac{x_2}{4}, x_3, .., x_n, ..).$$

It is obvious that $\|A\| = 1$ and A^{-1} attains norm. We consider also the two following operators,

$$E_1 : X \to X \text{ by } E_1(x_1, x_2, x_3, .., x_n, ..) = (-\frac{x_1}{2}, 0, 0, .., 0, ..)$$

and

$$E_2 : X \to X \text{ by } E_2(x_1, x_2, x_3, .., x_n, ..) = (0, -\frac{x_2}{4}, 0, .., 0, ..).$$

It's clear that $0 \in \sigma(A + E_1)$ and $\|E_1\| = \epsilon$ and hence $0 \in LU_\epsilon(A)$. Consider the unit vector $v_1 = (1, 0, 0, .., 0, ..)$. Clearly, $\|Av_1\| = \epsilon$ and hence $0 \in LV_\epsilon(A)$. Since, $0 \in \sigma(A + E_2)$ with $\|E_2\| < \epsilon$, by Lemma 1.2, $0 \notin L_\epsilon(A)$.

Corollary 2.5. *Let $\epsilon > 0$ and $A \in \mathcal{B}(X)$. If $(A - \lambda I)^{-1}$ is norm attaining for all $\lambda \in L_\epsilon(A)$, then*

$$\sigma_\epsilon(A) = \mathcal{U}_\epsilon(A).$$

Proof. Follows from Theorem 2.1, Lemma 1.2 and Corollary 2.4. Q.E.D.

Corollary 2.6. *Let $\epsilon > 0$ and $A \in \mathcal{B}(X)$. If $(A - \lambda I)^{-1}$ is norm attaining for all $\lambda \in L_\epsilon(A)$, then*

$$\sigma_\epsilon(A) \setminus \sigma(A) = \mathcal{V}_\epsilon(A) \setminus \sigma(A).$$

Proof. The direct inclusion comes from Corollary 2.3 and Theorem 2.1, but the reverse inclusion follows from Lemma 1.1 and Lemma 2.4. Q.E.D.

3. Main results

The main goal of this section is to give a necessary and sufficient condition for the characterization of pseudospectrum of a linear operator A which is defined on Banach spaces which belongs to class $\mathcal{W}$, where $A \in \mathcal{K}(X)$.

Definition 3.1. [13] *A complex Banach space X is called "belongs to class $\mathcal{W}$" if for each $A \in \mathcal{K}(X)$ such that $\|I + A\| > 1$, $I + A$ attains norm.*

Lemma 3.1. *Let $\epsilon > 0$, $A \in \mathcal{K}(X)$. If $\lambda \in L_\epsilon(A)$ such that $\|(A - \lambda I)^{-1}\| = \dfrac{1}{|\lambda - \delta_0|}$ for some $\delta_0 \in \sigma(A)$. Then, $\lambda \in \mathcal{U}_\epsilon(A)$.*

Proof. Consider

$$E : X \to X \; defined \; by \; E(s) = (\lambda - \delta_0)s.$$

Obviously, $\lambda \in \sigma(A + E)$ and

$$\|E\| = |\lambda - \delta_0| = \|(A - \lambda I)^{-1}\|^{-1} = \epsilon.$$

Consequently, $\lambda \in \mathcal{U}_\epsilon(A)$. Q.E.D.

Lemma 3.2. *Let $\epsilon > 0$, $A \in \mathcal{K}(X)$. If $\lambda \in L_\epsilon(A)$ such that $\|(A - \lambda I)^{-1}\| = \dfrac{1}{|\lambda - \delta_0|}$ for some $\delta_0 \in \sigma(A) \setminus \{0\}$. Then, $\lambda \in \mathcal{V}_\epsilon(A)$.*

Proof. We notice that each non zero element of $\sigma(A)$ is an eigenvalue of A. It is evident that, if $\delta_0 \in \sigma(A) \setminus \{0\}$ then there exists $x \in X$ with $\|x\| = 1$ such that $Ax = \delta_0 x$. Consequently,

$$\|(A - \lambda I)x\| = \|(\delta_0 - \lambda)x\| = |\delta_0 - \lambda| = \frac{1}{\|(A - \delta_0 I)^{-1}\|} = \epsilon.$$

Hence, $\lambda \in \mathcal{V}_\epsilon(A)$. Q.E.D.

The following example proves that, for $X = l^2(\mathbb{N})$ there exists $A \in \mathcal{K}(X)$, and $\lambda \in L_\epsilon(A)$ with $\|(A-\lambda I)^{-1}\| = \frac{1}{|\lambda|}$ such that $\lambda \notin \mathcal{V}_\epsilon(A)$. This example satisfies the conditions of the previous lemma, except that we take $\delta_0 = 0$, which reduces the problem to proving that $\lambda \notin \mathcal{V}_\epsilon(A)$.

Example 3.1. *Let $X = l^2(\mathbb{N})$ and $\epsilon = \frac{1}{2}$. Define*

$$A : X \to X \; by \; A(x_1, x_2, x_3, .., x_n, ..) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, .., \frac{x_n}{n}, ..).$$

Obviously, A is compact. It is proved in [11] that

$$\sigma(A) = \{0\} \bigcup \{\frac{1}{n}, n \in \mathbb{N}\},$$

and

$$\|(A - \lambda I)^{-1}\| = \frac{1}{inf_{k\in\mathbb{N}}|\lambda - \frac{1}{k}|} \; for \; all \; \lambda \notin \sigma(A).$$

For $\lambda = -\frac{1}{2}$, we obtain

$$\|(A + \frac{1}{2})^{-1}\| = \frac{1}{inf_{k\in\mathbb{N}}|(-\frac{1}{2}) - \frac{1}{k}|} = \frac{1}{\frac{1}{2}} = \frac{1}{|\lambda|}.$$

Also, since $\|(A+\frac{1}{2})^{-1}\| = \frac{1}{\frac{1}{2}} = \frac{1}{\epsilon}$, *then* $(-\frac{1}{2}) \in L_\epsilon(A)$. *For any* $x = (x_1, x_2, x_3, .., x_n, ..) \in X$ *with* $\|x\| = 1$, *we notice the following,*

$$\begin{aligned} \|(A+\frac{1}{2})x\| &= \sqrt{\sum_{k=1}^{\infty} |\frac{x_k}{k} + \frac{x_k}{2}|^2} \\ &= \sqrt{\sum_{k=1}^{\infty} |x_k(\frac{1}{k} + \frac{1}{2})|^2} \\ &> \frac{1}{2} \\ &= \epsilon. \end{aligned}$$

As a consequence, $(-\frac{1}{2}) \notin \mathcal{V}_\epsilon(A)$.

Lemma 3.3. *Let* $\epsilon > 0$, $A \in \mathcal{K}(X)$. *If* $(A - \lambda I)^{-1}$ *attains norm for all* $\lambda \in L_\epsilon(A)$, *then*

$$\sigma_\epsilon(A) = \mathcal{V}_\epsilon(A).$$

Proof. It comes directly from the fact that every non zero element of $\sigma(A)$ is an eigenvalue and zero is the element in the approximate point spectrum and Corollary 2.6. Q.E.D.

The following Theorem gives a sufficient condition for the equality between the sets $\sigma_\epsilon(A)$ and $\mathcal{U}_\epsilon(A)$.

Theorem 3.1. *Let* $A \in \mathcal{K}(X)$. *Assume A have the property: "If* $\lambda \in L_\epsilon(A)$ *with* $|\lambda| > \epsilon$, *then* $(A - \lambda I)^{-1}$ *attains norm". If* $\epsilon > 0$, *then*

$$\sigma_\epsilon(A) = \mathcal{U}_\epsilon(A).$$

Proof. Theorem 2.1 proves the reverse inclusion. Let's prove direct inclusion: Consider $\lambda \in \sigma_\epsilon(A)$, then either $\lambda \in \Sigma_\epsilon(A)$ or $\lambda \in L_\epsilon(A)$. If $\lambda \in \Sigma_\epsilon(A)$, then by Lemma 1.2, $\lambda \in \mathcal{U}_\epsilon(A)$. Suppose $\lambda \in L_\epsilon(A)$, then one of the following three cases holds,

$$\|(A - \lambda I)^{-1}\| = \frac{1}{\epsilon} > \frac{1}{|\lambda|}. \quad (1)$$

$$\|(A - \lambda I)^{-1}\| = \frac{1}{\epsilon} < \frac{1}{|\lambda|}. \quad (2)$$

$$\|(A - \lambda I)^{-1}\| = \frac{1}{\epsilon} = \frac{1}{|\lambda|}. \quad (3)$$

As A is compact, so by Corollary 2.1

$$\|(A - \lambda I)^{-1}\| \geqslant \frac{1}{|\lambda|}. \quad (4)$$

If (3) holds, then $\|(A - \lambda I)^{-1}\| = \frac{1}{|\lambda|}$, applying Lemma 3.1 we obtain $\lambda \in \mathcal{U}_\epsilon(A)$. Inequality (2) is impossible, because it gives $\|(A - \lambda I)^{-1}\| < \frac{1}{|\lambda|}$, which contradicts (4). Suppose inequality (1) holds then by our hypothesis $(A - \lambda I)^{-1}$ attains its norm. Using Corollary 2.4, we get $\lambda \in \mathcal{U}_\epsilon(A)$. Q.E.D.

The following example demonstrates the necessity of the norm-attaining property.

Example 3.2. *Let* $0 < s < 1$ *and* $X = l^1(\mathbb{N})$. *Define* $A : X \to X$ *by*

$$A(x_1, x_2, x_3, .., x_n, ..) = ((1+2s)x_1 - \sum_{m=2}^{\infty} x_m, -x_2, \frac{-x_3}{2}, \frac{-x_4}{3}, .., \frac{-x_n}{n-1}, ..).$$

Clearly, A is a compact operator. Also, it is proved in [12] *that*

$$\|(A-2sI)x\| > s \quad (*)$$

for all unit vector $x \in X$ and

$$inf_{\|x\|=1}\|(A-2sI)x\| = s. \quad (**)$$

Equation $(*)$ *gives* $(A-2sI)^{-1}$ *doesn't attain norm and by equation* $(**)$,

$$\|(A-2sI)^{-1}\| = \frac{1}{s} > \frac{1}{2s}.$$

Hence, $2s \in L_s(A) \subseteq \sigma_s(A)$. *Clearly,* $2s > s$.

Suppose $2s \in \mathcal{U}_s(A)$ *which means that* $2s \in \sigma(A+E)$ *such that* $\|E\| \leqslant s$ *for some* $E \in \mathcal{B}(X)$. *Then*

$$\|E\| \leqslant s < 2s.$$

Thus, $E-2sI$ *is an invertible operator. As A is compact,* $A+E-2sI$ *is a Fredholm operator with index zero. Since* $A+E-2sI$ *is not invertible and has index zero, there exists* $v \in X$ *with* $\|v\| = 1$ *such that*

$$(A+E-2sI)(v) = 0.$$

It implies, $(A-2sI)(v) = -E(v)$. *By equation* $(*)$, $\|-E(v)\| > s$ *which contradicts the fact that* $\|E\| \leqslant s$. *Consequently,* $2s \notin \sigma(A+E)$ *and then* $2s \notin \mathcal{U}_s(A)$.

Corollary 3.1. *Let* $A \in \mathcal{K}(X)$. *Assume A have the property as indicated in Theorem* 3.1. *If* $\epsilon > 0$, *then*

$$L_\epsilon(A) \subseteq LU_\epsilon(A).$$

Proof. By Theorem 3.1, we obtain that $\sigma_\epsilon(A) = \mathcal{U}_\epsilon(A)$. Therefore, $L_\epsilon(A) \subseteq LU_\epsilon(A)$. Q.E.D.

The following Theorem is kind of converse of Theorem 3.1.

Theorem 3.2. *Let* $A \in \mathcal{K}(X)$ *and* $\epsilon > 0$. *Assume that* $\sigma_\epsilon(A) = \mathcal{U}_\epsilon(A)$. *If* $\lambda \in L_\epsilon(A)$ *such that* $|\lambda| > \epsilon$, *then* $(A-\lambda I)^{-1}$ *attains norm.*

Proof. Let $\lambda \in L_\epsilon(A)$ such that $|\lambda| > \epsilon$. Our hypothesis gives that there exists $E \in \mathcal{B}(X)$ such that $\lambda \in \sigma(A+E)$ and $\|E\| \leqslant \epsilon$. Hence, $\|E\| < |\lambda|$. Therfore, $-\lambda I + E$ is an invertible operator. As every invertible operator is Fredholm and A is compact, $A-\lambda I+E$ is a Fredholm operator with index zero. Obviously, $0 \in \sigma(A-\lambda I+E)$, then $A-\lambda I+E$ is not invertible. Since $A-\lambda I+E$ has index zero, it has a non trivial kernel. As a consequence, there is a unit vector $x \in X$ such that

$$(A-\lambda I+E)x = 0 \Rightarrow -E(x) = (A-\lambda I)(x) \Rightarrow -(A-\lambda I)^{-1}E(x) = x.$$

As $\lambda \in L_\epsilon(A)$ and $\|E\| \leqslant \epsilon$, we obtain that

$$1 = \|x\| = \|-(A-\lambda I)^{-1}E(x)\| \leqslant \|(A-\lambda I)^{-1}\|\|E(x)\| \leqslant \|(A-\lambda I)^{-1}\|\|E\| \leqslant 1.$$

Thus, $\|E\| = \|E(x)\| = \epsilon$. Let

$$y = \frac{1}{\epsilon}E(x) \textit{ be a unit vector.}$$

We note that

$$-(A-\lambda I)^{-1}E(x) = x$$

equivalent to,

$$-(A-\lambda I)^{-1}y\epsilon = x$$

if, and only if

$$-(A-\lambda I)^{-1}y = [\epsilon]^{-1}x$$

Reducing to

$$\|-(A-\lambda I)^{-1}y\| = \frac{1}{\epsilon}\|x\| = \|(A-\lambda I)^{-1}\|.$$

Consequently, $(A-\lambda I)^{-1}$ attains its norm. Q.E.D.

Corollary 3.2. *Let $A \in \mathcal{K}(X)$. We assume one of the following relationships holds*

(1) $L_\epsilon(A) \subseteq LU_\epsilon(A)$.
(2) $\sigma_\epsilon(A) = \mathcal{V}_\epsilon(A)$.
(3) $L_\epsilon(A) \subseteq LV_\epsilon(A)$.

If $\lambda \in L_\epsilon(A)$ such that $|\lambda| > \epsilon$, then $(A-\lambda I)^{-1}$ attains norm.

Proof. Obviously, if one of last relationships holds then $\sigma_\epsilon(A) = \mathcal{U}_\epsilon(A)$. Hence, by Theorem 3.2 we get directly the result. Q.E.D.

Theorem 3.3. *Consider X an infinite complex Banach space with the property that "for any $K \in \mathcal{K}(X)$ such that $\|I+K\| > 1, I+K$ is norm attaining" and $\epsilon > 0$. If $A \in \mathcal{K}(X)$. Then*

$$\sigma_\epsilon(A) = \mathcal{U}_\epsilon(A).$$

Proof.

In view of Theorem 2.1, we have $\mathcal{U}_\epsilon(A) \subseteq \sigma_\epsilon(A)$; hence, it remains to show that $\sigma_\epsilon(A) \subseteq \mathcal{U}_\epsilon(A)$. Let $\lambda \in \sigma_\epsilon(A)$ then either $\lambda \in \Sigma_\epsilon(A)$ or $\lambda \in L_\epsilon(A)$. Suppose $\lambda \in \Sigma_\epsilon(A)$ then by Lemma 1.2, $\lambda \in \mathcal{U}_\epsilon(A)$. Suppose $\lambda \in L_\epsilon(A)$, since A is compact then, by Corollary 2.1

$$\|(A-\lambda I)^{-1}\| \geqslant \frac{1}{|\lambda|}.$$

If $\|(A-\lambda I)^{-1}\| = \frac{1}{|\lambda|}$, then by Lemma 3.1, $\lambda \in \mathcal{U}_\epsilon(A)$.

Suppose that $\|(A-\lambda I)^{-1}\| > \frac{1}{|\lambda|}$. To show that $\lambda \in \mathcal{U}_\epsilon(A)$, it suffices, according to Corollary 2.5, that $(A-\lambda I)^{-1}$ attains norm. We note that

$$(A-\lambda I)^{-1} = -\lambda^{-1}(I - \lambda(A-\lambda I)^{-1} - I) = -\lambda^{-1}(I+F). \tag{3.1}$$

Where $F = -\lambda(A-\lambda I)^{-1} - I$. As A is compact and

$$\begin{aligned} F &= -\lambda(A-\lambda I)^{-1} - I \\ &= (A-\lambda I)^{-1}[-\lambda I - (A-\lambda I)] \\ &= -(A-\lambda I)^{-1}A, \end{aligned}$$

we conclude F is compact. by 3.1 we have

$$\|I+F\| = |-\lambda|\|(A-\lambda I)^{-1}\| > |\lambda|\frac{1}{|\lambda|} = 1.$$

By our hypothesis $I+F$ attains its norm, and so, by 3.1 $(A-\lambda I)^{-1}$ attains norm. Hence, $\lambda \in \mathcal{U}_\epsilon(A)$. Q.E.D.

Corollary 3.3. *Let X be an infinite complex Banach space with the property as indicated in Theorem 3.3 and $\epsilon > 0$. If $A \in \mathcal{K}(X)$. Then*

$$L_\epsilon(A) \subseteq LU_\epsilon(A).$$

Proof. The proof of Theorem 3.3 gives, if $\lambda \in L_\epsilon(A)$ then $(A-\lambda I)^{-1}$ attains norm. Consequently, by Corollary 2.4, $L_\epsilon(A) \subseteq LU_\epsilon(A)$. Q.E.D.

Theorem 3.4. *Let X be an infinite complex Banach space with the property that "for any $K \in \mathcal{K}(X)$ such that $\|I+K\| \geqslant 1, I+K$ is norm attaining" and $\epsilon > 0$. If $A \in \mathcal{K}(X)$. Then*

$$\sigma_\epsilon(A) = \mathcal{V}_\epsilon(A).$$

Proof. The proof is similar to the proof of Theorem 3.3. Q.E.D.

Corollary 3.4. *Let X be an infinite complex Banach space with the property as indicated in Theorem 3.4 and $\epsilon > 0$. If $A \in \mathcal{K}(X)$. Then*

$$L_\epsilon(A) \subseteq LV_\epsilon(A).$$

Proof. The proof of Theorem 3.3 gives, if $\lambda \in L_\epsilon(A)$ then $(A-\lambda I)^{-1}$ attains norm. Consequently, by Corollary 2.5, $L_\epsilon(A) \subseteq LV_\epsilon(A)$. Q.E.D.

Remark 3.1. *The class of Banach spaces which have the property given in the Theorem 3.3, has been introduced by Shkarin in [13].*

In [13] Shkarin has classified completely the Banach spaces belonging to the class $\mathcal{W}$. In our case, we have this property "if the underlying Banach space X belongs to class $\mathcal{W}$, then the sets $\sigma_\epsilon(A)$ and $\mathcal{U}_\epsilon(A)$ are equal for any $A \in \mathcal{K}(X)$ and $\epsilon > 0$".

Next theorem is the partial converse of Theorem 3.3.

Theorem 3.5. *Let X be a Banach space possess the property that "$\sigma_\epsilon(T) = \mathcal{U}_\epsilon(T)$ for any $\epsilon > 0$ and for any $T \in \mathcal{K}(X)$". If $A \in \mathcal{K}(X)$ such that $I+A$ is invertible and $\|I+A\| > 1$ then $I+A$ attains its norm.*

Proof. Let $A \in \mathcal{K}(X)$, $I+A$ is invertible and $\|I+A\| > 1$. Putting

$$T = (I+A)^{-1} - I.$$

Since

$$T = (I+A)^{-1} - I = (I+A)^{-1}(I-(I+A)) = -(I+A)^{-1}A.$$

and A is compact, then T is compact. By the definition of T

$$I+T = (I+A)^{-1}.$$

Take $\delta = (\|I+A\|)^{-1}$. Implies, $\frac{1}{\delta} = \|I+A\| = \|(I+T)^{-1}\|$. Hence, $-1 \in L_\delta(T) \subseteq \sigma_\delta(T)$. By our hypothesis, $-1 \in \mathcal{U}_\epsilon(T)$. We can see easily that $\delta < 1$ (because $\|I+A\| > 1 \Rightarrow \delta = \|I+A\|)^{-1} < 1$). By Theorem 3.2 $(I+T)^{-1}$ attains its norm and since $(I+T)^{-1} = (I+A)$, thus $I+A$ attains norm. Q.E.D.

Corollary 3.5. *Let X be a complex Banach space. We suppose one of the following relationships holds, $\forall \epsilon > 0, \forall T \in \mathcal{K}(X)$*

(1) $L_\epsilon(T) \subseteq LU_\epsilon(T)$.
(2) $\sigma_\epsilon(T) = \mathcal{V}_\epsilon(T)$.
(3) $L_\epsilon(T) \subseteq LV_\epsilon(T)$.

If $A \in \mathcal{K}(X)$ such that $I+A$ is invertible and $\|I+A\| > 1$ then $I+A$ attains its norm.

Proof.

If one of the above relationships holds then we get $\sigma_\epsilon(T) = \mathcal{U}_\epsilon(T)$ for any $\epsilon > 0$ and any $T \in \mathcal{K}(X)$. By Theorem 3.5, we obtain directly the result.

Q.E.D.

4. Conclusion

In this paper, we investigated the ϵ-pseudospectrum of bounded and compact linear operators acting on Banach spaces, with a particular emphasis on the interplay between pseudospectral sets and norm-attaining properties of resolvent operators. By introducing and comparing the sets $\sigma_\epsilon(A)$, $\mathcal{U}_\epsilon(A)$ and $\mathcal{V}_\epsilon(A)$, we established several inclusion and equality results that extend and refine previously known characterizations in the literature.

Our main contribution lies in proving that, for Banach spaces belonging to the class $\mathcal{W}$, these pseudospectral sets coincide for every compact operator and every $\epsilon > 0$. This result highlights the strong connection between the geometry of the underlying Banach space and the stability properties of the spectrum under compact perturbations. Moreover, we obtained a converse-type result showing that the equality $\sigma_\epsilon(T) = \mathcal{U}_\epsilon(T)$ for all compact operators T characterizes the norm-attaining property of invertible operators of the form $I + A$ with $\|I + A\| > 1$.

The results presented in this work generalize and complement earlier studies by Shkarin and by Jeribi et al., and they emphasize the distinctive role played by compact operators in Banach spaces beyond the Hilbert space framework. Possible directions for future research include the study of essential pseudospectra, extensions to semi-Fredholm or multivalued operator matrices, and applications to spectral stability problems arising in applied analysis.

Compliance with ethical standards
Conflict of interest:
The authors declare that they have no conflict of interest.
Research involving human participants and/or animals:
This paper does not contain any studies involving with human participants/ animals.

References

[1] A. B. Ali and B. Saadaoui (2022), On the condition spectrum of linear operator pencils. Rendiconti del Circolo Matematico di Palermo Series 2. Doi:10.1007/s12215-022-00756-5.
[2] A. Ammar, A. Jeribi and B. Saadaoui, On some classes of demicompact linear relation and some results of essential pseudospectra, Mat. Stud. 52 (2019), 195–210.
[3] A. Ammar, A. Jeribi and B. Saadaoui, A characterization of essential pseudospectra of the multivalued operator matrix, Anal. Math. Phys. 8 (2018), no. 3, 325–350.
[4] A. Ammar, A. Jeribi and K. Mahfoudhi, A characterization of the condition pseudospectrum on Banach space, Funct. Anal. Approx. Comput. 10 (2) (2018), 13–21.
[5] A. Ammar, A. Jeribi and K. Mahfoudhi, On the Left (Right) Condition Pseudospectrum of Linear Operators, Filomat 36:10 (2022), 3207–3215.
[6] Davies, E.B.: Linear Operators and Their Spectra. Cambridge Studies in Advanced Mathematics, vol. 106. Cambridge University Press, Cambridge (2007).
[7] J. Cui, V. Forstall, C.-K. Li, and V. Yannello (2012). Properties and preservers of the pseudospectrum. Linear Algebra Appl., 436(2), 316–325.
[8] A. Harrabi (1998). Pseudospectre d'une suite d'opérateurs bornés, RAIRO Modél. Math. Anal. Numér., 32, 671–680.
[9] A. Jeribi (2015). Spectral Theory and Applications of Linear Operators and Block Operator Matrices. Springer-Verlag, New York.
[10] A. Jeribi (2018). Linear Operators and Their Essential Pseudospectra. CRC Press, Boca Raton.
[11] S. H. Kulkarni and D. Sukumar. The condition spectrum. Acta Sci. Math. (Szeged), 74(3–4):625–641, 2008.
[12] E. Shargorodsky. On the definition of pseudospectra. Bull. Lond. Math. Soc., 41(3):524–534, 2009.
[13] S. Shkarin. Norm attaining operators and pseudospectrum. Integral Equations Operator Theory, 64(1):115–136, 2009.
[14] D. Sukumar. Some comparative results on eigenvalues, pseudospectra and condition spectra. J. Anal., 2019.
[15] Trefethen, L.N.: Pseudospectra of linear operators. SIAM Rev. 39(3), 383–406 (1997).
[16] Trefethen, L.N., Embree, M.: Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators. Princeton University Press, Princeton (2005).
[17] Varah, J.M.: The computation of bounds for the invariant subspaces of a general matrix operator. Ph.D. thesis, Stanford University, ProQuest LLC, Ann Arbor (1967).
[18] M. P. H. Wolff (2001). Discrete approximation of unbounded operators and approximation of their spectra, J. Approx. Theory, 113, 229–244.