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On a new generalization of Stirling numbers and Bell polynomials

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Abstract. In this paper, we introduce and study a novel class of generalized Stirling numbers depending on a real parameter and two sequences of real numbers. This class extends several known families of numbers, including the Hsu-Shiue generalized Stirling numbers $S(n, k; \alpha, \beta, \gamma)$. We also present a combinatorial interpretation of these numbers, which allows us to derive a variety of combinatorial identities. Additionally, we establish several identities for the associated polynomials and, under suitable conditions on the defining parameters, analyze the real roots of these polynomials. Finally, we establish some properties of the differential operator acting on these polynomials.

1. Introduction

The Stirling numbers of the second kind, often denoted by $\left\{ {n \atop k} \right\}$, counts the number of partitions of the set $[n] = \{1, 2, \ldots, n\}$ into k non-empty subsets. These numbers satisfy the recurrence relation [4]

$$\left\{ {n \atop k} \right\} = k \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}, \; n \geq k \geq 1,$$

with the boundary conditions $\left\{ {n \atop 0} \right\} = \delta_{n,0}$ and $\left\{ {0 \atop k} \right\} = \delta_{0,k}$ for all $n, \; k \in \mathbb{N}$. An alternative definition is given by the formula

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} j^n.$$

In the literature, several generalizations of these numbers have been studied in different directions [3, 4, 8, 10, 11, 15, 17]. For example, Hsu and Shiue [8] introduced the generalized Stirling numbers $S(n, k; \alpha, \beta, \gamma)$, defined by the recurrence relation

$$S(n+1, k; \alpha, \beta, \gamma) = S(n, k-1; \alpha, \beta, \gamma) + (\beta k + \gamma - n\alpha) S(n, k; \alpha, \beta, \gamma), \; n \geq k \geq 1,$$

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with the boundary conditions $S(n,0;\alpha,\beta,\gamma) = (\gamma \mid \alpha)_n$ for all $n,\ k \in \mathbb{N}$, where α,β and γ are given real or complex numbers, with $(\alpha,\beta,\gamma) \neq (0,0,0)$, where

$$(t \mid \alpha)_n = \prod_{i=0}^{n-1} (t - i\alpha),\ n \geq 1, \text{ with } (t \mid \alpha)_0 = 1.$$

An alternative definition is given by the formula [5]

$$S(n,k;\alpha,\beta,\gamma) = \frac{1}{\beta^k k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(\beta j + \gamma \mid \alpha)_n,$$

These numbers and their associated polynomials have been investigated in depth in the works of [2, 5–8, 13].

In the same direction, this paper introduces and studies a novel class of generalized Stirling numbers depending on a real parameter and two sequences of real numbers. The paper is organized as follows. In the next section, we introduce this class of generalized Stirling numbers, which extends the classical Stirling and and r-Stirling numbers, the Whitney and r-Whitney numbers, the Lah and r-Lah numbers, the $(r_1,\ldots,r_p)$-Stirling numbers and the Hsu-Shiue generalized Stirling numbers $S(n,k;\alpha,\beta,\gamma)$. It is defined by an explicit formula that generalizes the corresponding explicit formulas for these numbers. We also provide a combinatorial interpretation, from which we derive a number of combinatorial identities. In the final section, we establish several identities for the associated polynomials and, under suitable conditions on the defining parameters, analyze the real roots of these polynomials. Finally, we investigate some properties of the differential operator acting on them.

2. Generalized Stirling numbers

Let δ be a nonzero real number, $\mathbf{a} = (a_0, a_1, ...)$ and $\mathbf{b} = (b_0, b_1, ...)$ be two sequences of real numbers such that

$$a_n \neq 0 \text{ for all } n \geq 0.$$

The generalized Stirling numbers of second kind are defined by

$$\left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}} := \frac{\delta^k}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\prod_{i=0}^{n-1}(a_i j + b_i). \tag{1}$$

with convention $\prod_{i=0}^{-1}(a_i j + b_i) = 1$. By setting

$$g_{n,\mathbf{a},\mathbf{b}}(\lambda) := \prod_{i=0}^{n-1}(a_i\lambda + b_i),\ \ n \geq 1, \text{ and } g_{0,\mathbf{a},\mathbf{b}}(\lambda) := 1,$$

and simplifying the notation to $g_n(\lambda)$ instead of $g_{n,\mathbf{a},\mathbf{b}}(\lambda)$, the definition (1) becomes

$$\left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}} = \frac{\delta^k}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}g_n(j). \tag{2}$$

Recall the classic binomial transform inversion formula. For any sequences (s_n) and (q_n) we have

$$s_n = \sum_{k=0}^{n}\binom{n}{k}q_k \Longleftrightarrow q_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}s_k.$$

Applying this inversion formulae to identity (2), we obtain

$$g_n(k) = \sum_{j=0}^{k} \delta^{-j} \left\{ {n \atop j} \right\}_{\delta,\mathbf{a},\mathbf{b}} (k)_j . \tag{3}$$

where $(x)_n = x(x-1)\cdots(x-n+1)$, $n \geq 1$, and $(x)_0 = 1$.
For specific values of $\delta, (a_0, a_1, ...)$ and $(b_0, b_1, ...)$, the generalized Stirling numbers $\left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}}$ reduce to several well-known classical combinatorial numbers. We highlight some notable cases.

- When $\delta = \frac{1}{\beta}$, $\mathbf{a} = (\beta, \beta, \ldots)$ and $\mathbf{b} = (\gamma, \gamma - \alpha, \gamma - 2\alpha, \ldots)$, we recover the generalized Stirling numbers [8]

$$\left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}} = S(n, k; \alpha, \beta, \gamma).$$

- When $\delta = \frac{1}{2}$, $\mathbf{a} = (2, 2, \ldots)$ and $\mathbf{b} = (1, 1, \ldots)$, we recover the Stirling numbers of type B [9]

$$\left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}} = S_B(n, k).$$

- When $\delta = 1$, $\mathbf{a} = (1, 1, \ldots)$ and $\mathbf{b} = (0, 0, \ldots)$, we recover the Stirling numbers of the second kind [4]

$$\left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}} = \left\{ {n \atop k} \right\}.$$

- When $\delta = 1$, $\mathbf{a} = (1, 1, \ldots)$ and $\mathbf{b} = (r, r, \ldots)$, we recover the r-Stirling numbers of the second kind [3]

$$\left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}} = \left\{ {n+r \atop k+r} \right\}_r.$$

- When $\delta = 1$, $\mathbf{a} = (1, 1, \ldots)$ and $\mathbf{b} = (2r, 2r+1, 2r+2, \ldots)$, we recover numbers related to the r-Lah numbers [16]

$$\left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}} = \left\lfloor {n \atop k} \right\rfloor_r.$$

- When $\delta = \frac{1}{m}$, $\mathbf{a} = (m, m, \ldots)$ with $m > 0$, and $\mathbf{b} = (r, r, \ldots)$ with $r \geq 0$, we recover the r-Whitney numbers of the second kind [14]

$$\left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}} = W_{m,r}(n, k).$$

- When $\delta = 1$ and $g_{r_1+\cdots+r_{p-1}}(j) = \left(j + r_p\right)_{r_1} \cdots \left(j + r_p\right)_{r_{p-1}} r_p^{n-r_1-\cdots-r_p}$ with $0 \leq r_1 \leq \ldots \leq r_p$, we recover the $\left(r_1, \ldots, r_p\right)$-Stirling numbers [15]

$$\left\{ {r_1 + \cdots + r_{p-1} \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}} = \left\{ {n \atop k + r_p} \right\}_{r_1,\cdots,r_p}.$$

Proposition 2.1. *For any positive integers n and k, the following recurrence relation holds*

$$\left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}} = \delta a_{n-1} \left\{ {n-1 \atop k-1} \right\}_{\delta,\mathbf{a},\mathbf{b}} + (a_{n-1}k + b_{n-1}) \left\{ {n-1 \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}}, \quad \text{for } n \geq k \geq 1.$$

with boundary conditions $\left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}} = 0$, *for* $n < k$; *and* $\left\{ {n \atop 0} \right\}_{\delta,\mathbf{a},\mathbf{b}} = g_n(0)$, *for* $n \geq 0$.

Proof. We have

$$\begin{aligned}
&\delta a_{n-1}\begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}_{\delta,\mathbf{a},\mathbf{b}} + (a_{n-1}k+b_{n-1})\begin{Bmatrix} n-1 \\ k \end{Bmatrix}_{\delta,\mathbf{a},\mathbf{b}} \\
&= -\frac{a_{n-1}\delta^k}{(k-1)!}\sum_{j=0}^{k-1}(-1)^{k-j}\binom{k-1}{j}g_{n-1}(j) + \frac{(a_{n-1}k+b_{n-1})\delta^k}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}g_{n-1}(j), \\
&= -\frac{a_{n-1}\delta^k}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(k-j)\,g_{n-1}(j) + \frac{(a_{n-1}k+b_{n-1})\delta^k}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}g_{n-1}(j),
\end{aligned}$$

and this is exactly

$$\begin{aligned}
&\frac{\delta^k}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(-a_{n-1}(k-j)+a_{n-1}k+b_{n-1})\,g_{n-1}(j) \\
&= \frac{\delta^k}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(a_{n-1}j+b_{n-1})\,g_{n-1}(j), \\
&= \frac{\delta^k}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}g_n(j), \\
&= \begin{Bmatrix} n \\ k \end{Bmatrix}_{\delta,\mathbf{a},\mathbf{b}}.
\end{aligned}$$

□

Proposition 2.2. *If* $\mathbf{a} = (a, a, \ldots)$ *and* $\mathbf{b} = (b, b, \ldots)$, *then the following holds*

$$\sum_{n=0}^{\infty}\begin{Bmatrix} n \\ k \end{Bmatrix}_{\delta,\mathbf{a},\mathbf{b}} x^n = (\delta a x)^k\prod_{j=0}^{k}(1-(aj+b)\,x)^{-1}.$$

Proof. Let

$$F_k(t) = \sum_{n=0}^{\infty}\begin{Bmatrix} n \\ k \end{Bmatrix}_{\delta,\mathbf{a},\mathbf{b}} t^n.$$

From the Proposition 2.1, we have

$$\begin{aligned}
F_k(t) &= \delta a\sum_{n=0}^{\infty}\begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}_{\delta,\mathbf{a},\mathbf{b}} t^n + (ak+b)\sum_{n=0}^{\infty}\begin{Bmatrix} n-1 \\ k \end{Bmatrix}_{\delta,\mathbf{a},\mathbf{b}} t^n \\
&= \delta a t F_{k-1}(t) + (ak+b)\,t F_k(t).
\end{aligned}$$

Thus

$$F_k(t) = \frac{\delta a t}{1-(ak+b)\,t}F_{k-1}(x),$$

since

$$F_0(x) = \sum_{n=0}^{\infty}\begin{Bmatrix} n \\ 0 \end{Bmatrix}_{\delta,\mathbf{a},\mathbf{b}} t^n = \sum_{n=0}^{\infty} b^n t^n = \frac{1}{1-bt},$$

we obtain the desired identity. □

In the rest of this section we give combinatorial interpretations of the numbers $k!\left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}}$ and $\frac{k!}{\delta^k}\left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}}$ in terms of distributing balls into cells under certain constraints. This interpretation is an extension of the interpretation presented in [7]. Based on this interpretation, we also derive several combinatorial identities for $\left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}}$. Throughout this section, we let $\mathbf{a} = (a_0, a_1, ...)$ and $\mathbf{b} = (b_0, b_1, ...)$ denote two sequences of nonnegative integers.

Definition 2.3. *The $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D}, n, k)$-set distribution refers to the distribution of n distinct balls labelled $1, \ldots, n$ into the $k + 1$ distinct cells. Each of the first k cells contains $\sum_{i=0}^{n-1} a_i$ distinct compartments, and the last cell contains $\sum_{i=0}^{n-1} b_i$ distinct compartments. Each compartment in the $k + 1$ cells can hold at most one ball. The balls are placed one at a time, in the order of their labels, such that the following conditions are satisfied*
($\boldsymbol{A}$): For each i with $1 \le i \le k$, the first a_0 compartments of cell i are each labelled with 1, the next a_1 compartments are each labelled with 2, and so on, until the last a_{n-1} compartments are each labelled with n. Similarly, the first b_0 compartments of $(k + 1)$-th cell are each labelled with 1, the next b_1 compartments are each labelled with 2, and so on, until the last b_{n-1} compartments are each labelled with n.
($\boldsymbol{B}$): For each j with $1 \le j \le n$, each compartment in the $k + 1$ cells labelled with j can hold only the ball also labelled with j.
($\boldsymbol{C}$): The first k cells are non-empty.
($\boldsymbol{D}$): Each of the first k cells will be colored by one of the δ colors.

Similarly, the $(\mathbf{A},\mathbf{B}, n, k)$-set distribution and the $(\mathbf{A},\mathbf{B},\mathbf{C}, n, k)$-set distribution are defined by requiring only the conditions **A** and **B**, and conditions **A**,**B** and **C**, respectively, to be satisfied. The structure of the $(\mathbf{A},\mathbf{B}, n, k)$-set distribution is illustrated in Figure 1. The colors in the figure are used solely for illustrative purposes.

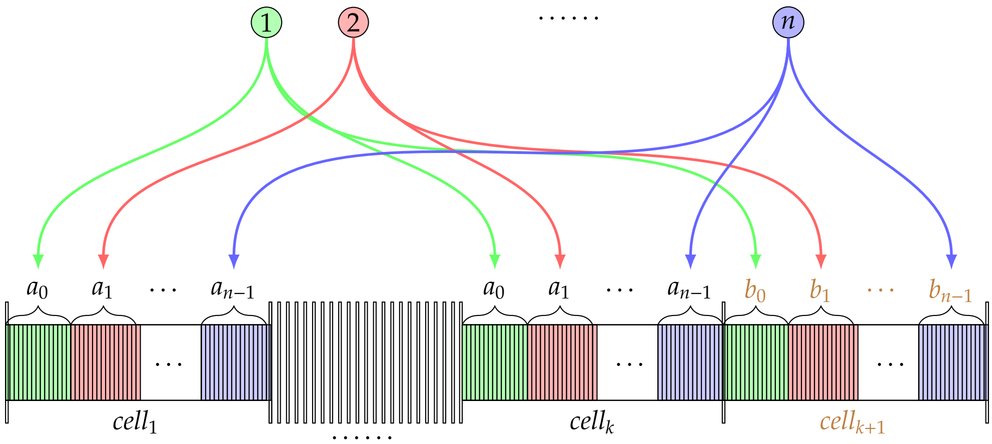

Figure 1: The $(\mathbf{A}, \mathbf{B}, n, k)$-set distribution.

Proposition 2.4. *Let n, k and δ be nonnegative integers. Then, the number $k!\left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}}$ denotes the cardinality of the $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D}, n, k)$-set distribution.*

Proof. Let Ω be the set of all ways to distribute n distinct balls labelled into the $k+1$ cells, one ball at a time, in the order of their labels, such that conditions **A** and **B** are satisfied, we have

$$|\Omega| = (a_0k + b_0)(a_1k + b_1)\cdots(a_{n-1}k + b_{n-1}) = g_n(k).$$

Let S_i, (for $1 \le i \le k$) denote the subset of distributions in Ω where the i-th cell is empty. We want to calculate the number of distributions in Ω where the first k cells are non-empty, i.e., $\left|\bigcap_{i=1}^{k} \overline{S_i}\right|$. By the principle of inclusion-exclusion, we have

$$\left|\bigcap_{i=1}^{k} \overline{S_i}\right| = |\Omega| - \sum_{j=1}^{k} (-1)^{j-1} \sum_{\substack{T\subset\{1,\ldots,k\}\\ |T|=j}} \left|\bigcap_{i\in T} S_i\right|.$$

The term $\sum_{\substack{T\subset\{1,\ldots,k\}\\ |T|=j}} \left|\bigcap_{i\in T} S_i\right|$ counts the number of distribution in Ω where j cells are empty, then

$$\begin{aligned}\left|\bigcap_{i=1}^{k} \overline{S_i}\right| &= |\Omega| - \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} g_n(k-j) \\ &= \sum_{j=0}^{k} (-1)^{j} \binom{k}{j} g_n(k-j) \\ &= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} g_n(j).\end{aligned}$$

Moreover, the number of ways to color the first k cells using δ colors is δ^k. Then, by (2) we obtain the desired count. □

Now, in the same way as the previous Proposition, we obtain the following proposition.

Proposition 2.5. *Let n, k be nonnegative integers. Then, the number $\frac{k!}{\delta^k}\left\{ n \atop k \right\}_{\delta,\mathbf{a},\mathbf{b}}$ denotes the cardinality of the $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}, n, k)$-set distribution.*

Note that, according to Proposition 2.5, we have

$$\left\{ n \atop n \right\}_{\delta,\mathbf{a},\mathbf{b}} = \delta^n \prod_{i=0}^{n-1} a_i.$$

Remark 2.6 (Combinatorial proof of Proposition 2.1). *Based on the above definitions, Proposition 2.1 associated with $\left\{ n \atop k \right\}_{\delta,\mathbf{a},\mathbf{b}}$ can also be proven as follows: The cardinality of $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}, n, k)$-set distribution is given by $\frac{k!}{\delta^k}\left\{ n \atop k \right\}_{\delta,\mathbf{a},\mathbf{b}}$. This number can be obtained in another way by considering the ball that labelled n is alone in a cell or not. Indeed, if the ball n is alone in a cell then the number of ways to obtain this is $ka_{n-1}\left(\frac{(k-1)!}{\delta^{k-1}}\left\{ n-1 \atop k-1 \right\}_{\delta,\mathbf{a},\mathbf{b}}\right)$. Otherwise, the cardinality of $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}, n-1, k)$-set distribution is $\frac{k!}{\delta^k}\left\{ n-1 \atop k \right\}_{\delta,\mathbf{a},\mathbf{b}}$, and the ball that labelled n can be distributed into the $k+1$ cells in $a_{n-1}k + b_{n-1}$ ways. Therefore, the identity holds.*

Remark 2.7. *We note that Identity (3) can also be proven combinatorially as follows:*
Let Ω be the set of all ways to distribute n distinct balls labelled into the $k+1$ cells, one ball at a time, in the order of their labels, such that conditions $\boldsymbol{A}$ and $\boldsymbol{B}$ are satisfied, we have

$$|\Omega| = (a_0k + b_0)(a_1k + b_1)\cdots(a_{n-1}k + b_{n-1}) = g_n(k).$$

This number can also be obtained by choosing j cells of the first k cells to be nonempty and then distributing the n balls among those j cells according to an $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}, n, j)$-set distribution, leaving the remaining $k-j$ cells empty. This can be done in $\binom{k}{j}\left(\frac{j!}{\delta^j}\left\{ n \atop j \right\}_{\delta,\mathbf{a},\mathbf{b}}\right)$ summing over all valid j where $0 \le j \le k$, we obtain the identity holds.

Let $i_1, \ldots, i_l$ be distinct positive integers. For a sequence $\mathbf{x} = (x_0, x_1, \ldots)$ we define the deletion operator

$$\mathbf{x} - \{i_1, \ldots, i_l\}$$

as the sequence obtained from $\mathbf{x}$ by removing the entries with indices $i_1 - 1, \ldots, i_l - 1$. In particular,

$$\mathbf{x} - \{1\} = (x_1, x_2, \ldots)$$

and,

$$\mathbf{x} - \{i_1, i_2\} = (x_0, \ldots, x_{i_1-2}, x_{i_1}, \ldots, x_{i_2-2}, x_{i_2}, \ldots),$$

More generally, we write

$$\mathbf{x} - \{i_1, \ldots, i_l\},$$

for the sequence obtained by deleting the terms $x_{i_1-1}, \ldots, x_{i_l-1}$ from $\mathbf{x}$. For simplicity, we also write

$$\mathbf{x} - i_{1,l} := \mathbf{x} - \{i_1, \ldots, i_l\}.$$

Now, similarly to the identity given in [5, Theorem 4], we may state the following proposition.

Proposition 2.8. *For integer $n \geq k \geq 1$. There holds*

$$\frac{k^2}{\delta}\left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}} = \sum_{j=k-1}^{n-1}\left(\sum_{1\le i_1<\cdots<i_{n-j}\le n}\left(\prod_{l=1}^{n-j} a_{i_l-1}\right)\left\{ {j \atop k-1} \right\}_{\delta,\mathbf{a}-i_{1,n-j},\mathbf{b}-i_{1,n-j}}\right).$$

Proof. The cardinality of the set of $(\mathbf{A},\mathbf{B},\mathbf{C}, n, k)$-set distribution, in which one of the first k cells is marked, is given by $k\left(\frac{k!}{\delta^k}\left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}}\right)$. Alternatively, fix j with $k-1 \leq j \leq n-1$ and choose the $n-j$ balls labelled $i_1, \ldots, i_{n-j}$ to be placed in the marked cell. For this choice, there are $\prod_{l=1}^{n-j} a_{i_l-1}$ ways to place those balls in the marked cell. The remaining j balls are then distributed among the $k-1$ unmarked cells, which can be done in $\frac{(k-1)!}{\delta^{k-1}}\left\{ {j \atop k-1} \right\}_{\delta,\mathbf{a}-i_{1,n-j},\mathbf{b}-i_{1,n-j}}$ ways. Summing over all valid j and all choices $1 \leq i_1 < \cdots < i_{n-j} \leq n$. gives

$$\sum_{j=k-1}^{n-1}\sum_{1\le i_1<\cdots<i_{n-j}\le n}\left(\prod_{l=1}^{n-j} a_{i_l-1}\right)\frac{(k-1)!}{\delta^{k-1}}\left\{ {n \atop k-1} \right\}_{\delta,\mathbf{a}-i_{1,n-j},\mathbf{b}-i_{1,n-j}}.$$

Therefore, the identity holds. □

The following convolution formula is similar to the identity given in [8].

Proposition 2.9. *For integer $n \geq k_1 + k_2 \geq 1$. There holds*

$$\frac{((k_1+k_2)!)^2}{(k_1!)^2(k_2!)^2}\left\{ {n \atop k_1+k_2} \right\}_{\delta,\mathbf{a},2\mathbf{b}} = \sum_{m=0}^{n}\sum_{1\le i_1<\cdots<i_m\le n}\left\{ {m \atop k_1} \right\}_{\delta,\mathbf{a}-l_{1,n-m},\mathbf{b}-l_{1,n-m}}\left\{ {n-m \atop k_2} \right\}_{\delta,\mathbf{a}-i_{1,m},\mathbf{b}-i_{1,m}}.$$

Where $\{l_1, \ldots, l_{n-m}\} = \{1, \ldots, n\} - \{i_1, \ldots, i_m\}$.

Proof. The cardinality of $(\mathbf{A},\mathbf{B},\mathbf{C},n,k_1+k_2)$-set distribution, in which k_1 of the first k_1+k_2 cells are marked, is given by $\binom{k_1+k_2}{k_1}\left(\frac{(k_1+k_2)!}{\delta^{k_1+k_2}}\left\{ {n \atop k_1+k_2} \right\}_{\delta,\mathbf{a},2\mathbf{b}}\right)$. Alternatively, fix m with $0 \le m \le n$ and choose the m balls labelled $i_1,\ldots,i_m$ to be placed in the k_1 marked cells and the (k_1+k_2+1)-th cell. For each such ball i_r the last cell has exactly b_{i_r-1} compartments reserved. The number of ways to distribute these m balls is therefore

$$\frac{k_1!}{\delta^{k_1}}\left\{ {m \atop k_1} \right\}_{\delta,\mathbf{a}-l_{1,n-m},\mathbf{b}-l_{1,n-m}},$$

where $\{l_1,\ldots,l_{n-m}\}=\{1,\ldots,n\}-\{i_1,\ldots,i_m\}$. The remaining $n-m$ balls, labelled $l_1,\ldots,l_{n-m}$ must be distributed among the k_2 unmarked cells together with the last cell. Similarly, the number of such distributions is

$$\frac{k_2!}{\delta^{k_2}}\left\{ {n-m \atop k_2} \right\}_{\delta,\mathbf{a}-i_{1,m},\mathbf{b}-i_{1,m}}.$$

Summing over all valid m and all choices $1\le i_1<\cdots<i_m\le n$. gives

$$\sum_{m=0}^{n}\sum_{1\le i_1<\cdots<i_m\le n}\frac{k_1!}{\delta^{k_1}}\left\{ {m \atop k_1} \right\}_{\delta,\mathbf{a}-l_{1,n-m},\mathbf{b}-l_{1,n-m}}\frac{k_2!}{\delta^{k_2}}\left\{ {n-m \atop k_2} \right\}_{\delta,\mathbf{a}-i_{1,m},\mathbf{b}-i_{1,m}}.$$

Therefore, the identity holds. □

In the literature, several combinatorial interpretations have been given for the Hsu–Shiue generalized Stirling numbers $S(n,k;\alpha,\beta,\gamma)$, see, for example [2, 7, 13, 17]. From Proposition 2.5 we can derive a new combinatorial interpretation for $\beta^k k! S(n,k;\alpha,\beta,\gamma)$, stated in the following Corollary.

Corollary 2.10. *Let n,k,α,β,γ be nonnegative integers such that $\gamma>(n-1)\alpha$. Then, the number $\beta^k k! S(n,k;\alpha,\beta,\gamma)$ count the number distribution of n distinct balls labelled $1,\ldots,n$ into the $k+1$ distinct cells. Each of the first k cells contains $n\beta$ distinct compartments, and the last cell contains $\sum_{i=0}^{n-1}(\gamma-i\alpha)$ distinct compartments. Each compartment in the $k+1$ cells can hold at most one ball. The balls are placed one at a time, in the order of their labels, such that the following conditions are satisfied*
***(1)**: For each i with $1\le i\le k$, the first β compartments of cell i are each labelled with 1, the next β compartments are each labelled with 2, and so on, until the last β compartments are each labelled with n. Similarly, the first γ compartments of $(k+1)$-th cell are each labelled with 1, the next $(\gamma-\alpha)$ compartments are each labelled with 2, and so on, until the last $(\gamma-(n-1)\alpha)$ compartments are each labelled with n.*
***(2)**: For each j with $1\le j\le n$, each compartment in the $k+1$ cells labelled with j can hold only the ball also labelled with j.*
***(3)**: The first k cells are non-empty.*

3. Generalized Bell polynomials

In this section, we introduce the Generalized Bell polynomials $B_n(x;\delta,\mathbf{a},\mathbf{b})$ associated with $\left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}}$ and establish several identities for them. Specifically, we define

$$B_n(x;\delta,\mathbf{a},\mathbf{b}) := \sum_{k=0}^{n}\left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}} x^k.$$

The first few polynomials are given by

$$B_0(x;\delta,\mathbf{a},\mathbf{b}) = 1,$$
$$B_1(x;\delta,\mathbf{a},\mathbf{b}) = b_0+\delta a_0 x,$$
$$B_2(x;\delta,\mathbf{a},\mathbf{b}) = b_0b_1+(a_1b_0+a_0b_1+a_1a_0)\,\delta x+\delta^2 a_0a_1x^2.$$

These polynomials satisfy the following Dobinski-type formula

Proposition 3.1. *There holds*

$$B_n(x;\delta,\mathbf{a},\mathbf{b}) = \exp(-\delta x)\sum_{j\geq 0} g_n(j)\frac{(\delta x)^j}{j!}. \tag{4}$$

Proof. From (3) we have

$$g_n(j)\frac{1}{j!} = \sum_{k=0}^{j}\delta^{-k}\left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}}\frac{1}{(j-k)!}.$$

Now, we multiply both sides by $(\delta x)^j$ and sum from $j=0$ to ∞, we obtain

$$\begin{aligned}
\sum_{j=0}^{\infty} g_n(j)\frac{(\delta x)^j}{j!} &= \sum_{j=0}^{\infty}\sum_{k=0}^{j}\delta^{-k}\left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}}\frac{(\delta x)^j}{(j-k)!},\\
&= \sum_{k=0}^{n}\delta^{-k}\left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}}\sum_{j=0}^{\infty}\frac{(\delta x)^j}{(j-k)!},\\
&= \sum_{k=0}^{n}\delta^{-k}\left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}}\sum_{j=0}^{\infty}\frac{(\delta x)^{j+k}}{j!},\\
&= e^{\delta x}\left(\sum_{k=0}^{n}\left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}} x^k\right),\\
&= e^{\delta x}B_n(x;\delta,\mathbf{a},\mathbf{b}).
\end{aligned}$$

□

The polynomials $B_n(x;\delta,\mathbf{a},\mathbf{b})$ can be explicitly obtained from the following result.

Proposition 3.2. *We have*

$$B_{n+1}(x;\delta,\mathbf{a},\mathbf{b}) = (\delta a_n x + b_n)B_n(x;\delta,\mathbf{a},\mathbf{b}) + a_n x\frac{d}{dx}B_n(x;\delta,\mathbf{a},\mathbf{b}),$$

or equivalently

$$x^{\frac{b_n}{a_n}}\exp(\delta x)B_{n+1}(x;\delta,\mathbf{a},\mathbf{b}) = a_n x\frac{d}{dx}\left(x^{\frac{b_n}{a_n}}\exp(\delta x)B_n(x;\delta,\mathbf{a},\mathbf{b})\right).$$

Proof. From (4) we have

$$\begin{aligned}
&a_n x\frac{d}{dx}B_n(x;\delta,\mathbf{a},\mathbf{b})\\
&= -\delta a_n x B_n(x;\delta,\mathbf{a},\mathbf{b}) + a_n\exp(-\delta x)\sum_{j\geq 0} j g_n(j)\frac{(\delta x)^j}{j!}\\
&= -\delta a_n x B_n(x;\delta,\mathbf{a},\mathbf{b}) + \exp(-\delta x)\sum_{j\geq 0}(a_n j + b_n - b_n)g_n(j)\frac{(\delta x)^j}{j!}\\
&= -\delta a_n x B_n(x;\delta,\mathbf{a},\mathbf{b}) + \exp(-\delta x)\sum_{j\geq 0} g_{n+1}(j)\frac{(\delta x)^j}{j!} - b_n\exp(-\delta x)\sum_{j\geq 0} g_n(j)\frac{(\delta x)^j}{j!}
\end{aligned}$$

$$= B_{n+1}(x;\delta,\mathbf{a},\mathbf{b}) - (\delta a_n x + b_n) B_n(x;\delta,\mathbf{a},\mathbf{b}).$$

This identity can be written as

$$x^{\frac{b_n}{a_n}} \exp(\delta x) B_{n+1}(x;\delta,\mathbf{a},\mathbf{b}) = a_n x \frac{d}{dx}\left(x^{\frac{b_n}{a_n}} \exp(\delta x) B_n(x;\delta,\mathbf{a},\mathbf{b})\right).$$

□

We now apply the previous proposition and Rolle's theorem to establish the following proposition

Proposition 3.3. *If $\frac{b_k}{a_k} > 0$ for all $k = 1, \ldots, n-1$, then the polynomial $B_n(x;\delta,\mathbf{a},\mathbf{b})$ has only real roots.*

Proof. We proceed by induction on n. Indeed, for $n = 1$, we have

$$B_1(x;\delta,\mathbf{a},\mathbf{b}) = b_0 + \delta a_0 x,$$

which clearly has a real root. Suppose now that $B_{n-1}(x;\delta,\mathbf{a},\mathbf{b})$ has only real roots. If $\frac{b_{n-1}}{a_{n-1}} \geq 1$, then the function $x^{\frac{b_{n-1}}{a_{n-1}}} \exp(\delta x) B_{n-1}(x;\delta,\mathbf{a},\mathbf{b})$ vanishes n times. By Rolle's theorem, the function $x\frac{d}{dx}\left(x^{\frac{b_{n-1}}{a_{n-1}}} \exp(\delta x) B_{n-1}(x;\delta,\mathbf{a},\mathbf{b})\right)$ vanishes n times. Consequently, the polynomial $B_n(x;\delta,\mathbf{a},\mathbf{b})$ vanishes $n-1$ times, and since it is of degree n, the missing zero must be necessarily real. If $0 < \frac{b_{n-1}}{a_{n-1}} < 1$, the function $x^{\frac{b_{n-1}}{a_{n-1}}} \exp(\delta x) B_{n-1}(x;\delta,\mathbf{a},\mathbf{b})$ vanishes $n-1$ times. By Rolle's theorem, the function $x\frac{d}{dx}\left(x^{\frac{b_{n-1}}{a_{n-1}}} \exp(\delta x) B_{n-1}(x;\delta,\mathbf{a},\mathbf{b})\right)$ vanishes $n-1$ times, consequently, the polynomial $B_n(x;\delta,\mathbf{a},\mathbf{b})$ vanishes $n-1$ times, and since it is of degree n, the remaining zero must also be real. □

As consequence of the last Proposition, we conclude that the polynomials

$$\begin{aligned} &B_n(x) = \sum_{k=0}^{n} {n \brace k} x^k, && S_n(x;\alpha,\beta,\gamma) = \sum_{k=0}^{n} S(n,k;\alpha,\beta,\gamma) x^k, \\ &B_{n,r_1,\ldots,r_p}(x) = \sum_{k=0}^{n} {n \brace k+r_p}_{r_1,\ldots,r_p} x^k, && B_{n,r}(x) = \sum_{k=0}^{n} {n+r \brace k+r}_r x^k, \\ &B_n^{(B)}(x) = \sum_{k=0}^{n} S_B(n,k) x^k, && D_{n,r}(x) = \sum_{k=0}^{n} W_{m,r}(n,k) x^k, \end{aligned}$$

have only real roots, $n \geq 1$.
Now, following the same reasoning as in [6, Theorem 2.2], we may formulate the next result.

Proposition 3.4. *Let $\delta > 0$ and suppose that for all $i \geq 0$ we have $a_{i+1} \geq a_i \geq 0$, $b_{i+1} \geq b_i \geq 0$. Then the polynomial $B_n(x;\delta,\mathbf{a},\mathbf{b})$ satisfies the inequalities*

$$B_{n+1}(x;\delta,\mathbf{a},\mathbf{b}) \leq \frac{1}{2}\left(B_n(x;\delta,\mathbf{a},\mathbf{b}) + B_{n+2}(x;\delta,\mathbf{a},\mathbf{b})\right), \quad x \geq 0,$$

and

$$\left(B_{n+1}(x;\delta,\mathbf{a},\mathbf{b})\right)^2 \leq B_n(x;\delta,\mathbf{a},\mathbf{b}) B_{n+2}(x;\delta,\mathbf{a},\mathbf{b}), \quad x \geq 0.$$

Proof. Since $a_{i+1} \geq a_i \geq 0$ and $b_{i+1} \geq b_i \geq 0$ for all $i \geq 0$, we observe that

$$\begin{aligned} &B_n(x;\delta,\mathbf{a},\mathbf{b}) + B_{n+2}(x;\delta,\mathbf{a},\mathbf{b}) \\ &= \exp(-\delta x) \sum_{j\geq 0} g_n(j) \frac{(\delta x)^j}{j!} + \exp(-\delta x) \sum_{j\geq 0} g_{n+2}(j) \frac{(\delta x)^j}{j!}, \end{aligned}$$

$$= \exp(-\delta x) \sum_{j\geq 0} g_{n+1}(j) \left(\frac{1}{a_n j + b_n} + a_{n+1} j + b_{n+1} \right) \frac{(\delta x)^j}{j!},$$
$$\geq \exp(-\delta x) \sum_{j\geq 0} g_{n+1}(j) \left(\frac{1}{a_n j + b_n} + a_n j + b_n \right) \frac{(\delta x)^j}{j!},$$
$$\geq 2\exp(-\delta x) \sum_{j\geq 0} g_{n+1}(j) \frac{(\delta x)^j}{j!}.$$

Hence, the first inequality follows. For the second inequality, we have

$$B_n(x;\delta,\mathbf{a},\mathbf{b}) B_{n+2}(x;\delta,\mathbf{a},\mathbf{b})$$
$$= \exp(-2\delta x) \sum_{i,j\geq 0} g_n(i) g_{n+2}(j) \frac{(\delta x)^{i+j}}{i!j!},$$
$$= \exp(-2\delta x) \sum_{i,j\geq 0} g_{n+1}(i) g_{n+1}(j) \left(\frac{a_{n+1} j + b_{n+1}}{a_n i + b_n} \right) \frac{(\delta x)^{i+j}}{i!j!},$$
$$\geq \exp(-2\delta x) \sum_{i,j\geq 0} g_{n+1}(i) g_{n+1}(j) \left(\frac{a_n j + b_n}{a_n i + b_n} \right) \frac{(\delta x)^{i+j}}{i!j!},$$

and by symmetry, we get

$$B_n(x;\delta,\mathbf{a},\mathbf{b}) B_{n+2}(x;\delta,\mathbf{a},\mathbf{b})$$
$$\geq \frac{1}{2}\exp(-2\delta x) \sum_{i,j\geq 0} g_{n+1}(i) g_{n+1}(j) \left(\frac{a_n i + b_n}{a_n j + b_n} + \frac{a_n j + b_n}{a_n i + b_n} \right) \frac{(\delta x)^{i+j}}{i!j!},$$
$$\geq \exp(-2\delta x) \sum_{i,j\geq 0} g_{n+1}(i) g_{n+1}(j) \frac{(\delta x)^{i+j}}{i!j!},$$
$$= (B_{n+1}(x;\delta,\mathbf{a},\mathbf{b}))^2.$$

□

Note that the first identity of the Proposition 3.2 can be written as

$$B_{n+1}(x;\delta,\mathbf{a},\mathbf{b}) = (\delta a_n x + b_n + a_n x\mathbf{D}) B_n(x;\delta,\mathbf{a},\mathbf{b}),$$

where $\mathbf{D} := \frac{d}{dx}$, denotes the differential operator. By iterating this relation, we obtain the following result

Corollary 3.5. *Let n, m be positive integers, there hold*

$$B_{n+m}(x;\delta,\mathbf{a},\mathbf{b}) = (\delta a_{n+m-1} x + b_{n+m-1} + a_{n+m-1} x\mathbf{D}) \cdots (\delta a_m x + b_m + a_m x\mathbf{D}) B_m(x;\delta,\mathbf{a},\mathbf{b}).$$

In particular, for $m = 0$ we obtain

$$B_n(x;\delta,\mathbf{a},\mathbf{b}) = (\delta a_{n-1} x + b_{n-1} + a_{n-1} x\mathbf{D}) \cdots (\delta a_0 x + b_0 + a_0 x\mathbf{D}) 1.$$

Proposition 3.6. *For any n-times differentiable function f, we have*

$$\left(a_{n-1} x^{1-\frac{b_{n-1}}{a_{n-1}}+\frac{b_n}{a_n}} \mathbf{D} \right) \cdots \left(a_0 x^{1-\frac{b_0}{a_0}+\frac{b_1}{a_1}} \mathbf{D} \right) \left(x^{\frac{b_0}{a_0}} f(x) \right) = x^{\frac{b_n}{a_n}} \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_{\delta,\mathbf{a},\mathbf{b}} \delta^{-k} x^k \mathbf{D}^k f(x), \text{ for } n \geq 0.$$

Proof. We proceed by induction on n. The case $n = 0$ is easy to verify.
Assume now that the statement holds for $n-1$. Then we have

$$\begin{aligned}
&\left(a_{n-1}x^{1-\frac{b_{n-1}}{a_{n-1}}+\frac{b_n}{a_n}}\mathbf{D}\right)\cdots\left(a_0x^{1-\frac{b_0}{a_0}+\frac{b_1}{a_1}}\mathbf{D}\right)\left(x^{\frac{b_0}{a_0}}f(x)\right)\\
&=\left(a_{n-1}x^{1-\frac{b_{n-1}}{a_{n-1}}+\frac{b_n}{a_n}}\mathbf{D}\right)\left(\sum_{k=0}^{n-1}\left\{{n-1 \atop k}\right\}_{\delta,\mathbf{a},\mathbf{b}}\delta^{-k}x^{k+\frac{b_{n-1}}{a_{n-1}}}\mathbf{D}^kf(x)\right)\\
&=x^{\frac{b_n}{a_n}}\left(\sum_{k=0}^{n-1}(a_{n-1}k+b_{n-1})\left\{{n-1 \atop k}\right\}_{\delta,\mathbf{a},\mathbf{b}}\delta^{-k}x^k\mathbf{D}^kf(x)\right)+x^{\frac{b_n}{a_n}}\left(\sum_{k=0}^{n-1}(a_{n-1}\delta)\left\{{n-1 \atop k}\right\}_{\delta,\mathbf{a},\mathbf{b}}\delta^{-k-1}x^{k+1}\mathbf{D}^{k+1}f(x)\right)\\
&=x^{\frac{b_n}{a_n}}\left(\sum_{k=0}^{n-1}(a_{n-1}k+b_{n-1})\left\{{n-1 \atop k}\right\}_{\delta,\mathbf{a},\mathbf{b}}\delta^{-k}x^k\mathbf{D}^kf(x)\right)+x^{\frac{b_n}{a_n}}\left(\sum_{k=1}^{n}(a_{n-1}\delta)\left\{{n-1 \atop k-1}\right\}_{\delta,\mathbf{a},\mathbf{b}}\delta^{-k}x^k\mathbf{D}^kf(x)\right).
\end{aligned}$$

By applying Proposition 2.1 we obtain the desired identity. □

When $\delta = 1$, $\mathbf{a} = (1, 1, \ldots)$ and $\mathbf{b} = (0, 0, \ldots)$, we recover the well-known identity

$$(x\mathbf{D})^n f(x) = \sum_{k=0}^{n}\left\{{n \atop k}\right\}x^k\mathbf{D}^kf(x).$$

By setting $f(x) = e^{\delta x}$ in the previous proposition, we obtain the following Corollary, which can be regarded as a generalization of the Mellin derivative operator ($x\mathbf{D}$).

Corollary 3.7. *There holds*

$$B_n(x;\delta,\mathbf{a},\mathbf{b}) = e^{-\delta x}x^{-\frac{b_n}{a_n}}\left(a_{n-1}x^{1-\frac{b_{n-1}}{a_{n-1}}+\frac{b_n}{a_n}}\mathbf{D}\right)\cdots\left(a_0x^{1-\frac{b_0}{a_0}+\frac{b_1}{a_1}}\mathbf{D}\right)\left(x^{\frac{b_0}{a_0}}e^{\delta x}\right),\ \textit{for } n \geq 0.$$

Similarly, by choosing $f(x) = (x+r)^m$ in the previous proposition, we obtain the following Corollary

Corollary 3.8. *Let n, m, r be integers with $0 \leq n \leq m$. Then we have*

$$\sum_{j=0}^{m}\binom{m}{j}r^{m-j}g_n(j)\frac{x^j}{(x+r)^m} = \sum_{k=0}^{n}\left\{{n \atop k}\right\}_{\delta,\mathbf{a},\mathbf{b}}\delta^{-k}(m)_k\frac{x^k}{(x+r)^k}.$$

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