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On the order statistics from the XLindley distribution and associated inference with an application to fatigue data

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Abstract. In this paper, we consider the order statistics from a newly-introduced lifetime distribution, called the XLindley distribution. We have derived explicit closed form expressions for the single moments and product moments of order statistics from the XLindley distribution. Utilizing these expressions, we calculated the means, variances and covariances of order statistics for sample sizes ranging from $n = 1$ to $n = 10$ and arbitrarily selected parameter values. Additionally, these moments allow us to identify the best linear unbiased estimators (BLUEs) and best linear invariant estimators (BLIEs) for the location and scale parameters, based on both complete samples and Type-II right censored samples. We also address the linear prediction of unobserved order statistics based on Type-II right censored samples. We also explore the formulation of confidence intervals for both location and scale parameters, along with prediction intervals for unobserved order statistics. To provide comparison and illustration, we conduct a simulation study and analyze a real data example. Finally, we conclude with several remarks.

1. Introduction

The concept of order statistics arises from the arrangement of sample observations in ascending order. Order statistics have applications in reliability theory, engineering, quality control, environmental and medical sciences, economics, finance and auction theory. Order statistics have attracted the interest of many researchers, including Arnold et al. [7], Balakrishnan and Rao [10] and David and Nagaraja [17].

Lifetime data analysis is a crucial field within statistics, focusing on modeling and studying the time until failure or survival in various phenomena. The real-world applications of recent computational techniques in various fields like medicine, finance, biological engineering and statistics. Statistical analysis depends significantly on the fundamental probability model or distributions. With its memoryless property and

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constant failure rate function, the exponential distribution is one of the most popular distributions for lifetime data. Sometimes, the exponential distribution might not suffice for representing certain data types showcasing various forms of failure rate functions, including increasing, decreasing, unimodal, or bathtub-shaped distributions. Therefore, several alternative distributions are proposed for the limitation of the exponential distribution which are more flexible and have a better fit for lifetime data. The Lindley distribution, introduced by Lindley [24], can be derived by the mixture of exponential(ψ) and Gamma($2,\psi$) distributions with probability density functions (pdfs) $\mathfrak{f}_1(x) = \psi e^{-\psi x}$ and $\mathfrak{f}_2(x) = \psi^2 x e^{-\psi x}$, respectively, for $x > 0$ and $\psi > 0$. A generalization of the Lindley distribution with an extra shape parameter has been developed by Shanker and Mishra [36], who claimed that this distribution surpasses both the exponential and Lindley distributions, concerning failure rate and mean residual life functions. Shanker et al. [37] added a parameter in the Lindley distribution to introduce the two-parameter Lindley distribution which is more capable of fitting data than the Lindley distribution. On the other hand, the XLindley distribution is also a mixture of exponential(ψ) and Lindley(ψ) distributions, originally proposed by Chouia and Zeghdoudi [15], being straightforward and easily applicable, offers simple formulas for computing key statistical measures such as mean, variance, coefficient of variation, skewness, kurtosis, and index of dispersion. Chouia and Zeghdoudi [15] claimed that the XLindley distribution, which is a simpler distribution than more complicated ones, can be used in analyzing many real lifetime data sets such as Corona, Ebola and Nipah virus data.

The pdf of the XLindley distribution with one parameter ψ is given by

$$\mathfrak{f}(x) = \frac{\psi^2(2+\psi+x)}{(1+\psi)^2}\mathrm{e}^{-\psi x}, \quad x > 0,\ \psi > 0, \tag{1.1}$$

and therefore its cumulative distribution function (cdf) is

$$F(x) = 1 - \left(1 + \frac{\psi x}{(1+\psi)^2}\right)\mathrm{e}^{-\psi x}, \quad x > 0,\ \psi > 0. \tag{1.2}$$

Now, consider the three parameter XLindley distribution (location and scale parameters added), so the pdf and cdf of the three parameter XLindley distribution are

$$\mathfrak{f}(x;\varphi,\sigma,\psi) = \frac{\psi^2(2+\psi+(\frac{x-\varphi}{\sigma}))}{\sigma(1+\psi)^2}\exp\left(-\frac{\psi(x-\varphi)}{\sigma}\right), \quad x > \varphi, \tag{1.3}$$

and

$$F(x;\varphi,\sigma,\psi) = 1 - \left(1 + \frac{\psi(x-\varphi)}{\sigma(1+\psi)^2}\right)\exp\left(-\frac{\psi(x-\varphi)}{\sigma}\right), \quad x > \varphi, \tag{1.4}$$

respectively, where $\psi > 0$, $\sigma > 0$ and $\varphi \in \Re$ the shape, scale and location parameters, respectively. We write $X \sim XL(\varphi,\sigma,\psi)$ if the pdf of X can be written as (1.3).

Chouia and Zeghdoudi [15] studied various statistical properties and aspects of the XLindley distribution such as moments, skewness, kurtosis, quantile function, hazard rate function, maximum likelihood estimation, method of moments, and so on. This distribution has been the subject of numerous studies by other authors, who have explored its properties, characterization, statistical inference and applications. Some notable examples include Alotaibi et al. [6], Metiri et al. [29], Nassar et al. [33], and Zanjiran and MirMostafaee [39]. The various applications of moments of order statistics are extensively documented in the statistical literature. These moments play a crucial role in statistical modeling, inference, decision-making methods, reliability analysis, and so on. Several notable studies include Ahsanullah and Alzaatreh [1], Akhter et al. [3–5], Balakrishnan and Chan [8], Guan et al. [21], Mahmoud et al. [26], Raqab and Ahsanullah [35] and Sultan and AL-Thubyani [38], among others.

Nadarajah [31] provided explicit closed form expressions for the moments of order statistics from the normal, log-normal, gamma and beta distributions. For further insights in this area, one may refer to Akhter et al. [2], Birbiçer and Genç [11], Çetinkaya and Genç [14], Genç [18], MirMostafaee [30], Nagaraja [32] and Zghoul [40], and references therein.

The structure of this paper is laid out as follows: Section 2 outlines the fundamental concepts needed for the upcoming discussions. In Section 3, we present closed-form expressions for the moments of order statistics. Section 4 builds upon this by offering closed-form expressions for the product moments. Utilizing the results of the previous sections, Section 5 focuses on finding the means, variances and covariances of order statistics extracted from sample sizes of up to 10 when the shape parameter takes selected values. Section 6 delves into the derivation of the best linear unbiased estimators (BLUEs) and best linear invariant estimators (BLIEs) for the location and scale parameters of the XLindley distribution, based on both complete and Type-II censored data. Moving forward, Section 7 discusses two linear predictors, whereas Sections 8 and 9 provide a simulation study and an example, respectively, for comparative analysis and demonstration. The paper concludes with a summary of key insights in the final section.

2. Preliminaries

Suppose $X_1, \cdots, X_n$ denote a random sample of size n from the XLindley distribution with pdf $\mathfrak{f}(x)$ and cdf $F(x)$, given in (1.1) and (1.2), respectively, and let $X_{1:n} \leq \cdots \leq X_{n:n}$ denote the order statistics extracted from the mentioned sample. Then, the pdf of the rth order statistic $X_{r:n}$, say $\mathfrak{f}_{r:n}(x)$, for $1 \leq r \leq n$, is (Arnold et al. [7]; David and Nagaraja [17])

$$\mathfrak{f}_{r:n}(x) = C_{r:n}(F(x))^{r-1}\{1-F(x)\}^{n-r}\mathfrak{f}(x), \quad 0 < x < \infty$$

The joint pdf of $X_{r:n}$ and $X_{s:n}$ is given by

$$\mathfrak{f}_{r,s:n}(x,y) = C_{r,s:n}(F(x))^{r-1}\{F(y)-F(x)\}^{s-r-1}\{1-F(y)\}^{n-s}\mathfrak{f}(x)\mathfrak{f}(y), \tag{2.1}$$

for $0 < x < y < \infty$ and $1 \leq r < s \leq n$, where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!} \quad \text{and} \quad C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}. \tag{2.2}$$

Next, the kth single moment of $X_{r:n}$ takes the form

$$\varphi_{r:n}^{(k)} = \mathrm{E}(X_{r:n}^k) = \int_0^\infty x^k \mathfrak{f}_{r:n}(x)\mathrm{d}x, \quad 1 \leq r \leq n, \ k \in \mathbb{N}, \tag{2.3}$$

and the (k,l)th product moment of $X_{r:n}$ and $X_{s:n}$ is given by

$$\varphi_{r,s:n}^{(k,l)} = \mathrm{E}(X_{r:n}^k X_{s:n}^l) = \int_0^\infty \int_x^\infty x^k y^l \mathfrak{f}_{r,s:n}(x,y)\mathrm{d}y\mathrm{d}x, \quad 1 \leq r < s \leq n, \ k,l \in \mathbb{N}. \tag{2.4}$$

3. Single Moments

The following theorem establishes the main result of this section.

Theorem 3.1: For $1 \leq r \leq n$ and $k \in \mathbb{N}$, we have

$$\varphi_{r:n}^{(k)} = C_{r:n} \sum_{\epsilon=0}^{r-1} \sum_{\rho=0}^{n-r+\epsilon} \frac{\binom{r-1}{\epsilon}\binom{n-r+\epsilon}{\rho}(-1)^\epsilon \psi^{1-k}\Gamma(\rho+k+1)}{(n-r+\epsilon+1)^{\rho+k+1}(1+\psi)^{2(\rho+1)}} \left[\psi + 2 + \frac{\rho+k+1}{\psi(n-r+\epsilon+1)}\right], \tag{3.1}$$

where $\Gamma(.)$ is complete gamma function and $C_{r:n}$ is given in (2.2).

Proof: From (2.3), we can write

$$\begin{aligned}
\varphi_{r:n}^{(k)} &= C_{r:n}\int_0^\infty x^k[1-(1-F(x))]^{r-1}[1-F(x)]^{n-r}\mathfrak{f}(x)\mathrm{d}x\\
&= C_{r:n}\sum_{\epsilon=0}^{r-1}(-1)^\epsilon\binom{r-1}{\epsilon}\int_0^\infty x^k[1-F(x)]^{n-r+\epsilon}\mathfrak{f}(x)\,\mathrm{d}x\\
&= C_{r:n}\sum_{\epsilon=0}^{r-1}(-1)^\epsilon\binom{r-1}{\epsilon}\int_0^\infty x^k\,\frac{\psi^2(2+\psi+x)}{(1+\psi)^2}\left(1+\frac{\psi x}{(1+\psi)^2}\right)^{n-r+\epsilon}\mathrm{e}^{-\psi(n-r+\epsilon+1)x}\,\mathrm{d}x\\
&= C_{r:n}\sum_{\epsilon=0}^{r-1}\sum_{\rho=0}^{n-r+\epsilon}(-1)^\epsilon\binom{r-1}{\epsilon}\binom{n-r+\epsilon}{\rho}\frac{\psi^{2+\rho}}{(1+\psi)^{2(\rho+1)}}\int_0^\infty x^{\rho+k}(2+\psi+x)\,\mathrm{e}^{-\psi(n-r+\epsilon+1)x}\,\mathrm{d}x.
\end{aligned}$$

Since $\int_0^\infty x^{\nu-1}\mathrm{e}^{-\eta x}\mathrm{d}x=\frac{\Gamma(\nu)}{\eta^\nu}$, $\eta>0$, $\nu>0$ (see Gradshteyn and Ryzhik [20], Equation (3.381.4)), it follows that

$$\begin{aligned}
\varphi_{r:n}^{(k)} &= C_{r:n}\sum_{\epsilon=0}^{r-1}\sum_{\rho=0}^{n-r+\epsilon}\binom{r-1}{\epsilon}\binom{n-r+\epsilon}{\rho}(-1)^\epsilon\frac{\psi^{1-k}}{(1+\psi)^{2(\rho+1)}}\\
&\quad\times\left[\frac{(\psi+2)\Gamma(\rho+k+1)}{(n-r+\epsilon+1)^{\rho+k+1}}+\frac{\Gamma(\rho+k+2)}{\psi(n-r+\epsilon+1)^{\rho+k+2}}\right]\\
&= C_{r:n}\sum_{\epsilon=0}^{r-1}\sum_{\rho=0}^{n-r+\epsilon}\frac{\binom{r-1}{\epsilon}\binom{n-r+\epsilon}{\rho}(-1)^\epsilon\psi^{1-k}\Gamma(\rho+k+1)}{(n-r+\epsilon+1)^{\rho+k+1}(1+\psi)^{2(\rho+1)}}\left[\psi+2+\frac{\rho+k+1}{\psi(n-r+\epsilon+1)}\right].
\end{aligned}$$

Thus we have the result. □

Remark 3.1: (**a**) By setting $n=r=1$ in (3.1), we obtain

$$\varphi_{1:1}^{(k)}=\mathrm{E}[X^k]=\frac{\psi^2+2\psi+k+1}{\psi^k(1+\psi)^2}, \tag{3.2}$$

which is the kth moment of X reported by Chouia and Zeghdoudi [15].

(**b**) Upon setting $r=1$ in (3.1), we have

$$\varphi_{1:n}^{(k)}=\sum_{\rho=0}^{n-1}\binom{n-1}{\rho}\frac{\psi^{1-k}\Gamma(\rho+k+2)}{n^{\rho+k}(1+\psi)^{2(\rho+1)}}\left(\psi+2+\frac{\rho+k+1}{n\psi}\right), \tag{3.3}$$

and for $r=n$, in (3.1), we have

$$\varphi_{n:n}^{(k)}=n\sum_{\epsilon=0}^{n-1}\sum_{\rho=0}^{\epsilon}(-1)^\epsilon\binom{n-1}{\epsilon}\binom{\epsilon}{\rho}\frac{\psi^{1-k}\Gamma(\rho+k+2)}{(1+\psi)^{2(\rho+1)}(\epsilon+1)^{\rho+k+1}}\left(\psi+2+\frac{\rho+k+1}{\psi(\epsilon+1)}\right), \tag{3.4}$$

which are the kth moments of the minimum and the maximum order statistics, respectively.

4. Product Moments

In this section, we obtain the results for the product moments of order statistics from the XLindley distribution in the following theorem.

Theorem 4.1: For $1 \leq r < s \leq n$, $k, l \in \mathbb{N}$, we obtain the closed-form expression for product moments of order statistics;

$$\begin{aligned}\varphi_{r,s:n}^{(k,l)} &= C_{r,s:n}\sum_{\epsilon=0}^{r-1}\sum_{\rho=0}^{s-r-1}\sum_{g=0}^{n-r-1-\rho}\sum_{\eta=0}^{\epsilon+\rho}\binom{r-1}{\epsilon}\binom{s-r-1}{\rho}\binom{n-r-1-\rho}{g}\binom{\epsilon+\rho}{\eta}\\ &\times\frac{(l+g)!(-1)^{\epsilon+s-r-1-\rho}}{(n-r-\rho)(n-r+\epsilon+1)^{k+\eta+1}(1+\psi)^{2(2+\eta+g)}\psi^{k+l-2}}\\ &\times\Bigg[\frac{\left(\psi+2+\frac{k+\eta+l+g+2}{\psi(n-r+\epsilon+1)}\right)\Gamma(k+\eta+l+g+2)}{\psi(n-r+\epsilon+1)^{l+g+1}(l+g)!}\\ &+\left(\psi+2+\frac{l+g+1}{\psi(n-r-\rho)}\right)\sum_{\delta=0}^{l+g}\frac{\left(\psi+2+\frac{k+\eta+\delta+1}{\psi(n-r+\epsilon+1)}\right)\Gamma(k+\eta+\delta+1)}{(n-r-\rho)^{l+g-\delta}\,(n-r+\epsilon+1)^{\delta}\,\delta!}\Bigg],\end{aligned} \tag{4.1}$$

where $C_{r,s:n}$ is defined as before.

Proof: From (2.4), we have

$$\begin{aligned}\varphi_{r,s:n}^{(k,l)} &= C_{r,s:n}\int_0^\infty\int_x^\infty x^k y^l[1-(1-F(x))]^{r-1}[1-F(y)]^{n-s}[(1-F(x))-(1-F(y))]^{s-r-1}\\ &\qquad\times\mathfrak{f}(x)\mathfrak{f}(y)\mathrm{d}y\mathrm{d}x.\end{aligned} \tag{4.2}$$

Utilizing the binomial expansion, (4.2) can be reexpressed as,

$$\varphi_{r,s:n}^{(k,l)} = C_{r,s:n}\sum_{\epsilon=0}^{r-1}\sum_{\rho=0}^{s-r-1}(-1)^{\epsilon+s-r-1-\rho}\binom{r-1}{\epsilon}\binom{s-r-1}{\rho}\int_0^\infty x^k[1-F(x)]^{\epsilon+\rho}\mathfrak{f}(x)\mathrm{I}(x)\mathrm{d}x,$$

where

$$\begin{aligned}\mathrm{I}(x) &= \int_x^\infty y^l[1-F(y)]^{n-r-1-\rho}\mathfrak{f}(y)\mathrm{d}y\\ &= \frac{\psi^2}{(1+\psi)^2}\int_x^\infty y^l\left(1+\frac{\psi y}{(1+\psi)^2}\right)^{n-r-1-\rho}(\psi+2+y)\mathrm{e}^{-\psi(n-r-\rho)y}\mathrm{d}y\\ &= \sum_{g=0}^{n-r-1-\rho}\binom{n-r-1-\rho}{g}\frac{\psi^{2+g}}{(1+\psi)^{2(1+g)}}\int_x^\infty y^{l+g}(\psi+2+y)\mathrm{e}^{-\psi(n-r-\rho)y}\mathrm{d}y.\end{aligned}$$

Note that $\int_u^\infty x^\nu \mathrm{e}^{-\eta x}\mathrm{d}x = \mathrm{e}^{-\eta u}\sum_{p=0}^{\nu}\frac{\nu!}{p!}\frac{u^p}{\eta^{\nu-p+1}}$ for $\eta > 0$, $u > 0$ and $\nu = 0, 1, 2, \ldots$ (see Gradshteyn and Ryzhik [20], Equation (3.351.2)), it follows that

$$
\begin{aligned}
\mathrm{I}(x) \;=\; & \sum_{g=0}^{n-r-1-\rho}\binom{n-r-1-\rho}{g}\frac{\psi^{2+g}}{(1+\psi)^{2(1+g)}}\,\mathrm{e}^{-\psi(n-r-\rho)x} \\
& \times\left[(\psi+2)\sum_{\delta=0}^{l+g}\frac{x^{\delta}(l+g)!}{(\psi(n-r-\rho))^{l+g-\delta+1}\delta!}+\sum_{\delta=0}^{l+g+1}\frac{x^{\delta}(l+g+1)!}{(\psi(n-r-\rho))^{l+g-\delta+2}\delta!}\right] \\
= \; & \sum_{g=0}^{n-r-1-\rho}\binom{n-r-1-\rho}{g}\frac{\psi^{2+g}(l+g)!}{(1+\psi)^{2(1+g)}}\,\mathrm{e}^{-\psi(n-r-\rho)x} \\
& \times\left[(\psi+2)\sum_{\delta=0}^{l+g}\frac{x^{\delta}}{(\psi(n-r-\rho))^{l+g-\delta+1}\delta!}+\sum_{\delta=0}^{l+g+1}\frac{x^{\delta}(l+g+1)}{(\psi(n-r-\rho))^{l+g-\delta+2}\delta!}\right] \\
= \; & \sum_{g=0}^{n-r-1-\rho}\binom{n-r-1-\rho}{g}\frac{\psi^{1+g}(l+g)!\mathrm{e}^{-\psi(n-r-\rho)x}}{(1+\psi)^{2(1+g)}(n-r-\rho)}\left[\frac{x^{l+g+1}}{(l+g)!}+\sum_{\delta=0}^{l+g}\frac{\left(\psi+2+\frac{l+g+1}{\psi(n-r-\rho)}\right)x^{\delta}}{[\psi(n-r-\rho)]^{l+g-\delta}\delta!}\right].
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
& \int_0^{\infty}x^{k}[1-F(x)]^{\epsilon+\rho}\mathfrak{f}(x)\mathrm{I}(x)\mathrm{d}x=\sum_{g=0}^{n-r-1-\rho}\binom{n-r-1-\rho}{g}\frac{\psi^{1+g}(l+g)!}{(1+\psi)^{2(1+g)}(n-r-\rho)} \\
& \times\int_0^{\infty}x^{k}[1-F(x)]^{\epsilon+\rho}\mathfrak{f}(x)\mathrm{e}^{-\psi(n-r-\rho)x}\left[\frac{x^{l+g+1}}{(l+g)!}+\sum_{\delta=0}^{l+g}\frac{\left(\psi+2+\frac{l+g+1}{\psi(n-r-\rho)}\right)x^{\delta}}{[\psi(n-r-\rho)]^{l+g-\delta}\delta!}\right]\mathrm{d}x.
\end{aligned}
$$

Applying the result $\int_0^{\infty}x^{\nu-1}\mathrm{e}^{-\eta x}\mathrm{d}x=\frac{\Gamma(\nu)}{\eta^{\nu}}$ for $\eta>0$, $\nu>0$ (see Gradshteyn and Ryzhik [20], Equation (3.381.4)), we obtain

$$
\begin{aligned}
& \int_0^{\infty}x^{k}[1-F(x)]^{\epsilon+\rho}\mathfrak{f}(x)\mathrm{e}^{-\psi(n-r-\rho)x}\left[\frac{x^{l+g+1}}{(l+g)!}+\sum_{\delta=0}^{l+g}\frac{\left(\psi+2+\frac{l+g+1}{\psi(n-r-\rho)}\right)x^{\delta}}{[\psi(n-r-\rho)]^{l+g-\delta}\delta!}\right]\mathrm{d}x \\
= \; & \frac{\psi^{2}}{(1+\psi)^{2}}\int_0^{\infty}x^{k}\left(1+\frac{\psi x}{(1+\psi)^{2}}\right)^{\epsilon+\rho}(\psi+2+x)\mathrm{e}^{-\psi(n-r+\epsilon+1)x} \\
& \times\left[\frac{x^{l+g+1}}{(l+g)!}+\sum_{\delta=0}^{l+g}\frac{\left(\psi+2+\frac{l+g+1}{\psi(n-r-\rho)}\right)x^{\delta}}{[\psi(n-r-\rho)]^{l+g-\delta}\delta!}\right]\mathrm{d}x \\
= \; & \sum_{\eta=0}^{\epsilon+\rho}\binom{\epsilon+\rho}{\eta}\frac{\psi^{2+\eta}}{(1+\psi)^{2(1+\eta)}}\int_0^{\infty}x^{k+\eta}(\psi+2+x)\mathrm{e}^{-\psi(n-r+\epsilon+1)x} \\
& \times\left[\frac{x^{l+g+1}}{(l+g)!}+\sum_{\delta=0}^{l+g}\frac{\left(\psi+2+\frac{l+g+1}{\psi(n-r-\rho)}\right)x^{\delta}}{[\psi(n-r-\rho)]^{l+g-\delta}\delta!}\right]\mathrm{d}x \\
= \; & \sum_{\eta=0}^{\epsilon+\rho}\binom{\epsilon+\rho}{\eta}\frac{\psi^{2+\eta}}{(1+\psi)^{2(1+\eta)}}\Bigg[\frac{\left(\psi+2+\frac{k+\eta+l+g+2}{\psi(n-r+\epsilon+1)}\right)\Gamma(k+\eta+l+g+2)}{[\psi(n-r+\epsilon+1)]^{k+\eta+l+g+2}(l+g)!} \\
& +\left(\psi+2+\frac{l+g+1}{\psi(n-r-\rho)}\right)\sum_{\delta=0}^{l+g}\frac{\left(\psi+2+\frac{k+\eta+\delta+1}{\psi(n-r+\epsilon+1)}\right)\Gamma(k+\eta+\delta+1)}{[\psi(n-r-\rho)]^{l+g-\delta}\delta!\,[\psi(n-r+\epsilon+1)]^{k+\eta+\delta+1}}\Bigg].
\end{aligned}
$$

Finally, we can write

$$\begin{aligned}
\varphi^{(k,l)}_{r,s:n} &= C_{r,s:n}\sum_{\epsilon=0}^{r-1}\sum_{\rho=0}^{s-r-1}\sum_{g=0}^{n-r-1-\rho}\sum_{\eta=0}^{\epsilon+\rho}\binom{r-1}{\epsilon}\binom{s-r-1}{\rho}\binom{n-r-1-\rho}{g}\binom{\epsilon+\rho}{\eta}\\
&\quad\times(-1)^{\epsilon+s-r-1-\rho}\frac{\psi^{1+g}(l+g)!}{(1+\psi)^{2(1+g)}(n-r-\rho)}\frac{\psi^{2+\eta}}{(1+\psi)^{2(1+\eta)}}\\
&\quad\times\Bigg[\frac{\left(\psi+2+\frac{k+\eta+l+g+2}{\psi(n-r+\epsilon+1)}\right)\Gamma(k+\eta+l+g+2)}{[\psi(n-r+\epsilon+1)]^{k+\eta+l+g+2}(l+g)!}\\
&\quad+\left(\psi+2+\frac{l+g+1}{\psi(n-r-\rho)}\right)\sum_{\delta=0}^{l+g}\frac{\left(\psi+2+\frac{k+\eta+\delta+1}{\psi(n-r+\epsilon+1)}\right)\Gamma(k+\eta+\delta+1)}{[\psi(n-r-\rho)]^{l+g-\delta}\delta!\,[\psi(n-r+\epsilon+1)]^{k+\eta+\delta+1}}\Bigg]\\
&= C_{r,s:n}\sum_{\epsilon=0}^{r-1}\sum_{\rho=0}^{s-r-1}\sum_{g=0}^{n-r-1-\rho}\sum_{\eta=0}^{\epsilon+\rho}\binom{r-1}{\epsilon}\binom{s-r-1}{\rho}\binom{n-r-1-\rho}{g}\binom{\epsilon+\rho}{\eta}\\
&\quad\times\frac{\psi^{3+g+\eta}(l+g)!(-1)^{\epsilon+s-r-1-\rho}}{(1+\psi)^{2(2+\eta+g)}(n-r-\rho)}\\
&\quad\times\Bigg[\frac{\left(\psi+2+\frac{k+\eta+l+g+2}{\psi(n-r+\epsilon+1)}\right)\Gamma(k+\eta+l+g+2)}{[\psi(n-r+\epsilon+1)]^{k+\eta+l+g+2}(l+g)!}\\
&\quad+\left(\psi+2+\frac{l+g+1}{\psi(n-r-\rho)}\right)\sum_{\delta=0}^{l+g}\frac{\left(\psi+2+\frac{k+\eta+\delta+1}{\psi(n-r+\epsilon+1)}\right)\Gamma(k+\eta+\delta+1)}{[\psi(n-r-\rho)]^{l+g-\delta}\delta!\,[\psi(n-r+\epsilon+1)]^{k+\eta+\delta+1}}\Bigg].
\end{aligned}$$

Hence, the result (4.1) can be obtained after some algebraic manipulation. □

5. Computations of Means, Variances and Covariances

Here, we build upon the results derived in the preceding sections to calculate the means, variances and covariances of order statistics for the XLindley distribution.

To find the means of the order statistics, we substitute $k = 1$ into (3.1). This allows us to calculate the means for sample sizes ranging from $n = 1$ to $n = 10$ and for selected values of the shape parameter. Table 1 includes the computed mean values.

Table 1: Means of order statistics

n	r	$\psi = 1$	$\psi = 2$	$\psi = 3$	$\psi = 4$
1	1	1.25000	0.55556	0.35417	0.26000
2	1	0.64062	0.27932	0.17741	0.13010
	2	1.85938	0.83179	0.53092	0.38990
3	1	0.43171	0.18661	0.11835	0.08676
	2	1.05845	0.46475	0.29552	0.21679
	3	2.25984	1.01531	0.64862	0.47646
4	1	0.32578	0.14011	0.08879	0.06508
	2	0.74953	0.32609	0.20702	0.15180
	3	1.36737	0.60340	0.38403	0.28178
	4	2.55733	1.15261	0.73682	0.54135
5	1	0.26166	0.11217	0.07105	0.05207
	2	0.58225	0.25189	0.15977	0.11712
	3	1.00045	0.43740	0.27790	0.20381
	4	1.61199	0.71407	0.45477	0.33375

(*Continued*)

	5	2.79366	1.26225	0.80734	0.59325
6	1	0.21866	0.09352	0.05922	0.04339
	2	0.47666	0.20542	0.13022	0.09544
	3	0.79342	0.34483	0.21888	0.16048
	4	1.20748	0.52996	0.33693	0.24715
	5	1.81424	0.80613	0.51370	0.37705
	6	2.98954	1.35348	0.86606	0.63649
7	1	0.18781	0.08018	0.05076	0.03719
	2	0.40375	0.17351	0.10994	0.08057
	3	0.65894	0.28519	0.18090	0.13261
	4	0.97273	0.42435	0.26951	0.19763
	5	1.38354	0.60916	0.38750	0.28428
	6	1.98653	0.88491	0.56417	0.41416
	7	3.15671	1.43157	0.91638	0.67355
8	1	0.16460	0.07018	0.04442	0.03255
	2	0.35030	0.15022	0.09516	0.06973
	3	0.56408	0.24338	0.15430	0.11310
	4	0.81703	0.35488	0.22523	0.16513
	5	1.12843	0.49382	0.31379	0.23014
	6	1.53660	0.67837	0.43173	0.31677
	7	2.13650	0.95376	0.60832	0.44662
	8	3.30246	1.49983	0.96039	0.70596
9	1	0.14649	0.06239	0.03949	0.02893
	2	0.30941	0.13246	0.08389	0.06147
	3	0.49341	0.21237	0.13460	0.09865
	4	0.70543	0.30540	0.19372	0.14201
	5	0.95653	0.41674	0.26461	0.19403
	6	1.26595	.555490	0.35314	0.25903
	7	1.67193	0.73981	0.47102	0.34564
	8	2.26924	1.01489	0.64755	0.47547
	9	3.43161	1.56045	0.99949	0.73477
10	1	0.13198	0.05616	0.03554	0.02604
	2	0.27711	0.11847	0.07501	0.05496
	3	0.43864	0.18843	0.11939	0.08750
	4	0.62121	0.26824	0.17008	0.12467
	5	0.83175	0.36114	0.22918	0.16802
	6	1.08130	0.47234	0.30004	0.22003
	7	1.38905	0.61093	0.38853	0.28502
	8	1.79317	0.79504	0.50638	0.37162
	9	2.38825	1.06985	0.68284	0.50144
	10	3.54754	1.61496	1.03468	0.76070

We see that the condition $\sum_{r=1}^{n} \varphi_{r:n} = n\mathrm{E}(X)$, is satisfied (David and Nagaraja [17])

By setting $k = 1$ and $k = 2$ in (3.1), respectively, we can systematically calculate the first two moments $\varphi_{r:n}^{(1)}$ and $\varphi_{r:n}^{(2)}$. These moments facilitate the computation of the variances of the order statistics. We have compiled tables of variances for sample sizes ranging from $n = 1$ to $n = 10$ and for selected values of the shape parameter.

By setting $k = l = 1$ in (4.1), we first compute all product moments. Utilizing these product moments in conjunction with the previously calculated first two moments allows for straightforward computation of the covariances of order statistics. We calculated the covariances for for sample sizes ranging from $n = 1$ to $n = 10$ and for selected values of the shape parameter. The resulting variances and covariances are reported in Table 2.

Table 2: Variances and covariances of order statistics.

n	s	r	$\psi = 1$	$\psi = 2$	$\psi = 3$	$\psi = 4$
1	1	1	1.43750	0.30247	0.12457	0.06740
2	1	1	0.38647	0.07707	0.03135	0.01690
	2	1	0.37134	0.07631	0.03124	0.01687
	2	2	1.74585	0.37525	0.15529	0.08415
3	1	1	0.17782	0.03452	0.01397	0.00752
	2	1	0.17302	0.03432	0.01394	0.00751
	3	1	0.16681	0.03399	0.01390	0.00750
	2	2	0.54192	0.11061	0.04519	0.02439
	3	2	0.52320	0.10957	0.04504	0.02435
	3	3	1.86670	0.40654	0.16879	0.09156
4	1	1	0.10211	0.01950	0.00787	0.00423
	2	1	0.09995	0.01942	0.00786	0.00423
	3	1	0.09749	0.01931	0.00784	0.00423
	4	1	0.09425	0.01913	0.00782	0.00422
	2	2	0.27029	0.05364	0.02179	0.01174
	3	2	0.26379	0.05334	0.02175	0.01173
	4	2	0.25518	0.05285	0.02168	0.01171
	3	3	0.62267	0.12912	0.05293	0.02859
	4	3	0.60312	0.12797	0.05275	0.02855
	4	4	1.92738	0.42360	0.17629	0.09570
5	1	1	0.06625	0.01251	0.00504	0.00271
	2	1	0.06509	0.01247	0.00503	0.00271
	3	1	0.06384	0.01242	0.00503	0.00271
	4	1	0.06238	0.01235	0.00502	0.00270
	5	1	0.06044	0.01224	0.00500	0.00270
	2	2	0.16330	0.03182	0.01288	0.00694
	3	2	0.16019	0.03170	0.01287	0.00693
	4	2	0.15659	0.03153	0.01284	0.00693
	5	2	0.15177	0.03124	0.01280	0.00692
	3	3	0.32584	0.06571	0.02677	0.01444
	4	3	0.31868	0.06536	0.02672	0.01442
	5	3	0.30904	0.06478	0.02664	0.01440
	4	4	0.67097	0.14078	0.05785	0.03127
	5	4	0.65142	0.13956	0.05766	0.03123
	5	5	1.96222	0.43421	0.18104	0.09834
6	1	1	0.04646	0.00870	0.00350	0.00188
	2	1	0.04577	0.00868	0.00350	0.00188
	3	1	0.04503	0.00865	0.00350	0.00188
	4	1	0.04422	0.00862	0.00349	0.00188
	5	1	0.04328	0.00857	0.00348	0.00188
	6	1	0.04201	0.00850	0.00347	0.00188
	2	2	0.10972	0.02111	0.00853	0.00459
	3	2	0.10798	0.02104	0.00852	0.00459
	4	2	0.10606	0.02096	0.00851	0.00458
	5	2	0.10382	0.02085	0.00849	0.00458
	6	2	0.10079	0.02067	0.00847	0.00457
	3	3	0.20356	0.04029	0.01636	0.00881
	4	3	0.20000	0.04014	0.01634	0.00881
	5	3	0.19584	0.03993	0.01631	0.00880
	6	3	0.19017	0.03958	0.01625	0.00879
	4	4	0.36239	0.07399	0.03022	0.01631
	5	4	0.35500	0.07361	0.03016	0.01629
	6	4	0.34492	0.07298	0.03007	0.01627

(*Continued*)

	5	5	0.70254	0.14875	0.06125	0.03313
	6	5	0.68330	0.14750	0.06106	0.03308
	6	6	1.98393	0.44137	0.18430	0.10017
7	1	1	0.03439	0.00640	0.00257	0.00138
	2	1	0.03394	0.00639	0.00257	0.00138
	3	1	0.03347	0.00637	0.00257	0.00138
	4	1	0.03297	0.00635	0.00257	0.00138
	5	1	0.03241	0.00633	0.00256	0.00138
	6	1	0.03176	0.00629	0.00256	0.00138
	7	1	0.03087	0.00624	0.00255	0.00138
	2	2	0.07894	0.01504	0.00607	0.00326
	3	2	0.07785	0.01500	0.00606	0.00326
	4	2	0.07669	0.01496	0.00606	0.00326
	5	2	0.07541	0.01490	0.00605	0.00326
	6	2	0.07391	0.01482	0.00604	0.00326
	7	2	0.07185	0.01470	0.00602	0.00325
	3	3	0.14018	0.02737	0.01108	0.00597
	4	3	0.13811	0.02729	0.01107	0.00596
	5	3	0.13583	0.02719	0.01106	0.00596
	6	3	0.13315	0.02705	0.01104	0.00596
	7	3	0.12946	0.02682	0.01101	0.00595
	4	4	0.23181	0.04645	0.01890	0.01019
	5	4	0.22803	0.04628	0.01888	0.01018
	6	4	0.22357	0.04604	0.01884	0.01017
	7	4	0.21744	0.04566	0.01878	0.01016
	5	5	0.38800	0.08001	0.03274	0.01768
	6	5	0.38057	0.07961	0.03268	0.01766
	7	5	0.37032	0.07895	0.03258	0.01764
	6	6	0.72447	0.15452	0.06373	0.03449
	7	6	0.70564	0.15326	0.06354	0.03445
	7	7	1.99822	0.44648	0.18668	0.10150
8	1	1	0.02648	0.00491	0.00197	0.00106
	2	1	0.02617	0.00490	0.00197	0.00106
	3	1	0.02585	0.00489	0.00197	0.00106
	4	1	0.02552	0.00487	0.00197	0.00106
	5	1	0.02516	0.00486	0.00197	0.00106
	6	1	0.02476	0.00484	0.00196	0.00106
	7	1	0.02429	0.00482	0.00196	0.00106
	8	1	0.02364	0.00478	0.00195	0.00105
	2	2	0.05957	0.01127	0.00454	0.00244
	3	2	0.05884	0.01124	0.00454	0.00244
	4	2	0.05808	0.01121	0.00453	0.00244
	5	2	0.05727	0.01118	0.00453	0.00244
	6	2	0.05637	0.01114	0.00452	0.00244
	7	2	0.05530	0.01108	0.00452	0.00243
	8	2	0.05382	0.01099	0.00450	0.00243
	3	3	0.10276	0.01986	0.00803	0.00432
	4	3	0.10144	0.01981	0.00802	0.00432
	5	3	0.10003	0.01975	0.00801	0.00432
	6	3	0.09846	0.01968	0.00800	0.00431
	7	3	0.09660	0.01958	0.00799	0.00431
	8	3	0.09403	0.01941	0.00796	0.00430
	4	4	0.16256	0.03213	0.01304	0.00702
	5	4	0.16032	0.03203	0.01302	0.00702
	6	4	0.15783	0.03191	0.01301	0.00701

(*Continued*)

	7	4	0.15487	0.03175	0.01299	0.00701
	8	4	0.15078	0.03149	0.01294	0.00700
	5	5	0.25257	0.05113	0.02084	0.01124
	6	5	0.24869	0.05094	0.02081	0.01123
	7	5	0.24409	0.05069	0.02078	0.01122
	8	5	0.23770	0.05027	0.02071	0.01121
	6	6	0.40678	0.08457	0.03466	0.01872
	7	6	0.39940	0.08416	0.03461	0.01871
	8	6	0.38912	0.08347	0.03450	0.01869
	7	7	0.74040	0.15888	0.06563	0.03553
	8	7	0.72200	0.15761	0.06543	0.03549
	8	8	2.00797	0.45030	0.18848	0.10252
9	1	1	0.02102	0.00388	0.00156	0.00084
	2	1	0.02080	0.00387	0.00156	0.00084
	3	1	0.02057	0.00387	0.00156	0.00084
	4	1	0.02034	0.00386	0.00155	0.00084
	5	1	0.02009	0.00385	0.00155	0.00084
	6	1	0.01982	0.00384	0.00155	0.00084
	7	1	0.01952	0.00382	0.00155	0.00083
	8	1	0.01917	0.00380	0.00155	0.00083
	9	1	0.01867	0.00377	0.00154	0.00083
	2	2	0.04658	0.00876	0.00353	0.00189
	3	2	0.04607	0.00874	0.00352	0.00189
	4	2	0.04554	0.00872	0.00352	0.00189
	5	2	0.04499	0.00870	0.00352	0.00189
	6	2	0.04439	0.00867	0.00351	0.00189
	7	2	0.04372	0.00864	0.00351	0.00189
	8	2	0.04293	0.00860	0.00350	0.00189
	9	2	0.04182	0.00853	0.00349	0.00189
	3	3	0.07871	0.01509	0.00609	0.00328
	4	3	0.07781	0.01505	0.00609	0.00327
	5	3	0.07687	0.01502	0.00608	0.00327
	6	3	0.07585	0.01497	0.00607	0.00327
	7	3	0.07472	0.01492	0.00607	0.00327
	8	3	0.07337	0.01484	0.00606	0.00327
	9	3	0.07148	0.01472	0.00604	0.00326
	4	4	0.12090	0.02364	0.00957	0.00515
	5	4	0.11944	0.02358	0.00956	0.00515
	6	4	0.11788	0.02351	0.00956	0.00515
	7	4	0.11612	0.02342	0.00954	0.00515
	8	4	0.11403	0.02331	0.00953	0.00514
	9	4	0.11112	0.02312	0.00950	0.00514
	5	5	0.17962	0.03586	0.01457	0.00785
	6	5	0.17728	0.03575	0.01456	0.00785
	7	5	0.17467	0.03562	0.01454	0.00785
	8	5	0.17155	0.03545	0.01452	0.00784
	9	5	0.16719	0.03516	0.01447	0.00783
	6	6	0.26838	0.05479	0.02237	0.01207
	7	6	0.26448	0.05459	0.02234	0.01206
	8	6	0.25981	0.05433	0.02230	0.01205
	9	6	0.25327	0.05389	0.02224	0.01204
	7	7	0.42104	0.08814	0.03618	0.01955
	8	7	0.41375	0.08771	0.03612	0.01954
	9	7	0.40352	0.08701	0.03601	0.01951
	8	8	0.75236	0.16228	0.06712	0.03635

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	9	8	0.73439	0.16101	0.06691	0.03631
	9	9	2.01480	0.45323	0.18988	0.10332
10	1	1	0.01709	0.00315	0.00126	0.00068
	2	1	0.01693	0.00314	0.00126	0.00068
	3	1	0.01676	0.00313	0.00126	0.00068
	4	1	0.01659	0.00313	0.00126	0.00068
	5	1	0.01641	0.00312	0.00126	0.00068
	6	1	0.01622	0.00311	0.00126	0.00068
	7	1	0.01601	0.00311	0.00126	0.00068
	8	1	0.01578	0.00309	0.00126	0.00068
	9	1	0.01551	0.00308	0.00125	0.00068
	10	1	0.01512	0.00305	0.00125	0.00067
	2	2	0.03743	0.00700	0.00282	0.00151
	3	2	0.03706	0.00699	0.00282	0.00151
	4	2	0.03668	0.00698	0.00281	0.00151
	5	2	0.03628	0.00696	0.00281	0.00151
	6	2	0.03586	0.00695	0.00281	0.00151
	7	2	0.03541	0.00693	0.00281	0.00151
	8	2	0.03490	0.00690	0.00280	0.00151
	9	2	0.03429	0.00687	0.00280	0.00151
	10	2	0.03344	0.00681	0.00279	0.00151
	3	3	0.06229	0.01186	0.00478	0.00257
	4	3	0.06165	0.01184	0.00478	0.00257
	5	3	0.06098	0.01181	0.00478	0.00257
	6	3	0.06028	0.01178	0.00477	0.00257
	7	3	0.05952	0.01175	0.00477	0.00257
	8	3	0.05867	0.01171	0.00476	0.00257
	9	3	0.05765	0.01165	0.00475	0.00256
	10	3	0.05621	0.01156	0.00474	0.00256
	4	4	0.09369	0.01816	0.00734	0.00395
	5	4	0.09269	0.01812	0.00734	0.00395
	6	4	0.09162	0.01808	0.00733	0.00395
	7	4	0.09047	0.01802	0.00733	0.00395
	8	4	0.08918	0.01796	0.00732	0.00395
	9	4	0.08764	0.01787	0.00730	0.00394
	10	4	0.08547	0.01773	0.00728	0.00394
	5	5	0.13511	0.02668	0.01082	0.00583
	6	5	0.13357	0.02661	0.01081	0.00583
	7	5	0.13191	0.02653	0.01080	0.00582
	8	5	0.13004	0.02644	0.01079	0.00582
	9	5	0.12780	0.02631	0.01077	0.00582
	10	5	0.12464	0.02610	0.01074	0.00581
	6	6	0.19298	0.03885	0.01582	0.00853
	7	6	0.19060	0.03874	0.01580	0.00852
	8	6	0.18792	0.03861	0.01578	0.00852
	9	6	0.18470	0.03842	0.01576	0.00851
	10	6	0.18017	0.03812	0.01571	0.00850
	7	7	0.28076	0.05773	0.02360	0.01274
	8	7	0.27686	0.05753	0.02358	0.01273
	9	7	0.27218	0.05726	0.02354	0.01272
	10	7	0.26557	0.05680	0.02347	0.01271
	8	8	0.43216	0.09100	0.03740	0.02022
	9	8	0.42499	0.09057	0.03734	0.02021
	10	8	0.41485	0.08986	0.03723	0.02018
	9	9	0.76159	0.16499	0.06832	0.03702

(*Continued*)

	10	9	0.74402	0.16373	0.06811	0.03697
	10	10	2.01965	0.45554	0.19101	0.10397

Here, the condition $\sum_{r=1}^{n}\sum_{s=1}^{n}\sigma_{r,s:n} = n\sigma^2$ (David and Nagaraja, [17]), is satisfied, where $\sigma_{r,s:n} = \mathrm{Cov}(X_{r:n}, X_{s:n})$ and $\sigma^2 = \mathrm{Var}(X)$.

6. Linear Estimation

Suppose $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n-m:n}$, $m = 0, 1, \ldots, n-1$, denote Type-II censored order statistics from $XL(\varphi, \sigma, \psi)$. Besides, let $Z_{r:n} = \dfrac{X_{r:n} - \varphi}{\sigma}$, $E(Z_{r:n}) = \varphi_{r:n}^{(1)}$, $1 \leq r \leq n-m$, and $\mathrm{Cov}(Z_{r:n}, Z_{s:n}) = \sigma_{r,s:n} = \varphi_{r,s:n}^{(1,1)} - \varphi_{r:n}^{(1)}\varphi_{s:n}^{(1)}$, $1 \leq r < s \leq n-m$.

We shall use the following notations:

$$\mathbf{X} = (X_{1:n}, X_{2:n}, \ldots, X_{n-m:n})^{\mathrm{T}},$$

$$\boldsymbol{\varphi} = (\varphi_{1:n}, \varphi_{2:n}, \ldots, \varphi_{n-m:n})^{\mathrm{T}},$$

$$\mathbf{1} = (1, 1, \ldots, 1)^{\mathrm{T}}_{1\times(n-m)},$$

and

$$\boldsymbol{\Sigma} = ((\sigma_{r,s:n}))\,,\ 1 \leq r,\ s \leq n-m.$$

Then, the BLUEs of φ and σ, denoted by $\widehat{\varphi}$ and $\widehat{\sigma}$, respectively, are given by

$$\widehat{\varphi} = \left\{\frac{\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\varphi}\mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1} - \boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}}{(\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\varphi})(\mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1})^2}\right\}\mathbf{X} = \sum_{r=1}^{n-m} a_r X_{r:n}, \tag{6.1}$$

and

$$\widehat{\sigma} = \left\{\frac{\mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1} - \mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\varphi}\mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}}{(\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\varphi})(\mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1})^2}\right\}\mathbf{X} = \sum_{r=1}^{n-m} b_r X_{r:n}. \tag{6.2}$$

The variances and covariances of these BLUEs are given by

$$\mathrm{Var}(\widehat{\varphi}) = \sigma^2\left\{\frac{\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\varphi}}{(\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\varphi})(\mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1})^2}\right\} = \sigma^2 V_1,$$

$$\mathrm{Var}(\widehat{\sigma}) = \sigma^2\left\{\frac{\mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}}{(\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\varphi})(\mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1})^2}\right\} = \sigma^2 V_2$$

and

$$\mathrm{Cov}(\widehat{\varphi}, \widehat{\sigma}) = \sigma^2\left\{\frac{-\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}}{(\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\varphi})(\mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1})^2}\right\} = \sigma^2 V_3,$$

respectively. See for details, Arnold et al. [7], Balakrishnan and Cohen [9], David and Nagaraja [17] and Lloyd [25].

We proceed to derive the BLIEs of φ and σ. In view of Mann [27] (see also Bondesson [12]), the BLIEs of φ and σ are given by

$$\widetilde{\varphi} = \left\{ \frac{\left(\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\varphi} + 1\right)\mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1} - \boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}}{(\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\varphi})(\mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1})^2 + \mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}} \right\} \mathbf{X} = \widehat{\varphi} - \frac{V_3}{1+V_2}\widehat{\sigma} = \sum_{r=1}^{n-m} a'_r X_{r:n}, \tag{6.3}$$

$$\widetilde{\sigma} = \left\{ \frac{\mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1} - \mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\varphi}\mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}}{(\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\varphi})(\mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1})^2 + \mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}} \right\} \mathbf{X} = \frac{\widehat{\sigma}}{1+V_2} = \sum_{r=1}^{n-m} b'_r X_{r:n}, \tag{6.4}$$

where $V_1 = \frac{1}{\sigma^2}\mathrm{Var}(\widehat{\varphi})$, $V_2 = \frac{1}{\sigma^2}\mathrm{Var}(\widehat{\sigma})$ and $V_3 = \frac{1}{\sigma^2}\mathrm{Cov}(\widehat{\varphi}, \widehat{\sigma})$.

Next, using (6.1) and (6.2), respectively, we calculated the coefficients of the BLUEs for location and scale parameters for the Type-II right censored samples, presented in Tables 3 and 4 for $n = 6$ and 10 with m ranging from 0 to $[n/2] - 1$ and $\psi = 1, \cdots, 4$. Moreover, Tables 5 and 6 present the coefficients of the BLIEs using equations (6.3) and (6.4), respectively.

Note that the following conditions verify the validity of the computed coefficients:

$$\sum_{r=1}^{n-m} a_r = 1 \quad \text{and} \quad \sum_{r=1}^{n-m} b_r = 0.$$

Based on the BLUEs for the location and scale parameters, confidence intervals (CIs) for φ and σ can be derived using the following pivotal quantities:

$$T_1 = \frac{\hat{\varphi} - \varphi}{\hat{\sigma}\sqrt{V_1}}, \quad \text{and} \quad T_2 = \frac{\hat{\sigma} - \sigma}{\sigma\sqrt{V_2}}.$$

To construct these CIs, we need the percentage points of T_1 and T_2, which can be calculated using the BLUEs $\hat{\varphi}$ and $\hat{\sigma}$ through the Monte Carlo method. Table 7 presents these percentage points, determined from 10,000 simulations across various values of n and ψ. Using these simulated percentage points, we can construct a $100(1-\Upsilon)\%$ CI for φ based on the pivotal quantity T_1 as follows:

$$\mathrm{P}\left(\hat{\varphi} - \hat{\sigma}\sqrt{V_1}T_1(1-\Upsilon/2) \le \varphi \le \hat{\varphi} - \hat{\sigma}\sqrt{V_1}T_1(\Upsilon/2)\right) = 1 - \Upsilon,$$

where $T_1(\tau)$ denotes the left percentage point of T_1 at τ, i.e. $\mathrm{P}(T_1 < T_1(\tau)) = \tau$.

Similarly, a $100(1-\Upsilon)\%$ CI for σ can be constructed using the pivotal quantity T_2 as follows:

$$\mathrm{P}\left(\frac{\hat{\sigma}}{1 + \sqrt{V_2}T_2(1-\Upsilon/2)} \le \sigma \le \frac{\hat{\sigma}}{1 + \sqrt{V_2}T_2(\Upsilon/2)}\right) = 1 - \Upsilon,$$

where $T_2(\tau)$ is the left percentage point of T_2 at τ, i.e. $\mathrm{P}(T_2 < T_2(\tau)) = \tau$.

Based on the BLIEs, we can construct confidence intervals (CIs) for the location and scale parameters using the following pivotal quantities:

$$T_3 = \frac{\widetilde{\varphi} - \varphi}{\widetilde{\sigma}\sqrt{V_1 - \frac{V_3^2(2+V_2)}{(1+V_2)^2}}}, \qquad \text{and} \quad T_4 = \frac{\widetilde{\sigma} - \sigma}{\sigma\frac{\sqrt{V_2}}{1+V_2}}.$$

Table 8 shows the percentage points of T_3 and T_4 based on 10,000 simulations with various choices of n and ψ. Using the BLIEs and the values from Table 8, we can construct a $100(1-\Upsilon)\%$ CI for φ through the pivotal quantity T_3 as follows:

$$P\left(\widetilde{\varphi} - \widetilde{\sigma}\sqrt{V_1 - \frac{V_3^2(2+V_2)}{(1+V_2)^2}}\,T_3(1-\Upsilon/2) \le \varphi \le \widetilde{\varphi} - \widetilde{\sigma}\sqrt{V_1 - \frac{V_3^2(2+V_2)}{(1+V_2)^2}}\,T_3(\Upsilon/2)\right) = 1-\Upsilon,$$

where $T_3(\tau)$ represents the left percentage point of T_3 at τ, i.e., $P(T_3 < T_3(\tau)) = \tau$.

Similarly, a $100(1-\Upsilon)\%$ CI for σ can be constructed using the pivotal quantity T_4 as follows:

$$P\left(\frac{\widetilde{\sigma}}{1 + \frac{\sqrt{V_2}}{1+V_2}T_4(1-\Upsilon/2)} \le \sigma \le \frac{\widetilde{\sigma}}{1 + \frac{\sqrt{V_2}}{1+V_2}T_4(\Upsilon/2)}\right) = 1-\Upsilon,$$

where $T_4(\tau)$ denotes the left percentage point of T_4 at τ, i.e., $P(T_4 < T_4(\tau)) = \tau$.

Table 3: Coefficients for the BLUEs of the location parameter.

ψ	n	m	$a_i, i = 1,2,3,\ldots,(n-m)$				
1	6	0	1.16480	−0.02847	−0.03056	−0.03268	−0.03497
			−0.03813				
		1	1.20851	−0.03658	−0.03907	−0.04158	−0.09128
		2	1.28036	−0.05002	−0.05317	−0.17717	
	10	0	1.09713	−0.00887	−0.00933	−0.00980	−0.01025
			−0.01072	−0.01119	−0.01168	−0.01224	−0.01306
		1	1.11032	−0.01019	−0.01070	−0.01120	−0.01170
			−0.01221	−0.01272	−0.01326	−0.02834	
		2	1.12709	−0.01188	−0.01245	−0.01301	−0.01356
			−0.01412	−0.01469	−0.04738		
		3	1.14925	−0.01413	−0.01477	−0.01541	−0.01604
			−0.01666	−0.07225			
		4	1.18006	−0.01727	−0.01802	−0.01876	−0.01948
			−0.10654				
2	6	0	1.16595	−0.03235	−0.03265	−0.03301	−0.03348
			−0.03447				
		1	1.20814	−0.04066	−0.04103	−0.04147	−0.08498
		2	1.27808	−0.05447	−0.05494	−0.16867	
	10	0	1.09930	−0.01072	−0.01078	−0.01084	−0.0109
			−0.01097	−0.01106	−0.01116	−0.01129	−0.01158
		1	1.11200	−0.01211	−0.01217	−0.01224	−0.01231
			−0.01239	−0.01248	−0.01259	−0.02571	
		2	1.12824	−0.01388	−0.01395	−0.01403	−0.01411
			−0.01420	−0.01430	−0.04376		
		3	1.14982	−0.01625	−0.01633	−0.01641	−0.0165
			−0.01661	−0.06772			
		4	1.17996	−0.01955	−0.01964	−0.01974	−0.01985
			−0.10118				
3	6	0	1.16638	−0.03301	−0.03309	−0.0332	−0.03335
			−0.03373				
		1	1.20824	−0.04135	−0.04145	−0.04158	−0.08387
		2	1.27785	−0.05521	−0.05535	−0.16729	
	10	0	1.09975	−0.01099	−0.01100	−0.01102	−0.01104
			−0.01106	−0.01108	−0.01112	−0.01116	−0.01128
		1	1.11233	−0.01238	−0.01240	−0.01242	−0.01244
			−0.01246	−0.01249	−0.01252	−0.02524	
		2	1.12846	−0.01417	−0.01418	−0.01420	−0.01423

(*Continued*)

			−0.01425	−0.01428	−0.04315		
		3	1.149940	−0.01654	−0.01656	−0.01658	−0.01661
			−0.01664	−0.06699			
		4	1.17998	−0.01987	−0.01989	−0.01992	−0.01995
			−0.10036				
4	6	0	1.16654	−0.0332	−0.03323	−0.03327	−0.03333
			−0.03350				
		1	1.20829	−0.04154	−0.04158	−0.04162	−0.08356
		2	1.27780	−0.05542	−0.05547	−0.16692	
	10	0	1.09989	−0.01106	−0.01107	−0.01107	−0.01108
			−0.01109	−0.01110	−0.01111	−0.01113	−0.01118
		1	1.11243	−0.01245	−0.01246	−0.01246	−0.01247
			−0.01248	−0.01249	−0.01251	−0.02510	
		2	1.12852	−0.01424	−0.01424	−0.01425	−0.01426
			−0.01427	−0.01428	−0.04298		
		3	1.149970	−0.01662	−0.01663	−0.01663	−0.01664
			−0.01665	−0.06680			
			1.17999	−0.01995	−0.01996	−0.01997	−0.01998
			−0.10014				

Table 4: Coefficients for the BLUEs of the scale parameter.

ψ	n	m	$b_i, i = 1, 2, 3, \dots, (n-m)$				
1	6	0	−0.76791	0.13836	0.14548	0.15269	0.16041
			0.17098				
		1	−0.96393	0.17471	0.18364	0.19262	0.41296
		2	−1.28900	0.23553	0.24744	0.80603	
	10	0	−0.75637	0.07371	0.07623	0.07872	0.08118
			0.08366	0.08618	0.08882	0.09177	0.09611
		1	−0.85342	0.08342	0.08627	0.08908	0.09186
			0.09465	0.09748	0.10042	0.21025	
		2	−0.97783	0.09597	0.09925	0.10247	0.10566
			0.10884	0.11205	0.35360		
		3	−1.14321	0.11276	0.11659	0.12036	0.12409
			0.12780	0.54161			
		4	−1.37418	0.13631	0.14093	0.14547	0.14994
			0.80153				
2	6	0	−1.77763	0.34748	0.35034	0.35377	0.35831
			0.36774				
		1	−2.22777	0.43621	0.43978	0.44404	0.90775
		2	−2.97486	0.58368	0.58841	1.80277	
	10	0	−1.77207	0.19201	0.19286	0.19379	0.19481
			0.19595	0.19727	0.19886	0.20099	0.20551
		1	−1.99741	0.21662	0.21758	0.21863	0.21978
			0.22106	0.22254	0.22431	0.45688	
		2	−2.28585	0.24817	0.24927	0.25047	0.25178
			0.25324	0.25492	0.77801		
		3	−2.66951	0.29016	0.29145	0.29284	0.29437
			0.29607	1.20461			
		4	−3.20583	0.34890	0.35044	0.35211	0.35394
			1.80043				
3	6	0	−2.81058	0.55792	0.55922	0.56088	0.56322
			0.56933				
		1	−3.51722	0.69863	0.70026	0.70231	1.41602
		2	−4.69249	0.93273	0.93489	2.82487	
	10	0	−2.80782	0.30953	0.30990	0.31032	0.31079

(*Continued*)

			0.31134	0.31199	0.31282	0.31401	0.31712
		1	−3.16153	0.34864	0.34906	0.34953	0.35006
			0.35067	0.35141	0.35233	0.70983	
		2	−3.61518	0.39882	0.39930	0.39983	0.40044
			0.40114	0.40197	1.21369		
		3	−4.21937	0.46566	0.46622	0.46684	0.46754
			0.46836	1.88476			
		4	−5.06468	0.55918	0.55985	0.56060	0.56144
			2.82361				
4	6	0	−3.83839	0.76529	0.76598	0.76688	0.76818
			0.77206				
		1	−4.80059	0.95740	0.95826	0.95937	1.92556
		2	−6.40255	1.27726	1.27841	3.84688	
	10	0	−3.83686	0.42495	0.42514	0.42536	0.42562
			0.42591	0.42627	0.42674	0.42742	0.42945
		1	−4.31826	0.47834	0.47856	0.47880	0.47909
			0.47942	0.47982	0.48034	0.96389	
		2	−4.93642	0.54690	0.54715	0.54743	0.54776
			0.54813	0.54859	1.65045		
		3	−5.76016	0.63827	0.63856	0.63889	0.63927
			0.63971	2.56546			
		4	−6.91305	0.76615	0.76650	0.76690	0.76735
			3.84616				

Table 5: Coefficients for the BLIEs of the location parameter.

ψ	n	m	$a'_i,\ i = 1, 2, 3, \ldots, (n-m)$				
1	6	0	1.13734	−0.02352	−0.02535	−0.02722	−0.02923
			−0.03201				
		1	1.16691	−0.02904	−0.03114	−0.03327	−0.07346
		2	1.21050	−0.03726	−0.03976	−0.13349	
	10	0	1.08734	−0.00791	−0.00835	−0.00878	−0.0092
			−0.00963	−0.01007	−0.01053	−0.01105	−0.01182
		1	1.09800	−0.00898	−0.00945	−0.00992	−0.01038
			−0.01084	−0.01132	−0.01181	−0.0253	
		2	1.11115	−0.01031	−0.01083	−0.01134	−0.01184
			−0.01235	−0.01286	−0.04162		
		3	1.12790	−0.01202	−0.01259	−0.01316	−0.01372
			−0.01428	−0.06214			
		4	1.15005	−0.01429	−0.01494	−0.01558	−0.01621
			−0.08903				
2	6	0	1.13843	−0.02697	−0.02722	−0.02753	−0.02793
			−0.02877				
		1	1.16664	−0.03254	−0.03284	−0.0332	−0.06807
		2	1.20869	−0.04086	−0.04122	−0.12662	
	10	0	1.08942	−0.00965	−0.00970	−0.00976	−0.00982
			−0.00988	−0.00996	−0.01005	−0.01017	−0.01044
		1	1.09960	−0.01076	−0.01082	−0.01088	−0.01094
			−0.01102	−0.01110	−0.01120	−0.02288	
		2	1.11225	−0.01215	−0.01221	−0.01228	−0.01235
			−0.01243	−0.01252	−0.03832		
		3	1.12846	−0.01393	−0.01399	−0.01407	−0.01415
			−0.01424	−0.05808			
		4	1.15001	−0.01629	−0.01637	−0.01645	−0.01655
			−0.08436				
3	6	0	1.13871	−0.02752	−0.02759	−0.02768	−0.0278

(*Continued*)

			−0.02812				
		1	1.16665	−0.03309	−0.03317	−0.03327	−0.06712
		2	1.20844	−0.04142	−0.04152	−0.12550	
	10	0	1.08980	−0.00989	−0.00991	−0.00992	−0.00994
			−0.00996	−0.00998	−0.01001	−0.01005	−0.01015
		1	1.09987	−0.01101	−0.01102	−0.01104	−0.01106
			−0.01108	−0.01110	−0.01113	−0.02244	
		2	1.11242	−0.01240	−0.01241	−0.01243	−0.01245
			−0.01247	−0.01250	−0.03776		
		3	1.12854	−0.01418	−0.0142	−0.01422	−0.01424
			−0.01427	−0.05743			
		4	1.15000	−0.01656	−0.01658	−0.01660	−0.01663
			−0.08365				
4	6	0	1.13881	−0.02767	−0.0277	−0.02773	−0.02778
			−0.02793				
		1	1.16666	−0.03323	−0.03326	−0.0333	−0.06685
		2	1.20838	−0.04156	−0.04160	−0.12521	
	10	0	1.08991	−0.00996	−0.00996	−0.00997	−0.00997
			−0.00998	−0.00999	−0.01000	−0.01002	−0.01007
		1	1.09994	−0.01107	−0.01108	−0.01108	−0.01109
			−0.01110	−0.01110	−0.01112	−0.02231	
		2	1.11247	−0.01246	−0.01246	−0.01247	−0.01248
			−0.01249	−0.0125	−0.03761		
		3	1.12856	−0.01424	−0.01425	−0.01426	−0.01427
			−0.01428	−0.05726			
		4	1.15000	−0.01662	−0.01663	−0.01664	−0.01665
			−0.08346				

Table 6: Coefficients for the BLIEs of the scale parameter.

ψ	n	m	$b'_i,\ i = 1, 2, 3, \ldots, (n-m)$				
1	6	0	−0.64800	0.11675	0.12276	0.12885	0.13536
			0.14428				
		1	−0.78128	0.14161	0.14884	0.15612	0.33471
		2	−0.97992	0.17906	0.18811	0.61276	
	10	0	−0.68605	0.06685	0.06914	0.07140	0.07364
			0.07588	0.07816	0.08056	0.08324	0.08717
		1	−0.7647	0.07475	0.07730	0.07982	0.08231
			0.08481	0.08735	0.08998	0.18839	
		2	−0.86267	0.08467	0.08756	0.09040	0.09321
			0.09602	0.09886	0.31195		
		3	−0.98819	0.09747	0.10078	0.10404	0.10726
			0.11047	0.46816			
		4	−1.15506	0.11457	0.11846	0.12227	0.12603
			0.67373				
2	6	0	−1.48617	0.29051	0.29289	0.29576	0.29956
			0.30744				
		1	−1.78762	0.35003	0.35289	0.35631	0.72840
	6	2	−2.23759	0.43902	0.44258	1.35598	
	10	0	−1.59801	0.17315	0.17392	0.17476	0.17568
			0.17671	0.17790	0.17933	0.18125	0.18533
		1	−1.77882	0.19291	0.19377	0.19470	0.19573
			0.19687	0.19818	0.19976	0.40688	
		2	−2.00375	0.21754	0.21851	0.21956	0.22071
			0.22199	0.22346	0.68200		
		3	−2.29219	0.24915	0.25026	0.25145	0.25276

(*Continued*)

			0.25422	1.03435			
		4	−2.67611	0.29125	0.29254	0.29393	0.29546
			1.50294				
3	6	0	−2.34481	0.46546	0.46655	0.46793	0.46989
			0.47498				
		1	−2.81661	0.55947	0.56077	0.56241	1.13396
		2	−3.52263	0.70020	0.70181	2.12061	
	10	0	−2.52877	0.27877	0.27910	0.27948	0.27990
			0.28040	0.28098	0.28173	0.28280	0.28560
		1	−2.81202	0.31010	0.31047	0.31089	0.31136
			0.31190	0.31256	0.31338	0.63136	
		2	−3.16515	0.34917	0.34959	0.35006	0.35059
			0.35120	0.35193	1.06260		
		3	−3.61863	0.39936	0.39984	0.40037	0.40098
			0.40167	1.61641			
		4	−4.22284	0.46623	0.46679	0.46742	0.46812
			2.35428				
4	6	0	−3.20022	0.63805	0.63863	0.63938	0.64047
			0.64370				
		1	−3.84209	0.76624	0.76693	0.76782	1.54110
		2	−4.80374	0.95831	0.95917	2.88626	
	10	0	−3.45419	0.38257	0.38274	0.38294	0.38317
			0.38343	0.38376	0.38418	0.38479	0.38662
		1	−3.83947	0.42530	0.42550	0.42572	0.42597
			0.42626	0.42662	0.42708	0.85702	
		2	−4.32042	0.47866	0.47887	0.47912	0.47940
			0.47973	0.48014	1.44450		
		3	−4.93842	0.54721	0.54746	0.54775	0.54807
			0.54844	2.19948			
		4	−5.76214	0.63860	0.63889	0.63922	0.63959
			3.20583				

Next, we intend to check the efficiency of the linear estimators suggested in this section. The mean squared errors (MSEs) of the BLUEs and BLIEs of φ may be given by

$$\mathrm{MSE}(\widehat{\varphi}) = \sigma^2 V_1,$$

and

$$\mathrm{MSE}(\widetilde{\varphi}) = \sigma^2 \left\{ \frac{\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\varphi} + 1}{(\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\varphi})(\mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1})^2 + \mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}} \right\} = \sigma^2 \left(V_1 - \frac{V_3^2}{1 + V_2} \right),$$

respectively.

The MSEs of the BLUEs and BLIEs of σ are

$$\mathrm{MSE}(\widehat{\sigma}) = \sigma^2 V_2,$$

and

$$\mathrm{MSE}(\widetilde{\sigma}) = \sigma^2 \left\{ \frac{\mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}}{(\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\varphi})(\mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1})^2 + \mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}} \right\} = \frac{\sigma^2 V_2}{1 + V_2},$$

respectively.

Thus, the relative efficiency criterions (RECs) of the BLIEs of φ with respect to the BLUEs of φ is given by

$$\mathrm{REC}(\widetilde{\varphi}, \widehat{\varphi}) = \frac{V_1}{V_1 - \frac{V_3^2}{1+V_2}}. \tag{6.5}$$

Similarly, the RECs of the BLIEs of σ with respect to the BLUEs of σ is given by

$$\mathrm{REC}(\widetilde{\sigma},\widehat{\sigma}) = \frac{\mathrm{MSE}(\widehat{\sigma})}{\mathrm{MSE}(\widetilde{\sigma})} = 1 + V_2. \tag{6.6}$$

It is evident that both RECs are greater than 1 which verifies the superiority of the BLIEs.

We calculated the variances and covariances for the BLUEs of the location and scale parameters in terms of σ^2. Additionally, we calculated the RECs using (6.5) and (6.6) for Type-II right censored samples of sizes $n = 6$ and 10 with m ranging from 0 to $[n/2]-1$ and $\psi = 1, \cdots, 4$. The results are summarized in Table 9.

Table 7: Simulated quantiles of T_1 and T_2.

ψ	n	m	T_1				T_2			
			0.025	0.05	0.95	0.975	0.025	0.05	0.95	0.975
1	6	0	-0.8978	-0.8737	2.8295	3.9428	-1.5398	-1.3839	1.8656	2.3050
		1	-0.8807	-0.8568	2.9112	4.1003	-1.4721	-1.3298	1.8152	2.3110
		2	-0.8521	-0.8288	3.5163	5.3046	-1.4026	-1.2924	1.8865	2.4439
	10	0	-0.9321	-0.9082	2.4709	3.5555	-1.6687	-1.4476	1.8083	2.2341
		1	-0.9250	-0.8990	2.4891	3.6256	-1.6572	-1.4439	1.8288	2.3033
		2	-0.9173	-0.8950	2.5101	3.5760	-1.6216	-1.4252	1.8541	2.2587
		3	-0.9099	-0.8845	2.6774	3.7372	-1.5690	-1.3974	1.8148	2.2806
		4	-0.8978	-0.8746	2.8998	4.1416	-1.5393	-1.3829	1.8490	2.3085
2	6	0	-0.8897	-0.8663	2.7817	3.9359	-1.5009	-1.3672	1.7675	2.2691
		1	-0.8726	-0.8489	3.0881	4.5173	-1.4785	-1.3157	1.8894	2.4233
		2	-0.8440	-0.8227	3.6865	5.5509	-1.3804	-1.2664	1.8961	2.4077
	10	0	-0.9260	-0.9012	2.3966	3.3919	-1.6343	-1.4574	1.7925	2.2208
		1	-0.9197	-0.8964	2.6083	3.5056	-1.6053	-1.4251	1.8601	2.3783
		2	-0.9137	-0.8867	2.5283	3.4814	-1.5701	-1.4034	1.8431	2.2891
		3	-0.9039	-0.8822	2.6312	3.7844	-1.5539	-1.3877	1.8838	2.3184
		4	-0.8917	-0.8693	2.8245	4.0445	-1.4962	-1.3358	1.9245	2.3871
3	6	0	-0.8925	-0.8639	2.7511	4.0257	-1.5196	-1.3476	1.8517	2.3827
		1	-0.8723	-0.8503	3.1442	4.6298	-1.4449	-1.3173	1.8643	2.3721
		2	-0.8436	-0.8197	3.6674	5.5449	-1.3846	-1.2760	1.8893	2.4688
	10	0	-0.9263	-0.9023	2.3569	3.2632	-1.6431	-1.4281	1.8186	2.2366
		1	-0.9199	-0.8936	2.4461	3.3830	-1.6130	-1.4119	1.8098	2.2553
		2	-0.9090	-0.8865	2.4581	3.5173	-1.5609	-1.3930	1.8141	2.2439
		3	-0.9022	-0.8780	2.6821	3.8591	-1.5594	-1.3945	1.8051	2.2466
		4	-0.8886	-0.8637	2.9251	4.1946	-1.4937	-1.3422	1.8371	2.3322
4	6	0	-0.8924	-0.8671	2.8063	4.2672	-1.5281	-1.3619	1.8677	2.3828
		1	-0.8722	-0.8487	3.0416	4.3763	-1.4421	-1.3071	1.9235	2.3853
		2	-0.8423	-0.8199	3.7080	5.6595	-1.3789	-1.2573	1.8838	2.4782
	10	0	-0.9266	-0.9012	2.3821	3.3501	-1.6292	-1.4180	1.7917	2.2129
		1	-0.9197	-0.8945	2.4726	3.4174	-1.6321	-1.4259	1.8601	2.2985
		2	-0.9107	-0.8869	2.5691	3.5821	-1.5857	-1.3812	1.8747	2.2729
		3	-0.9025	-0.8795	2.7359	3.7134	-1.5375	-1.3813	1.8365	2.3429
		4	-0.8885	-0.8644	2.8087	4.1154	-1.5091	-1.3436	1.8384	2.3835

Table 8: Simulated quantiles of T_3 and T_4.

ψ	n	m	T_3				T_4			
			0.025	0.05	0.95	0.975	0.025	0.05	0.95	0.975
1	6	0	-0.9080	-0.8786	3.6233	4.9767	-1.9699	-1.8141	1.4354	1.8748
		1	-0.8991	-0.8685	3.9549	5.4769	-1.9556	-1.8133	1.3317	1.8275
		2	-0.8854	-0.8528	5.2013	7.6929	-1.9642	-1.8540	1.3248	1.8823

(*Continued*)

	10	0	-0.9329	-0.9063	2.8540	4.0611	-1.9889	-1.7678	1.4881	1.9139
		1	-0.9271	-0.8977	2.9274	4.2105	-1.9978	-1.7845	1.4882	1.9627
		2	-0.9214	-0.8957	3.0208	4.2469	-1.9870	-1.7906	1.4887	1.8933
		3	-0.9175	-0.8875	3.3126	4.5623	-1.9651	-1.7935	1.4187	1.8846
		4	-0.9106	-0.8822	3.7262	5.2424	-1.9749	-1.8184	1.4134	1.8730
2	6	0	-0.9062	-0.8774	3.6002	5.0169	-1.9437	-1.8100	1.3247	1.8263
		1	-0.8977	-0.8671	4.2248	6.0733	-1.9747	-1.8119	1.3932	1.9271
		2	-0.8841	-0.8540	5.4973	8.1232	-1.9544	-1.8404	1.3221	1.8337
	10	0	-0.9311	-0.9033	2.7885	3.9027	-1.9644	-1.7874	1.4625	1.8908
		1	-0.9264	-0.8999	3.0818	4.1011	-1.9559	-1.7756	1.5095	2.0277
		2	-0.9228	-0.8915	3.0621	4.1656	-1.9453	-1.7786	1.4679	1.9139
		3	-0.9163	-0.8906	3.2806	4.6497	-1.9596	-1.7934	1.4780	1.9127
		4	-0.9095	-0.8820	3.6599	5.1599	-1.9411	-1.7807	1.4796	1.9422
3	6	0	-0.9112	-0.8760	3.5713	5.1393	-1.9653	-1.7933	1.4060	1.9370
		1	-0.8989	-0.8704	4.3071	6.2327	-1.9437	-1.8160	1.3656	1.8734
		2	-0.8853	-0.8515	5.4818	8.1318	-1.9609	-1.8523	1.3130	1.8925
	10	0	-0.9322	-0.9054	2.7482	3.7642	-1.9753	-1.7602	1.4864	1.9044
		1	-0.9275	-0.8975	2.9016	3.9675	-1.9656	-1.7645	1.4572	1.9027
		2	-0.9184	-0.8923	2.9850	4.2127	-1.9379	-1.7701	1.4370	1.8669
		3	-0.9153	-0.8866	3.3453	4.7445	-1.9669	-1.8019	1.3976	1.8391
		4	-0.9067	-0.8760	3.7883	5.3511	-1.9402	-1.7887	1.3906	1.8857
4	6	0	-0.9115	-0.8803	3.6420	5.4406	-1.9747	-1.8085	1.4211	1.9363
		1	-0.8992	-0.8688	4.1768	5.9079	-1.9415	-1.8065	1.4240	1.8858
		2	-0.8837	-0.8522	5.5424	8.2984	-1.9558	-1.8342	1.3069	1.9013
	10	0	-0.9329	-0.9044	2.7778	3.8633	-1.9620	-1.7508	1.4588	1.8801
		1	-0.9275	-0.8988	2.9331	4.0083	-1.9852	-1.7790	1.5070	1.9454
		2	-0.9206	-0.8930	3.1149	4.2895	-1.9633	-1.7588	1.4971	1.8953
		3	-0.9159	-0.8887	3.4105	4.5729	-1.9454	-1.7892	1.4286	1.9350
		4	-0.9068	-0.8772	3.6464	5.2555	-1.9561	-1.7905	1.3915	1.9366

Table 9: Variances and covariances for the BLUEs of the location and scale parameters in terms of σ^2 and the RECs of the BLIEs with respect to the BLUEs.

ψ	n	m	V_1	V_2	V_3	REC($\widetilde{\varphi}, \widehat{\varphi}$)	REC($\widetilde{\sigma}, \widehat{\sigma}$)
1	6	0	0.05615	0.18505	−0.04238	1.02774	1.18505
		1	0.05857	0.23378	−0.05325	1.04084	1.23378
		2	0.06256	0.31541	−0.07129	1.06582	1.31541
	10	0	0.01908	0.10250	−0.01428	1.00979	1.10250
		1	0.01933	0.11602	−0.01612	1.01219	1.11602
		2	0.01964	0.13350	−0.01847	1.01556	1.13349
		3	0.02006	0.15688	−0.02161	1.02052	1.15688
		4	0.02065	0.18970	−0.02598	1.02826	1.18970
2	6	0	0.01045	0.19612	−0.01852	1.02820	1.19612
		1	0.01089	0.24622	−0.02321	1.04134	1.24622
		2	0.01162	0.32949	−0.03101	1.06637	1.32949
	10	0	0.00350	0.10892	−0.00618	1.00996	1.10892
		1	0.00354	0.12288	−0.00697	1.01237	1.12288
		2	0.00360	0.14078	−0.00798	1.01576	1.14078
		3	0.00367	0.16461	−0.00932	1.02072	1.16461
		4	0.00378	0.19794	−0.01119	1.02847	1.19794
3	6	0	0.00420	0.19864	−0.01180	1.02843	1.19864
		1	0.00438	0.24874	−0.01477	1.04155	1.24874
		2	0.00467	0.33210	−0.01971	1.06656	1.33210
	10	0	0.00140	0.11035	−0.00394	1.01005	1.11035
		1	0.00142	0.12429	−0.00443	1.01245	1.12429
		2	0.00144	0.14218	−0.00507	1.01583	1.14218

(*Continued*)

		3	0.00147	0.16601	−0.00591	1.02079	1.16601
		4	0.00152	0.19936	−0.00710	1.02854	1.19936
4	6	0	0.00226	0.19942	−0.00866	1.02851	1.19940
		1	0.00235	0.24948	−0.01084	1.04161	1.24947
		2	0.00251	0.33283	−0.01445	1.06662	1.33283
	10	0	0.00075	0.11078	−0.00289	1.01008	1.11078
		1	0.00076	0.12470	−0.00325	1.01248	1.12470
		2	0.00078	0.14258	−0.00372	1.01586	1.14258
		3	0.00079	0.16640	−0.00434	1.02082	1.16640
		4	0.00081	0.19974	−0.00521	1.02856	1.19974

7. Linear Prediction

Let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n-m:n}$, $m = 1, ..., n-1$, represent a Type-II right censored sample from $XL(\varphi, \sigma, \psi)$. Furthermore, define $Z_{r:n} = \dfrac{X_{r:n} - \varphi}{\sigma}$, where $\mathrm{E}(Z_{r:n}) = \varphi^{(1)}_{r:n}$, $1 \leq r \leq n-m$, and $\mathrm{Cov}(Z_{r:n}, Z_{s:n}) = \sigma_{r,s:n} = \varphi^{(1,1)}_{r,s:n} - \varphi^{(1)}_{r:n}\varphi^{(1)}_{s:n}$, $1 \leq r < s \leq n-m$.

Suppose $Z_{q:n} = \dfrac{X_{q:n} - \varphi}{\sigma}$ for $n-m+1 \leq q \leq n$. So, the best linear unbiased predictor (BLUP) of $X_{q:n}$ can be expressed as (see Christensen [16], Goldberger [19] and Kaminski and Nelson [23])

$$\widehat{X}_{q:n} = \widehat{\varphi} + \widehat{\sigma}\varphi_{q:n} + \boldsymbol{w}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{X} - \widehat{\varphi}\mathbf{1} - \widehat{\sigma}\boldsymbol{\varphi}\right),$$

where $\varphi_{q:n}$ is the mean of $Z_{q:n}$, $\widehat{\varphi}$ and $\widehat{\sigma}$ are the BLUEs of the location and scale parameters, respectively, and

$$\boldsymbol{w}^{\mathrm{T}} = \Big(\mathrm{Cov}(Z_{1:n}, Z_{q:n}), \cdots, \mathrm{Cov}(Z_{n-m:n}, Z_{q:n})\Big).$$

The mean squared prediction error (MSPE) of $\widehat{X}_{q:n}$ is (see, for example, Burkschat [13])

$$\begin{aligned}\mathrm{MSPE}(\widehat{X}_{q:n}) &= \mathrm{E}\left[\left(\widehat{X}_{q:n} - X_{q:n}\right)^2\right]\\ &= \sigma^2\Big[\left(1 - \boldsymbol{w}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}\right)^2 V_1 + \left(\varphi_{q:n} - \boldsymbol{w}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\varphi}\right)^2 V_2 - \boldsymbol{w}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{w}\\ &\quad +2\left(1 - \boldsymbol{w}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}\right)\left(\varphi_{q:n} - \boldsymbol{w}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\varphi}\right) V_3 + \mathrm{Var}(Z_{q:n})\Big].\end{aligned}$$

According to the findings of Mann [27], the best linear invariant predictor (BLIP) of $X_{q:n}$ can be expressed as

$$\widetilde{X}_{q:n} = \widetilde{\varphi} + \widetilde{\sigma}\varphi_{q:n} + \boldsymbol{w}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{X} - \widetilde{\varphi}\mathbf{1} - \widetilde{\sigma}\boldsymbol{\varphi}\right) = \widehat{X}_{q:n} - \left(\frac{V_4}{1+V_2}\right)\widehat{\sigma},$$

where $\widetilde{\varphi}$ and $\widetilde{\sigma}$ are the BLIEs of the location and scale parameters, respectively, and

$$V_4 = (1 - \boldsymbol{w}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1})V_3 + \left(\varphi_{q:n} - \boldsymbol{w}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\varphi}\right)V_2.$$

The MSPE of $\widetilde{X}_{q:n}$ is given by (see, for example, Burkschat [13])

$$\begin{aligned}\mathrm{MSPE}(\widetilde{X}_{q:n}) &= \mathrm{E}\left[\left(\widetilde{X}_{q:n} - X_{q:n}\right)^2\right]\\ &= \sigma^2\Big[\left(1 - \boldsymbol{w}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}\right)^2\left(\frac{V_1}{1+V_2} + \frac{1}{\Delta}\right) + \left(\varphi_{q:n} - \boldsymbol{w}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\varphi}\right)^2 \frac{V_2}{1+V_2} - \boldsymbol{w}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{w}\\ &\quad +2\left(1 - \boldsymbol{w}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}\right)\left(\varphi_{q:n} - \boldsymbol{w}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\varphi}\right)\frac{V_3}{1+V_2} + \mathrm{Var}(Z_{q:n})\Big],\end{aligned}$$

where

$$\Delta = (\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\varphi})(\mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1})^2 + \mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}.$$

To compare the BLUP and BLIP, we consider the REC of $\widetilde{X}_{q:n}$ with respect to $\widehat{X}_{q:n}$, which is given by

$$\mathrm{REC}(\widetilde{X}_{q:n}, \widehat{X}_{q:n}) = \frac{\mathrm{MSPE}(\widehat{X}_{q:n})}{\mathrm{MSPE}(\widetilde{X}_{q:n})}.$$

Table 10 displays the computed values of $\mathrm{REC}(\widetilde{X}_{q:n}, \widehat{X}_{q:n})$ and coefficients V_4 for $n = 6$ and 10 with m ranging from 1 to $([n/2] - 1)$ and $\psi = 1, \cdots, 4$. Table 10 verifies the superiority of the BLIPs over the BLUPs for the cases that are considered.

We now turn our attention to the construction of prediction intervals (PIs) for the order statistic for $X_{q:n}$. These PIs can be formulated using specific pivotal quantities (see, for example, Balakrishnan and Chan [8]):

$$T_1^* = \frac{X_{q:n} - X_{(n-m):n}}{\hat{\sigma}},$$

and

$$T_2^* = \frac{X_{q:n} - X_{(n-m):n}}{\widetilde{\sigma}}.$$

To establish these PIs, it is essential to determine the percentage points of T_1^* and T_2^*. The simulated percentage points for these pivotal quantities are presented in Table 11, calculated from a Monte Carlo simulation comprising 10,000 iterations, with different values of n and ψ.

By employing the pivotal quantity T_1^*, we can express a $100(1-\Upsilon)\%$ PI for $X_{q:n}$ as follows:

$$\mathrm{P}\left(X_{(n-m):n} + \hat{\sigma}T_1^*(\Upsilon/2) \le X_{q:n} \le X_{(n-m):n} + \hat{\sigma}T_1^*(1-\Upsilon/2)\right) = 1 - \Upsilon,$$

where $T_1^*(\tau)$ denotes the left percentage point of T_1^* at τ, i.e., $\mathrm{P}(T_1^* < T_1^*(\tau)) = \tau$.

Similarly, by employing the pivotal quantity T_2^*, we can express a $100(1-\Upsilon)\%$ PI for $X_{q:n}$ as follows:

$$\mathrm{P}\left(X_{(n-m):n} + \tilde{\sigma}T_1^*(\Upsilon/2) \le X_{q:n} \le X_{(n-m):n} + \tilde{\sigma}T_1^*(1-\Upsilon/2)\right) = 1 - \Upsilon,$$

where $T_2^*(\tau)$ denotes the left percentage point of T_2^* at τ, i.e., $\mathrm{P}(T_2^* < T_2^*(\tau)) = \tau$.

Table 10: Coefficients V_4 and the RECs of BLIPs with respect to the BLUPs.

ψ	n	m	$(n-m+1) \le q \le n$							
			V_4				RECs			
1	6	1	0.28499				1.04112			
		2	0.19768	0.57698			1.06617	1.10296		
	10	1	0.14063				1.01226			
		2	0.08314	0.24312			1.01566	1.02592		
		3	0.06613	0.16277	0.34927		1.02063	1.03558	1.0432	
		4	0.06059	0.13974	0.25561	0.47973	1.02837	1.04889	1.06281	1.06737
2	6	1	0.13624				1.04189			
		2	0.09174	0.27331			1.06682	1.10406		
	10	1	0.06795				1.01259			
		2	0.03918	0.11673			1.01594	1.02645		
		3	0.03063	0.0763	0.16674		1.02089	1.03607	1.04383	

(*Continued*)

		4	0.02767	0.0644	0.11919	0.22773	1.02862	1.04937	1.06344	1.06798
3	6	1	0.088				1.0418			
		2	0.05887	0.17618			1.06675	1.10377		
	10	1	0.04397				1.01256			
		2	0.0252	0.07543			1.01591	1.02636		
		3	0.01963	0.04903	0.10761		1.02086	1.036	1.04365	
		4	0.01769	0.04125	0.07652	0.14681	1.0286	1.04931	1.06331	1.06767
4	6	1	0.06484				1.04174			
		2	0.04329	0.12973			1.06671	1.10361		
	10	1	0.03241				1.01253			
		2	0.01854	0.05558			1.01589	1.0263		
		3	0.01443	0.03606	0.07926		1.02085	1.03596	1.04356	
		4	0.013	0.03031	0.05627	0.10811	1.02858	1.04928	1.06324	1.06753

Table 11: Simulated quantiles of T_1^* and T_2^*.

ψ	n	m	q	T_1^*				T_2^*			
1	6	1	6	0.0312	0.0607	4.9687	6.6338	0.0312	0.0607	4.9687	6.6338
		2	5	0.0167	0.0325	3.1080	4.4073	0.0167	0.0325	3.1080	4.4073
			6	0.2009	0.2892	8.0494	11.3947	0.2009	0.2892	8.0494	11.3947
	10	1	10	0.0296	0.0572	4.1384	5.2123	0.0296	0.0572	4.1384	5.2123
		2	9	0.0146	0.0299	2.2299	2.8513	0.0146	0.0299	2.2299	2.8513
			10	0.1977	0.2941	5.5721	6.9617	0.1977	0.2941	5.5721	6.9617
		3	8	0.0113	0.0206	1.5785	1.9908	0.0113	0.0206	1.5785	1.9908
			9	0.1086	0.1586	3.1891	4.0463	0.1086	0.1586	3.1891	4.0463
			10	0.3741	0.5194	6.4680	8.0829	0.3741	0.5194	6.4680	8.0829
		4	7	0.0075	0.0157	1.2673	1.6964	0.0075	0.0157	1.2673	1.6964
			8	0.0821	0.1201	2.4486	3.1193	0.0821	0.1201	2.4486	3.1193
			9	0.2342	0.3170	4.2819	5.4436	0.2342	0.3170	4.2819	5.4436
			10	0.5415	0.6866	7.6615	9.9841	0.5415	0.6866	7.6615	9.9841
2	6	1	6	0.0171	0.0312	2.3784	3.1912	0.0171	0.0312	2.3784	3.1912
		2	5	0.0068	0.0142	1.3616	1.9148	0.0068	0.0142	1.3616	1.9148
			6	0.0870	0.1252	3.7318	5.0208	0.0870	0.1252	3.7318	5.0208
	10	1	10	0.0134	0.0270	2.0137	2.6044	0.0134	0.0270	2.0137	2.6044
		2	9	0.0068	0.0138	1.0655	1.3676	0.0068	0.0138	1.0655	1.3676
			10	0.0918	0.1376	2.5513	3.1611	0.0918	0.1376	2.5513	3.1611
		3	8	0.0053	0.0104	0.7227	0.9525	0.0053	0.0104	0.7227	0.9525
			9	0.0513	0.0764	1.4957	1.8966	0.0513	0.0764	1.4957	1.8966
			10	0.1753	0.2309	3.0790	3.9131	0.1753	0.2309	3.0790	3.9131
		4	7	0.0038	0.0075	0.5666	0.7532	0.0038	0.0075	0.5666	0.7532
			8	0.0380	0.0554	1.1184	1.4284	0.0380	0.0554	1.1184	1.4284
			9	0.1012	0.1418	1.9145	2.4569	0.1012	0.1418	1.9145	2.4569
			10	0.2412	0.3169	3.5842	4.4294	0.2412	0.3169	3.5842	4.4294
3	6	1	6	0.0081	0.0187	1.5986	2.1286	0.0081	0.0187	1.5986	2.1286
		2	5	0.0043	0.0093	0.9151	1.2752	0.0043	0.0093	0.9151	1.2752
			6	0.0556	0.0838	2.5309	3.4847	0.0556	0.0838	2.5309	3.4847
	10	1	10	0.0089	0.0185	1.2697	1.6020	0.0089	0.0185	1.2697	1.6020
		2	9	0.0048	0.0096	0.6572	0.8626	0.0048	0.0096	0.6572	0.8626
			10	0.0578	0.0851	1.6583	2.1177	0.0578	0.0851	1.6583	2.1177
		3	8	0.0031	0.0066	0.4591	0.6062	0.0031	0.0066	0.4591	0.6062
			9	0.0336	0.0491	0.9747	1.2620	0.0336	0.0491	0.9747	1.2620
			10	0.1079	0.1481	2.0206	2.5351	0.1079	0.1481	2.0206	2.5351
		4	7	0.0023	0.0048	0.3673	0.5027	0.0023	0.0048	0.3673	0.5027
			8	0.0240	0.0346	0.7319	0.9273	0.0240	0.0346	0.7319	0.9273
			9	0.0661	0.0891	1.2601	1.5855	0.0661	0.0891	1.2601	1.5855
			10	0.1535	0.1997	2.3646	2.9617	0.1535	0.1997	2.3646	2.9617

(*Continued*)

4	6	1	6	0.0071	0.0143	1.1443	1.6182	0.0071	0.0143	1.1443	1.6182
		2	5	0.0035	0.0072	0.6715	0.9427	0.0035	0.0072	0.6715	0.9427
			6	0.0410	0.0617	1.7516	2.4405	0.0410	0.0617	1.7516	2.4405
	10	1	10	0.0064	0.0130	0.9314	1.2081	0.0064	0.0130	0.9314	1.2081
		2	9	0.0035	0.0070	0.4874	0.6370	0.0035	0.0070	0.4874	0.6370
			10	0.0409	0.0621	1.2365	1.5721	0.0409	0.0621	1.2365	1.5721
		3	8	0.0021	0.0043	0.3262	0.4301	0.0021	0.0043	0.3262	0.4301
			9	0.0229	0.0346	0.7195	0.9092	0.0229	0.0346	0.7195	0.9092
			10	0.0841	0.1124	1.4966	1.8947	0.0841	0.1124	1.4966	1.8947
		4	7	0.0016	0.0035	0.2679	0.3548	0.0016	0.0035	0.2679	0.3548
			8	0.0172	0.0256	0.5232	0.6678	0.0172	0.0256	0.5232	0.6678
			9	0.0483	0.0671	0.9351	1.1675	0.0483	0.0671	0.9351	1.1675
			10	0.1144	0.1475	1.7556	2.2218	0.1144	0.1475	1.7556	2.2218

8. A Simulation Study

Here, we conduct a simulation to evaluate the suggested estimators and predictors, aiming to determine whether the simulation outcomes align with the theoretical findings presented in the previous sections. The parameters considered here include $\psi = 1, 2$ and $\varphi = 0, \sigma = 1$ with sample sizes $n = 6$ and 10. For $n = 6$, the values of m are set to 1 and 2, and for $n = 10$, m takes on the values of $1, 2, 3$ and 4. We perform $N = 1000$ replications for the simulation. We derive the BLUEs and BLIEs of the location and scale parameters and the BLUPs and BLIPs of $X_{q:n}$ where $q = n - m + 1$. Assume $\widehat{Y}$ symbolizes a predictor of Y and $\widehat{Y_i}$ is a prediction of Y_i, where $\widehat{Y_i}$ is derived in the i-th replication and Y_i is the real value of Y generated at the same replication. Now, the estimated bias (bias for short) and the estimated MSPE (EMSPE) of $\widehat{Y}$ are defined as follows

$$\text{bias}(\widehat{Y}) = \frac{1}{N}\sum_{i=1}^{N}(\widehat{Y_i} - Y_i), \qquad \text{and} \qquad \text{EMSPE}(\widehat{Y}) = \frac{1}{N}\sum_{i=1}^{N}(\widehat{Y_i} - Y_i)^2,$$

respectively.

We compute and present the averages of the BLUEs and BLIEs for the location and scale parameters, along with the BLUPs and BLIPs for $X_{q:n}$, and their biases and EMSPEs in Table 12. Additionally, we compare the performance of the 95% CIs and PIs using the average width (AW) and estimated coverage probability (CP for short). The results for 95% CIs of the location and scale parameters and 95% PIs are reported in Table 13. Table 12 reveals that 1) the means of the BLUEs are more closely aligned with the true parameter value than those of BLIEs, which is consistent with our expectations 2) The EMSPEs of the BLIPs are lower than those of the corresponding BLUPs, which supports the previous theoretical findings.

From Table 13, it is observed that 1) the 95% CIs for φ based on T_1 and T_3 display identical performance, with one exception, 2) the 95% CIs for σ based on T_2 and T_4 show very little difference in their performance, 3) the 95% PIs based on T_2^* exhibit smaller AWs compared to those based on T_1^*, whereas the corresponding CPs are rather close to each other.

9. A Real Data Example

The real data considered in this section are related to the fatigue life (in hours) for 10 bearings of a certain type, due to McCool [28]. The data, arranged in order, are:

152.7, 172.0, 172.5, 173.3, 193.0, 204.7, 216.5, 234.9, 262.6, 422.6.

We computed the coefficient of correlation between the data presented above and the means shown in Table 1 for $n = 10$ and $\psi = 4$. The result exceeded 0.963, allowing us to conclude that the data likely originates from the XLindley distribution with $\psi = 4$.

We calculated the BLUEs and BLIEs of the location and scale parameters along with their respective MSEs expressed in terms of σ^2. Besides, we determined the predictions for $X_{10:10}$ and the MSPEs in terms of σ^2, see Table 14.

Table 12: Averages of the BLUEs and BLIEs of φ and σ and BLUPs and BLIPs of $X_{q:n}$ (Y for short) along with their biases and EMSPEs for $XL(0,1,\psi)$.

ψ	n	m	q	$\widehat{\varphi}$	$\widetilde{\varphi}$	$\widehat{\sigma}$	$\widetilde{\sigma}$	$\widehat{Y}$	$\widetilde{Y}$	bias $\widehat{Y}$	bias $\widetilde{Y}$	EMSPE $\widehat{Y}$	EMSPE $\widetilde{Y}$
1	6	1	6	0.0132	0.0474	0.7920	0.6419	2.35	2.1670	0.2991	0.1161	0.7240	0.5655
		2	5	0.0072	0.0516	0.8191	0.6227	1.4839	1.3608	0.0656	-0.0575	0.2695	0.2321
			6	0.0072	0.0516	0.8191	0.6227	2.4469	2.0876	0.3960	0.0367	1.4803	1.0031
	10	1	10	0.0144	0.0266	0.8442	0.7564	2.934	2.8276	0.4212	0.3148	0.5331	0.4332
		2	9	0.0119	0.0260	0.8626	0.7610	2.0412	1.9780	0.0875	0.0242	0.2184	0.2037
			10	0.0119	0.0260	0.8626	0.7610	3.0419	2.8569	0.5291	0.3441	1.0213	0.7936
		3	8	0.0094	0.0259	0.8809	0.7614	1.5793	1.5289	0.0519	0.0015	0.1178	0.1091
			9	0.0094	0.0259	0.8809	0.7614	2.1037	1.9797	0.1499	0.0260	0.4481	0.3921
			10	0.0094	0.0259	0.8809	0.7614	3.1251	2.8591	0.6123	0.3464	1.5230	1.1181
		4	7	0.0080	0.0275	0.8919	0.7497	1.2435	1.1980	0.0203	-0.0251	0.0766	0.0722
			8	0.0080	0.0275	0.8919	0.7497	1.6039	1.4992	0.0765	-0.0283	0.2490	0.2181
			9	0.0080	0.0275	0.8919	0.7497	2.1347	1.9431	0.1810	-0.0106	0.6608	0.5453
			10	0.0080	0.0275	0.8919	0.7497	3.1688	2.8091	0.6560	0.2963	2.0108	1.3850
2	6	1	6	0.0105	0.0246	0.7526	0.6039	1.0091	0.9268	0.1432	0.0609	0.1489	0.1167
		2	5	0.0066	0.0251	0.7950	0.5980	0.6437	0.5888	0.0467	-0.0082	0.0437	0.0350
			6	0.0066	0.0251	0.7950	0.5980	1.0789	0.9154	0.2129	0.0495	0.2870	0.1844
	10	1	10	0.0053	0.0104	0.8272	0.7367	1.3044	1.2543	0.1715	0.1214	0.1306	0.1107
		2	9	0.0042	0.0101	0.8472	0.7426	0.8969	0.8678	0.0437	0.0146	0.0444	0.0409
			10	0.0042	0.0101	0.8472	0.7426	1.3588	1.2721	0.2259	0.1392	0.2296	0.1818
		3	8	0.0034	0.0103	0.8609	0.7392	0.6816	0.6590	0.0176	-0.005	0.0237	0.0223
			9	0.0034	0.0103	0.8609	0.7392	0.9183	0.8619	0.0651	0.0087	0.0831	0.0722
			10	0.0034	0.0103	0.8609	0.7392	1.3876	1.2643	0.2547	0.1315	0.3107	0.2314
		4	7	0.0024	0.0106	0.8786	0.7334	0.5378	0.5175	0.0147	-0.0056	0.0145	0.0134
			8	0.0024	0.0106	0.8786	0.7334	0.6996	0.6523	0.0356	-0.0117	0.0468	0.0410
			9	0.0024	0.0106	0.8786	0.7334	0.9410	0.8536	0.0878	0.0004	0.1293	0.1049
			10	0.0024	0.0106	0.8786	0.7334	1.4199	1.2529	0.2871	0.1201	0.4200	0.2916
3	6	1	6	0.0050	0.0140	0.7686	0.6155	0.6604	0.6063	0.0948	0.0406	0.0598	0.0461
		2	5	0.0031	0.0149	0.8006	0.6010	0.4122	0.3768	0.0226	-0.0128	0.0198	0.0169
			6	0.0031	0.0149	0.8006	0.6010	0.6943	0.5884	0.1287	0.0228	0.1141	0.0750
	10	1	10	0.0023	0.0055	0.8255	0.7342	0.8245	0.7922	0.1147	0.0824	0.0510	0.0424
		2	9	0.0013	0.0051	0.8529	0.7467	0.5726	0.5538	0.0386	0.0198	0.0172	0.0152
			10	0.0013	0.0051	0.8529	0.7467	0.8727	0.8164	0.1628	0.1065	0.0975	0.0753
		3	8	0.0007	0.0051	0.8698	0.7459	0.4360	0.4213	0.0139	-0.0008	0.0096	0.0090
			9	0.0007	0.0051	0.8698	0.7459	0.5895	0.5529	0.0555	0.0189	0.0339	0.0284
			10	0.0007	0.0051	0.8698	0.7459	0.8955	0.8152	0.1856	0.1053	0.1387	0.1014
		4	7	0.0001	0.0054	0.8856	0.7384	0.3419	0.3288	0.0084	-0.0047	0.0064	0.0059
			8	0.0001	0.0054	0.8856	0.7384	0.4462	0.4158	0.0241	-0.0063	0.0200	0.0174
			9	0.0001	0.0054	0.8856	0.7384	0.6025	0.5460	0.0685	0.0120	0.0536	0.0418
			10	0.0001	0.0054	0.8856	0.7384	0.9141	0.8057	0.2042	0.0958	0.1885	0.1288
4	6	1	6	0.0035	0.0102	0.7701	0.6163	0.4871	0.4471	0.0672	0.0272	0.0304	0.0231
		2	5	0.0024	0.0110	0.7947	0.5962	0.3000	0.2742	0.0128	-0.013	0.0117	0.0100
			6	0.0024	0.0110	0.7947	0.5962	0.5062	0.4289	0.0863	0.0089	0.0635	0.0418
	10	1	10	0.0010	0.0034	0.8137	0.7234	0.6006	0.5772	0.0867	0.0632	0.0267	0.0221
		2	9	0.0003	0.0031	0.8399	0.7351	0.4169	0.4033	0.0273	0.0136	0.0095	0.0086
			10	0.0003	0.0031	0.8399	0.7351	0.6347	0.5938	0.1207	0.0799	0.0493	0.0378
		3	8	0.0000	0.0032	0.8529	0.7312	0.3158	0.3052	0.0079	-0.0027	0.0050	0.0048
			9	0.0000	0.0032	0.8529	0.7312	0.4265	0.4001	0.0368	0.0104	0.0175	0.0149
			10	0.0000	0.0032	0.8529	0.7312	0.6476	0.5897	0.1336	0.0757	0.0682	0.0493
		4	7	0.0000	0.0037	0.8537	0.7115	0.2422	0.2329	0.0003	-0.009	0.0033	0.0032
			8	0.0000	0.0037	0.8537	0.7115	0.3161	0.2945	0.0082	-0.0133	0.0105	0.0096
			9	0.0000	0.0037	0.8537	0.7115	0.4269	0.3869	0.0373	-0.0028	0.0269	0.0220
			10	0.0000	0.0037	0.8537	0.7115	0.6483	0.5713	0.1343	0.0574	0.0882	0.0607

Table 13: The AWs and CPs of the 95% CIs and PIs for $XL(0, 1, \psi)$.

ψ	n	m	q	Average Width						Coverage Probability					
				T_1	T_2	T_3	T_4	T_1^*	T_2^*	T_1	T_2	T_3	T_4	T_1^*	T_2^*
1	6	1	6	0.9548	2.3738	0.9548	2.3737	5.5748	4.5185	0.930	0.936	0.930	0.936	0.943	0.942
		2	5	1.2614	3.5134	1.2614	3.5132	3.4676	2.6361	0.945	0.951	0.945	0.951	0.946	0.942
			6	1.2614	3.5134	1.2614	3.5132	8.9030	6.7682	0.945	0.951	0.945	0.951	0.939	0.939
	10	1	10	0.5340	1.4652	0.5341	1.4652	4.3934	3.9367	0.942	0.936	0.942	0.936	0.962	0.965
		2	9	0.5432	1.6441	0.5432	1.6441	2.4874	2.1944	0.940	0.948	0.940	0.948	0.958	0.954
			10	0.5432	1.6441	0.5432	1.6441	5.6851	5.0155	0.940	0.948	0.940	0.948	0.946	0.952
		3	8	0.5799	1.8642	0.5799	1.8643	1.7409	1.5048	0.944	0.946	0.944	0.946	0.942	0.938
			9	0.5799	1.8642	0.5799	1.8643	3.5865	3.1001	0.944	0.946	0.944	0.946	0.955	0.951
			10	0.5799	1.8642	0.5799	1.8643	6.9970	6.0482	0.944	0.946	0.944	0.946	0.943	0.953
		4	7	0.6459	2.2615	0.6459	2.2617	1.5206	1.2781	0.948	0.957	0.948	0.957	0.954	0.941
			8	0.6459	2.2615	0.6459	2.2617	2.6303	2.2109	0.948	0.957	0.948	0.957	0.949	0.941
			9	0.6459	2.2615	0.6459	2.2617	4.6472	3.9062	0.948	0.957	0.948	0.957	0.950	0.948
			10	0.6459	2.2615	0.6459	2.2617	8.1863	6.8810	0.948	0.957	0.948	0.957	0.947	0.950
2	6	1	6	0.4234	2.4838	0.4234	2.4838	2.4241	1.9452	0.936	0.934	0.936	0.934	0.955	0.959
		2	5	0.5481	3.4951	0.5481	3.4950	1.6197	1.2183	0.943	0.944	0.943	0.944	0.963	0.958
			6	0.5481	3.4951	0.5481	3.4950	4.0698	3.0612	0.943	0.944	0.943	0.944	0.947	0.956
	10	1	10	0.2178	1.4407	0.2178	1.4407	2.0821	1.8542	0.943	0.933	0.943	0.933	0.948	0.946
		2	9	0.2233	1.6061	0.2233	1.6061	1.1211	0.9827	0.935	0.930	0.935	0.929	0.945	0.947
			10	0.2233	1.6061	0.2233	1.6061	2.7398	2.4017	0.935	0.930	0.935	0.929	0.937	0.941
		3	8	0.2446	1.8860	0.2446	1.8859	0.7709	0.6620	0.947	0.953	0.947	0.953	0.934	0.925
			9	0.2446	1.8860	0.2446	1.8859	1.6350	1.4039	0.947	0.953	0.947	0.953	0.966	0.969
			10	0.2446	1.8860	0.2446	1.8859	3.1688	2.7209	0.947	0.953	0.947	0.953	0.943	0.955
		4	7	0.2666	2.2018	0.2666	2.2018	0.6655	0.5556	0.953	0.948	0.953	0.948	0.952	0.951
			8	0.2666	2.2018	0.2666	2.2018	1.2485	1.0422	0.953	0.948	0.953	0.948	0.955	0.955
			9	0.2666	2.2018	0.2666	2.2018	2.1234	1.7726	0.953	0.948	0.953	0.948	0.945	0.956
			10	0.2666	2.2018	0.2666	2.2018	3.9378	3.2871	0.953	0.948	0.953	0.948	0.945	0.952
3	6	1	6	0.2798	2.3992	0.2798	2.3995	1.6665	1.3346	0.941	0.937	0.941	0.937	0.941	0.945
		2	5	0.3496	3.6315	0.3496	3.6317	0.9740	0.7312	0.951	0.963	0.951	0.963	0.954	0.947
			6	0.3496	3.6315	0.3496	3.6317	2.6200	1.9668	0.951	0.963	0.951	0.963	0.949	0.952
	10	1	10	0.1339	1.4540	0.1339	1.4540	1.3167	1.1711	0.959	0.932	0.959	0.932	0.962	0.962
		2	9	0.1434	1.6110	0.1434	1.6109	0.7595	0.6650	0.952	0.931	0.952	0.931	0.954	0.957
			10	0.1434	1.6110	0.1434	1.6109	1.7906	1.5677	0.952	0.931	0.952	0.931	0.953	0.956
		3	8	0.1589	1.9312	0.1589	1.9314	0.5197	0.4457	0.961	0.953	0.961	0.953	0.953	0.949
			9	0.1589	1.9312	0.1589	1.9314	1.0608	0.9098	0.961	0.953	0.961	0.953	0.949	0.954
			10	0.1589	1.9312	0.1589	1.9314	2.0789	1.7829	0.961	0.953	0.961	0.953	0.944	0.955
		4	7	0.1752	2.2249	0.1752	2.2250	0.4169	0.3476	0.958	0.946	0.958	0.946	0.946	0.937
			8	0.1752	2.2249	0.1752	2.2250	0.8036	0.6700	0.958	0.946	0.958	0.946	0.953	0.949
			9	0.1752	2.2249	0.1752	2.2250	1.3313	1.1100	0.958	0.946	0.958	0.946	0.939	0.935
			10	0.1752	2.2249	0.1752	2.2250	2.5249	2.1052	0.958	0.946	0.958	0.946	0.927	0.947
4	6	1	6	0.1960	2.4017	0.1960	2.4014	1.1909	0.9531	0.937	0.930	0.937	0.930	0.952	0.951
		2	5	0.2588	3.5588	0.2588	3.5587	0.7468	0.5603	0.946	0.944	0.946	0.944	0.951	0.937
			6	0.2588	3.5588	0.2588	3.5587	1.9510	1.4638	0.946	0.944	0.946	0.944	0.954	0.957
	10	1	10	0.0974	1.4715	0.0974	1.4714	0.9702	0.8626	0.934	0.920	0.934	0.920	0.950	0.950
		2	9	0.1050	1.6413	0.1050	1.6413	0.5212	0.4561	0.933	0.921	0.933	0.921	0.960	0.957
			10	0.1050	1.6413	0.1050	1.6413	1.2709	1.1123	0.933	0.921	0.933	0.921	0.946	0.956
		3	8	0.1107	1.8516	0.1107	1.8516	0.3653	0.3132	0.940	0.929	0.940	0.929	0.963	0.958
			9	0.1107	1.8516	0.1107	1.8516	0.7327	0.6281	0.940	0.929	0.940	0.929	0.949	0.950
			10	0.1107	1.8516	0.1107	1.8516	1.5240	1.3066	0.940	0.929	0.940	0.929	0.951	0.956
		4	7	0.1218	2.2089	0.1218	2.2092	0.3025	0.2522	0.935	0.936	0.935	0.937	0.955	0.948
			8	0.1218	2.2089	0.1218	2.2092	0.5754	0.4796	0.935	0.936	0.935	0.937	0.955	0.949
			9	0.1218	2.2089	0.1218	2.2092	0.9307	0.7757	0.935	0.936	0.935	0.937	0.951	0.956
			10	0.1218	2.2089	0.1218	2.2092	1.7496	1.4583	0.935	0.936	0.935	0.937	0.961	0.970

Table 14: The BLUEs and BLIEs of the location and scale parameters and the computed MSEs in terms of σ^2, and the predictions of $X_{10:10}$ (Y for short) and the MSPEs in terms of σ^2 for the example.

	$\widehat{\varphi}$	$\widehat{\sigma}$	$\widetilde{\varphi}$	$\widetilde{\sigma}$	MSE($\widehat{\varphi}$)	MSE($\widehat{\sigma}$)	$Cov(\widehat{\varphi},\widehat{\sigma})$
Complete Sample	145.1533	289.8381	145.9070	260.9310	0.0008	0.1108	-0.0029
$c = 1, q = 10$	146.2211	248.8363	146.9404	221.2461	0.0008	0.1247	-0.0033

	MSE($\widetilde{\varphi}$)	MSE($\widetilde{\sigma}$)	$\widehat{Y}$	$\widetilde{Y}$	V_4	MSPE($\widehat{Y}$)	MSPE($\widetilde{Y}$)
Complete Sample	0.0007	0.0997					
$c = 1, q = 10$	0.0008	0.1109	327.1256	319.9548	0.0324	0.0755	0.0745

10. Conclusion

We studied the order statistics of the XLindley distribution and derived expressions for their single and product moments. We proceeded to calculate the means, variances and covariances of order statistics for specific cases. Following that, our focus shifted to deriving the BLUEs and BLIEs of the location and scale parameters. We focused on the methodologies for obtaining the BLUP and BLIP of a future unobserved order statistic. We additionally explored the construction of confidence intervals for both the location and scale parameters, alongside prediction intervals for an unobserved order statistic. Additionally, we conducted a simulation study and included an example. The results highlight that the BLIE outperforms the BLUE in terms of MSE and the BLIP surpasses the BLUP in terms of MSPE. Furthermore, prediction intervals based on the BLIE of the scale parameter showed better performance compared to those derived based on the BLUE of the scale parameter in terms of AW. In terms of confidence intervals, those for the location parameter exhibited approximately identical performance, while the confidence intervals for the scale parameter showed close performance in relation to both AW and CP. All computations were performed using the R software [34] and the package `nleqslv` (Hasselman [22]).

The findings of this paper are based on the assumption that the shape parameter ψ is known. This assumption has also been adopted by several other researchers. If this parameter is unknown, it may be replaced by an appropriate estimate, thereby yielding approximate BLUEs and BLIEs, as well as approximate BLUPs and BLIPs for future order statistics.

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Compliance with ethical standards

Conflict of interest: On behalf of all authors, the corresponding author states that there is no conflict of interest.

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