



Complete asymptotic expansion of integral operators based on Charlier polynomials

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Abstract. A significant extension of the Szász–Mirakyan operators involves their formulation using Charlier polynomials. In this paper, we derive the complete asymptotic expansion of the Kantorovich and Durrmeyer variants associated with these Charlier-type operators.

This work is dedicated to Professor Gradimir V. Milovanović for his significant contributions in the field of approximation theory

1. Introduction

The field of approximation theory frequently explores linear positive operators, given their extensive applications in computational techniques and approximating functions. Recent progress in this field has been thoroughly documented in various works, including detailed discussions like those found in [4] and [6]. Among these, operators involving Charlier polynomials form an important class. Charlier polynomials play a significant role in various fields of mathematics and applied sciences. These polynomials exhibit orthogonality with respect to the Poisson distribution and are widely applied in probabilistic modeling and the discrete data analysis. They are widely utilized in numerical computations, combinatorial analysis, and quantum mechanics.

The well-known Meixner polynomials $M_k(x; \beta, c)$ ($0 < c < 1, \beta > 0$) are orthogonal with respect to a discrete measure with jumps $c^x(\beta)_x/x!$ at the integer points $x = 0, 1, 2, \dots$, i.e.,

$$\sum_{x=0}^{\infty} M_n(x; \beta, c) M_m(x; \beta, c) c^x \frac{(\beta)_x}{x!} = \frac{n!}{c^n (1-c)^\beta (\beta)_n} \delta_{n,m},$$

where $(\beta)_x$ is the Pochhammer symbol defined by

$$(\beta)_x = (\beta)(\beta+1)\cdots(\beta+x-1) = \frac{\Gamma(\beta+x)}{\Gamma(\beta)}.$$

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Meixner polynomials can also be expressed in terms of the Gauss hypergeometric function as

$$M_k(x; \beta, c) = {}_2F_1\left(-k, -x; \beta; 1 - \frac{1}{c}\right).$$

Beside the limiting relation

$$\lim_{c \rightarrow 1} M_k\left(\frac{x}{1-c}; \alpha + 1, c\right) = \frac{k!}{(\alpha + 1)_k} L_k^\alpha(x),$$

where $L_k^\alpha(x)$ are the generalized Laguerre polynomials of degree k and the parameter $\alpha > -1$, there is another limiting case which leads to the Charlier (or Poisson-Charlier) polynomials $C_k(x; a)$,

$$\lim_{\beta \rightarrow \infty} M_k\left(x; \beta, \frac{a}{a + \beta}\right) = {}_2F_0\left(-k, -x; -\frac{1}{a}\right) =: C_k(x; a).$$

The generating functions corresponding to Charlier polynomials [7] can be expressed as

$$e^y \left(1 - \frac{y}{a}\right)^x = \sum_{k=0}^{\infty} \frac{C_k^{(a)}(x)}{k!} y^k, \quad |y| < a. \tag{1}$$

In the recent years, the polynomials have found new and important applications in approximation theory. Varma and Taşdelen in the important paper [12] introduced a class of Szász-type operators constructed using Charlier polynomials defined as follows

$$(R_m^a f)(x) = \sum_{k=0}^{\infty} c_{m,k}^a(x) f\left(\frac{k}{m}\right), \tag{2}$$

where $a > 1, x \geq 0$ and

$$c_{m,k}^a(x) = e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)mx} \frac{C_k^{(a)}(-(a-1)mx)}{k!}.$$

Although the operators mentioned in (2) are of non-exponential types, their special instances yield exponential forms that satisfy specific partial differential equations. Comprehensive discussions related to exponential operators have been reported in [5] and [6]. Also, some properties and composition of these operators with other operators have been discussed in [3], [9].

As the operators defined in (2) are not suitable for approximating integrable functions, we therefore proceed to consider their Kantorovich and Durrmeyer variants. The Kantorovich type generalization of the Charlier-based operators (see [12],[10]) is defined as

$$(K_m^a f)(x) = m \sum_{k=0}^{\infty} c_{m,k}^a(x) \int_{k/m}^{(k+1)/m} f(t) dt, \tag{3}$$

where $a > 1$ and $x \geq 0$.

The Durrmeyer-type generalization for the Charlier-based operators (see [2]) is defined as

$$(D_{m,\nu}^a f)(x) = m \sum_{k=0}^{\infty} c_{m,k}^a(x) \int_0^{\infty} s_{k+\nu}(mt) f(t) dt, \tag{4}$$

where, $s_{k+\nu}(t) = e^{-t} \frac{t^{k+\nu}}{(k+\nu)!}$, ν is some non-negative integer, $a > 1$ and $x \geq 0$.

The Stirling number of first kind denoted by $s_{r,i}$ is defined by

$$x^{\underline{r}} = \sum_{i=0}^r s_{r,i} x^i, \quad x^{\bar{r}} = \sum_{i=0}^r (-1)^{r-i} s_{r,i} x^i.$$

The Stirling numbers of the second kind can be defined by

$$x^{\nu} = \sum_{i=0}^{\nu} \left\{ \begin{matrix} \nu \\ i \end{matrix} \right\} x^i.$$

Here $x^{\underline{s}} = \prod_{i=0}^{s-1} (x - i)$ denotes the falling factorial and the monomial is given by $e_j(t) = t^j$, where j is a non-negative integer. Also, $E_{\beta}[0, \infty)$ denotes the class of all continuous functions on $[0, \infty)$ with the growth $|f(x)| \leq M e^{\beta x}$, $M > 0, \beta > 0$ and $x \geq 0$.

This study investigates the complete asymptotic expansion corresponding to the Kantorovich and Durrmeyer Charlier-type operators. While various approximation properties of these operators have been explored in earlier works (see [12], [10], [2]), their asymptotic expansion has not yet been examined in detail. In this work, we begin by deriving lemmas for the computation of moments and central moments, and subsequently obtain their complete asymptotic expansion.

2. Asymptotic expansion for Charlier-Kantorovich operators

In this section, we present the complete asymptotic expansion for K_m^a , beginning with the derivation of expression for moments and central moments.

Lemma 2.1. For some non-negative integer r , the image of monomials e_r under K_m^a have the following representation:

$$(K_m^a e_r)(x) = \frac{1}{(r+1)} \sum_{t=0}^r m^{-t} (x(1-a))^{r-t} \sum_{l=r-t}^r (1-a)^{-l} s_{l,r-t} \sum_{i=l}^r \left\{ \begin{matrix} r+1 \\ i+1 \end{matrix} \right\} (i+1).$$

Proof. It is clear that, $(K_m^a e_0)(x) = 1$. Proceeding with reference to (3), for $r \in \mathbb{N}$, we get

$$\begin{aligned} (K_m^a e_r)(x) &= m \sum_{k=0}^{\infty} e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)mx} \frac{C_k^{(a)}(- (a-1)mx)}{k!} \int_{k/m}^{(k+1)/m} t^r dt \\ &= m \sum_{k=0}^{\infty} e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)mx} \frac{C_k^{(a)}(- (a-1)mx)}{k! m^{r+1}(r+1)} ((k+1)^{r+1} - k^{r+1}). \end{aligned}$$

By definition of Stirling number of second kind, it follows that

$$\begin{aligned} (K_m^a e_r)(x) &= \frac{m^{-r}}{(r+1)} e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)mx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(- (a-1)mx)}{k!} \sum_{i=0}^{r+1} \left\{ \begin{matrix} r+1 \\ i \end{matrix} \right\} ((k+1)^i - k^i) \\ &= \frac{m^{-r}}{(r+1)} e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)mx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(- (a-1)mx)}{k!} \sum_{i=0}^r \left\{ \begin{matrix} r+1 \\ i+1 \end{matrix} \right\} (i+1) k^i \\ &= \frac{m^{-r}}{(r+1)} e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)mx} \sum_{i=0}^r \left\{ \begin{matrix} r+1 \\ i+1 \end{matrix} \right\} (i+1) \left[\frac{d^i}{dz^i} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(- (a-1)mx)}{k!} z^k \right]_{z=1}. \end{aligned}$$

Using generating function of Charlier polynomial mentioned in (1) and Leibniz rule of higher order differentiation, we get

$$(K_m^a e_r)(x) = \frac{m^{-r}}{(r+1)} \sum_{i=0}^r \left\{ \begin{matrix} r+1 \\ i+1 \end{matrix} \right\} (i+1) \sum_{l=0}^i \binom{i}{l} (1-a)^{-l} (-(a-1)mx)^l.$$

Using Stirling number of first kind, we get

$$\begin{aligned} (K_m^a e_r)(x) &= \frac{m^{-r}}{(r+1)} \sum_{i=0}^r \left\{ \begin{matrix} r+1 \\ i+1 \end{matrix} \right\} (i+1) \sum_{l=0}^i \binom{i}{l} (1-a)^{-l} \sum_{j=0}^l (-(a-1)mx)^j s_{l,j} \\ &= \frac{m^{-r}}{(r+1)} \sum_{i=0}^r \left\{ \begin{matrix} r+1 \\ i+1 \end{matrix} \right\} (i+1) \sum_{j=0}^i (-(a-1)mx)^j \sum_{l=j}^i \binom{i}{l} (1-a)^{-l} s_{l,j} \\ &= \frac{m^{-r}}{(r+1)} \sum_{j=0}^r (-(a-1)mx)^j \sum_{i=j}^r \left\{ \begin{matrix} r+1 \\ i+1 \end{matrix} \right\} (i+1) \sum_{l=j}^i \binom{i}{l} (1-a)^{-l} s_{l,j} \\ &= \frac{1}{(r+1)} \sum_{t=0}^r m^{-t} (-(a-1)x)^{r-t} \sum_{i=r-t}^r \left\{ \begin{matrix} r+1 \\ i+1 \end{matrix} \right\} (i+1) \sum_{l=r-t}^i \binom{i}{l} (1-a)^{-l} s_{l,r-t} \\ &= \frac{1}{(r+1)} \sum_{t=0}^r m^{-t} (x(1-a))^{r-t} \sum_{l=r-t}^r (1-a)^{-l} s_{l,r-t} \sum_{i=l}^r \binom{i}{l} \left\{ \begin{matrix} r+1 \\ i+1 \end{matrix} \right\} (i+1). \end{aligned}$$

Thus, the lemma has been established. \square

Remark 2.2. The first few moments of K_m^a are as follows:

$$\begin{aligned} (K_m^a e_0)(x) &= 1, \\ (K_m^a e_1)(x) &= \frac{3}{2m} + x, \\ (K_m^a e_2)(x) &= \frac{10}{3m^2} + \left(4 + \frac{1}{a-1}\right) \frac{x}{m} + x^2, \\ (K_m^a e_3)(x) &= \frac{37}{4m^3} + \left(31 + \frac{15}{a-1} + \frac{4}{(a-1)^2}\right) \frac{x}{2m^2} + \left(15 + \frac{6}{a-1}\right) \frac{x^2}{2m} + x^3, \\ (K_m^a e_4)(x) &= \frac{151}{5m^4} + \left(64 + \frac{45}{a-1} + \frac{24}{(a-1)^2} + \frac{6}{(a-1)^3}\right) \frac{x}{m^3} + \left(45 + \frac{36}{a-1} + \frac{11}{(a-1)^2}\right) \frac{x^2}{m^2} \\ &\quad + \left(12 + \frac{6}{a-1}\right) \frac{x^3}{m} + x^4. \end{aligned}$$

Lemma 2.3. For $s \in \mathbb{N} \cup \{0\}$, the central moments corresponding to K_m^a can be expressed as

$$(K_m^a \psi_x^s)(x) = \sum_{t=\lfloor \frac{s+1}{2} \rfloor}^s m^{-t} x^{s-t} \sum_{r=0}^{s-t} \binom{s}{r+t} (-1)^{s-r-t} \varrho(r, t),$$

where $(\psi_x^s)(t) := (e_1 - xe_0)^s(t)$, $\lfloor \cdot \rfloor$ represents the greatest integer function and $\varrho(r, t)$ is specified as,

$$\varrho(r, t) = \frac{1}{(r+t+1)} \sum_{l=0}^t (1-a)^{-l} s_{l+r,r} \sum_{i=l+r}^{r+t} \binom{i}{l+r} \left\{ \begin{matrix} r+t+1 \\ i+1 \end{matrix} \right\} (i+1). \tag{5}$$

Moreover, as $m \rightarrow \infty$,

$$(K_m^a \psi_x^s)(x) = O\left(m^{-\lfloor \frac{s+1}{2} \rfloor}\right).$$

Proof. Clearly, above result holds for $s = 0$. For $s \in \mathbb{N}$, we have

$$(K_m^a \psi_x^s)(x) = \sum_{r=0}^s \binom{s}{r} (-x)^{s-r} (K_m^a e_r)(x). \tag{6}$$

Using Lemma 2.1 in (6),

$$(K_m^a \psi_x^s)(x) = \sum_{r=0}^s \binom{s}{r} \frac{(-x)^{s-r}}{(r+1)} \sum_{t=0}^r m^{-t} (x(1-a))^{r-t} \sum_{l=r-t}^r (1-a)^{-l} s_{l,r-t} \sum_{i=l}^r \binom{i}{l} \left\{ \begin{matrix} r+1 \\ i+1 \end{matrix} \right\} (i+1).$$

Thus,

$$\begin{aligned} (K_m^a \psi_x^s)(x) &= \sum_{t=0}^s m^{-t} x^{s-t} \sum_{r=t}^s \binom{s}{r} \frac{(-1)^{s-r}}{(r+1)} \sum_{l=r-t}^r (1-a)^{r-t-l} s_{l,r-t} \sum_{i=l}^r \binom{i}{l} \left\{ \begin{matrix} r+1 \\ i+1 \end{matrix} \right\} (i+1) \\ &= \sum_{t=0}^s m^{-t} x^{s-t} \sum_{r=0}^{s-t} \binom{s}{r+t} \frac{(-1)^{s-r-t}}{(r+t+1)} \sum_{l=r}^{r+t} (1-a)^{r-l} s_{l,r} \sum_{i=l}^{r+t} \binom{i}{l} \left\{ \begin{matrix} r+t+1 \\ i+1 \end{matrix} \right\} (i+1) \\ &= \sum_{t=0}^s m^{-t} x^{s-t} \sum_{r=0}^{s-t} \binom{s}{r+t} \frac{(-1)^{s-r-t}}{(r+t+1)} \sum_{l=0}^t (1-a)^{-l} s_{l+r,r} \sum_{i=l+r}^{r+t} \binom{i}{l+r} \left\{ \begin{matrix} r+t+1 \\ i+1 \end{matrix} \right\} (i+1) \\ &= \sum_{t=0}^s m^{-t} x^{s-t} \sum_{r=0}^{s-t} \binom{s}{r+t} (-1)^{s-r-t} \varrho(r,t), \end{aligned}$$

where

$$\begin{aligned} \varrho(r,t) &= \frac{1}{(r+t+1)} \sum_{l=0}^t (1-a)^{-l} s_{l+r,r} \sum_{i=l+r}^{r+t} \binom{i}{l+r} \left\{ \begin{matrix} r+t+1 \\ i+1 \end{matrix} \right\} (i+1) \\ &= \frac{1}{(r+t+1)} \sum_{l=0}^t (1-a)^{-l} s_{l+r,r} \sum_{i=0}^{t-l} \binom{i+l+r}{i} \left\{ \begin{matrix} r+t+1 \\ i+l+r+1 \end{matrix} \right\} (i+l+r+1). \end{aligned}$$

The Stirling numbers of the first kind [8, p. 151, Eq. (5)] can be expressed in the form

$$s_{l+r,r} = \sum_{v=0}^l M_{l,l-v} \binom{l+r}{l+v}_r,$$

where the coefficient $M_{l,l-v}$ is independent of r . The Stirling numbers of second kind [1, p. 226-227, Ex. 16] possess the representation

$$\left\{ \begin{matrix} r+t+1 \\ i+l+r+1 \end{matrix} \right\} = \sum_{j=t-i-l+1}^{2(t-i-l)} \binom{r+t+1}{j} \left\{ \begin{matrix} j \\ j-t+i+l \end{matrix} \right\}_2,$$

where $\left\{ \begin{matrix} j \\ j-t+i+l \end{matrix} \right\}_2$ denote the associated Stirling numbers of second kind. It follows that, $s_{l+r,r}$ is polynomial in r of degree $2l$ and $\sum_{i=0}^{t-l} \binom{i+l+r}{i} \left\{ \begin{matrix} r+t+1 \\ i+l+r+1 \end{matrix} \right\} (i+l+r+1)$ is polynomial in r of degree $2(t-l)+1$. Thus, $\varrho(r,t)$ is a polynomial in r of degree $2t$. It follows that the inner sum vanishes for $s > 2t$ and hence,

$$(K_m^a \psi_x^s)(x) = \sum_{t=\lfloor \frac{s+1}{2} \rfloor}^s m^{-t} x^{s-t} \sum_{r=0}^{s-t} \binom{s}{r+t} (-1)^{s-r-t} \varrho(r,t).$$

Accordingly, the proof is completed. \square

Remark 2.4. The first few central moments of K_m^a are as follows:

$$\begin{aligned} (K_m^a \psi_x^0)(x) &= 1, \\ (K_m^a \psi_x^1)(x) &= \frac{3}{2m}, \\ (K_m^a \psi_x^2)(x) &= \frac{a}{(a-1)} \frac{x}{m} + \frac{10}{3m^2}, \\ (K_m^a \psi_x^3)(x) &= \left(11 + \frac{15}{a-1} + \frac{4}{(a-1)^2} \right) \frac{x}{2m^2} + \frac{37}{4m^3}, \\ (K_m^a \psi_x^4)(x) &= \frac{3a^2}{(a-1)^2} \frac{x^2}{m^2} + \left(27 + \frac{45}{a-1} + \frac{24}{(a-1)^2} + \frac{6}{(a-1)^3} \right) \frac{x}{m^3} + \frac{151}{5m^4}. \end{aligned}$$

Remark 2.5. We may point out that the third and fourth moments and central moments are not obtained correctly in [11, Lemma 1, Lemma 2]. The correct estimates are provided in Remark 2.2 and 2.4.

Theorem 2.6. Let $q \in \mathbb{N}$ and $x \in [0, \infty)$. For each $f \in E_\beta[0, \infty)$, such that f is $2q$ times differentiable at x , the operators K_m^a have the following asymptotic expansion:

$$(K_m^a f)(x) = f(x) + \sum_{t=1}^q m^{-t} \sum_{s=t}^{2t} \frac{f^{(s)}(x)}{s!} x^{s-t} \sum_{r=0}^{s-t} \binom{s}{r+t} (-1)^{s-r-t} \varrho(r, t) + o(m^{-q}),$$

where $\varrho(r, t)$ is defined in (5).

Proof. Following Sikkema’s result [11, Theorem 3], we obtain for $m \rightarrow \infty$

$$(K_m^a f)(x) = \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} (R_m^a \psi_x^s)(x) + o(m^{-q}).$$

Substituting value of $(K_m^a \psi_x^s)(x)$ from Lemma 2.3, it yields

$$\begin{aligned} (K_m^a f)(x) &= \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} \sum_{t=\lfloor \frac{s+1}{2} \rfloor}^s m^{-t} x^{s-t} \sum_{r=0}^{s-t} \binom{s}{r+t} (-1)^{s-r-t} \varrho(r, t) + o(m^{-q}) \\ &= f(x) + \sum_{t=1}^q m^{-t} \sum_{s=t}^{2t} \frac{f^{(s)}(x)}{s!} x^{s-t} \sum_{r=0}^{s-t} \binom{s}{r+t} (-1)^{s-r-t} \varrho(r, t) + o(m^{-q}). \end{aligned}$$

This completes the result. \square

Corollary 2.7. For $x \in [0, \infty)$, and for each $f \in E_\beta[0, \infty)$, that is twice differentiable at x , the operators K_m^a satisfy

$$\lim_{m \rightarrow \infty} m((K_m^a f)(x) - f(x)) = \frac{3}{2} f'(x) + \frac{ax}{2(a-1)} f''(x).$$

Remark 2.8. In a more definite manner, for $q = 3$, as $m \rightarrow \infty$,

$$\begin{aligned} (K_m^a f)(x) &= f(x) + \frac{1}{m} \left(\frac{3}{2} f'(x) + \frac{ax}{2(a-1)} f''(x) \right) \\ &\quad + \frac{1}{m^2} \left(\frac{5}{3} f''(x) + \frac{ax(11a-7)}{12(a-1)^2} f^{(3)}(x) + \frac{a^2 x^2}{8(a-1)^2} f^{(4)}(x) \right) \\ &\quad + \frac{1}{m^3} \left(\frac{37}{24} f^{(3)}(x) + \frac{ax(9a^2-12a+5)}{8(a-1)^3} f^{(4)}(x) \right. \\ &\quad \left. + \frac{x^2 a^2 (13a-5)}{48(a-1)^3} f^{(5)}(x) + \frac{x^3 a^3}{48(a-1)^3} f^{(6)}(x) \right) + o(m^{-3}). \end{aligned}$$

Remark 2.9. If $f(x) = Ax^{\frac{3-2a}{a}} + B$, where A and B are absolute constants. Under the conditions $a \neq 1$, $a \neq \frac{3}{2}$ and $a \neq 0$, we have

$$\lim_{m \rightarrow \infty} m^2[(K_m^a f)(x) - f(x)] = \frac{A(2a - 3)x^{\frac{3-4a}{a}}(12a^2 - 39a + 25)}{8a^2(a - 1)}$$

for such functions we may have better order of approximation.

3. Asymptotic expansion for Charlier-Durrmeyer operators

In this section, we present the complete asymptotic expansion for $D_{m,v}^a$, beginning with the derivation of expression for moments and central moments.

Lemma 3.1. For some non-negative integer r , the image of monomials e_r under $D_{m,v}^a$ have the following representation:

$$(D_{m,v}^a e_r)(x) = \sum_{j=0}^r m^{-j}((1-a)x)^{r-j} \sum_{l=r-j}^r \binom{r}{l} (1-a)^{-l} s_{l,r-j} \sum_{i=l}^r \binom{r-l}{i-l} (v+r)^{r-i}.$$

Proof. It is clear that, $(D_{m,v}^a e_0)(x) = 1$. Proceeding with reference to (4), for $r \in \mathbb{N}$, we get

$$\begin{aligned} (D_{m,v}^a e_r)(x) &= m \sum_{k=0}^{\infty} e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)mx} \frac{C_k^{(a)}(- (a-1)mx)}{k!} \int_0^{\infty} s_{k+v}(mt)t^r dt \\ &= m \sum_{k=0}^{\infty} e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)mx} \frac{C_k^{(a)}(- (a-1)mx)}{k!} (k+v+r)^r \\ &= m^{-r} e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)mx} \left[\frac{d^r}{dz^r} z^{(v+r)} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(- (a-1)mx)}{k!} z^k \right]_{z=1}. \end{aligned}$$

Using generating function mentioned in (1) and Leibniz rule of higher order differentiation, we obtain

$$\begin{aligned} (D_{m,v}^a e_r)(x) &= m^{-r} e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)mx} \left[\frac{d^r}{dz^r} z^{(v+r)} e^z \left(1 - \frac{z}{a}\right)^{-(a-1)mx} z^k \right]_{z=1} \\ &= m^{-r} \sum_{i=0}^r \binom{r}{i} (v+r)^{r-i} \sum_{l=0}^i \binom{i}{l} (1-a)^{-l} (-(a-1)mx)^l. \end{aligned}$$

Using Stirling number of first kind, we get

$$\begin{aligned} (D_{m,v}^a e_r)(x) &= m^{-r} \sum_{i=0}^r \binom{r}{i} (v+r)^{r-i} \sum_{l=0}^i \binom{i}{l} (1-a)^{-l} \sum_{j=0}^l (-(a-1)mx)^j s_{l,j} \\ &= m^{-r} \sum_{i=0}^r \binom{r}{i} (v+r)^{r-i} \sum_{j=0}^i (-(a-1)mx)^j \sum_{l=j}^i \binom{i}{l} (1-a)^{-l} s_{l,j} \\ &= \sum_{j=0}^r m^{-j} (-(a-1)x)^{r-j} \sum_{i=r-j}^r \binom{r}{i} (v+r)^{r-i} \sum_{l=r-j}^i \binom{i}{l} (1-a)^{-l} s_{l,r-j} \\ &= \sum_{j=0}^r m^{-j} (-(a-1)x)^{r-j} \sum_{l=r-j}^r (1-a)^{-l} s_{l,r-j} \sum_{i=l}^r \binom{i}{l} \binom{r}{i} (v+r)^{r-i} \\ &= \sum_{j=0}^r m^{-j} ((1-a)x)^{r-j} \sum_{l=r-j}^r \binom{r}{l} (1-a)^{-l} s_{l,r-j} \sum_{i=l}^r \binom{r-l}{i-l} (v+r)^{r-i}. \end{aligned}$$

Thus, the lemma has been established. \square

Remark 3.2. The first few moments of $D_{m,\nu}^a$ are as follows:

$$\begin{aligned} (D_{m,\nu}^a e_0)(x) &= 1, \\ (D_{m,\nu}^a e_1)(x) &= \frac{\nu + 2}{m} + x, \\ (D_{m,\nu}^a e_2)(x) &= \frac{x(2a(\nu + 3) - 2\nu - 5)}{(a - 1)m} + \frac{\nu(\nu + 5) + 7}{m^2} + x^2, \\ (D_{m,\nu}^a e_3)(x) &= \frac{3x^2(a(\nu + 4) - \nu - 3)}{(a - 1)m} + \frac{x(3(a - 1)^2(\nu(\nu + 7) + 13) + 3(a - 1)(\nu + 4) + 2)}{(a - 1)^2 m^2} \\ &\quad + \frac{\nu(\nu(\nu + 9) + 29) + 34}{m^3} + x^3, \\ (D_{m,\nu}^a e_4)(x) &= \frac{2x^3(2a(\nu + 5) - 2\nu - 7)}{(a - 1)m} + \frac{x^2(6a^2(\nu(\nu + 9) + 21) - 12a(\nu + 4)^2 + 6\nu(\nu + 7) + 77)}{(a - 1)^2 m^2} \\ &\quad + \frac{2x(2(a - 1)^3(\nu(\nu^2 + 12\nu + 50) + 73) + 3(a - 1)^2(\nu^2 + 9\nu + 21) + 4(a - 1)(\nu + 5) + 3)}{(a - 1)^3 m^3} \\ &\quad + \frac{\nu(\nu^3 + 14\nu^2 + 77\nu + 200) + 209}{m^4} + x^4. \end{aligned}$$

Lemma 3.3. For $s \in \mathbb{N} \cup \{0\}$, the central moments corresponding to $D_{m,\nu}^a$ can be expressed as

$$(D_{m,\nu}^a \psi_x^s)(x) = \sum_{j=\lfloor \frac{s+1}{2} \rfloor}^s m^{-j} x^{s-j} \sum_{r=0}^{s-j} \binom{s}{r+j} (-1)^{s-r-j} \xi(r, j),$$

where $(\psi_x^s)(t) := (e_1 - xe_0)^s(t)$, $\lfloor \cdot \rfloor$ represents the greatest integer function and $\xi(r, t)$ is specified as,

$$\xi(r, j) = \sum_{l=0}^j \binom{r+j}{l+r} (1-a)^{-l} s_{l+r,r} \sum_{i=l+r}^{r+j} \binom{j-l}{i-l-r} (\nu+r+j)^{r+j-i}. \tag{7}$$

Moreover, as $m \rightarrow \infty$,

$$(D_{m,\nu}^a \psi_x^s)(x) = O\left(m^{-\lfloor \frac{s+1}{2} \rfloor}\right).$$

Proof. Clearly, above result holds for $s = 0$. For $s \in \mathbb{N}$, we have

$$(D_{m,\nu}^a \psi_x^s)(x) = \sum_{r=0}^s \binom{s}{r} (-x)^{s-r} (D_{m,\nu}^a e_r)(x). \tag{8}$$

Using Lemma 3.1 in (8),

$$(D_{m,\nu}^a \psi_x^s)(x) = \sum_{r=0}^s \binom{s}{r} (-x)^{s-r} \sum_{j=0}^r m^{-j} ((1-a)x)^{r-j} \sum_{l=r-j}^r \binom{r}{l} (1-a)^{-l} s_{l,r-j} \sum_{i=l}^r \binom{r-l}{i-l} (\nu+r)^{r-i}.$$

Thus

$$\begin{aligned}
 (D_{m,v}^a \psi_x^s)(x) &= \sum_{j=0}^s m^{-j} x^{s-j} \sum_{r=j}^s \binom{s}{r} (-1)^{s-r} \sum_{l=r-j}^r \binom{r}{l} (1-a)^{r-j-l} s_{l,r-j} \sum_{i=l}^r \binom{r-l}{i-l} (v+r)^{r-i} \\
 &= \sum_{j=0}^s m^{-j} x^{s-j} \sum_{r=0}^{s-j} \binom{s}{r+j} (-1)^{s-r-j} \sum_{l=r}^{r+j} \binom{r+j}{l} (1-a)^{r-l} s_{l,r} \sum_{i=l}^{r+j} \binom{r+j-l}{i-l} (v+r+j)^{r+j-i} \\
 &= \sum_{j=0}^s m^{-j} x^{s-j} \sum_{r=0}^{s-j} \binom{s}{r+j} (-1)^{s-r-j} \sum_{l=0}^j \binom{r+j}{l+r} (1-a)^{-l} s_{l+r,r} \sum_{i=l+r}^{r+j} \binom{j-l}{i-l-r} (v+r+j)^{r+j-i} \\
 &= \sum_{j=0}^s m^{-j} x^{s-j} \sum_{r=0}^{s-j} \binom{s}{r+j} (-1)^{s-r-j} \xi(r, j),
 \end{aligned}$$

where,

$$\begin{aligned}
 \xi(r, j) &= \sum_{l=0}^j \binom{r+j}{l+r} (1-a)^{-l} s_{l+r,r} \sum_{i=l+r}^{r+j} \binom{j-l}{i-l-r} (v+r+j)^{r+j-i} \\
 &= \sum_{l=0}^j \binom{r+j}{j-l} (1-a)^{-l} s_{l+r,r} \sum_{i=0}^{j-l} \binom{j-l}{i} (v+r+j)^{j-i-l}.
 \end{aligned}$$

Since, $s_{l+r,r}$ is polynomial in r of degree $2l$, $\binom{r+j}{j-l}$ is polynomial in r of degree $j-l$ and $\sum_{i=0}^{j-l} \binom{j-l}{i} (v+r+j)^{j-i-l}$ is polynomial in r of degree $j-i-l$. Thus, $\xi(r, j)$ is a polynomial in r of degree $2j$. It follows that the inner sum vanishes for $s > 2j$ and hence,

$$(D_{m,v}^a \psi_x^s)(x) = \sum_{j=\lfloor \frac{s+1}{2} \rfloor}^s m^{-j} x^{s-j} \sum_{r=0}^{s-j} \binom{s}{r+j} (-1)^{s-r-j} \xi(r, j).$$

Accordingly, the proof is completed. \square

Remark 3.4. The first few central moments of $D_{m,v}^a$ are as follows:

$$\begin{aligned}
 (D_{m,v}^a \psi_x^0)(x) &= 1, \\
 (D_{m,v}^a \psi_x^1)(x) &= \frac{v+2}{m}, \\
 (D_{m,v}^a \psi_x^2)(x) &= \frac{(2a-1)x}{(a-1)m} + \frac{v(v+5)+7}{m^2}, \\
 (D_{m,v}^a \psi_x^3)(x) &= \frac{x(3a(2a(v+3)-3v-8)+3v+8)}{(a-1)^2 m^2} + \frac{v(v^2+9v+29)+34}{m^3}, \\
 (D_{m,v}^a \psi_x^4)(x) &= \frac{x(12a^3(v(v+7)+13)-6a^2(v(5v+33)+57)+8a(v(3v+19)+32)-2v(3v+19)-64)}{(a-1)^3 m^3} \\
 &\quad + \frac{3(1-2a)^2 x^2}{(a-1)^2 m^2} + \frac{v(v^3+14v^2+77v+200)+209}{m^4}.
 \end{aligned}$$

Theorem 3.5. For each $f \in E_{\beta}[0, \infty)$, such that f is $2q$ times differentiable at x , the operators $D_{m,v}^a$ have the following asymptotic expansion:

$$(D_{m,v}^a f)(x) = f(x) + \sum_{j=1}^q m^{-j} \sum_{s=j}^{2j} \frac{f^{(s)}(x)}{s!} x^{s-j} \sum_{r=0}^{s-j} \binom{s}{r+j} (-1)^{s-r-j} \xi(r, j) + o(m^{-q}),$$

where $\xi(r, j)$ is defined in (7).

Proof. Following Sikkema's result [11, Theorem 3], we obtain for $m \rightarrow \infty$

$$(D_{m,\nu}^a f)(x) = \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} (D_{m,\nu}^a \psi_x^s)(x) + o(m^{-q}).$$

Substituting value of $(D_{m,\nu}^a \psi_x^s)(x)$ from Lemma 3.3, it yields

$$\begin{aligned} (D_{m,\nu}^a f)(x) &= \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} \sum_{j=\lfloor \frac{s+1}{2} \rfloor}^s m^{-j} x^{s-j} \sum_{r=0}^{s-j} \binom{s}{r+j} (-1)^{s-r-j} \xi(r, j) + o(m^{-q}) \\ &= f(x) + \sum_{j=1}^q m^{-j} \sum_{s=j}^{2j} \frac{f^{(s)}(x)}{s!} x^{s-j} \sum_{r=0}^{s-j} \binom{s}{r+j} (-1)^{s-r-j} \xi(r, j) + o(m^{-q}). \end{aligned}$$

This completes the result. \square

Corollary 3.6. For each $f \in E_\beta[0, \infty)$, that is twice differentiable at x , the operators $D_{m,\nu}^a$ satisfy

$$\lim_{m \rightarrow \infty} m((D_{m,\nu}^a f)(x) - f(x)) = (2 + \nu)f'(x) + \frac{x(2a-1)}{2(a-1)} f''(x).$$

Remark 3.7. In a more definite manner, for $q = 3$, as $m \rightarrow \infty$,

$$\begin{aligned} (D_{m,\nu}^a f)(x) &= f(x) + \frac{1}{m} \left((2 + \nu)f'(x) + \frac{x(2a-1)}{2(a-1)} f''(x) \right) + \frac{1}{m^2} \left(\frac{(\nu^2 + 5\nu + 7)}{2} f''(x) \right. \\ &\quad \left. + \frac{x(8 + 3\nu + 3a(-8 - 3\nu + 2a(3 + \nu)))}{6(a-1)^2} f^{(3)}(x) + \frac{(1 - 2a)^2 x^2}{8(a-1)^2} f^{(4)}(x) \right) \\ &\quad + \frac{1}{m^3} \left(\frac{(\nu^3 + 9\nu^2 + 29\nu + 34)}{6} f^{(3)}(x) + \frac{x(8a(3\nu^2 + 19\nu + 32) - 64)}{24(a-1)^3} f^{(4)}(x) \right. \\ &\quad \left. + \frac{x(12(\nu^2 + 7\nu + 13)a^3 - 2\nu(3\nu + 19) - 6(\nu(5\nu + 33) + 57)a^2)}{24(a-1)^3} f^{(4)}(x) \right. \\ &\quad \left. + \frac{x^2(2a-1)(10 + 3\nu + 3a(-10 - 3\nu + 2a(4 + \nu)))}{24(a-1)^3} f^{(5)}(x) + \right. \\ &\quad \left. \frac{x^3(2a-1)^3}{48(a-1)^3} f^{(6)}(x) \right) + o(m^{-3}). \end{aligned}$$

4. Conflict of interest

The authors declare that they have no conflict of interest.

Data Availability Statement

The author confirms that there is no associated data.

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