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Error estimation of Gaussian quadrature formulae for some modifications of Jacobi weights in the class of analytic functions

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Abstract. For analytic functions in a neighborhood of the real interval $[-1, 1]$, we study the remainder terms of Gauss quadrature rules with respect to some modifications of Jacobi weight functions, namely

$$\omega_1(x) = \frac{(1-x)^\alpha(1+x)^\beta}{v-x} \quad \omega_2(x) = (v-x)\,(1-x)^\alpha(1+x)^\beta$$

where $\alpha, \beta > -1$ and $v \in \mathbb{R}, |v| > 1$. Quadrature formulas concerned with this kind of weight functions have been considered recently by D. Lj Djukić et al. [Appl. Comput. Math. 22 (2023), 426–442].

1. Introduction

In this paper the remainder term $R_n(f)$ corresponding to the Gaussian quadrature formula

$$\int_{-1}^{1} f(t)w(t)\,dt = G_n[f] + R_n(f), \quad G_n[f] = \sum_{\nu=1}^{n} \lambda_\nu f(\tau_\nu) \quad (n \in \mathbb{N}) \tag{1}$$

will be studied. In particular, two modified classes of Jacobi weight functions will be handled, namely

$$\omega_1(x) = \frac{(1-x)^\alpha(1+x)^\beta}{v-x}, \quad \omega_2(x) = (v-x)\,(1-x)^\alpha(1+x)^\beta, \tag{2}$$

where $\alpha, \beta > -1$, and $v \in \mathbb{R}$ such that $|v| > 1$, in such a way that it may be expressed in the form $v = \frac{1}{2}\left(c + \frac{1}{c}\right)$, with $c \in (-1, 1) \setminus \{0\}$. The weight functions (2) have been studied extensively in [3, §2.4], and then in [1]. As

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shown in [3, §2.4], they are useful to compute the so-called Hilbert transform of the Jacobi weight, by means of the corresponding modification algorithms. Moreover, in [1] the internality properties of the averaged and generalized averaged Gaussian quadrature rules relative to the given weight functions is studied, as well as the asymptotic behavior of the recurrence coefficients used to express orthogonal polynomials associated with those modified weight functions.

For the sake of completeness, let us briefly remind the expression of the remainder (or error) term for the Gaussian quadrature formula. Thus, let Γ be a simple closed curve in the complex plane surrounding the interval $[-1, 1]$ and $\mathcal{D} = \operatorname{int}\Gamma$ its interior. If the integrand f is analytic in $\mathcal{D}$ and continuous on $\overline{\mathcal{D}}$, then the remainder term $R_n(f)$ in (1) admits the contour integral representation

$$R_n(f) = \frac{1}{2\pi i}\oint_{\Gamma} K_n(z) f(z) dz. \tag{3}$$

The *kernel* is given by

$$K_n(z) \equiv K_n(z, \omega) = \frac{\varrho_n(z)}{\pi_n(z)}, \quad z \notin [-1, 1],$$

where π_n denotes the nth-orthogonal polynomial with respect to the weight function ω and

$$\varrho_n(z) \equiv \varrho_{n,\omega}(z) = \int_{-1}^{1} \frac{\pi_n(t)}{z - t}\,\omega(t) dt.$$

The modulus of the kernel is symmetric with respect to the real axis, that is, $|K_n(\overline{z})| = |K_n(z)|$. If in addition the weight function ω in (1) is even, the modulus of the kernel is symmetric with respect to both axes, that is, $|K_n(-\overline{z})| = |K_n(z)|$ (see [5]).

The integral representation (3) leads to the error estimate

$$\left|R_n(f)\right| \le \frac{\ell(\Gamma)}{2\pi}\left(\max_{z\in\Gamma}|K_n(z)|\right)\left(\max_{z\in\Gamma}|f(z)|\right), \tag{4}$$

where $\ell(\Gamma)$ is the length of the contour Γ. For a different approach to the estimation of $R_n(f)$, see [6].

In the current paper, we consider as usual integrands being analytic inside elliptical contours with foci at ∓ 1 and the sum of semi-axes $\rho > 1$ of the form,

$$\mathcal{E}_\rho = \left\{ z \in \mathbb{C} \mid z = \frac{1}{2}\left(u + u^{-1}\right),\ 0 \le \theta \le 2\pi \right\}, \quad u = \rho\, e^{i\theta}, \tag{5}$$

where w is a nonnegative and integrable function on the interval $(-1, 1)$. This formula is exact for all algebraic polynomials of degree at most $2n - 1$. The nodes τ_ν in (1) are zeros of the orthogonal polynomials π_n with respect to the weight function w.

As it is well known in the literature, the reason to select those contours is that as $\rho \to 1$ the ellipse (5) shrinks to the interval $[-1, 1]$, while for increasing values of ρ it becomes more and more a circle-like contour. The advantage of elliptical contours, compared to circular ones, for example, is that such a choice requires the analyticity of f in a smaller region of the complex plane, especially when ρ is close to 1.

Since in this paper we take $\Gamma = \mathcal{E}_\rho$, where the ellipse $\mathcal{E}_\rho$ is given by (5), the estimate (4) reduces to

$$\left|R_n(f)\right| \le \frac{\ell(\mathcal{E}_\rho)}{2\pi}\left(\max_{z\in\mathcal{E}_\rho}|K_n(z)|\right)\left(\max_{z\in\mathcal{E}_\rho}|f(z)|\right). \tag{6}$$

In order to obtain suitable upper bounds for the quadrature error $|R_n(f)|$ from (6) we need to properly estimate $\max_{z\in\mathcal{E}_\rho}|K_n(z)|$; in particular, to figure out the location of the extremal point $\eta \in \mathcal{E}_\rho$ at which $|K_n|$ attains its maximum is very useful.

In general, this may be not an easy task, but in the case of the Gauss-type quadrature formula (1) there exist effective algorithms for the calculation of $K_n(z)$ at any point z outside $[-1, 1]$ (see [5]). In [4] the analytic form of the corresponding orthogonal polynomials is determined for the weight functions under consideration; in addition, analytical expressions for the coefficients of the corresponding three-term recurrence relation are derived, which enabled a numerically stable construction of the corresponding Gaussian quadrature formulas. The method presented here was discussed in detail for the Gauss quadrature rules with the four Chebyshev weight functions [5], and later generalized in some sense to the Gauss quadrature rules with general symmetric weight functions in Schira [13]. Methods related to specific cases of Bernstein–Szegő-type quadrature rules are presented in [10], [11], [12] and [14]. Another approach is given in Hunter [6]. The case of general weight functions is discussed in [17] for Gauss quadrature rules, and for quadrature rules with multiple nodes, and for positive interpolatory ones, by Spalević in [15], [16], respectively.

In the next section the main theoretical results of our method for the Gauss quadrature rules for these modified Jacobi weight functions are given, and then in the last section the accuracy of the proposed error bounds is shown by means of numerical examples.

2. Main results

First, we will focus on the weight ω_1 in (2).

Theorem 1. *Consider the Gauss quadrature formula* (1), $n \in \mathbb{N}$, *for the weight function* ω_1 *given in (2), where* $\alpha, \beta > -1$, $c \in (-1,1) \setminus \{0\}$. *Then, for n large enough there exists* $\rho^* \in (1, +\infty)$ ($\rho^* = \rho_n^* = \rho^*(n, \alpha, \beta, c)$) *such that for each* $\rho \geq \rho^*$ *the modulus of the kernel* $\left|K^{(1)}_{n,\alpha,\beta,c}(z)\right|$ *attains its maximum value on the positive real semi-axis* ($\theta = 0$) *if* $c > 2(\alpha - \beta)$, *and on the negative real semi axis* ($\theta = \pi$) *if* $c < 2(\alpha - \beta)$*; that is,*

$$\max_{z \in \mathcal{E}_\rho} \left|K^{(1)}_{n,\alpha,\beta,c}(z)\right| = \left|K^{(1)}_{n,\alpha,\beta,c}\left(\frac{1}{2}(\rho + \rho^{-1})\right)\right|$$

for $c > 2(\alpha - \beta)$, *and*

$$\max_{z \in \mathcal{E}_\rho} \left|K^{(1)}_{n,\alpha,\beta,c}(z)\right| = \left|K^{(1)}_{n,\alpha,\beta,c}\left(-\frac{1}{2}(\rho + \rho^{-1})\right)\right|$$

for $c < 2(\alpha - \beta)$.

Proof. We denote by $\pi_n^{(1)}$ the n-th degree orthogonal polynomial for the weight function ω_1. Then, the following expression holds

$$\pi_n^{(1)}(t) = Q_n(t) - \gamma_{n-1} Q_{n-1}(t),$$

where Whang and Zhang in [18] showed that

$$Q_n(u) = P_n^{(\alpha,\beta)}(z) = \sum_{k=-n}^{n} a_{n,|k|} u^k = 2 \sum_{k=0}^{n} a_{n,k} T_k(z), \quad z = \frac{u + \frac{1}{u}}{2}, \tag{7}$$

with the explicit expressions for the coefficients $a_{n,k}$ also derived; it is also proved in [1]

$$\gamma_n = \frac{c}{2}(1 + Gn^{-2} + o(n^{-2})), \quad n \to +\infty,$$

where

$$G = \frac{1 - 2(1+c)\alpha^2 - 2(1-c)\beta^2}{4(1-c^2)}.$$

Now, using the identity [19, eq. (4.3)],

$$\frac{1}{\frac{1}{2}\left(u + \frac{1}{u}\right) - t} = \frac{4}{(1-u^2)} \sum_{k=0}^{+\infty} T_k(t) u^{-k-1}, \quad |u| > 1,$$

we have that

$$\varrho_n^{(1)}(z) \equiv \varrho_{n,\omega}^{(1)}(z) = \int_{-1}^{1} \frac{\pi_n^{(1)}(t)}{\frac{1}{2}\left(u+\frac{1}{u}\right)-t}\,\omega_1(t)dt = \frac{4}{1-u^{-2}} \sum_{k=0}^{+\infty} \frac{1}{u^{k+1}} \int_{-1}^{1} T_k(t)\pi_n^{(1)}(t)\omega_1(t)\,dt, \tag{8}$$

and taking into account that, by orthogonality, $\int_{-1}^{1} T_k(t)\pi_n^{(1)}(t)\omega_1(t)\,dt = 0$ for each $k \in \mathbb{N}_0$, $k < n$, we get

$$\begin{aligned} \varrho_n^{(1)}(z) &= \sum_{k=n}^{+\infty} \frac{1}{u^{k+1}} \int_{-1}^{1} T_k(t)\pi_n^{(1)}(t)\omega_1(t)\,dt = \sum_{k=0}^{+\infty} \frac{1}{u^{n+k+1}} \int_{-1}^{1} T_{n+k}(t)\pi_n^{(1)}(t)\omega_1(t)\,dt \\ &= \frac{4}{1-u^{-2}} \sum_{k=0}^{+\infty} \frac{\mu_{n,k}^{(1)}}{u^{n+k+1}} = \frac{4u^{-(n+1)}}{1-u^{-2}} \sum_{k=0}^{+\infty} \frac{\mu_{n,k}^{(1)}}{u^k}, \end{aligned} \tag{9}$$

where we are setting

$$\mu_{n,k}^{(1)} = \int_{-1}^{1} T_{n+k}(t)\pi_n^{(1)}(t)\omega_1(t)\,dt, \quad k \in \mathbb{N}_0.$$

Moreover,

$$\begin{aligned} &\varrho_n^{(1)}(z) = 4u^{-(n+1)} \textstyle\sum_{l=0}^{+\infty} u^{-2l} \sum_{k=0}^{+\infty} \dfrac{\mu_{n,k}^{(1)}}{u^k} \\ &= 4u^{-(n+1)} \left(\mu_{n,0}^{(1)} + \mu_{n,1}^{(1)}u^{-1} + \ldots\right) = 4u^{-(n+1)} \left(\mu_{n,0}^{(1)}\right)\left(1 + \left(\mu'_{n,1}\right)^{(1)} u^{-1} + \ldots\right) \end{aligned}$$

where we denote $\left(\mu'_{n,k}\right)^{(1)} = \dfrac{\mu_{n,k}^{(1)}}{\mu_{n,0}^{(1)}}$ for each $k \in \mathbb{N}$ and, also

$$\begin{aligned} 0 &\neq \int_{-1}^{1} \left(\pi_n^{(1)}\right)^2(t)\omega_1(t)\,dt = \int_{-1}^{1} \pi_n^{(1)}(t)\left(Q_n(t) - \gamma_{n-1}Q_{n-1}(t)\right)\omega_1(t)\,dt \\ &= \int_{-1}^{1} \pi_n^{(1)}(t)Q_n(t)\omega_1(t)\,dt = 2a_{n,n} \int_{-1}^{1} T_n(t)\pi_n^{(1)}(t)\omega_1(t)\,dt = 2a_{n,n}\mu_{n,0}, \end{aligned}$$

which implies $\mu_{n,0} \neq 0$.

Let us now start from

$$\int_{-1}^{1} \pi_{n+1}^{(1)}(t)\pi_n^{(1)}(t)\omega_1(t)\,dt = 0.$$

This further means that

$$\begin{aligned} 0 &= \int_{-1}^{1} \left(Q_{n+1}(t) - \gamma_n Q_n(t)\right)\pi_n^{(1)}(t)\omega_1(t)\,dt \\ &= \int_{-1}^{1} \left(2a_{n+1,n+1}T_{n+1}(t) + 2\sum_{k=0}^{n} \left(a_{n+1,k} - \gamma_n a_{n,k}\right) T_k(t)\right) \pi_n^{(1)}(t)\omega_1(t)\,dt \\ &= 2a_{n+1,n+1} \int_{-1}^{1} T_{n+1}(t)\pi_n^{(1)}(t)\omega_1(t)\,dt + 2\left(a_{n+1,n} - \gamma_n a_{n,n}\right) \int_{-1}^{1} T_n(t)\pi_n^{(1)}(t)\omega_1(t)\,dt \\ &= 2a_{n+1,n+1}\mu_{n,1}^{(1)} + 2\left(a_{n+1,n} - \gamma_n a_{n,n}\right)\mu_{n,0}^{(1)}, \end{aligned}$$

which directly implies

$$\left(\mu'_{n,1}\right)^{(1)} = \frac{\mu_{n,1}^{(1)}}{\mu_{n,0}^{(1)}} = \frac{\gamma_n a_{n,n} - a_{n+1,n}}{a_{n+1,n+1}}.$$

Since

$$\begin{aligned}\pi_n^{(1)}(u) &= Q_n(z) - \gamma_{n-1}Q_{n-1}(z) = a_{n,n}u^n + (a_{n,n-1} - \gamma_{n-1}a_{n-1,n-1})\,u^{n-1} + \dots \\ &= a_{n,n}u^n\left(1 + \frac{a_{n,n-1} - s_{n-1}a_{n-1,n-1}}{a_{n,n}}u^{-1} + \dots\right) = a_{n,n}u^n\left(1 + \left(a'_{n,n-1}\right)^{(1)} u^{-1} + \dots\right),\end{aligned}$$

it holds, by the same principle as in [19, just after eq. (4.9)], that

$$K_n^{(1)}(z) = \frac{\varrho_n^{(1)}(u)}{\pi_n^{(1)}(u)} = 4u^{-2n-1}\frac{\mu_{n,0}^{(1)}}{a_{n,n}}\left(1 + \left(\left(\mu'_{n,1}\right)^{(1)} - \left(a'_{n,n-1}\right)^{(1)}\right)u^{-1} + \dots\right).$$

Thus, the asymptotic behavior of the modulus of the kernel $K_n^{(1)}(z)$ for sufficiently large ρ depends on the sign of the expression

$$D^{(1)} = \left(\mu'_{n,1}\right)^{(1)} - \left(a'_{n,n-1}\right)^{(1)} = \frac{\gamma_n a_{n,n} - a_{n+1,n}}{a_{n+1,n+1}} - \frac{a_{n,n-1} - \gamma_{n-1}a_{n-1,n-1}}{a_{n,n}}.$$

We cannot directly determine its sign for specific values of n since explicit expressions for γ_n are not available. However, its asymptotic behavior for large n is known (see [1]). In particular, it follows that

$$\lim_{n\to+\infty} \gamma_n = \frac{c}{2}.$$

Furthermore, following ([19, eqs. (4.14–4.15)], we have

$$\lim_{n\to+\infty} \frac{a_{n,n-1}}{a_{n,n}} = \lim_{n\to+\infty} \frac{2(\alpha-\beta)n}{2n+\alpha+\beta} = \alpha - \beta = \lim_{n\to+\infty} \frac{a_{n+1,n}}{a_{n+1,n+1}}. \tag{10}$$

Additionally, we have

$$\begin{aligned}\frac{a_{n,n}}{a_{n+1,n+1}} &= \frac{\Gamma(2n+\alpha+\beta+1)}{\Gamma(2n+2+\alpha+\beta+1)}\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+\beta+1)}\frac{\Gamma(n+2)}{\Gamma(n+1)}\frac{2^{2n+2}}{2^{2n}} \\ &= 4\,\frac{(n+\alpha+\beta+1)(n+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)} \Rightarrow \lim_{n\to+\infty}\frac{a_{n,n}}{a_{n+1,n+1}} = 1.\end{aligned}$$

This directly leads us to conclude that $\lim_{n\to+\infty} D^{(1)} = c - 2(\alpha-\beta)$.

In cases where $\alpha = \beta$ the asymptotic behavior of the modulus of this kernel depends directly on the sign of c. This completes the proof. □

Now, we focus on the case of ω_2.

Theorem 2. *For the Gauss quadrature formula (1), $n \in \mathbb{N}$, with the weight function ω_2 given in (2), $\alpha, \beta > -1$ and $c \in (-1,1)\setminus\{0\}$, we have that for n large enough there exists a $\rho^* \in (1,+\infty)$ ($\rho^* = \rho_n^* = \rho^*(n,\alpha,\beta,c)$) such that for each $\rho \geq \rho^*$ the modulus of the kernel $\left|K_{n,\alpha,\beta,c}^{(2)}(z)\right|$ attains its maximum value on the positive real semi axis ($\theta = 0$) if $\frac{1}{2c} - 2c > 2(\alpha-\beta)$ and on the negative real semi axis ($\theta = \pi$) if $\frac{1}{2c} - 2c < 2(\alpha-\beta)$, i. e.,*

$$\max_{z\in\mathcal{E}_\rho}\left|K_{n,\alpha,\beta,c}^{(2)}(z)\right| = \left|K_{n,\alpha,\beta,c}^{(2)}\left(\frac{1}{2}(\rho+\rho^{-1})\right)\right|$$

for $\frac{1}{2c} - 2c > 2(\alpha - \beta)$, *and*

$$\max_{z\in\mathcal{E}_\rho}\left|K^{(2)}_{n,\alpha,\beta,c}(z)\right| = \left|K^{(2)}_{n,\alpha,\beta,c}\left(-\frac{1}{2}(\rho+\rho^{-1})\right)\right|$$

for $\frac{1}{2c} - 2c < 2(\alpha - \beta)$.

Proof. Here, the corresponding orthogonal polynomial $\pi_n^{(2)}(t)$ satisfies the relation

$$(t-v)\,\pi_n^{(2)}(t) = Q_{n+1}(t) - \epsilon_n Q_n(t),$$

where

$$\epsilon_n = \frac{Q_{n+1}(v)}{Q_n(v)}.$$

Using (7) and the results from [1], it has been proven that

$$\epsilon_n = \frac{1}{2c}(1 + En^{-2} + o(n^{-2})), \quad n \to +\infty,$$

where

$$E = \frac{c}{c^2-1}\left(\frac{\beta^2-\alpha^2}{2} + \frac{c\left(1-2\alpha^2-2\beta^2\right)}{4}\right).$$

Analogously to (8), the following relation holds:

$$\varrho_n^{(2)}(z) \equiv \varrho_{n,\omega}^{(2)}(z) = \int_{-1}^{1}\frac{\pi_n^{(2)}(t)}{\frac{1}{2}\left(u+\frac{1}{u}\right)-t}\,\omega_2(t)dt = \frac{4u^{-(n+1)}}{1-u^{-2}}\sum_{k=0}^{+\infty}\frac{\mu^{(2)}_{n,k}}{u^k} \quad . \tag{11}$$

Here, we define

$$\mu^{(2)}_{n,k} = \int_{-1}^{1} T_{n+k}(t)\pi_n^{(2)}(t)\omega_2(t)\,dt, \quad k \in \mathbb{N}_0.$$

In addition,

$$\begin{aligned}\varrho_n^{(2)}(u) &= 4u^{-(n+1)}\textstyle\sum_{l=0}^{+\infty} u^{-2l}\sum_{k=0}^{+\infty}\dfrac{\mu^{(2)}_{n,k}}{u^k}\\ &= 4u^{-(n+1)}\left(\mu^{(2)}_{n,0} + \mu^{(2)}_{n,1}u^{-1} + \ldots\right) = 4u^{-(n+1)}\left(\mu_{n,0}\right)^{(2)}\left(1 + \left(\mu'_{n,1}\right)^{(2)}u^{-1} + \ldots\right)\end{aligned}$$

where $\left(\mu'_{n,k}\right)^{(2)} = \dfrac{\mu^{(2)}_{n,k}}{\mu^{(2)}_{n,0}}$ for each $k \in \mathbb{N}$.

It holds

$$\mu^{(2)}_{n,0} = \int_{-1}^{1} T_n(t)\pi_n^{(2)}(t)\,(v-t)\,(1-t)^\alpha(1+t)^\beta\,dt = \int_{-1}^{1} T_n(t)\left(-Q_{n+1}(t) + \epsilon_n Q_n(t)\right)(1-t)^\alpha(1+t)^\beta\,dt. \tag{12}$$

From (7), it follows that $T_n(t) - \frac{1}{2a_{n,n}}Q_n(t)$ is polynomial of degree $n-1$. This implies the existence of unique coefficients $\alpha_{n,0}, \ldots, \alpha_{n,n-1}$ such that

$$T_n(t) - \frac{1}{2a_{n,n}}Q_n(t) = \sum_{k=0}^{n-1}\alpha_k Q_k(t), \quad T_n(t) = \frac{1}{2a_{n,n}}Q_n(t) + \sum_{k=0}^{n-1}\alpha_{n,k}Q_k(t).$$

Consequently, we can express

$$\begin{aligned}\mu_{n,0}^{(2)} = & \int_{-1}^{1}\left(\frac{1}{2a_{n,n}}Q_n(t)+\sum_{k=0}^{n-1}\alpha_{n,k}Q_k(t)\right)(-Q_{n+1}(t)+\epsilon_n Q_n(t))\,(1-t)^{\alpha}(1+t)^{\beta}\,dt \\ = & \frac{\epsilon_n}{2a_{n,n}}\int_{-1}^{1}(Q_n(t))^2\,(1-t)^{\alpha}(1+t)^{\beta}\,dt = \frac{2^{\alpha+\beta}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\epsilon_n}{a_{n,n}\,(2n+\alpha+\beta+1)\,n!\Gamma(n+\alpha+\beta+1)} > 0\end{aligned} \tag{13}$$

according to [18, eq. (2.2)],

$$\begin{aligned}\mu_{n,1}^{(2)} = & \int_{-1}^{1} T_{n+1}(t)\pi_n^{(2)}(t)\,(v-t)\,(1-t)^{\alpha}(1+t)^{\beta}\,dt \\ = & \int_{-1}^{1} T_{n+1}(t)\,(-Q_{n+1}(t)+\epsilon_n Q_n(t))\,(1-t)^{\alpha}(1+t)^{\beta}\,dt \\ = & -\int_{-1}^{1} T_{n+1}(t)Q_{n+1}(t)(1-t)^{\alpha}(1+t)^{\beta}\,dt + \epsilon_n\int_{-1}^{1} T_{n+1}(t)Q_n(t)(1-t)^{\alpha}(1+t)^{\beta}\,dt \\ = & -\mu_{n+1,0}+\epsilon_n\mu_{n,1},\end{aligned}$$

where $\mu_{n,k}$ are the same as in [19]. From the other side, note that from (13) it follows

$$\mu_{n,0}^{(2)} = \epsilon_n\int_{-1}^{1} T_n(t)Q_n(t)(1-t)^{\alpha}(1+t)^{\beta}\,dt = u_n\mu_{n,0}$$

and that

$$\mu_{n,0} = \frac{2^{\alpha+\beta}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{a_{n,n}\,(2n+\alpha+\beta+1)\,n!\Gamma(n+\alpha+\beta+1)}. \tag{14}$$

Finally, using [19, eq. (4.15)],

$$\begin{aligned}\left(\mu'_{n,1}\right)^{(2)} &= \frac{\mu_{n,1}^{(2)}}{\mu_{n,0}^{(2)}} = \frac{-\mu_{n+1,0}+\epsilon_n\mu_{n,1}}{\epsilon_n\mu_{n,0}} = -\frac{1}{u_n}\frac{\mu_{n+1,0}}{\mu_{n,0}}+\frac{\mu_{n,1}}{\mu_{n,0}} \\ &= -\frac{1}{(n+1)\epsilon_n}\frac{a_{n,n}}{a_{n+1,n+1}}\frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+1)}{(n+\alpha+\beta+1)(2n+\alpha+\beta+3)} - \frac{2(\alpha-\beta)(n+1)}{2n+2+\alpha+\beta} \\ &= -\frac{1}{(n+1)\epsilon_n}\,4\,\frac{(n+\alpha+\beta+1)(n+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)}\frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+1)}{(n+\alpha+\beta+1)(2n+\alpha+\beta+3)} \\ &\quad -\frac{2(\alpha-\beta)(n+1)}{2n+2+\alpha+\beta} = -\frac{1}{\epsilon_n}\frac{4(n+\alpha)(n+\beta)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+3)} - \frac{2(\alpha-\beta)(n+1)}{2n+2+\alpha+\beta},\end{aligned}$$

and, consequently,

$$\lim_{n\to+\infty}\left(\mu'_{n,1}\right)^{(2)} = -2c-(\alpha-\beta). \tag{15}$$

Therefore,

$$\begin{aligned}\pi_n^{(2)}(z) &= \frac{Q_{n+1}(z)-\dfrac{Q_{n+1}(v)}{Q_n(v)}Q_n(z)}{z-v} = \frac{Q_{n+1}(z)Q_n(v)-Q_{n+1}(v)Q_n(z)}{Q_n(v)(z-v)} \\ &= \frac{Q_{n+1}(z)Q_n(v)-Q_{n+1}(z)Q_n(z)}{Q_n(v)(z-v)} \\ &= \frac{(Q_{n+1}(z)-Q_{n+1}(v))\,Q_n(v)-Q_{n+1}(v)\,(Q_n(z)-Q_n(v))}{Q_n(v)(z-v)} \\ &= \frac{Q_{n+1}(z)-Q_{n+1}(v)}{z-v}-\epsilon_n\frac{Q_n(z)-Q_n(v)}{z-v} \\ &= 2\sum_{k=0}^{n+1}a_{n+1,k}\frac{T_k(z)-T_k(v)}{z-v}-2\epsilon_n\sum_{k=0}^{n}a_{n,k}\frac{T_k(z)-T_k(v)}{z-v}.\end{aligned}$$

Since for each $k \in \mathbb{N}$, T_k has the form $T_k(z) = 2^{k-1}z^k + a_{k-2}z^{k-2} + ...$, we have that

$$\frac{T_k(z) - T_k(v)}{z - v} = 2^{k-1}z^{k-1} + o(z^{k-2}), \quad |z| \to +\infty$$

and this expression can be written in terms of u, yielding

$$2^{k-1}\left(\frac{u + \frac{1}{u}}{2}\right)^{k-1} + o\left(\frac{u + \frac{1}{u}}{2}\right)^{k-2} = u^{k-1} + o(u^{k-2}), \quad |u| \to +\infty$$

and so,

$$\pi_n^{(2)}(u) = 2\left(a_{n+1,n+1}u^n + (a_{n+1,n} - u_n a_{n,n})\, u^{n-1} + o(u^{n-1})\right) \quad |u| \to +\infty.$$

Hence,

$$\begin{aligned} \pi_n^{(2)}(u) &= 2a_{n+1,n+1}u^n\left(1 + \frac{a_{n+1,n} - s_n a_{n,n}}{a_{n+1,n+1}}u^{-1} + ...\right) \\ &= 2a_{n+1,n+1}u^n\left(1 + \left(a'_{n,n-1}\right)^{(2)} u^{-1} + ...\right), \end{aligned}$$

and it holds by the same principle as in [19, just after eq. (4.9)] that

$$K_n^{(2)}(u) = \frac{\varrho_n^{(2)}(u)}{\pi_n^{(2)}(u)} = 4u^{-2n-1}\frac{\mu_{n,0}^{(2)}}{a_{n+1,n+1}}\left(1 + \left(\left(\mu'_{n,1}\right)^{(2)} - \left(a'_{n,n-1}\right)^{(2)}\right)u^{-1} + ...\right).$$

Since, using (10) yields

$$\lim_{n\to+\infty}\left(a'_{n,n-1}\right)^{(2)} = \lim_{n\to\infty}\frac{a_{n+1,n}}{a_{n+1,n+1}} - \lim_{n\to+\infty}\epsilon_n \lim_{n\to+\infty}\frac{a_{n,n}}{a_{n+1,n+1}} = \alpha - \beta - \frac{1}{2c},$$

we conclude that the asymptotic behavior of the modulus of the kernel $K_n^{(2)}$ for sufficiently large ρ depends on the sign of the expression

$$D^{(2)} = \left(\mu'_{n,1}\right)^{(2)} - \left(a'_{n,n-1}\right)^{(2)}.$$

Again, we cannot determine the sign for specific values of ϵ_n due to the lack of explicit expressions for ϵ_n. However, we know their asymptotic behavior for sufficiently large n (see [1]), where we have already established that $\lim_{n\to+\infty}\epsilon_n = \frac{1}{2c}$. Indeed, (15) and the previous expressions, it follows that

$$\lim_{n\to+\infty} D^{(2)} = -2c - (\alpha - \beta) - \left(\alpha - \beta - \frac{1}{2c}\right) = \frac{1}{2c} - 2c - 2(\alpha - \beta).$$

As in our previous result, we conclude that when $\alpha = \beta$ the asymptotic behavior of the modulus of this kernel directly depends on the sign of the expression $\dfrac{1 - 4c^2}{2c}$. The proof is given. □

The results given in Theorems 1 and 2, as well as the subsequent upper bounds for the quadrature error, will be tested in the next section for different examples.

3. Numerical performance

Throughout this section, we first display several graphic representations of the modulus of the kernel K_n, illustrating the different scenarios depicted in Theorem 1 (for the weight ω_1) and Theorem 2 (for ω_2).

Second, the estimations of the quadrature error are tested by means of some numerical examples for randomly selected values of n, α, β and c.

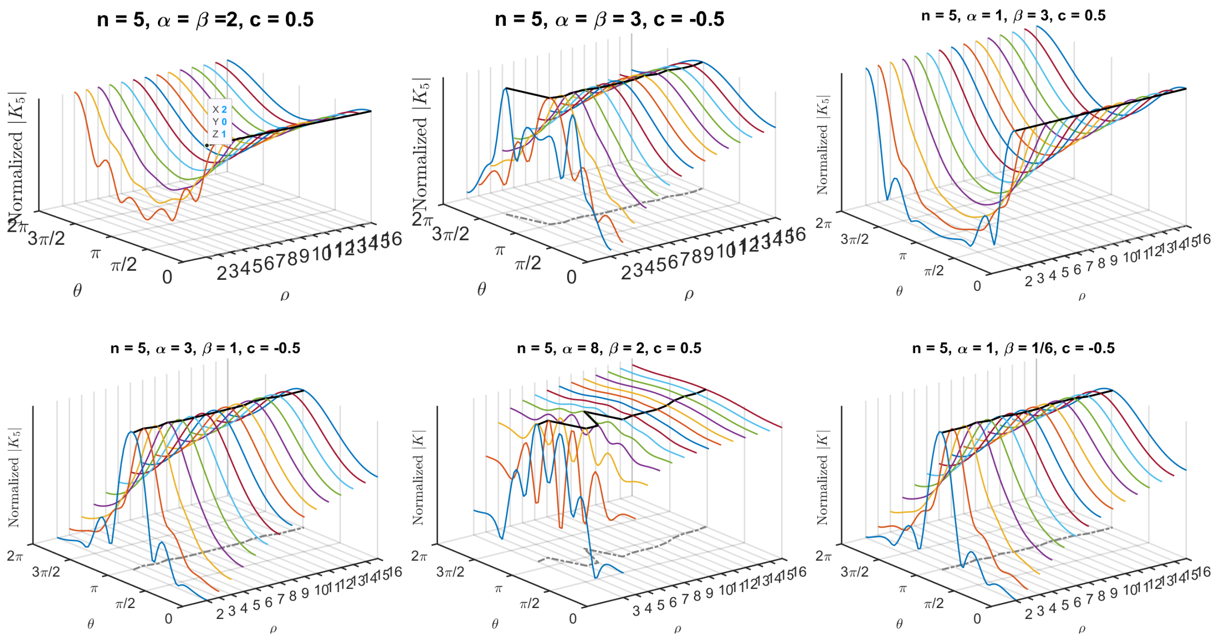

Figure 1: Graphical representation of the modulus of the kernel for various values, illustrating scenarios from Theorem 1.

Indeed, Figure 1 illustrates all possible scenarios depicted in Theorem 1, while Figure 2 corresponds to those from Theorem 2, both for fixed values of n, namely $n = 5$ and $n = 6$, respectively. In both cases, different values of α, β, and c are considered. The modulus of the kernel is normalized to ensure a clearer visualization of its behavior for different values of ρ and θ. The results in every scenario match the predictions of both theorems. The choice of n is arbitrary; a small value was randomly selected to allow faster calculations. The black line in the graphic represents the maximum value of the modulus of the kernel for each ρ and θ, while the gray line is its projection onto the (ρ, θ) plane.

Under the assumption that f is analytic inside the region $\mathcal{E}_{\rho_{\max}}$, the error bound of the corresponding quadrature formula can be estimated by

$$|R_n(f)| \leq r_n(f),$$

where, based on equation (5), with $\Gamma = \mathcal{E}_\rho$, (4), we have

$$r_n(f) \leq \inf_{\rho^*<\rho<\rho_{\max}} \left[\frac{\ell(\mathcal{E}_\rho)}{2\pi} \Big(\max_{z\in\mathcal{E}_\rho} |K_n(z)| \Big) \Big(\max_{z\in\mathcal{E}_\rho} |f(z)| \Big) \right].$$

Here, ρ^* is the smallest possible corresponding value of ρ^* obtained empirically (starting from the value 1.01), $\rho_{\max}$ depends on the integrand (it will be specified in the following examples), and $\ell(\mathcal{E}_\rho)$ represents the length of the ellipse $\mathcal{E}_\rho$ (5), which can be estimated by (see [2])

$$\ell(\mathcal{E}_\rho) \leq 2\pi a_1 \left(1 - \frac{1}{4} a_1^{-2} - \frac{3}{64} a_1^{-4} - \frac{5}{256} a_1^{-6} \right),$$

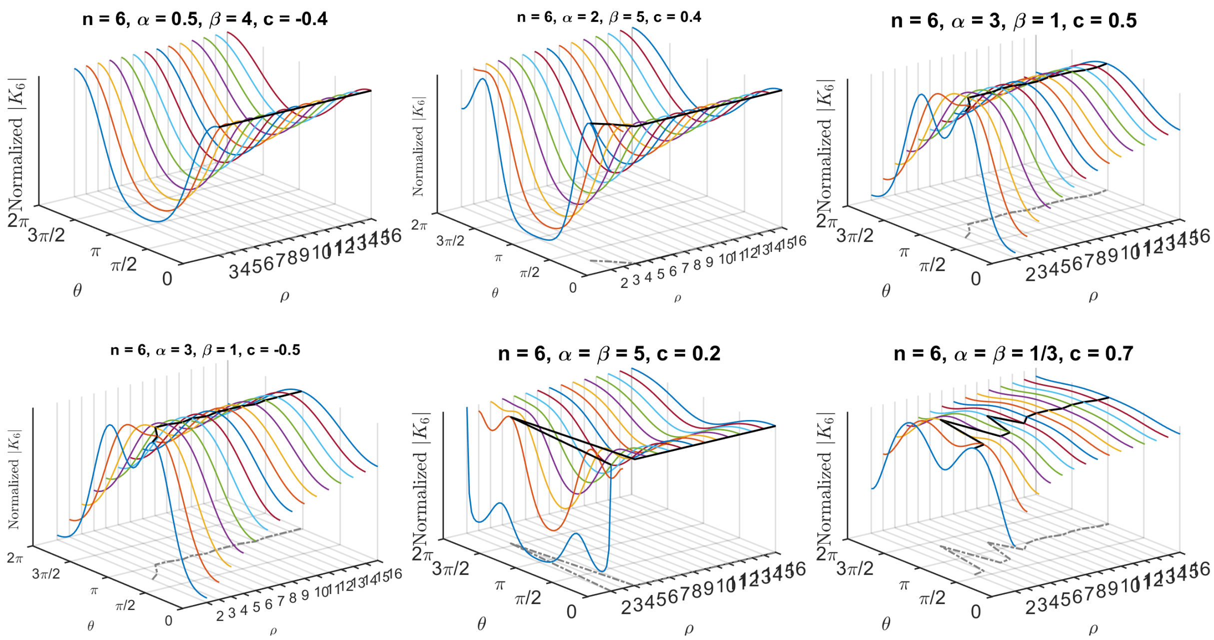

Figure 2: Graphic representation of the modulus of the kernel for various values, illustrating scenarios from Theorem 2.

where $a_1 = (\rho + \rho^{-1})/2$. Thus, the error bound $r_n(f) = r_n(f, \omega)$ reduces to

$$r_n(f) \le \inf_{\rho^* < \rho < \rho_{\max}} \left[a_1 \left(1 - \frac{1}{4} a_1^{-2} - \frac{3}{64} a_1^{-4} - \frac{5}{256} a_1^{-6} \right) \left(\max_{z \in \mathcal{E}_\rho} |K_n(z)| \right) \left(\max_{z \in \mathcal{E}_\rho} |f(z)| \right) \right]. \tag{16}$$

To verify our claims and the proposed error bound, we performed multiple tests and compared the results with the exact or actual errors, here denoted by R_n. The value of the exact error has been approximated by

$$R_n = |I_{150}(f) - I_n(f)|,$$

that is, using the result of the 150-point Gauss rule as a (reliable) estimation of the exact value of the integral.

On the other hand, we implemented the algorithms proposed by Gautschi to compute the recurrence coefficients and nodes corresponding to the modified weight functions[3, eqs. (2.4.12–-2.4.13)].

Finally, for the evaluation of the error bound, instead of obtaining an expression of the Kernel, we have preferred to numerically evaluate it using the 100-point Gauss-Jacobi quadrature rule with a precision of 50 significant digits to ensure, thus, more than 30 significant digits of accuracy. Our approach follows the methodology described in [19]. We use the computed kernel to estimate r_n in this way.

The results have been summarized in Tables 1 to 4, and correspond to the randomly selected values of α, β, c, and the given function f.

Example 1. *Let* $f_1(z) = e^{\omega z^2}$, $\omega > 0$.
The function f_1 *is entire and it is known that*

$$\max_{z \in \mathcal{E}_\rho} |f_1(z)| = e^{\omega a_1^2}, \quad a_1 = (\rho + \rho^{-1})/2, \quad \rho_{\max} = +\infty.$$

n, w, α, β, c	r_n	ρ_{opt}	R_n
(5, 0.1, 2, 1, -0.5)	4.469(-10)	10.51	8.287(-11)
(10, 0.1, 2, 1, -0.5)	5.720(-22)	19.51	1.313(-23)
(15, 0.1, 2, 1, -0.5)	2.163(-35)	24.01	3.362(-37)
(20, 0.1, 2, 1, -0.5)	1.452(-49)	28.01	1.711(-51)
(3, 0.5, 1, 3, 0.2)	1.087(-03)	5.01	8.316(-05)
(6, 0.5, 1, 3, 0.2)	1.909(-08)	7.01	9.813(-10)
(9, 0.5, 1, 3, 0.2)	8.250(-14)	8.51	3.261(-15)
(12, 0.5, 1, 3, 0.2)	1.349(-18)	9.51	7.942(-20)
(15, 0.5, 1, 3, 0.2)	3.351(-24)	11.01	1.640(-25)

Table 1: Comparison of the estimated and actual error for the function f_1 in case of the weight function ω_1.

n, w, α, β, c	r_n	ρ_{opt}	R_n
(5, 0.1, 2, 1, -0.5)	8.202(-11)	10.01	4.564(-11)
(10, 0.1, 2, 1, -0.5)	5.836(-23)	20.01	1.297(-23)
(15, 0.1, 2, 1, -0.5)	7.455(-37)	25.01	3.335(-37)
(20, 0.1, 2, 1, -0.5)	7.338(-51)	28.51	1.711(-51)
(3, 0.5, 1, 3, 0.2)	1.040(-03)	5.51	4.706(-04)
(6, 0.5, 1, 3, 0.2)	1.082(-08)	7.51	5.781(-09)
(9, 0.5, 1, 3, 0.2)	3.310(-14)	9.01	1.952(-14)
(12, 0.5, 1, 3, 0.2)	4.197(-20)	10.01	2.660(-20)
(15, 0.5, 1, 3, 0.2)	2.725(-26)	11.01	1.801(-26)

Table 2: Comparison of the estimated and actual error for the function f_1 in case of ω_2.

Example 2.

$$f_2(z) = \frac{\cos z}{w^2 + z^2},\ \omega > 0.$$

The function f_2 has two poles in $\pm i\omega$. Thus, in this case we have that

$$\rho_{\max} = w + \sqrt{w^2 - 1}.$$

On the other hand,

$$\max_{z\in\mathcal{E}_\rho} |f_2(z)| = \frac{\cosh b_1}{w^2 - b_1^2}, \quad b_1 = (\rho - \rho^{-1})/2.$$

n, w, α, β, c	r_n	ρ_{opt}	R_n
(3, 5, 1, 8, 0.2)	7.810(-07)	9.91	3.307(-07)
(6, 5, 1, 8, 0.2)	4.528(-13)	9.91	9.219(-14)
(9, 5, 1, 8, 0.2)	2.419(-18)	9.91	3.783(-20)
(12, 5, 1, 8, 0.2)	9.261(-25)	9.91	2.226(-25)
(3, 10, 8, 1, 0.2)	7.277(-08)	20.01	3.829(-08)
(6, 10, 8, 1, 0.2)	1.207(-15)	20.01	9.270(-16)
(9, 10, 8, 1, 0.2)	5.092(-23)	20.01	9.281(-24)
(12, 10, 8, 1, 0.2)	7.778(-27)	20.01	2.800(-30)

Table 3: Comparison of the estimated and actual error for the function f_2 in case of w_1.

n, w, α, β, c	r_n	ρ_{opt}	R_n
(3, 5, 1, 8, 0.2)	9.595(-06)	9.91	1.814(-06)
(6, 5, 1, 8, 0.2)	6.182(-13)	9.91	5.645(-13)
(9, 5, 1, 8, 0.2)	3.092(-19)	9.91	2.460(-19)
(12, 5, 1, 8, 0.2)	5.810(-25)	9.91	1.390(-25)
(3, 10, 8, 1, 0.2)	6.649(-07)	20.01	2.948(-07)
(6, 10, 8, 1, 0.2)	1.273(-14)	20.01	6.467(-15)
(9, 10, 8, 1, 0.2)	5.994(-23)	20.01	5.881(-23)
(12, 10, 8, 1, 0.2)	9.947(-31)	20.01	5.772(-31)

Table 4: Comparison of the estimated and actual error for the function f_2 in case of ω_2.

The results in Tables 1-4 above show the sharpness of the error bound r_n. Indeed, in most of the rows the order of r_n is the same as that of R_n (the actual error).

4. Possible Directions for Future Research

It would be of interest to investigate the applicability of the proposed approach to broader classes of quadrature rules, as well as to other classes of weight functions. A natural next step would be to consider the proposed approach for Jacobi weight functions modified by quadratic factors or by more general rational functions, in the context of Gauss quadrature rules. Such modifications could also be examined for Bernstein–Szegő weight functions, and subsequently for the corresponding Gauss quadrature formulas.

Another interesting class is provided by the Gauss–Turán rules. In addition to function values at the nodes, these rules also require values of the derivatives of the integrand. They have already been studied for four classes of Chebyshev-type weight functions and for the Gori–Micchelli weight functions; see [7, 8, 9]. It would therefore be of interest to extend the proposed approach to certain rational modifications of these weight functions. The most general case involving Jacobi weight functions would also be of considerable interest. However, since the corresponding orthogonal polynomials, the so-called s-orthogonal polynomials, are not known in closed analytic form, the application of the proposed approach in this setting currently appears to be beyond reach.

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