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New results on the existence of positive periodic solutions for second-order nonlinear neutral differential equations with mixed delays

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Abstract. The present paper is devoted to study second-order neutral differential equations with variable coefficients and mixed delays. By using the theory of fixed point index in cones as well as some properties of a suitable Green function, several sufficient conditions for the existence of one and twin positive periodic solutions are obtained. Our results are a further expansion of previous research results of Khemis *et al.* (Ural Mathematical Journal, 8(2), (2022), 71-80). Finally, two examples are exhibited to show the efficiency and application of our findings which are completely new and enrich the existing literature.

1. Introduction

Neutral differential equations have attracted much attention from many researchers due to the fact that they depend on the present and past states, as well as the delay argument occurring in the highest order derivative of the state variable. This type of equation has been applied to describe numerous intricate dynamical systems, such as population dynamics, heat conduction, economics, engineering, etc. The basis of the theory of neutral functional differential equations can be found in [7],[8],[11], and the references therein.

In recent years, there has been ongoing research interest in both first and second-order differential equations with variable coefficients, including existence of solutions, stability, and oscillation, which have been published, for instance in [1]-[5],[10],[12],[13],[18],[15],[19],[24],[25],[27]-[32], and related sources. In particular, the existence of periodic solutions has become one of the most important qualitative properties in the study of delay differential equations, and much work on the existence of solutions was obtained by applying the fixed point theorem in cones [1]-[3],[5],[10],[20],[21],[22],[23],[27], [32]-[34], Krasnoselskii's fixed point theorem [13],[14],[29],[31], coincidence degree theory [16]. As it is well known, there has been

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a good amount of work on the existence of single and multiple positive periodic solutions for functional equations (see [10],[20],[21],[23],[24],[27],[28],[32], and references therein). Compared with the progress in the field of studying the existence of positive periodic solutions, less progress has been made in studying the existence of multiple positive periodic solutions. Although there are many valuable results about neutral differential equations, they are mainly concerned with the first-order case. In many cases, it is advantageous to treat the second-order differential equations directly rather than to convert them to first-order equations. One of the effective approaches to fulfilling such a problem is to use the fixed point theorem, and some a priori estimates of possible periodic solutions are obtained. However, only a few papers have been published on the existence of positive periodic solutions of the nonlinear second-order neutral delay population models (see [13],[14]).

To the best of the authors' knowledge, there have been no results about the existence of multiple solutions of second-order nonlinear neutral differential equations with mixed delays that are considered in the present paper. Therefore, our work is the first attempt to study the existence of positive periodic solutions for the present model by means of the fixed point theorem index in cones together with the Green functions, as well as some useful functional analysis tools. For this reason, in this paper we make a first attempt to close this gap and provide a set of novel sufficient conditions for the existence of single and multiple positive periodic solutions. Our approach enables us to obtain some results that are not covered by the existing literature. A comparison between the obtained results and some existing results is given. To the best of our knowledge, most results of the present paper are new. The main motivation of our analysis is raised from the results of Khemis *et al* [13],[14] and some related results in the literature [10], [19]-[21],[23],[27], [28],[32].

Recently, by means of the fixed point theorem index, Z. Dai and B. Du [5] established sufficient conditions for the existence of positive periodic solutions of two types of second-order nonlinear differential equations with variable coefficients and mixed delays of the following form:

$$\frac{d^2}{dt^2}y(t) + p(t)\frac{d}{dt}y(t) + q(t)y(t) = g(t, y(t - \tau(t))) + \int_0^{\infty} D(s) f(x(t-s))\, ds, \tag{1.1}$$

and

$$\frac{d^2}{dt^2}(Ty)(t) + p(t)\frac{d}{dt}y(t) + q(t)y(t) = g(t, y(t - \tau(t))) + \int_0^{\infty} D(s) f(x(t-s))\, ds, \tag{1.2}$$

where $p, q, \tau \in C(\mathbb{R}, (0, +\infty))$ are $\omega-$periodic functions, $f, g \in C(\mathbb{R}, \mathbb{R})$,

$$(Ty)(t) = y(t) - c(t)\, y(t-\tau),$$

$c \in C^1(\mathbb{R}, \mathbb{R})$ is an ω-periodic function with $c(t) \neq 1$, $\tau > 0$ is a constant, and D is a continuous and integrable function on $[0, +\infty)$ with $\int_0^{\infty} D(s)\, ds = 1$.

In [29], Wang *et al.* considered the following non-neutral second order delay differential equation with periodic coefficients:

$$\frac{d^2}{dt^2}y(t) + p(t)\frac{d}{dt}y(t) + q(t)y(t) = r(t)\, y'(t - \tau(t)) + f(t, y(t), y(t - \tau(t))). \tag{1.3}$$

By using Krasnoselskii's fixed point theorem and the contraction mapping principle combined with some useful properties of a Green function, they established some criteria for the existence and uniqueness of periodic solutions for the considered equation.

Very recently, Khemis *et al.* [14] generalized the same method of Wang *et al.* to the following second order linear neutral differential equations with multiple delays:

$$\frac{d^2}{dt^2}y(t) + p(t)\frac{d}{dt}y(t) + q(t)y(t) + \frac{d^2}{dt^2}\left[k(t)\, y(t) - \sum_{j=1}^{n} c_j(t)\, y\left(t - \tau_j(t)\right)\right] = e(t). \tag{1.4}$$

The authors obtained a set of easily verifiable sufficient conditions to only guarantee the existence and uniqueness of periodic solutions to equation (1.4), rather than positive solutions must exist. However, for many problems in real life, we need to consider the properties of its positive periodic solution. This situation motivates our present research.

Motivated by the papers mentioned above, using the theory of fixed point index in cones, we aim to study the existence of single and multiple periodic solutions. We proceed to develop more results in second-order nonlinear neutral differential equations with distributed and multiple delays, which are formulated as follows:

$$\frac{d^2}{dt^2}y(t)+p(t)\frac{d}{dt}y(t)+q(t)y(t)+\frac{d^2}{dt^2}\left[k(t)\,y(t)-\sum_{j=1}^{n}c_j(t)\,g\left(y\left(t-\tau_j(t)\right)\right)\right]$$
$$=e(t)\int_{-\infty}^{0}D(r)\,f(t,y(t+r))\,dr, \tag{1.5}$$

where, $p,q,e\in C(\mathbb{R},\mathbb{R}^+)\,,k,c_j,\tau_j\in C^1(\mathbb{R},\mathbb{R}^+)\,,j=\overline{1,n},$ the functions $f\in C(\mathbb{R}\times\mathbb{R},\mathbb{R}^+)\,,\ g\in C(\mathbb{R},\mathbb{R})$. Moreover, D is a continuous and integrable function on $]-\infty,0]$ with $\int_{-\infty}^{0}D(r)\,dr=1$.

As it is seen from the literature review given above, the qualitative behaviors of solutions of numerous second-order delay differential equations have been investigated recently by using various methods. Also, it is noticed that the neutral term

$$\frac{d^2}{dt^2}\left[k(t)\,y(t)-\sum_{j=1}^{n}c_j(t)\,g\left(y\left(t-\tau_j(t)\right)\right)\right], \tag{1.6}$$

of Eq. (1.5) produces nonlinearity in the derivative term, which is different from Eq. (1.4) that has been studied in [14], which enters linearly. In view of these, we would like to point out that Eq. (1.4) is a new mathematical model, and it includes some of the second-order delay differential equations above. In addition, the papers [13],[14] listed above obtained the existence and uniqueness results of periodic solutions by using Krasnosel'skii's fixed point theorem and contraction mapping principle, which seem to be restrictive in some cases.

Stimulated by the above discussions, the fixed point theorem index in cones is an ideal tool used to prove the existence of single and multiple positive periodic solutions to equation (1.5), which distinguishes our work from the already established literature. As is well known, the existence of multiple positive periodic solutions is significant and challenging in both theory and practice. However, the presence of the nonlinear neutral term defined as in (1.6) makes us face new challenges that have not been covered by the previous literature. Generalization and improvement of some previous relative results of the literature are also the novel contributions to the qualitative theory of second-order nonlinear neutral differential equations. The techniques used in this paper are completely different from the previous ones and can be applied to other functional differential equations.

For convenience, we first introduce some related definitions and lemmas which will be used in this paper.

Definition 1.1. (See [9]) *Let X be a Banach space and let K be a closed, nonempty subset of X. K is said to be a cone if*

a) $\alpha x+\beta y\in K$ for all $x,y\in K$ and all $\alpha,\beta\geq 0$;

b) $y,-y\in K$ imply $y=0$.

Theorem 1.1. (See [6, Theorem 2.3.3, p. 93]). *Let X be a Banach space, and let $K\subset X$ be a cone in X. Assume that Ω_1 and Ω_2 are open subsets of X with $0\in\Omega_1,\overline{\Omega}_1\subset\Omega_2$ and let*

$$\Phi:K\cap\left(\overline{\Omega}_2\setminus\Omega_1\right)\to K,$$

be a completely continuous operator such that, either

i) $\|\Phi y\| \leq \|y\|$ *for* $y \in K \cap \partial\Omega_1$ *and* $\|\Phi y\| \geq \|y\|$ *for* $y \in K \cap \partial\Omega_2$*; or*
ii) $\|\Phi y\| \geq \|y\|$ *for* $y \in K \cap \partial\Omega_1$ *and* $\|\Phi y\| \leq \|y\|$ *for* $y \in K \cap \partial\Omega_2$.
Then Φ *has a fixed point in* $K \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.

In order to use Theorem 1.1 to prove the existence of positive periodic solutions of equation (1.5), we shall consider the following space:

For $\omega > 0$. Let C_ω be the Banach space such that

$$C_\omega = \{y \in C(\mathbb{R}, \mathbb{R}),\ y(t+\omega) = y(t) \text{ for } t \in \mathbb{R}\},$$

with the supremum norm

$$\|y\| = \max_{t\in\mathbb{R}} |y(t)| = \max_{t\in[0,\omega]} |y(t)|, \text{ for } y \in C_\omega.$$

Throughout this paper, we assume that p, q, k, e and c_j, τ_j, $j = \overline{1,n}$ are continuous real-valued functions such that

$$\begin{aligned} &p(t+\omega) = p(t), q(t+\omega) = q(t), k(t+\omega) = k(t),\ e(t+\omega) = e(t) \\ &c_j(t+\omega) = c_j(t), \tau_j(t+\omega) = \tau_j(t), \tau_j(t) \geq \tau_j^* > 0, j = \overline{1,n}, \end{aligned} \tag{1.7}$$

and

$$\int_0^\omega q(u)\,du > 0, \int_0^\omega p(u)\,du > 0. \tag{1.8}$$

The function $f(t,y)$ is assumed to be periodic in t with period ω. That is

$$f(t+\omega, y) = f(t,y). \tag{1.9}$$

The organization of this paper is as follows. In Section 2, we introduce some basic preliminaries, which play an important role in this work. In Section 3, existence theorems for one and two positive periodic solutions of (1.5) are established by using a well-known fixed-point index theorem due to Krasnoselskii. Finally, in Section 4, we compare our results with some recent results and provide two examples to illustrate the effectiveness of the obtained results.

2. Green's Function and Periodicity

This section is devoted to setting up our problem and to obtaining some preliminary results that will be used in our further analysis.

Lemma 2.1. (See [17]). *Suppose that* (1.7) *and* (1.8) *hold and*

$$\frac{\widetilde{\mathfrak{K}}_1\left[\exp\left(\int_0^\omega p(u)\,du\right) - 1\right]}{Q_1\omega} \geq 1, \tag{2.1}$$

where

$$\widetilde{\mathfrak{K}}_1 = \max_{t\in[0,\omega]} \left| \int_t^{t+\omega} \frac{\exp\left(\int_t^s p(u)\,du\right)}{\exp\left(\int_0^\omega p(u)\,du\right) - 1} q(s)\,ds \right|,$$

$$Q_1 = \left(1 + \exp\left(\int_0^\omega p(u)\,du\right)\right)^2 \widetilde{\mathfrak{K}}_1^2.$$

Then, there exist $\omega-$periodic functions $a,b \in C^1(\mathbb{R},\mathbb{R})$, such that

$$b(t) > 0, \qquad \int_0^{\omega} a(u)\,du > 0,$$

and

$$a(t) + b(t) = p(t),\ b'(t) + a(t)b(t) = q(t),\ \text{for all } t \in \mathbb{R}.$$

Lemma 2.2. (See [29]). *Suppose the conditions of Lemma 2.1 hold and $y \in C_{\omega}$. Then, equation*

$$y''(t) + p(t)y'(t) + q(t)y(t) = \phi(t),$$

possesses an ω-periodic solution. Moreover, the periodic solution can be expressed by

$$y(t) = \int_t^{t+\omega} G(t,s)\,\phi(t)\,ds, \tag{2.2}$$

where

$$\begin{aligned} G(t,s) \;=\; & \frac{\int_t^s \exp\left[\int_t^u b(v)\,dv + \int_u^s a(v)\,dv\right] du}{\left[\exp\left(\int_0^{\omega} a(u)\,du\right) - 1\right]\left[\exp\left(\int_0^{\omega} b(u)\,du\right) - 1\right]} \\ & + \frac{\int_s^{t+\omega} \exp\left[\int_t^u b(v)\,dv + \int_u^{s+\omega} a(v)\,dv\right] du}{\left[\exp\left(\int_0^{\omega} a(u)\,du\right) - 1\right]\left[\exp\left(\int_0^{\omega} b(u)\,du\right) - 1\right]}. \end{aligned} \tag{2.3}$$

Corollary 2.1. (See [29]). *The Green function G satisfies the following properties:*

$$\begin{aligned} & G(t+\omega, s+\omega) = G(t,s),\ G(t, t+\omega) = G(t,t), \\ & \frac{\partial}{\partial s} G(t,s) = a(s)G(t,s) - \frac{\exp\left(\int_t^s b(v)\,dv\right)}{\exp\left(\int_0^{\omega} b(v)\,dv\right) - 1}, \\ & \frac{\partial}{\partial t} G(t,s) = -b(t)G(t,s) + \frac{\exp\left(\int_t^s a(v)\,dv\right)}{\exp\left(\int_0^{\omega} a(v)\,dv\right) - 1}, \\ & \frac{\partial^2}{\partial s^2} G(t,s) = \left((a'(s) + a(s))\,G(t,s) - (a(s) + b(s))\right) \frac{\exp\left(\int_t^s b(v)\,dv\right)}{\exp\left(\int_0^{\omega} b(v)\,dv\right) - 1}. \end{aligned}$$

Lemma 2.3 (See [29]). *Let $A = \int_0^{\omega} p(u)du$ and $B = \omega^2 \exp\left(\frac{1}{\omega}\int_0^{\omega} \ln(q(u))\,du\right)$. If*

$$A^2 \geq 4B, \tag{2.4}$$

then

$$\begin{aligned} \min\left\{\int_0^{\omega} a(u)du, \int_0^{\omega} b(u)du\right\} &\geq l, \\ \max\left\{\int_0^{\omega} a(u)du, \int_0^{\omega} b(u)du\right\} &\leq L, \end{aligned}$$

where

$$l = \frac{1}{2}\left(A - \sqrt{A^2 - 4B}\right),\ L = \frac{1}{2}\left(A + \sqrt{A^2 - 4B}\right). \tag{2.5}$$

The following lemma is fundamental to our discussion. Since the method is similar to that in the literature [14], we thus omit its proof.

Lemma 2.4. *Assume that* (1.7) – (1.9) *and* (2.1) *hold. For* $y \in C_\omega \cap C^1(\mathbb{R}, \mathbb{R})$, *the function* $y(\cdot)$ *is an* $\omega-$*periodic solution of equation* (1.5) *if and only if* $y(\cdot)$ *is an* $\omega-$*periodic solution of the following equation:*

$$\begin{aligned} y(t) &= \frac{1}{k(t)+1}\sum_{j=1}^{n} c_j(t)\, y\left(t-\tau_j(t)\right) \\ &+\frac{1}{k(t)+1}\int_t^{t+\omega} G(t,s)\, e(s)\int_{-\infty}^{0} D(r)\, f(s, y(s+r))\, dr ds \\ &+\frac{1}{k(t)+1}\int_t^{t+\omega} Z(t,s)\left[k(s)\, y(s) - \sum_{j=1}^{n} c_j(s)\, g\left(y\left(s-\tau_j(s)\right)\right)\right] ds, \end{aligned} \tag{2.6}$$

where

$$Z(t,s) = (a(s)+b(s))\, E(t,s) - (a(s)+a'(s))\, G(t,s), \tag{2.7}$$

and

$$E(t,s) = \frac{\exp\left(\int_t^s b(v)\, dv\right)}{\exp\left(\int_0^\omega b(v)\, dv\right)-1}.$$

Corollary 2.2 (See [29]). *Functions* G and E *satisfy*

$$0 < N_1 \le G(t,s) \le M_1, \ t, s \in [0, \omega], \tag{2.8}$$

and

$$|E(t,s)| \le \widehat{M}_2^*, \ t, s \in [0, \omega], \tag{2.9}$$

where

$$N_1 = \frac{\omega}{(\exp(L)-1)^2},\ M_1 = \frac{\omega\left(\exp\int_0^\omega p(u)du\right)}{(\exp(l)-1)^2}, \widehat{M}_2^* = \frac{e^L}{e^l-1}, \tag{2.10}$$

and

$$e^l \neq 1, \ \exp(L) \neq 1,$$

where L, l *are given by* (2.5).

Thanks to (2.6), we define operator $\Phi : C_\omega \to C_\omega$ by:

$$\begin{aligned} (\Phi y)(t) &= \frac{1}{k(t)+1}\sum_{j=1}^{n} c_j(t)\, y\left(t-\tau_j(t)\right) \\ &+\frac{1}{k(t)+1}\int_t^{t+\omega} G(t,s)\, e(s)\int_{-\infty}^{0} D(r)\, f(s, y(s+r))\, dr ds \\ &+\frac{1}{k(t)+1}\int_t^{t+\omega} ((a(s)+b(s))\, E(t,s) - (a(s)+a'(s))\, G(t,s)) \\ &\times\left[k(s)\, y(s) - \sum_{j=1}^{n} c_j(s)\, g\left(y\left(s-\tau_j(s)\right)\right)\right] ds. \end{aligned} \tag{2.11}$$

Therefore, the problem of discussing the existence of one and twin positive periodic solutions can be reduced to the existence of fixed points of Φ.
Now we will list the following assumptions, which will be imposed along our paper:
(A1) For all $t \in \mathbb{R}, y \in C_\omega(\mathbb{R}, \mathbb{R}^+)$

$$k(t)\, y(t) - \sum_{j=1}^{n} c_j(t)\, g\left(y\left(t - \tau_j(t)\right)\right) \geq 0.$$

(A2) There exists a nonnegative constant $\widehat{N_2^*} < \widehat{M_2^*}$, such that

$$E(t,s) \geq \widehat{N_2^*}, t, s \in [0, \omega],$$

where $\widehat{M_2^*}$ is as in (2.9).
Thanks to (2.8),(2.9),(A2), there exist nonnegative constants N_2, M_2, such that

$$0 \leq N_2 \leq Z(t,s) \leq M_2, \text{ for all } t, s \in [0, \omega]. \tag{2.12}$$

For any $t, s \in [0, \omega]$, from (2.8) and (2.12), we have

$$1 > \frac{G(t,s)}{\vartheta_2} \geq \frac{\vartheta_1}{\vartheta_2} = \sigma > 0, \tag{2.13}$$

where

$$\vartheta_1 = \min\{N_1, N_2\}, \vartheta_2 = \max\{M_1, M_2\}. \tag{2.14}$$

Since Eq. (2.6) is equivalent to Eq. (1.5), we just have to study the existence of positive periodic solutions to (2.6). To this end, we will use Theorem 1.1 where we consider $(X, \|.\|) = (C_\omega, \|.\|)$.

Now, define K as a cone in C_ω by

$$K = \left\{y(\cdot) \in C_\omega : y(t) \geq 0, \text{ and } y(t) \geq \sigma \left\|y\right\|, t \in [0, \omega]\right\}.$$

It is not difficult to check that K is a cone in C_ω.
Lemma 2.5. *Assume that (A1),(A2), and* (2.1), (2.4) *hold. Then* Φ: $K \to K$ defined by (2.11) is well defined, namely, $\Phi(K) \subset K$.

Proof. Since $x(\cdot) \in C_\omega$, and the functions $f(\cdot,\cdot) \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R}, \mathbb{R})$, $a \in C^1(\mathbb{R}, \mathbb{R})$, and $b, k, e, c_j, \tau_j, j = 1, n$ are continuous, it follows that the mapping $t \mapsto (\Phi x)(t)$ is continuous on $\mathbb{R}$, thanks to (2.11), and it is easy to verify that $(\Phi y)(t + \omega) = (\Phi y)(t)$. For all $(t,s) \in \mathbb{R}^2$ and by (A1),(A2),(2.12), it follows from (2.11), for any $y \in K$, that

$$(\Phi y)(t) \geq 0.$$

Also, for $y \in K$, by using (2.8),(2.12),(2.13), and (A1), we have that

$$\begin{aligned}\left|(\Phi y)(t)\right| &\leq \frac{1}{k(t)+1}\sum_{j=1}^{n} c_j(t)\left|y\left(t-\tau_j(t)\right)\right| \\ &+\frac{\vartheta_2}{k(t)+1}\int_t^{t+\omega} e(s)\int_{-\infty}^{0} D(r)\, f(s, y(s+r))\, drds \\ &+\frac{\vartheta_2}{1+k(t)}\int_t^{t+\omega}\left[k(s)\, y(s) - \sum_{j=1}^{n} c_j(s)\, g\left(y\left(s-\tau_j(s)\right)\right)\right] ds.\end{aligned}$$

Noticing that, we obtain

$$\begin{aligned}(\Phi y)(t) &\geq \frac{\vartheta_1}{\vartheta_2}\left[\frac{1}{k(t)+1}\int_t^{t+\omega}\vartheta_2\left(e(s)\int_{-\infty}^{0}D(r)f(s,y(s+r))\,dr\right)ds\right.\\ &+\frac{1}{k(t)+1}\int_t^{t+\omega}\vartheta_2\left[k(s)y(s)-\sum_{j=1}^{n}c_j(s)g\left(y\left(s-\tau_j(s)\right)\right)\right]ds\\ &\left.+\frac{1}{k(t)+1}\sum_{j=1}^{n}c_j(t)y\left(t-\tau_j(t)\right)\right]\\ &\geq \sigma\left\|(\Phi y)(t)\right\|.\end{aligned}$$

That is, $\Phi(K)$ is contained in K. This completes the proof.
For any $Y(t)=(y(t),y(t-\tau_1(t)),y(t-\tau_2(t)),...,y(t-\tau_n(t)))\in\mathbb{R}^{n+1}$, $t\geq 0$, we define $\left\|y_i\right\|=\sup_{t\in[0,\omega]}\left|y(t-\tau_i(t))\right|$, for all $i=\overline{0,n}$, where $\tau_0(t)=0$, and $\|Y\|=\max_{0\leq i\leq n}\left\|y_i\right\|$,

$$h(t,Y(t))=k(t)y(t)-\sum_{j=1}^{n}c_j(t)g\left(y\left(t-\tau_j(t)\right)\right), \tag{2.15}$$

$$\widehat{E}(t,s)=\frac{\exp\left(\int_t^s a(v)\,dv\right)}{\exp\left(\int_0^\omega b(v)\,dv\right)-1},\ t,s\in[0,\omega], \tag{2.16}$$

and

$$\varsigma(t)=\sum_{j=1}^{n}c_j(t). \tag{2.17}$$

To simplify our description, we introduce the following constants:

$$\begin{aligned}\widehat{\rho^*} &= \min_{0\leq t\leq\omega}\frac{1}{1+k(t)},\ \rho^*=\max_{0\leq t\leq\omega}\frac{1}{1+k(t)},\ e^*=\max_{0\leq t\leq\omega}e(t),\\ \widehat{c^*} &= \min_{0\leq t\leq\omega}\varsigma(t),\ c^*=\max_{0\leq t\leq\omega}\varsigma(t),\ \kappa_\circ=\max_{0\leq t\leq\omega}k'(t),\ \widehat{\kappa}=\max_{0\leq t\leq\omega}k(t)\\ \delta &= \max_{0\leq t\leq\omega}\sum_{j=1}^{n}\left|c_j'(t)\right|,\ \gamma=\max_{0\leq t\leq\omega}\left|1-\tau_j'(t)\right|,\\ \mu &= \max_{0\leq t\leq\omega}\left|k'(t)-b(t)(k(t)+1)\right|,\ \beta_\circ=\frac{\exp\left(\int_0^\omega a(v)\,dv\right)}{\exp\left(\int_0^\omega b(v)\,dv\right)-1},\\ \widehat{\omega} &= \max_{0\leq t\leq\omega}(a(t)+b(t)),\ \widehat{\xi}=\max_{0\leq t\leq\omega}\left|a(t)+a'(t)\right|.\end{aligned} \tag{2.18}$$

Lemma 2.6. *Assume that (2.1),(2.4) and (A1),(A2) hold. Then $\Phi: K\to K$ is completely continuous.*
Proof. We first show that Φ is continuous. For any $\varrho>0$ and $\varepsilon>0$, there exists $\eta_1>0$, such that for $x,y\in C_\omega$, $\|x\|\leq\varrho$, $\left\|y\right\|\leq\varrho$, and $\left\|x-y\right\|<\eta_1$, we have

$$\sup_{0\leq t\leq\omega}\left|\sum_{j=1}^{n}c_j(t)x\left(t-\tau_j(t)\right)-\sum_{j=1}^{n}c_j(t)y\left(t-\tau_j(t)\right)\right|<\frac{\varepsilon}{3\rho^*\vartheta_2}. \tag{2.19}$$

For any $\varrho > 0$ and $\varepsilon > 0$, there exists $\eta_2 > 0$, such that for $x, y \in C_\omega$, $\|x\| \le \varrho$, $\|y\| \le \varrho$, and $\|x - y\| < \eta_2$, we have

$$\sup_{0 \le t \le \omega} \left| e(t) \int_{-\infty}^{0} K(r) \left(f(t, x(t+r)) - f(t, y(t+r)) \right) dr \right| < \frac{\varepsilon}{3\rho^* \omega \vartheta_2}. \tag{2.20}$$

For any $\varrho > 0$ and $\varepsilon > 0$, there exists $\eta_3 > 0$, such that for $X, Y \in C_\omega^{n+1}$, $\|X\| \le \varrho$, $\|Y\| \le \varrho$, and $\|X - Y\| < \eta_3$, we have

$$\sup_{0 \le t \le \omega} |h(s, X(s)) - h(s, Y(s))| < \frac{\varepsilon}{3\omega \vartheta_2 \mu \rho^*}. \tag{2.21}$$

If $x, y \in C_\omega$ with $\|x\| \le \varrho$, $\|y\| \le \varrho$, and $\|x - y\| < \eta$, where $\eta = \max\{\eta_1, \eta_2, \eta_3\}$. From (2.19) – (2.21), we have

$$\begin{aligned}
& \left| (\Phi x)(t) - (\Phi y)(t) \right| \\
\le & \left| \sum_{j=1}^{n} c_j(t) \left(x\left(t - \tau_j(t)\right) - y\left(t - \tau_j(t)\right) \right) \right| \\
& + \vartheta_2 \rho^* \int_t^{t+\omega} \left(e(s) \int_{-\infty}^{0} D(r) \left| f(s, x(s+r)) - f(s, y(s+r)) \right| dr \right) ds \\
& + \vartheta_2 \rho^* \int_t^{t+\omega} |h(s, X(s)) - h(s, Y(s))| \, ds \\
< & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

which implies that Φ is continuous on K.
Now, we show that Φ is completely continuous. We let

$$\widetilde{\mathcal{D}} = \left\{ y(\cdot) \in K : \|y\| \le \ell, \|y'\| \le \widehat{\ell} \right\},$$

be a bounded set in K, where $\ell, \widehat{\ell}$ are nonnegative constants.
It follows from (2.11), for $t \in [0, \omega]$, $y \in D$,

$$\begin{aligned}
\left| (\Phi y)(t) \right| \le & \frac{1}{k(t) + 1} \sum_{j=1}^{n} c_j(t) \max_{t \in [0,\omega]} \left| y\left(t - \tau_j(t)\right) \right| \\
& + \frac{\vartheta_2}{k(t) + 1} \int_t^{t+\omega} e(s) \int_{-\infty}^{0} D(r) \max_{s \in [0,\omega] \|y\| \le \ell} f(s, y(s+r)) \, dr ds \\
& + \frac{\vartheta_2}{1 + k(t)} \int_t^{t+\omega} \max_{s \in [0,\omega] \|Y\| \le \ell} h(s, Y(s)) \, ds \\
\le & \rho^* \ell c^* + \vartheta_2 \rho^* e^* \omega \max_{s \in [0,\omega] \|y\| \le \ell} f(s, y(s+r)) + \vartheta_2 \rho^* \omega \max_{s \in [0,\omega] \|Y\| \le \ell} h(s, Y(s)) \\
= & : \Theta.
\end{aligned}$$

Therefore, for any $y \in D$,

$$\|\Phi y\| \le \Theta,$$

which implies that $\Phi\left(\widetilde{\mathcal{D}}\right)$ is a uniformly bounded set. By taking the derivative in (2.11), we have

$$\frac{d}{dt} (\Phi y)(t) = \frac{-k'(t)}{(k(t) + 1)^2} \sum_{j=1}^{n} c_j(t) \, y\left(t - \tau_j(t)\right)$$

$$
\begin{aligned}
&+\frac{1}{k(t)+1}\sum_{j=1}^{n}c_j'(t)\,y\left(t-\tau_j(t)\right)+\frac{1}{k(t)+1}\sum_{j=1}^{n}c_j(t)\,y'\left(t-\tau_j(t)\right)\left(1-\tau_j'(t)\right)\\
&+\frac{1}{k(t)+1}\int_t^{t+\omega}E(t,s)\,e(s)\int_{-\infty}^{0}D(r)\,f(s,y(s+r))\,dr\\
&+\frac{k'(t)-b(t)(k(t)+1)}{(k(t)+1)^2}\int_t^{t+\omega}e(s)\int_{-\infty}^{0}D(r)\,f(s,y(s+r))\,drG(t,s)\,ds\\
&+\frac{a(t)+b(t)}{k(t)+1}h(s,Y(s))\\
&+\frac{k'(t)-b(t)(k(t)+1)}{(k(t)+1)^2}\int_t^{t+\omega}h(s,Y(s))\,Z(t,s)\,ds\\
&-\frac{1}{k(t)+1}\int_t^{t+\omega}(a(s)+a'(s))\,h(s,Y(s))\,\widehat{E}(t,s)\,ds.
\end{aligned}
$$

The function $\widehat{E}$ satisfies

$$
\left|\widehat{E}(t,s)\right|\leq\beta_0, \tag{2.22}
$$

where β_0 is given as in (2.18) .
Consequently, by invoking (2.5), (2.14), (2.15), (2.19), (2.22), (3.2), we have

$$
\begin{aligned}
\frac{d}{dt}(\Phi y)(t) &\leq c^*\kappa_\circ\rho^{*2}\max_{t\in[0,\omega]}\left|y\left(t-\tau_j(t)\right)\right|+\delta\rho^*\max_{t\in[0,\omega]}\left|y\left(t-\tau_j(t)\right)\right|\\
&+c^*\rho^*\gamma\max_{t\in[0,\omega]}\left|y'\left(t-\tau_j(t)\right)\right|+\omega\rho^*e^*\widehat{M}_2^*\max_{s\in[0,\omega]\|y\|\leq\ell}f(s,y(s+r))\\
&+\omega\rho^*e^*M_1\mu\max_{s\in[0,\omega]\|y\|\leq\ell}f(s,y(s+r))\\
&+\omega\left(\mu\vartheta_2\rho^{*2}+\rho^*\widehat{\xi}\beta_0+\rho^*\varpi\right)\max_{s\in[0,\omega]\|Y\|\leq\ell}h(s,Y(s))\\
&\leq c^*\kappa_\circ\rho^{*2}\ell+\delta\rho^*\ell++c^*\rho^*\gamma\ell'\\
&+\left(\omega\rho^*e^*\widehat{M}_2^*++\omega\rho^*e^*M_1\mu\right)\max_{s\in[0,\omega]\|y\|\leq\ell}f(s,y(s+r))\\
&+\omega\left(\mu\vartheta_2\rho^{*2}+\rho^*\widehat{\xi}\beta_0+\rho^*\varpi\right)\max_{s\in[0,\omega]\|Y\|\leq\ell}h(s,Y(s))\\
&=:\Xi.
\end{aligned}
$$

Thus, the above expression yields $\left|\frac{d(\Phi y)(t)}{dt}\right|\leq\Xi$, which implies that $\frac{d(\Phi y)(t)}{dt}$ is also uniformly bounded, for any $y\in\widetilde{\mathcal{D}}$. Hence, $\Phi(D)\subset C_\omega$ is a family of uniformly bounded and equi-continuous functions. By the Arzelà-Ascoli theorem (see Royden [26]), the operator Φ is completely continuous. The proof of Lemma 2.6 is complete.

3. Existence of positive periodic solutions

This section is devoted to establishing several theorems, which are the main results of our paper. By means of the theory of fixed point index in cones, various sufficient conditions for the existence of one or twin positive periodic solutions for Eq. (1.5) are established. For this, we will use a hybrid technique that combines the fixed point theory in cones, Green's functions method, and some functional analysis tools to achieve our purposes. The solution continuously depends on the functions k,e,c_j, $j=\overline{1,n}$ and f,g. The derived results are new and generalize some previous studies.

We are now in a position to state and prove our results of the existence of positive ω-periodic solutions for (1.5).

Theorem 3.1. *Suppose that $g(y) \leq 0$ for $y \in \mathbb{R}^+$. If conditions (A2),(2.1), (2.4), (2.8), (2.12) hold, and further we assume that the following assumptions are fulfilled:*

(A3)

$$\begin{aligned} \liminf_{y\to 0^+} \frac{f(t,y)}{y} &= \alpha_1(t), \ \liminf_{y\to 0^+} \frac{-g(y)}{y} = \beta_1, \text{ and} \\ \limsup_{y\to\infty} \frac{f(t,y)}{y} &= \alpha_2(t), \ \limsup_{y\to\infty} \frac{-g(y)}{y} = \beta_2, \\ \text{with } \beta_1, \beta_2 &\in \mathbb{R}^+, \ \alpha_1, \alpha_2 \in C(\mathbb{R}, \mathbb{R}^+). \end{aligned}$$

(A4)

$$\int_0^\omega (e(s)\alpha_1(s) + k(s))\,ds > \frac{1}{\sigma\vartheta_1}\left(\frac{1}{\widehat{\rho^*}} - \sigma\widehat{c^*}\right) - \beta_1 \int_0^\omega \varsigma(s)\,ds, \tag{3.1}$$

and

$$\int_0^\omega (e(s)\alpha_2(s) + k(s))\,ds < \frac{1}{\vartheta_2}\left(\frac{1}{\rho^*} - c^*\right) - \beta_2 \int_0^\omega \varsigma(s)\,ds, \tag{3.2}$$

where $\vartheta_1, \vartheta_2, \varsigma$ are respectively given as in (2.14) and (2.17). *Then, equation (1.5) possesses at least one positive ω-periodic solution.*

Proof. We first construct two sets, Λ_1 and Λ_2 in order to apply Theorem 1.1. Since $\liminf_{y\to 0^+} \frac{f(t,y)}{y} = \alpha_1(t)$, $\liminf_{y\to 0^+} \frac{-g(y)}{y} = \beta_1$, there exists $S_1^* > 0$ such that

$$f(t,y) \geq \alpha_1(t)\,y, \text{ and } -g(y) \geq \beta_1 y, \text{ for } 0 < y \leq S_1^*, \ t \in [0,\omega].$$

Define

$$\Lambda_1 = \left\{ y \in C_\omega : \|y\| \leq S_1^* \right\}.$$

For all $(t,s) \in \mathbb{R}^2$, by (2.12) and $g(y) \leq 0$ for $y \in \mathbb{R}^+$, we have

$$Z(t,s)\left(k(t)\,y(t) - \sum_{j=1}^n c_j(t)\,g\left(y\left(t-\tau_j(t)\right)\right)\right) \geq 0,$$

as a consequence, condition (A1) is satisfied.

Any ω-periodic function y of $K \cap \partial\Lambda_1$ satisfies $\sigma S_1^* = \sigma\|y\| \leq y(t) \leq \|y\| = S_1^*$, for $t \in \mathbb{R}$. Hence, we have

$$\sigma S_1^* \leq y\left(t-\tau_j(t)\right) \leq S_1^*, \text{ for } t \in \mathbb{R} \text{ and } j = \overline{1,n}. \tag{3.3}$$

From the definition of Φ, and (3.1), (3.3), (2.8), (2.12), (2.13), we obtain

$$\begin{aligned} (\Phi y)(t) \geq\ & \frac{1}{k(t)+1}\sum_{j=1}^n c_j(t)\,y\left(t-\tau_j(t)\right) \\ & + \frac{N_1}{k(t)+1}\int_t^{t+\omega} e(s)\int_{-\infty}^0 D(r)\,f(s,y(s+r))\,drds \end{aligned}$$

$$
\begin{aligned}
& +\frac{N_2}{k(t)+1}\int_t^{t+\omega}\left(k(s)\,y(s)-\sum_{j=1}^{n}c_j(s)\,g\left(y\left(s-\tau_j(s)\right)\right)\right)ds \\
\geq\ & \sigma\widehat{c^*}\widehat{\rho^*}S_1^*+\sigma S_1^*\widehat{\rho^*}\vartheta_1\int_0^{\omega}\left(\left(e(s)\,\alpha_1(s)+k(s)\right)+\beta_1\varsigma(s)\right)ds \\
>\ & S_1^*=\left\|y\right\|.
\end{aligned}
$$

This proves

$$\left\|\Phi y\right\|\geq\left\|y\right\|,\ \text{for } y\in\partial\Lambda_1\cap K.$$

Next we construct the set Λ_2. Since $\limsup\limits_{y\to\infty}\dfrac{f(t,y)}{y}=\alpha_2(t)$, $\limsup\limits_{y\to\infty}\dfrac{-g(y)}{y}=\beta_2$, there exists a constant d with $0<S_1^*<d$, such that

$$f(t,y)\leq\alpha_2(t)\,y,\ \text{and } -g(y)\leq\beta_2 y,\ \text{for } y\geq d.$$

Let

$$S_2^*>\max\left\{\frac{\widehat{\theta}\vartheta_2}{1-\rho^*c^*-\rho^*\vartheta_2\int_0^{\omega}\left(e(s)\,\alpha_2(s)+k(s)+\beta_2\varsigma(s)\right)ds},d\right\}>S_1^*,$$

where

$$\widehat{\theta}=\rho^*\omega\left(e^*\xi_1(f)+\widehat{\kappa}d+c^*\xi_2(g)\right),$$

and

$$\xi_1(f)=\max_{s\in[0,\omega],y\in[0,d]}f(s,y),\ \xi_2(g)=\max_{y\in[0,d]}\left(-g(y)\right).$$

Define now the open set $\Lambda_2=\left\{y(.)\in C_\omega:\left\|y\right\|<S_2^*\right\}$, $P_1=\{y(.)\in C_\omega:y\leq d\}$, $P_2=\{y(.)\in C_\omega:y>d\}$, obviously $\overline{\Lambda_1}\subset\Lambda_2$. If $y\in K\cap\partial\Lambda_2$, by (3.3),(2.18), and (3.2), we have

$$
\begin{aligned}
(\Phi y)(t)\ \leq\ & \rho^*\sum_{j=1}^{n}c_j(t)\,y\left(t-\tau_j(t)\right) \\
& +\rho^*M_1\int_{P_1}e(s)\int_{-\infty}^{0}D(r)\,f(s,y(s+r))\,drds \\
& +\rho^*M_2\int_{P_1}\left[k(s)\,y(s)-\sum_{j=1}^{n}c_j(s)\,g\left(y\left(s-\tau_j(s)\right)\right)\right]ds \\
& +\rho^*M_1\int_{P_2}e(s)\int_{-\infty}^{0}D(r)\,f(s,y(s+r))\,drds \\
& +\rho^*M_2\int_{P_2}\left[k(s)\,y(s)-\sum_{j=1}^{n}c_j(s)\,g\left(y\left(s-\tau_j(s)\right)\right)\right]ds \\
\leq\ & \rho^*c^*S_2^*+\vartheta_2\widehat{\theta}+S_2^*\rho^*\vartheta_2\int_0^{\omega}\left(e(s)\,\alpha_2(s)+k(s)+\beta_2\varsigma(s)\right)ds \\
\leq\ & S_2^*=\left\|y\right\|,
\end{aligned}
$$

which implies

$$\left\|\Phi y\right\| \leq \left\|y\right\|, \text{ for } y \in K \cap \partial\Lambda_2.$$

From the above arguments, we know $\Phi : K \cap \left(\overline{\Lambda}_2 \setminus \Lambda_1\right) \to K$, is a completely continuous operator. In view of Theorem 1.1, Φ has a fixed point in $K \cap \left(\overline{\Lambda}_2 \setminus \Lambda_1\right)$. As a result, (1.5) has a positive ω-periodic solution with $S_1^* \leq \left\|y\right\| \leq S_2^*$ and $y(t) \geq \sigma S_1^* > 0$.

For any $\nu = (\nu_0, \nu_1, ..., \nu_n) \in \mathbb{R}^{n+1}$, define $\|\nu\| = \max\limits_{0\leq i\leq n} |\nu_i|$ and put

$$\begin{aligned}
\widehat{h}^{\infty} &= \limsup_{\|\nu\|\to\infty} \max_{t\in[0,\omega]} \frac{h(t,\nu)}{\|\nu\|},\ \widehat{f}^{\infty} = \limsup_{\|y\|\to\infty} \max_{t\in[0,\omega]} \frac{f(t,y)}{\left\|y\right\|},\\
h^{\infty} &= \liminf_{\|\nu\|\to\infty} \min_{t\in[0,\omega]} \frac{h(t,\nu)}{\|\nu\|},\ f^{\infty} = \liminf_{\|y\|\to\infty} \min_{t\in[0,\omega]} \frac{f(t,y)}{\left\|y\right\|},\\
\widehat{h}^{0} &= \limsup_{\|\nu\|\to 0} \max_{t\in[0,\omega]} \frac{h(t,\nu)}{\|\nu\|},\ \widehat{f}^{0} = \limsup_{\|y\|\to 0} \max_{t\in[0,\omega]} \frac{f(t,y)}{\left\|y\right\|},\\
h^{0} &= \liminf_{\|\nu\|\to 0} \min_{t\in[0,\omega]} \frac{h(t,\nu)}{\|\nu\|},\ f^{0} = \liminf_{\|y\|\to 0} \min_{t\in[0,\omega]} \frac{f(t,y)}{\left\|y\right\|},
\end{aligned}$$

where $h(t,\nu)$ is defined as in (2.15).

To simplify our description, we will use the following notations:

$$\begin{aligned}
\widehat{m}_1 &= \frac{1}{\omega}\int_0^{\omega} e(s)\,ds, \widehat{m}_2 = \frac{1}{\omega}\int_0^{\omega} \varsigma(s)\,ds, \widehat{m}_3 = \frac{1}{\omega}\int_0^{\omega} k(s)\,ds,\\
\widehat{m_0^*} &= \min\{\widehat{m}_1, 1\}, m_0^* = \max\{\widehat{m}_1, 1\}.
\end{aligned} \tag{3.4}$$

Theorem 3.2. *Assume that conditions (2.1),(2.4), and (A1),(A2) hold, and further assume that:*

A5) $\widehat{f}^{0} \leq \dfrac{\lambda}{4\vartheta_2\omega m_0^*\rho^*}, \widehat{h}^{0} \leq \dfrac{\lambda}{4\vartheta_2\omega m_0^*\rho^*}, \lambda < 1;$

A6) $\widehat{f}^{\infty} \leq \dfrac{\lambda}{4\vartheta_2\omega m_0^*\rho^*}, \widehat{h}^{\infty} \leq \dfrac{\lambda}{4\vartheta_2\omega m_0^*\rho^*}, \lambda < 1;$

A7) *For all,* $t \in [0,\omega]$, $\varsigma(t) \leq \dfrac{\lambda}{2\rho^*}, \lambda < 1;$

A8) *There exists a constant* $W > 0$, *such that*

$$f(t,y) > \frac{W}{2\omega\vartheta_1\widehat{m_0^*}\widehat{\rho^*}},\ h(t,\nu) > \frac{W}{2\omega\vartheta_1\widehat{m_0^*}\widehat{\rho^*}}, t \in [0,\omega],$$

for all $y, \nu_i \in [\sigma W, W]$, $i = \overline{0,n}$, where $\nu = (\nu_0, \nu_1, ..., \nu_n) \in \mathbb{R}^{n+1}$. *Then, equation* (1.5) *has at least two positive periodic solutions.*

Proof. In view of (A5), we can choose $\underline{W}$, where $0 < \underline{W} < W$, such that

$$h(t,\nu) \leq \left(\frac{\lambda}{4\omega\vartheta_2 m_0^*\rho^*} + \varepsilon\right)\|\nu\|, \text{ for all } \|\nu\| < \underline{W},$$

and,

$$f(t,y) \leq \left(\frac{\lambda}{4\omega\vartheta_2 m_0^*\rho^*} + \varepsilon\right)\left\|y\right\|, \text{ for all } \left\|y\right\| < \underline{W},$$

for all $t \in [0,\omega]$, where the constant $\varepsilon > 0$, satisfies $0 < \varepsilon < \frac{(1-\lambda)}{2\omega\vartheta_2 m_0^* \rho^*}$. We define

$$\Omega_1 =: \left\{ y(.) \in C_\omega : \left\| y \right\| < \underline{W} \right\},$$

as an open, bounded subset of the cone K. Obviously, while $y \in \partial\Omega_1 \cap K$, then $0 < \sigma\underline{W} = \sigma \left\| y \right\| \leq y \leq \left\| y \right\| = \underline{W}$. By combining (A7) and (2.8), (2.12), (2.13), we deduce

$$\begin{aligned}
(\Phi y)(t) &\leq \frac{1}{k(t)+1} \sum_{j=1}^{n} c_j(t)\, y\left(t - \tau_j(t)\right) \\
&\quad + \frac{M_1}{k(t)+1} \int_t^{t+\omega} e(s) \int_{-\infty}^{0} D(r)\, f(s, y(s+r))\, dr ds \\
&\quad + \frac{M_2}{k(t)+1} \int_t^{t+\omega} h(s, Y(s))\, ds \\
&\leq \rho^* \sum_{j=1}^{n} c_j(t) \max_{1 \leq j \leq n} \left\{ y\left(t - \tau_j(t)\right) \right\} + \widehat{m}_1 \times \omega\rho^* \vartheta_2 \left(\frac{\lambda}{4\omega\rho^* \vartheta_2 m_0^*} + \varepsilon \right) \left\| y \right\| \\
&\quad + 1 \times \omega\rho^* \vartheta_2 \left(\frac{\lambda}{4\omega\rho^* \vartheta_2 m_0^*} + \varepsilon \right) \| Y \| \\
&\leq \rho^* \sum_{j=1}^{n} c_j(t) \max_{1 \leq j \leq n} \left\{ \left| y\left(t - \tau_j(t)\right) \right| \right\} + \widehat{m}_1 \times \omega\rho^* \vartheta_2 \left(\frac{\lambda}{4\omega\rho^* \vartheta_2 m_0^*} + \varepsilon \right) \left\| y \right\| \\
&\quad + 1 \times \omega\rho^* \vartheta_2 \left(\frac{\lambda}{4\omega\rho^* \vartheta_2 m_0^*} + \varepsilon \right) \| Y \| \\
&\leq \rho^* \varsigma(t)\, \underline{W} + 2\omega m_0^* \rho^* \vartheta_2 \left(\frac{\lambda}{4\omega m_0^* \rho^* \vartheta_2} + \varepsilon \right) \underline{W} \\
&\leq \frac{\lambda}{2} \underline{W} + \frac{\lambda}{2} \underline{W} + 2\varepsilon\omega m_0^* \rho^* \vartheta_2 \underline{W} \\
&\leq \lambda \underline{W} + 2\varepsilon\omega m_0^* \rho^* \vartheta_2 \underline{W} \leq \underline{W}.
\end{aligned}$$

This yields,

$$\left\| \Phi y \right\| \leq \left\| y \right\|, \text{ for any } y \in \partial\Omega_1 \cap K. \tag{3.5}$$

Moreover, thanks to assumption (A6) we can choose $\overline{W} > \sigma W$, so that

$$f(t, y(t)) \leq \left(\frac{\lambda}{4\vartheta_2 \omega m_0^* \rho^*} + \varepsilon \right) \left\| y \right\|, \text{ for all } \left\| y \right\| > \overline{W},$$

and

$$h(t, v(t)) \leq \left(\frac{\lambda}{4\vartheta_2 \omega m_0^* \rho^*} + \varepsilon \right) \| v \|, \text{ for all } \| v \| > \overline{W},$$

for all $t \in [0,\omega]$, where the constant $\varepsilon > 0$, satisfies $0 < \varepsilon < \frac{(1-\lambda)}{2\vartheta_2 \omega m_0^* \rho^*}$.

Define $\overline{W}^* = \frac{\overline{W}}{\sigma}$ and

$$\Omega_2 =: \left\{ y(.) \in C_\omega : \left\| y \right\| < \overline{W}^* \right\},$$

then Ω_2 is an open subset of C_ω. For any $y \in K \cap \partial\Omega_2$, we have $0 < \sigma\overline{W}^* = \sigma\left\|y\right\| \le y \le \left\|y\right\| = \overline{W}^*$. By invoking (A7) and $(2.8), (2.12), (2.13)$, we have

$$\begin{aligned}
(\Phi y)(t) &\le \frac{1}{k(t)+1}\sum_{j=1}^{n} c_j(t)\, y\left(t-\tau_j(t)\right) \\
&+\frac{M_1}{k(t)+1}\int_t^{t+\omega} e(s)\int_{-\infty}^{0} D(r)\, f(s, y(s+r))\, drds + \frac{M_2}{k(t)+1}\int_t^{t+\omega} h(s, Y(s))\, ds \\
&\le \rho^*\sum_{j=1}^{n} c_j(t) \max_{1\le j\le n}\left\{\left|y\left(t-\tau_j(t)\right)\right|\right\} + m_0^* \times \omega\rho^*\vartheta_2\left(\frac{1}{4\omega\rho^*\vartheta_2 m_0^*}+\varepsilon\right)\left\|y\right\| \\
&+m_0^* \times \omega\rho^*\vartheta_2\left(\frac{\lambda}{4\omega\rho^*\vartheta_2 m_0^*}+\varepsilon\right)\|Y\| \\
&\le \varsigma(t)\,\rho^*\overline{W}^* + 2\omega m_0^*\rho^*\vartheta_2\left(\frac{\lambda}{4\omega m_0^*\rho^*\vartheta_2}+\varepsilon\right)\overline{W}^* \\
&\le \frac{\lambda}{2}\overline{W}^* + \frac{\lambda}{2}\overline{W}^* + 2\varepsilon\omega m_0^*\rho^*\vartheta_2\overline{W}^* \\
&\le \lambda\overline{W}^* + 2\varepsilon\omega m_0^*\rho^*\vartheta_2\overline{W}^* \le \overline{W}^*.
\end{aligned}$$

This yields,

$$\left\|\Phi y\right\| \le \left\|y\right\|, \text{ for any } y \in \partial\Omega_2 \cap K. \tag{3.6}$$

Moreover, we introduce

$$\Omega_3 =: \left\{y(.) \in C_\omega : \left\|y\right\| < W\right\}.$$

If $y \in \partial\Omega_3 \cap K$, then $0 < \sigma W = \sigma\left\|y\right\| \le y \le \left\|y\right\| = W$. In light of the above-given arguments $(2.8), (2.12), (2.13)$, and (A8), we derive

$$\begin{aligned}
(\Phi y)(t) \quad &\ge \quad \frac{1}{k(t)+1}\sum_{j=1}^{n} c_j(t)\, y\left(t-\tau_j(t)\right) \\
&\quad +\frac{N_1}{k(t)+1}\int_t^{t+\omega} e(s)\int_{-\infty}^{0} D(r)\, f(s, y(s+r))\, drds + \frac{N_2}{k(t)+1}\int_t^{t+\omega} h(s, Y(s))\, ds \\
&\ge \quad \omega\vartheta_1\frac{W}{2\vartheta_1\omega\widehat{m_0^*}\widehat{\rho^*}} \times \widehat{m_0^*}\widehat{\rho^*} + \omega\vartheta_1\frac{W}{2\vartheta_1\omega\widehat{m_0^*}\widehat{\rho^*}} \times \widehat{m_0^*}\widehat{\rho^*} \\
&= \quad \frac{W}{2} + \frac{W}{2} = W.
\end{aligned}$$

This means

$$\left\|\Phi y\right\| \ge \left\|y\right\| \text{ for all } y \in \partial\Omega_3 \cap K. \tag{3.7}$$

By Lemma 2.6, Φ is a completely continuous operator. Since $\underline{W} < W < \overline{W}^*$, and from (3.5)-(3.7), by Theorem 1.1 the mapping Φ has two fixed points $\widetilde{y}_1 \in K \cap \overline{\Omega}_2\backslash\Omega_1$ and $\widetilde{y}_2 \in K \cap \overline{\Omega}_3\backslash\Omega_2$. It follows that Φ has at least two $\omega-$positive periodic solutions of (1.5). That is $\underline{W} < \widetilde{y}_1 < W < \widetilde{y}_2 < \overline{W}^*$. This proves Theorem 3.2.

Corollary 3.1. Assume that conditions (2.1), (2.4) , (A1),(A2),(A7),(A8) hold, further if (A5), (or (A6)) holds, then Eq. (1.5) has at least one positive periodic solution.

Theorem 3.3. *In addition to conditions* (2.1), (2.4) , (A1),(A2) *hold, assume that the hypotheses below are fulfilled:*

A9) $f^0 \geq \frac{1}{2\sigma\omega\vartheta_1\widehat{m_0^*}\widehat{\rho^*}}$, and $h^0 \geq \frac{1}{2\sigma\omega\vartheta_1\widehat{m_0^*}\widehat{\rho^*}}$;

A10) $f^\infty \geq \frac{1}{2\sigma\omega\vartheta_1\widehat{m_0^*}\widehat{\rho^*}}$, and $h^\infty \geq \frac{1}{2\sigma\omega\vartheta_1\widehat{m_0^*}\widehat{\rho^*}}$;

A11) For all, $t \in [0,\omega]$, $\varsigma(t) \leq \frac{1}{2\rho^* c^*}$;

A12) There exists a constant $R > 0$ *such that:*

$$f(t,y) < \frac{R}{4\omega\vartheta_2 m_0^*\rho^*},\ h(t,\nu) < \frac{R}{4\omega\vartheta_2 m_0^*\rho^*}, t \in [0,\omega],$$

for all $y, \nu_i \in [\sigma R, R]$, $i = \overline{0,n}$, where $\nu = (\nu_0, \nu_1, ..., \nu_n) \in \mathbb{R}^{n+1}$. *Then the equation* (1.5) *has at least two positive periodic solutions.*

Proof. In view of (A9), we can choose $\underline{R}$, where $0 < \underline{R} < R$, such that

$$h(t,\nu(t)) \geq \left(\frac{1}{2\omega\sigma\vartheta_1\widehat{m_0^*}\widehat{\rho^*}} - \varepsilon\right)\|\nu\|,\ \ \|\nu\| \leq \underline{R},$$

and

$$f(t,y(t)) \geq \left(\frac{1}{2\omega\sigma\vartheta_1\widehat{m_0^*}\widehat{\rho^*}} - \varepsilon\right)\left\|y\right\|,\ \ \left\|y\right\| \leq \underline{R},$$

for all $t \in [0,\omega]$, where the constant $\varepsilon > 0$. We introduce

$$\widehat{\Omega}_1 =: \left\{y(.) \in C_\omega : \left\|y\right\| < \underline{R}\right\}.$$

Thus, for any $y \in \partial\widehat{\Omega}_1 \cap K$, $0 < \sigma\underline{R} = \sigma\left\|y\right\| \leq y \leq \left\|y\right\| = \underline{R}$. From (2.8) , (2.12) , (2.13), we have that

$$\begin{aligned}
\Phi y(t) &\geq \frac{1}{k(t)+1}\sum_{j=1}^{n} c_j(t)\, y\left(t-\tau_j(t)\right) \\
&\quad + \frac{N_1}{k(t)+1}\int_t^{t+\omega} e(s)\int_{-\infty}^{0} D(r)\, f(s, y(s+r))\, dr ds + \frac{N_2}{1+k(t)}\int_t^{t+\omega} h(s, Y(s))\, ds \\
&\geq \widehat{\rho^*}\vartheta_1\left(\frac{1}{2\omega\sigma\vartheta_1\widehat{m_0^*}\widehat{\rho^*}} - \varepsilon\right)\int_t^{t+\omega} e(s)\left\|y\right\| ds + \int_t^{t+\omega}\widehat{\rho^*}\vartheta_1\left(\frac{1}{2\omega\sigma\vartheta_1\widehat{m_0^*}\widehat{\rho^*}} - \varepsilon\right)\|Y\|\, ds \\
&\geq 2\omega\widehat{m_0^*}\widehat{\rho^*}\vartheta_1\left(\frac{1}{2\omega\sigma\vartheta_1\widehat{m_0^*}\widehat{\rho^*}} - \varepsilon\right)\sigma\underline{R} \\
&\geq \underline{R} - 2\varepsilon\sigma\underline{R}\omega\vartheta_1\widehat{m_0^*}\widehat{\rho^*} \geq \underline{R},
\end{aligned}$$

for the arbitrariness of ε, which yields,

$$\left\|\Phi y\right\| \geq \left\|y\right\|,\ y \in \partial\widehat{\Omega}_1 \cap K.$$

Moreover, from assumption (A10), for a sufficiently small $\varepsilon > 0$, there exists $\overline{R} > \sigma R$ such that

$$f(t,y(t)) \geq \left(\frac{1}{2\sigma\omega\vartheta_1\widehat{m_0^*}\widehat{\rho^*}} - \varepsilon\right)\left\|y\right\|,\ \text{for}\ \left\|y\right\| > \overline{R},$$

and

$$h(t,v(t)) \geq \left(\frac{1}{2\sigma\omega\vartheta_1\widehat{m_0^*\rho^*}} - \varepsilon\right)\|v\|, \text{ for } \|v\| > \overline{R},$$

for all $t \in [0,\omega]$.
Define $\overline{R}^* = \dfrac{\overline{R}}{\sigma}$ and

$$\widehat{\Omega}_2 =: \left\{y(.) \in C_\omega : \left\|y\right\| < \overline{R}^*\right\}.$$

Then for any

$$y \in \partial\widehat{\Omega}_2 \cap K, \text{ showing } \left\|\Phi y\right\| \leq \left\|y\right\|,$$

is similar to the procedure given above, and hence we omit it. Moreover, we introduce

$$\widehat{\Omega}_3 =: \left\{y(.) \in C_\omega : \left\|y\right\| < R\right\}.$$

If $y \in \partial\widehat{\Omega}_3 \cap K$, then $0 < \sigma R = \sigma\left\|y\right\| \leq y \leq \left\|y\right\| = R$. From (A11),(A12) and (2.8), (2.12), (2.13), we derive

$$\begin{aligned}
(\Phi y)(t) &\leq \frac{1}{k(t)+1}\sum_{j=1}^{n} c_j(t)\, y\left(t-\tau_j(t)\right) \\
&\quad + \frac{M_1}{k(t)+1}\int_t^{t+\omega} e(s)\int_{-\infty}^{0} D(r)\, f(s, y(s+r))\, dr ds \\
&\quad + \frac{M_2}{1+k(t)}\int_t^{t+\omega} h(s, Y(s))\, ds \\
&\leq \rho^*\frac{R}{2\rho^* c^*} + \vartheta_2\omega\rho^* \times m_1 \times \frac{R}{4m_0^*\vartheta_2\omega\rho^*} + \vartheta_2\omega\rho^* \times 1 \times \frac{R}{4m_0^*\vartheta_2\omega\rho^*} \\
&= c^*\rho^*\frac{R}{2\rho^* c^*} + 2\frac{R}{4m_0^*\vartheta_2\omega\rho^*} m_0^*\vartheta_2\omega\rho^* = \frac{R}{2} + \frac{R}{2} = R,
\end{aligned}$$

which yields

$$\left\|\Phi y\right\| \leq \left\|y\right\|, \text{ for } y \in \partial\widehat{\Omega}_3 \cap K.$$

This means $\left\|\Phi y\right\| \leq \left\|y\right\|$ for $y \in \partial\widehat{\Omega}_3 \cap K$. Since $\underline{R} < R < \overline{R}^*$, by Theorem 1.1 the mapping Φ has two fixed points $\widehat{y_1} \in K \cap \overline{\widehat{\Omega}_2}\backslash\widehat{\Omega}_1$ and $\widehat{y_2} \in K \cap \overline{\widehat{\Omega}_3}\backslash\widehat{\Omega}_2$. It follows that Φ has at least two $\omega-$positive periodic solutions of (1.5). That is $\underline{R} < \widehat{\overline{y}_1} < R < \widehat{\overline{y}_2} < \overline{R}^*$. The proof is complete.

As a consequence of Theorem 3.2, we state corollary omitting its proof.

Corollary 3.3. Assume that conditions (2.1),(2.4) , and (A1),(A2),(A11),(A12) hold, further if (A9), (or (A10)) holds, then Eq. (1.5) has at least one positive periodic solution.

For convenience and simplicity in the following discussion, we use the following notations:
For a positive constant r and $y \in K$, we set

$$\Re_r = \inf_{y\in K, \|y\|=r} \frac{-g(y)}{y}, \ \widehat{\Re}_r = \sup_{y\in K, \|y\|=r} \frac{-g(y)}{y},$$

$$\mathfrak{F}_r = \inf_{y\in K,\|y\|=r} \min_{t\in[0,\omega]} \frac{f(t,y)}{y}, \ \widehat{\mathfrak{F}}_r = \sup_{y\in K,\|y\|=r} \max_{t\in[0,\omega]} \frac{f(t,y)}{y}.$$

Theorem 3.4. *Suppose that $g(y) \le 0$ for $y \in \mathbb{R}^+$. If conditions (2.1), (2.4), (A2) hold, and we further assume that:*
A13)

$$\lim_{y\to 0^+} \frac{-g(y)}{y} = \lim_{y\to\infty} \frac{-g(y)}{y} = \lim_{y\to 0^+} \min_{t\in[0,\omega]} \frac{f(t,y)}{y} = \lim_{y\to\infty} \min_{t\in[0,\omega]} \frac{f(t,y)}{y} = +\infty.$$

A14) There exists a constant $E > 0$ such that, for all $y \in (0, E]$,

$$\rho^* c^* + \omega\rho^* \vartheta_2 \left(\widehat{m}_1 \widehat{\mathfrak{F}}_E + \widehat{m}_2 \widehat{\mathfrak{R}}_E + \widehat{m}_3\right) < 1,$$

where $\widehat{m}_1, \widehat{m}_2, \widehat{m}_3$ are given as in (3.4). Then, equation (1.5) has at least two positive periodic solutions.

Proof. Since $\lim_{y\to 0} \frac{-g(y)}{y} = \lim_{y\to 0} \min_{t\in[0,\omega]} \frac{f(t,y)}{y} = \infty$, we can choose E_*, where $0 < E_* < E$, such that

$$-g(y) \ge \widehat{\epsilon}_1 y, \text{ and } f(t,y) \ge \widehat{\epsilon}_1 y, \text{ for } y \in [0, E_*], t \in [0,\omega],$$

where the constant $\widehat{\epsilon}_1 > 0$, satisfies $\widehat{\epsilon}_1 > \frac{1 - \widehat{\rho}^* \omega\vartheta_1 \widehat{m}_3 \sigma}{\widehat{\rho}^* \omega\vartheta_1 \sigma (\widehat{m}_1 + \widehat{m}_2)}$, $(0 \le \widehat{\rho}^* \omega\vartheta_1 \widehat{m}_3 \sigma < 1)$. We introduce

$$\widetilde{\Lambda}_1 =: \left\{ y(.) \in C_\omega : \|y\| < E_* \right\}.$$

Thus, for any $y \in \partial\widetilde{\Lambda}_1 \cap K$, $\sigma\|y\| \le y(t) \le \|y\|$. From (2.8), (2.12), (2.13) and (3.4), we have

$$\begin{aligned}
\Phi y(t) &\ge \frac{1}{k(t)+1} \sum_{j=1}^n c_j(t)\, y\left(t - \tau_j(t)\right) + \frac{N_1}{k(t)+1} \int_t^{t+\omega} e(s) \int_{-\infty}^0 D(r) f(s, y(s+r))\, dr ds \\
&\quad + \frac{N_2}{1+k(t)} \int_t^{t+\omega} k(s)\, y(s)\, ds + \frac{N_2}{1+k(t)} \int_t^{t+\omega} \sum_{j=1}^n c_j(s) \times \left(-g\left(y\left(s - \tau_j(s)\right)\right)\right) ds \\
&\ge \widehat{\rho}^* \omega\vartheta_1 \widehat{\epsilon}_1 \widehat{m}_1 \sigma |y| + \widehat{\rho}^* \omega\vartheta_1 \widehat{m}_3 \sigma |y| + \widehat{m}_2 \widehat{\rho}^* \vartheta_1 \omega \widehat{\epsilon}_1 \sigma |y| \\
&= \left[\widehat{\rho}^* \omega\vartheta_1 \widehat{m}_3 \sigma + \widehat{\rho}^* \omega\vartheta_1 \sigma \widehat{\epsilon}_1 (\widehat{m}_1 + \widehat{m}_2)\right] |y| \\
&\ge |y|,
\end{aligned}$$

and thus

$$\|\Phi y\| \ge \|y\|, \ y \in \partial\widetilde{\Lambda}_1 \cap K.$$

Moreover, if $\lim_{y\to\infty} \frac{-g(y)}{y} = \lim_{y\to\infty} \min_{t\in[0,\omega]} \frac{f(t,y)}{y} = \infty$, then there exists $\widehat{E} > E > 0$ such that $g(y) > \widehat{\epsilon}_2 y$, and $f(t,y) \ge \widehat{\epsilon}_2 y$, for all $y > \widehat{E}$, $t \in [0,\omega]$, where the constant $\widehat{\epsilon}_2 > 0$, satisfies $\widehat{\epsilon}_2 > \frac{1 - \widehat{\rho}^* \omega\vartheta_1 \widehat{m}_3 \sigma}{\widehat{\rho}^* \omega\vartheta_1 \sigma (\widehat{m}_1 + \widehat{m}_2)}$, $(0 \le \widehat{\rho}^* \omega\vartheta_1 \widehat{m}_3 \sigma < 1)$.
Define $E^* = \frac{\widehat{E}}{\sigma} > E$ and

$$\widetilde{\Lambda}_2 =: \left\{ y(.) \in C_\omega : \|y\| < E^* \right\}.$$

Then, for any $y \in \partial\widetilde{\Lambda}_2 \cap K$, we obtain the result $\|\Phi y\| \geq \|y\|$ by applying the same procedure given above. Moreover, we introduce

$$\widetilde{\Lambda}_3 =: \left\{ y(.) \in C_\omega : \|y\| < E \right\}.$$

If $y \in \partial\widetilde{\Lambda}_3 \cap K$, then $\sigma \|y\| \leq y(t) \leq \|y\|$. From (2.8), (2.12), (2.13) and (A14), (3.4), we get

$$\begin{aligned}
(\Phi y)(t) &\leq \frac{1}{k(t)+1}\sum_{j=1}^{n} c_j(t)\, y\left(t-\tau_j(t)\right) + \frac{M_1}{k(t)+1}\int_t^{t+\omega} e(s)\int_{-\infty}^{0} D(r)\,\frac{f(s,y(s+r))}{y(s+r)}\, y(s+r)\, drds \\
&\quad + \frac{M_2}{1+k(t)}\int_t^{t+\omega} k(s)\, y(s)\, ds + \frac{M_2}{1+k(t)}\int_t^{t+\omega}\sum_{j=1}^{n} c_j(s) \times \frac{-g\left(y\left(s-\tau_j(s)\right)\right)}{y\left(t-\tau_j(t)\right)}\, y\left(t-\tau_j(t)\right) ds \\
&\leq \left(\rho^* c^* + \omega\rho^*\vartheta_2\left(\widehat{m}_1\widehat{\mathfrak{F}}_E + \widehat{m}_2\widehat{\mathfrak{K}}_E + \widehat{m}_3\right)\right)|y| \leq |y|,
\end{aligned}$$

and therefore

$$\|\Phi y\| \leq \|y\|.$$

This means $\|\Phi y\| \leq \|y\|$ for $y \in \partial\Lambda_3 \cap K$. Since $E_* < E < E^*$, by Theorem 1.1 the mapping K has two fixed points in $K \cap \overline{\widetilde{\Lambda}_3} \backslash \widetilde{\Lambda}_2$ and $K \cap \overline{\widetilde{\Lambda}_2} \backslash \widetilde{\Lambda}_1$. It follows that Φ has at least two positive periodic solutions of (1.5). That is $E_* < \widetilde{y}_1 < E < \widetilde{y}_2 < E^*$. The proof is complete.

Theorem 3.5. *Assume that $g(y) \leq 0$ for $y \in \mathbb{R}^+$, conditions* (A2), (2.1), (2.4), (2.6), (2.10) *hold, and the following conditions are fulfilled:*

A15)

$$\lim_{y\to 0^+} \frac{-g(y)}{y} = \lim_{y\to\infty} \frac{-g(y)}{y} = \lim_{y\to 0^+}\max_{t\in[0,\omega]} \frac{f(t,y)}{y} = \lim_{y\to\infty}\max_{t\in[0,\omega]} \frac{f(t,y)}{y} = 0.$$

A16) There exists a constant $\Gamma > 0$, such that

$$\widehat{\rho^*}\widehat{c^*} + \omega\widehat{\rho^*}\vartheta_1\left(\widehat{m}_3 + \widehat{m}_1\widehat{\mathfrak{F}}_\Gamma + \widehat{m}_2\mathfrak{K}_\Gamma\right) > 1,$$

where $\widehat{m}_1, \widehat{m}_2, \widehat{m}_3$ are given as in (3.4)*. Then, equation* (1.5) *has at least two positive ω-periodic solutions.*

Proof. The proof follows along the lines of the proof of Theorem 3.4 and hence we omit it.
Let

$$Y = (y_0, y_1, ..., y_n) \in [0, +\infty)^{n+1}, \widetilde{Y} = \max\{y_0, y_1, ..., y_n\}.$$

Theorem 3.6. *Assume that the conditions* (2.1), (2.4), (A1), (A2) *hold, and further assume that:*

A17)

$$\lim_{\widetilde{Y}\to 0^+}\min_{t\in[0,\omega]} \frac{h(t,Y)}{\widetilde{Y}} = \lim_{\widetilde{Y}\to\infty}\min_{t\in[0,\omega]} \frac{h(t,Y)}{\widetilde{Y}} = \lim_{y\to 0^+}\min_{t\in[0,\omega]} \frac{f(t,y)}{y} = \lim_{y\to\infty}\min_{t\in[0,\omega]} \frac{f(t,y)}{y} = +\infty.$$

A18) There exists a constant $F > 0$, such that

$$\begin{aligned}
h(t,Y) &\leq \frac{F}{4\omega m_0^*\vartheta_2\rho^*}, \text{for } \widetilde{Y} \leq F, \\
f(t,y) &\leq \frac{F}{4\omega m_0^*\vartheta_2\rho^*}, \text{ for } y \leq F,
\end{aligned}$$

and

$$\varsigma(t) \leq \frac{1}{2\rho^*}, \text{ for all } t \in [0,\omega].$$

Then, equation (1.5) *has at least two positive periodic solutions.*

Proof. Since $\lim_{\widetilde{Y}\to 0^+} \min_{t\in[0,\omega]} \frac{h(t,Y)}{\widetilde{Y}} = \lim_{y\to 0^+} \min_{t\in[0,\omega]} \frac{f(t,y)}{y} = +\infty$, we can choose $\underline{F}$, where $0 < \underline{F} < F$, such that

$$h(t,Y) \geq \epsilon_1^* \widetilde{Y}, \text{ and } f(t,y) \geq \epsilon_1^* y, \text{ for } \widetilde{Y} \in \left[0, \underline{F}\right], t \in [0,\omega],$$

where the constant $\epsilon_1^* > 0$, satisfies, $\epsilon_1^* \geq \frac{1}{2\sigma\omega\vartheta_1 \widehat{\rho^*} \widehat{m_0^*}}$. We introduce

$$\widetilde{\Omega}_1 =: \left\{y(.) \in C_\omega : \left\|y\right\| < \underline{F}\right\}.$$

Thus, for any $y \in \partial\widetilde{\Omega}_1 \cap K$, $\sigma\left\|y\right\| \leq \widetilde{Y} \leq \left\|y\right\|$ and thanks to (2.8), (2.12), (2.13), and (3.4), we deduce

$$\begin{aligned}
(\Phi y)(t) &\geq \frac{1}{k(t)+1}\sum_{j=1}^{n} c_j(t)\, y\left(t-\tau_j(t)\right) \\
&\quad + \frac{N_1}{k(t)+1}\int_t^{t+\omega} e(s)\int_{-\infty}^{0} D(r)\, f(s, y(s+r))\, drds \\
&\quad + \frac{N_2}{k(t)+1}\int_t^{t+\omega} h(t, Y(t))\, ds \\
&\geq \epsilon_1^* \omega \widehat{\rho^*} \widehat{m_0^*} \vartheta_1 y + \epsilon_1^* \omega \widehat{\rho^*} \widehat{m_0^*} \vartheta_1 \widetilde{Y} \\
&\geq 2\sigma\omega\epsilon_1 \widehat{\rho^*} \widehat{m_0^*} \vartheta_1 \left\|y\right\|,
\end{aligned}$$

and thus

$$\left\|\Phi y\right\| \geq \left\|y\right\|, \ y \in \partial\widetilde{\Omega}_1 \cap K.$$

Moreover, if $\lim_{\widetilde{Y}\to\infty} \min_{t\in[0,\omega]} \frac{h(t,Y)}{\widetilde{Y}} = \lim_{y\to\infty} \min_{t\in[0,\omega]} \frac{f(t,y)}{y} = +\infty$, then there exists $\overline{F} > F$ such that $h(t,Y) > \epsilon_2^* \widetilde{Y}$ and $f(t,y) \geq \epsilon_2^* y$ for all $y > \overline{F}$, $t \in [0,\omega]$, where the constant $\epsilon_2^* > 0$, satisfies $\epsilon_2^* > \frac{1}{2\sigma\omega\vartheta_1 \widehat{\rho^*} \widehat{m_0^*}}$.
Set

$$\widetilde{\Omega}_2 =: \left\{y(.) \in C_\omega : \left\|y\right\| < \overline{F}\right\}.$$

Then for any $y \in \partial\widetilde{\Omega}_2 \cap K$, we obtain the result $\left\|\Phi y\right\| \geq \left\|y\right\|$ by applying the same procedure given above. Furthermore, we introduce

$$\widetilde{\Omega}_3 =: \left\{y(.) \in C_\omega : \left\|y\right\| < F\right\}.$$

If $y \in \partial\widetilde{\Omega}_3 \cap K$, then $\sigma\left\|y\right\| \leq \widetilde{Y} \leq \left\|y\right\|$. From (3.4), and (2.8), (2.12), (2.13), we derive

$$(\Phi y)(t) \leq \frac{1}{k(t)+1}\sum_{j=1}^{n} c_j(t)\, y\left(t-\tau_j(t)\right)$$

$$
\begin{aligned}
&+\frac{M_1}{k(t)+1}\int_t^{t+\omega} e(s)\int_{-\infty}^{0} D(r)\, f(s, y(s+r))\, drds + \frac{M_2}{k(t)+1}\int_t^{t+\omega} h(t, Y(t))\, ds \\
\leq\ & \rho^* \times \frac{1}{3\rho^*}F + \omega\vartheta_2\rho^*\frac{1}{\omega}\int_t^{t+\omega} e(s)\frac{F}{4\omega m_0^*\vartheta_2\rho^*}ds + \vartheta_2\rho^*\int_t^{t+\omega}\frac{F}{4\omega m_0^*\vartheta_2\rho^*}ds \\
\leq\ & \rho^* \times \frac{1}{2\rho^*}F + 2\omega\vartheta_2\rho^* m_0^* \times \frac{F}{4\omega m_0^*\vartheta_2\rho^*} \\
=\ & \frac{F}{2} + \frac{F}{2} = F,
\end{aligned}
$$

and therefore

$$\left\|\Phi y\right\| \leq \left\|y\right\|.$$

This means $\left\|\Phi y\right\| \leq \left\|y\right\|$ for $y \in \partial\widetilde{\Omega}_3 \cap K$. Since $\underline{F} < F < \overline{F}$, by Theorem 1.1 the mapping Φ has two fixed points in $K \cap \overline{\widetilde{\Omega}}_3 \backslash \widetilde{\Omega}_2$ and $K \cap \overline{\widetilde{\Omega}}_2 \backslash \widetilde{\Omega}_1$. It follows that Φ has at least two positive periodic solutions of (1.5). That is $\underline{F} < \widetilde{y}_1 < F < \widetilde{y}_2 < \overline{F}$. The proof is complete. □

Corollary 3.4. *Assume conditions* (2.1) , (2.4) , (A1),(A2),(A18) *hold. Moreover, assume also that:*

A17)*

$$\lim_{\widetilde{Y}\to 0^+}\min_{t\in[0,\omega]}\frac{h(t,Y)}{\widetilde{Y}} = \lim_{y\to 0^+}\min_{t\in[0,\omega]}\frac{f(t,y)}{y} = \infty,$$

$$\left(\text{or } \lim_{\widetilde{Y}\to \infty}\min_{t\in[0,\omega]}\frac{h(t,Y)}{\widetilde{Y}} = \lim_{y\to \infty}\min_{t\in[0,\omega]}\frac{f(t,y)}{y} = \infty\ \right).$$

Then, equation (1.5) *has at least one positive periodic solution.*

The following result can be proved in a similar way as Theorem 3.6. Therefore, we omit its proof.

Theorem 3.7. *Assume that* $\varsigma(t) \leq \frac{1}{2\widehat{\rho^*}}$ *for* $t \in \mathbb{R}$ *and conditions* (2.1) , (2.4) , (A1),(A2) *hold. Furthermore, assume that:*

A19)

$$\lim_{\widetilde{Y}\to 0^+}\max_{t\in[0,\omega]}\frac{h(t,Y)}{\widetilde{Y}} = \lim_{\widetilde{Y}\to \infty}\max_{t\in[0,\omega]}\frac{h(t,Y)}{\widetilde{Y}} = \lim_{y\to 0^+}\max_{t\in[0,\omega]}\frac{f(t,y)}{y} = \lim_{y\to \infty}\max_{t\in[0,\omega]}\frac{f(t,y)}{y} = 0.$$

A20) There exists a constant $\chi > 0$ *such that,*

$$
\begin{aligned}
h(t,Y) &\geq \frac{\chi}{2\omega\widehat{m}_0^*\vartheta_1\widehat{\rho^*}}, \text{for } \widetilde{Y} \in [\chi\sigma, \chi], \\
f(t,y) &\geq \frac{\chi}{2\omega\widehat{m}_0^*\vartheta_1\widehat{\rho^*}}, \text{ for } y \in [\chi\sigma, \chi].
\end{aligned}
$$

Then, equation (1.5) *has at least two positive periodic solutions.*

Before establishing the following results, we first prove the result below:

Theorem 3.8. *Assume that* $g(y) \leq 0$ *for all* $y \in \mathbb{R}^+$, *and conditions* (A1), (A2), (2.1) , (2.4) , (2.6) , (2.10) *hold. Moreover, assume there are positive constants* R_1, R_2, R_3 *with* $R_1 < R_2 < R_3$ *such that:*

A21)

$$\rho^{*}c^{*}+\omega\rho^{*}\vartheta_{2}\left(\widehat{m}_{1}\widehat{\mathfrak{F}}_{R_{1}}+\widehat{m}_{2}\widehat{\mathfrak{K}}_{R_{1}}+\widehat{m}_{3}\right)\leq 1,$$

A22)

$$\widehat{\rho^{*}c^{*}}+\omega\widehat{\rho^{*}}\vartheta_{1}\left(\widehat{m}_{3}+\widehat{m}_{1}\mathfrak{F}_{R_{2}}+\widehat{m}_{2}\mathfrak{K}_{R_{2}}\right)\geq 1,$$

A23)

$$\rho^{*}c^{*}+\omega\rho^{*}\vartheta_{2}\left(\widehat{m}_{1}\widehat{\mathfrak{F}}_{R_{3}}+\widehat{m}_{2}\widehat{\mathfrak{K}}_{R_{3}}+\widehat{m}_{3}\right)\leq 1.$$

Then, (1.5) *possesses two positive ω-periodic solutions y_1, y_2 with $R_1 \leq y_1 \leq R_2 \leq y_2 \leq R_3$.*

Proof. Without loss of generality, we assume $R_1 < R_2$. Define an open set

$$\Upsilon_{1}=\left\{y\left(.\right)\in C_{\omega}:\left\|y\right\|<R_{1}\right\}.$$

Then, for any $y \in \partial\Upsilon_1 \cap K$, we have $\sigma\left\|y\right\| \leq y \leq \left\|y\right\| = R_1$. From this, the definition of Φ and $\widehat{\mathfrak{K}}_{R_1}, \widehat{\mathfrak{F}}_{R_1}$, it follows that

$$\begin{aligned}
\Phi y\left(t\right) &\leq \frac{1}{k\left(t\right)+1}\sum_{j=1}^{n}c_{j}\left(t\right)y\left(t-\tau_{j}\left(t\right)\right)\\
&\quad+\frac{M_{1}}{k\left(t\right)+1}\int_{t}^{t+\omega}e\left(s\right)\int_{-\infty}^{0}D\left(r\right)f\left(s,y\left(s+r\right)\right)drds\\
&\quad+\frac{M_{2}}{1+k\left(t\right)}\int_{t}^{t+\omega}k\left(s\right)y\left(s\right)ds\\
&\quad+\frac{M_{2}}{1+k\left(t\right)}\int_{t}^{t+\omega}\sum_{j=1}^{n}c_{j}\left(s\right)\times\left(-g\left(y\left(s-\tau_{j}\left(s\right)\right)\right)\right)ds\\
&\leq \left(\rho^{*}c^{*}+\omega\rho^{*}\vartheta_{2}\left(\widehat{m}_{1}\widehat{\mathfrak{F}}_{R_{1}}+\widehat{m}_{2}\widehat{\mathfrak{K}}_{R_{1}}+\widehat{m}_{3}\right)\right)\left\|y\right\|\\
&\leq \left\|y\right\|.
\end{aligned}$$

This yields $\left\|\Phi y\right\| \leq (\Phi y)(t) \leq \left\|y\right\|$ for $y \in \partial\Upsilon_1 \cap K$. Set

$$\Upsilon_{3}=\left\{y\left(.\right)\in C_{\omega}:\left\|y\right\|<R_{3}\right\}.$$

Then, for any $y \in \partial\Upsilon_3 \cap K$, we obtain the result $\left\|\Phi y\right\| \geq \left\|y\right\|$, by applying the same procedure given above.

Set

$$\Upsilon_{2}=\left\{y\left(.\right)\in C_{\omega}:\left\|y\right\|<R_{2}\right\},$$

for all $y \in \partial\Upsilon_2 \cap K$, then $R_2\sigma \leq y \leq R_2$. Hence, by virtue of (3.4), (2.8) , (2.13) , and (A22), we obtain

$$\begin{aligned}
\Phi y\left(t\right) &\geq \frac{1}{k\left(t\right)+1}\sum_{j=1}^{n}c_{j}\left(t\right)y\left(t-\tau_{j}\left(t\right)\right)\\
&\quad+\frac{N_{1}}{k\left(t\right)+1}\int_{t}^{t+\omega}e\left(s\right)\int_{-\infty}^{0}D\left(r\right)f\left(s,y\left(s+r\right)\right)drds\\
&\quad+\frac{N_{2}}{1+k\left(t\right)}\int_{t}^{t+\omega}k\left(s\right)y\left(s\right)ds
\end{aligned}$$

$$+\frac{N_2}{1+k(t)}\int_t^{t+\omega}\sum_{j=1}^{n}c_j(s)\times\left(-g\left(y\left(s-\tau_j(s)\right)\right)\right)ds$$
$$\geq \left(\rho^*c^*+\omega\rho^*\vartheta_1\left(\widehat{m}_1\widehat{\mathfrak{F}}_{R_2}+\widehat{m}_2\widehat{\mathfrak{K}}_{R_2}+\widehat{m}_3\right)\right)\left\|y\right\|$$
$$\geq \left\|y\right\|.$$

This implies that for any $y\in\partial\Upsilon_2\cap K$, $\left\|\Phi y\right\|\geq\left\|y\right\|$. Since $R_1<R_2<R_3$, by Theorem 1.1 the mapping Φ has at least two fixed points in $K\cap\overline{\Upsilon}_3\setminus\Upsilon_2$ and $K\cap\overline{\Upsilon}_2\setminus\Upsilon_1$. It follows that Φ possesses at least two positive periodic solutions of (1.5). That is $R_1<y_1<R_2<y_2<R_3$. The proof is complete.

Theorem 3.9. *Assume that $\varsigma(t)\leq\frac{1}{2\rho^*}$, for $t\in\mathbb{R}$, conditions* (2.1), (2.4), (A1),(A2) *hold, and there are positive constants L_1, L_2 and L_3, with $L_1<L_2<L_3$, such that*

A24) $\max_{t\in[0.\omega]}h(t,Y)\leq\frac{L_1}{4\omega\vartheta_2\rho^*m_0^*}$, $\max_{t\in[0.\omega]}f(t,y)\leq\frac{L_1}{4\omega\vartheta_2\rho^*m_0^*}$, $y\in[0,L_1]$, $\widetilde{Y}\in[0,L_1]$.

A25) $\min_{t\in[0.\omega]}h(t,Y)\geq\frac{L_2}{2\omega\vartheta_1\widehat{\rho}^*\widehat{m}_0^*}$, $\min_{t\in[0.\omega]}f(t,y)\geq\frac{L_2}{2\omega\vartheta_1\widehat{\rho}^*\widehat{m}_0^*}$, $y\in[\sigma L_2,L_2]$, $\widetilde{Y}\in[\sigma L_2,L_2]$.

A26) $\max_{t\in[0.\omega]}h(t,Y)\leq\frac{L_3}{4\omega\vartheta_2\rho^*m_0^*}$, $\max_{t\in[0.\omega]}f(t,y)\leq\frac{L_3}{4\omega\vartheta_2\rho^*m_0^*}$, $y\in[0,L_3]$, $\widetilde{Y}\in[0,L_3]$.

Then, equation (1.5) has two-positive ω-periodic solutions y_1^*, y_2^{**} with $L_1<y_1^*<L_2<y_2^{**}<L_3$.

Proof. Without loss of generality, we assume $L_1<L_2$. We define

$$\widehat{\Upsilon}_1=\left\{y(.)\in C_\omega:\left\|y\right\|<L_1\right\}.$$

If $y\in\partial\widehat{\Upsilon}_1\cap K$, then $\sigma\left\|y\right\|\leq y\leq\left\|y\right\|$. From (A24) and (2.8), (2.13), (3.4), we have

$$\begin{aligned}\left|\Phi y(t)\right| &\leq \rho^*\varsigma(t)\max_{1\leq j\leq n}\left\{\left|y\left(t-\tau_j(t)\right)\right|\right\}\\ &\quad+\vartheta_2\rho^*\int_t^{t+\omega}e(s)\int_{-\infty}^{0}D(r)f(s,y(s+r))drds+\vartheta_2\rho^*\int_t^{t+\omega}h(s,Y(s))ds\\ &\leq \rho^*\varsigma(t)L_1+\vartheta_2\rho^*\widehat{m}_1\omega\times\frac{L_1}{4\omega\vartheta_2\rho^*m_0^*}+\omega\vartheta_2\rho^*\times\frac{L_1}{4\omega\vartheta_2\rho^*m_0^*}\\ &\leq \rho^*\varsigma(t)L_1+2\omega\vartheta_2\rho^*m_0^*\times\frac{L_1}{4\omega\vartheta_2\rho^*m_0^*}\\ &\leq \frac{L_1}{2}+\frac{L_1}{2}=L_1.\end{aligned}$$

This means $||\Phi y||\leq||y||$ for $y\in\partial\widehat{\Upsilon}_1\cap K$. Set

$$\widehat{\Upsilon}_3=\left\{y(.)\in C_\omega:\left\|y\right\|<L_3\right\}.$$

Thus, for any $y\in\partial\widehat{\Upsilon}_3\cap K$, we obtain the result $||\Phi y||\leq||y||$ by applying the same procedure given above.

Let

$$\widehat{\Upsilon}_2=\left\{y(.)\in C_\omega:\left\|y\right\|<L_2\right\}.$$

If $y\in\partial\widehat{\Upsilon}_2\cap K$, then $\sigma L_2\leq y\leq L_2$. From (2.8), (2.13), (3.4), (3.25) we have

$$\left|\Phi y(t)\right| \geq \rho^*\varsigma(t)\max_{1\leq j\leq n}\left\{\left|y\left(t-\tau_j(t)\right)\right|\right\}$$

$$
\begin{aligned}
&+\vartheta_2\rho^*\int_t^{t+\omega} e(s)\int_{-\infty}^{0} D(r) f(s,y(s+r))\,drds + \vartheta_2\rho^*\int_t^{t+\omega} h(s,Y(s))\,ds\\
\geq\ & 2\omega\vartheta_2\rho^*\widehat{m}_0^* \times \frac{L_2}{4\omega\vartheta_2\rho^*\widehat{m}_0^*}\\
\geq\ & \frac{L_2}{2}+\frac{L_2}{2} = L_2,
\end{aligned}
$$

and therefore $\|\Phi y\| \geq \|y\|$.

Since $L_1 < L_2 < L_3$, by Theorem 1.1 the mapping Φ has at least two fixed points in $K \cap \widehat{\Upsilon}_3 \setminus \widehat{\Upsilon}_2$ and $K \cap \widehat{\Upsilon}_2 \setminus \widehat{\Upsilon}_1$. It follows that Φ has at least two positive periodic solutions of (1.5). That is, $L_1 < y_1^* < L_2 < y_2^{**} < L_3$. The proof is complete.

As consequence of Theorem 3.9, we state a corollary whose proof is similar to the proof of Theorem 3.9 and hence we omit it.

Corollary 3.5. *Assume* $\varsigma(t) \leq \dfrac{1}{2\rho^*}$, *for* $t \in \mathbb{R}$, *conditions* (A1),(A2), (2.4) , (2.1) *are fulfilled, and there exist positive constants* $\widehat{L}_1, \widehat{L}_2, \widehat{L}_3$ *with* $\widehat{L}_1 < \widehat{L}_2 < \widehat{L}_3$, *such that:*

A27) $\min\limits_{t\in[0.\omega]} h(t,Y) \geq \dfrac{\widehat{L}_1}{2\omega\vartheta_1\widehat{m}_0^*}, \min\limits_{t\in[0.\omega]} f(t,y) \geq \dfrac{\widehat{L}_1}{2\omega\vartheta_1\widehat{m}_0^*}, y \in \left[\sigma\widehat{L}_1, \widehat{L}_1\right], \widetilde{Y} \in \left[\sigma\widehat{L}_1, \widehat{L}_1\right].$

A28) $\max\limits_{t\in[0.\omega]} h(t,Y) \leq \dfrac{\widehat{L}_2}{4\omega\vartheta_2 m_0^*}, \max\limits_{t\in[0.\omega]} f(t,y) \leq \dfrac{\widehat{L}_2}{4\omega\vartheta_2 m_0^*}, y \in \left[0, \widehat{L}_2\right], \widetilde{Y} \in \left[0, \widehat{L}_2\right].$

A29) $\min\limits_{t\in[0.\omega]} h(t,Y) \geq \dfrac{\widehat{L}_3}{2\omega\vartheta_1\widehat{m}_0^*}, \min\limits_{t\in[0.\omega]} f(t,y) \geq \dfrac{\widehat{L}_3}{2\omega\vartheta_1\widehat{m}_0^*}, y \in \left[\sigma\widehat{L}_3, \widehat{L}_3\right], \widetilde{Y} \in \left[\sigma\widehat{L}_3, \widehat{L}_3\right].$

Then, there exist two $\omega-$ *periodic solutions which are fixed points of* Φ *and satisfy* $\widehat{L}_1 < \overset{*}{\widehat{y}} < \widehat{L}_2 < \overset{**}{\widehat{y}} < \widehat{L}_3$.

4. Examples

In this section, we provide two examples to illustrate the applicability of Theorems 3.2 and 3.9, respectively.

Example 4.1. Let us consider the following second-order nonlinear neutral differential equations with mixed delays:

$$
\begin{aligned}
&\frac{d^2}{dt^2}y(t) + p(t)\frac{d}{dt}y(t) + q(t)y(t) + \frac{d^2}{dt^2}\left[k(t)y(t) - \sum_{j=1}^{2} c_j(t)\, g\left(y\left(t-\tau_j(t)\right)\right)\right]\\
&\qquad = e(t)\int_{-\infty}^{0} D(r) f(t, y(t+r))\,dr. \qquad (4.1)
\end{aligned}
$$

Corresponding to equation (4.1), we let

$$
\begin{aligned}
p(t) &= 0.85,\ q(t) = 0.13, \omega = 0.02,\ k(t) = 0.004\sin^2 100\pi t + 0.02,\\
c_1(t) &= 0.02\left(\ln(0.001+\sin^2 100\pi t)\right)^4,\ c_2(t) = 0.001\left(\ln(0.003+\cos^4 200\pi t)\right)^6,\\
e(t) &= -0.4\cos 400\pi t + 40, \tau_1(t) = \sin^2 200\pi t, \tau_2(t) = \cos^4 600\pi t.
\end{aligned}
$$

Moreover $D \in C(\mathbb{R}^-, \mathbb{R}^+)$ is an arbitrary function satisfying $\int_{-\infty}^{0} D(r)\,dr = 1$.
Thanks to direct computations, we obtain

$$
\widetilde{\mathfrak{R}}_1 \simeq 0.1542, Q_1 \simeq 0.1542, \frac{\widetilde{\mathfrak{R}}_1\left[\exp\left(\int_0^{\omega} p(u)\,du\right) - 1\right]}{Q_1\omega} \simeq 1.3665 \geq 1, \qquad (4.2)
$$

$$\begin{aligned} A &= 0.017,\ B \simeq 52\times 10^{-6},\ 289\times 10^{-6} \simeq A^2 \geq 4B \simeq 208\times 10^{-6},\\ a(t) &= 0.2, b(t) = 0.65, l \simeq 0.00848, L \simeq 0.00851, \end{aligned}$$

and

$$\widehat{m}_0^* = \min\{\widehat{m}_1, 1\} = 0.8,\ m_0^* = \max\{\widehat{m}_1, 1\} = 1, \widehat{\rho}^* = 0.97, \rho^* = 0.98, \tag{4.3}$$

where $\widehat{m}_1, \widehat{\rho}^*, \rho^*$ are respectively as in $(2.18), (3.4)$. By straightforward computations, for $t, s \in [0, \omega]$, we derive

$$\begin{aligned} &0 < N_1 \leq G(t,s) \leq M_1,\\ &0 < \widehat{N}_2^* \leq E(t,s) \leq \widehat{M}_2^*, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} N_1 &= \frac{\omega}{(\exp(L)-1)^2} \simeq 273.8246,\ M_1 = \frac{\omega\left(\exp\int_0^\omega p(u)du\right)}{(\exp(l)-1)^2} \simeq 280.5019,\\ \widehat{N}_2^* &= \frac{\exp\left(\int_0^\omega b(v)\,dv\right)}{\exp\left(\int_0^\omega b(v)\,dv\right)-1} \simeq 77.42,\ \widehat{M}_2^* = \frac{e^L}{e^l-1} \simeq 118.4288. \end{aligned} \tag{4.5}$$

From (4.4) and (4.5), it is easy to deduce

$$0 < 9.71 = N_2 \leq Z(t,s) \leq M_2 = 45.9,$$

where $Z(t,s)$ is as in (2.7).
For $y \in K, y(t) \geq \sigma\|y\|$ for all $t \in [0,\omega]$, where

$$\sigma = \frac{\vartheta_1}{\vartheta_2} = \frac{\min\{N_1, N_2\}}{\max\{M_1, M_2\}} = \frac{9.71}{280.5019} \simeq 0.03461.$$

Also assume that

$$\begin{aligned} &f(t,y) = 0.002\sin^4(5y) + \frac{1}{\sigma}\ln\left|\left(0.001 + \cos^2 100t\right)\right| y,\\ &g(y) = -10\left(\sin^2\left(10y^2\right) + 0.5y\right). \end{aligned}$$

It is easy to see that

$$\forall\, t \in [0,\omega], \varsigma(t) = 1.996\times 10^{-14} \leq \frac{\lambda}{2\rho^*} = 0.4591,\ \text{where } \lambda = 0.9 < 1,$$

and

$$\begin{aligned} \widehat{f}^0 &= 0.03 \leq \frac{\lambda}{4\vartheta_2\omega m_0^*\rho^*} = 0.04,\ \widehat{f}^\infty = 0.03 \leq \frac{\lambda}{4\vartheta_2\omega m_0^*\rho^*} = 0.04,\\ \widehat{h}^0 &= 0.02 \leq \frac{\lambda}{4\vartheta_2\omega m_0^*\rho^*} = 0.04,\ \widehat{h}^\infty = 0.02 \leq \frac{\lambda}{4\vartheta_2\omega m_0^*\rho^*} = 0.04. \end{aligned}$$

There exists a constant $W = 1$ such that, for all $y, y_1, y_2 \in K \cap \partial\Omega_1$, we have

$$\begin{aligned} f(t,y) &\geq \frac{1}{\sigma}\left|\ln\left(0.001 + \cos^2 100t\right)\right| \times \sigma\|y\|\\ &= 6.908\\ &\geq \frac{W}{2\omega\vartheta_1\widehat{m}_0^*\widehat{\rho}^*} \simeq 3.319, \end{aligned}$$

and

$$
\begin{aligned}
h(t, Y) &\geq 0.02y + 5 \times 0.02 (\ln (0.001))^4 \sigma \left\| y_1 \right\| + 5 \times 0.001 \left(|\ln(0.003)|^6 \right) \sigma \left\| y_2 \right\| \\
&\simeq 14.53 \geq \frac{W}{2\omega \vartheta_1 \widehat{m_0^*} \widehat{\rho^*}} \simeq 3.319.
\end{aligned}
$$

Since assumption (A5) holds, for a sufficiently small $\varepsilon > 0$ satisfying

$$0 < \varepsilon < \frac{1 - \lambda}{2\vartheta_2 \omega m_0^* \rho^*} = 0.01,$$

we can choose $\underline{W} = 0.1$, where $0 < \underline{W} < W = 1$, we obtain

$$
\begin{aligned}
h(t, Y) &\leq 0.024y + 0.2 |\ln(1.001)|^4 \left(\left(10^2 y^4 + 0.5y \right) \right) \\
&\quad + 0.01 |\ln(1.003)|^6 \left(\left(10^2 y^4 + 0.5y \right) \right) \\
&\leq \Big(0.024 + 0.2 |\ln(1.001)|^4 \left(\left(10^2 y^3 + 0.5 \right) \right) \\
&\quad + 0.01 |\ln(1.003)|^6 \left(\left(10^2 y^3 + 0.5 \right) \right) \Big) \|Y\| \\
&= 0.02 \|Y\| \leq \left(\frac{\lambda}{4\vartheta_2 \omega m_0^* \rho^*} + \varepsilon \right) \|Y\| \\
&\leq 0.05 \|Y\|,
\end{aligned}
$$

and

$$
\begin{aligned}
f(t, y) &= 0.002 \sin^4 (5y) + \frac{1}{\sigma} \ln \left| \left(0.001 + \cos^2 100t \right) \right| y \\
&\leq 0.002 \left(5^4 \times \underline{W}^3 + \frac{1}{\sigma} |\ln (1.001)| \right) \times \left\| y \right\| \\
&= 0.001 \left\| y \right\| \leq \left(\frac{\lambda}{4\vartheta_2 \omega m_0^* \rho^*} + \varepsilon \right) \left\| y \right\| \\
&\leq 0.05 \left\| y \right\|.
\end{aligned}
$$

Since assumption (A6) holds, for a sufficiently small $\varepsilon > 0$ satisfying

$$0 < \varepsilon < \frac{1 - \lambda}{2\vartheta_2 \omega m_0^* \rho^*} = 0.01,$$

we can choose $\overline{W} = 0.2$, ($\overline{W} > \sigma W$). For all, $\left\| y \right\| > \overline{W}$, and $t \in [0, \omega]$, we obtain

$$
\begin{aligned}
h(t, Y) &\leq 0.024y + 0.2 |\ln(1.001)|^4 ((1 + 0.5y)) + 0.01 |\ln(1.003)|^6 ((1 + 0.5y)) \\
&\leq 0.024y + |\ln(1.001)|^4 y + 0.2 |\ln(1.001)|^4 0.5y \\
&\quad + \frac{0.01}{0.2} |\ln(1.003)|^6 y + 0.01 |\ln(1.003)|^6 0.5y \\
&\leq 0.02 \|Y\| \leq \left(\frac{\lambda}{4\vartheta_2 \omega m_0^* \rho^*} + \varepsilon \right) \|Y\| \\
&\leq 0.05 \|Y\|,
\end{aligned}
$$

and

$$f(t, y) \leq 0.002 + \frac{1}{\sigma} |\ln (1.001)| y$$

$$
\begin{aligned}
&\leq \left(0.01 + \frac{1}{\sigma}\left|\ln(1.001)\right|\right)\left\|y\right\| \\
&= 0.04\left\|y\right\| \leq \left(\frac{\lambda}{4\vartheta_2\omega m_0^*\rho^*} + \varepsilon\right)\left\|y\right\| \\
&\leq 0.05\left\|y\right\|.
\end{aligned}
$$

All hypotheses of Theorem 3.2 are fulfilled, therefore equation (4.1) has at least two positive 0.02−periodic solutions $\widetilde{y}_1 \in K \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$ and $\widetilde{y}_2 \in K \cap \left(\overline{\Omega}_3 \setminus \Omega_2\right)$ satisfying

$$0.1 = \underline{W} \leq \widetilde{y}_1 \leq W = 1 \leq \widetilde{y}_2 \leq \overline{W}^* = \frac{\overline{W}}{\sigma} = 5.78.$$

We should point out that, in order to apply Krasnoselskii's fixed point theorem in [13, Theorem 2.7] or [14, Theorem 2.7], we need to construct two mappings Φ_1, and Φ_2, Φ_1 is a contraction and Φ_2 is compact. Therefore, we express (4.1) as

$$(\Phi y)(t) = (\Phi_1 y)(t) + (\Phi_2 y)(t),$$

where $\Phi_1 : C_\omega \to C_\omega$ is given by

$$
\begin{aligned}
(\Phi_1 y)(t) &= \frac{1}{k(t)+1}\sum_{j=1}^{n} c_j(t)\, y\left(t - \tau_j(t)\right) \\
&\quad + \frac{1}{k(t)+1}\int_t^{t+\omega} G(t,s)\, e(s) \int_{-\infty}^{0} D(r)\, f(s, y(s+r))\, dr ds,
\end{aligned}
$$

and

$$(\Phi_2 y)(t) = \frac{1}{k(t)+1}\int_t^{t+\omega} Z(t,s)\, h(s, Y(s))\, ds.$$

However, notice that

$$\exists \widehat{\gamma} = 8.709 > 1 : \forall y, z \in C_\omega : \left|(\Phi_2 y)(t) - (\Phi_2 z)(t)\right| \leq \widehat{\gamma}\left|y - z\right|.$$

This implies that Φ_1 is not a contraction. In this case, the existence of 0.02-periodic solutions of equation (4.1) in C_ω cannot be proved using [13, Theorem 2.7]. However the results obtained in our work are quite significant compared to the ones in the above mentioned papers [13] and [14, Theorem 2.7].

Example 4.2. Let us consider the following second-order nonlinear neutral differential equation:

$$
\begin{aligned}
&\frac{d^2}{dt^2} y(t) + p(t)\frac{d}{dt} y(t) + q(t) y(t) + \frac{d^2}{dt^2}\left[k(t)\, y(t) - \sum_{j=1}^{2} c_j(t)\, g\left(y\left(t - \tau_j(t)\right)\right)\right] \\
&\qquad = e(t)\int_{-\infty}^{0} D(r)\, f(t, y(t+r))\, dr. \qquad (4.6)
\end{aligned}
$$

Corresponding to equation (4.6), we let

$$
\begin{aligned}
p(t) &= 0.85,\ q(t) = 0.13, \omega = 0.02,\ k(t) = \left(\ln\left(0.95\sin^2 100\pi t + 0.07\right)\right)^2, \\
c_1(t) &= 0.05 + 0.05\cos^2 200\pi t, c_2(t) = 0.05\sin^2 200\pi t, \\
\tau_1(t) &= \sin^2 400\pi t, \tau_2(t) = \cos^2 200\pi t, e(t) = 0.1\cos 100\pi t + 0.8.
\end{aligned}
$$

Also assume that

$$f(t, y) = \left(\ln\left(\sin^2 100\pi t + 0.04\right)\right)^4 \sqrt{10y},$$

$$g(y) = -\left(\ln\left(\left(\sin^2\sqrt{y}\right)+0.001\right)\right)^4. \tag{4.7}$$

Thanks to direct computations, we obtain

$$\widehat{m}_0^* = \min\{\widehat{m}_1, 1\} = 0.8,\ m_0^* = \max\{\widehat{m}_1, 1\} = 1,$$
$$\rho^* = 0.999, \widehat{\rho}^* = 0.124.$$

It is easy to see that

$$\varsigma(t) = \sum_{i=1}^{2} c_i(t) = 0.1 \leq \frac{1}{2\rho^*} = 0.5.$$

Since we use the same setting as in Example 4.1, where

$$p(t) = 0.85,\ q(t) = 0.13, \omega = 0.02,$$

we have

$$\sigma = 0.03461, \vartheta_1 = 9.71, \vartheta_2 = 280.5019.$$

There exist constants L_1, L_2, L_3, with $L_1 = 1 < L_2 = 1.5 < L_3 = 2$, such that, for all $t \in [0, 0.02]$, we have

$$\max_{t\in[0.\omega]} f(t,y) = 7.4828\times 10^{-6} \leq \frac{L_1}{4\omega\vartheta_2\rho^* m_0^*} = 0.0446, y \in [0, L_1],$$
$$\min_{t\in[0.\omega]} f(t,y) = 77.351 \geq \frac{L_2}{2\omega\vartheta_1\widehat{\rho}^*\widehat{m}_0^*} = 38.931, y \in [\sigma L_2, L_2],$$
$$\max_{t\in[0.\omega]} f(t,y) = 1.0582\times 10^{-5} \leq \frac{L_3}{4\omega\vartheta_2\rho^* m_0^*} = 0.089, y \in [0, L_3],$$

and

$$\max_{t\in[0.\omega]} h(t,Y) = 3.921\times 10^{-4} \leq \frac{L_1}{4\omega\vartheta_2\rho^* m_0^*} = 0.0446, y \in [0, L_1],$$
$$\min_{t\in[0.\omega]} h(t,Y) = 228.059 \geq \frac{L_2}{2\omega\vartheta_1\widehat{\rho}^*\widehat{m}_0^*} = 38.931, y \in [\sigma L_2, L_2],$$
$$\max_{t\in[0.\omega]} h(t,Y) = 7.843\times 10^{-4} \leq \frac{L_3}{4\omega\vartheta_2\rho^* m_0^*} = 0.089, y \in [0, L_3],$$

where $h(t, Y(t))$ is as in (2.15). Thus, all assumptions of Theorem 3.9 hold, and consequently, equation (4.6) possesses at least two positive 0.02-periodic solutions, $\widehat{y_1} \in K \cap \widehat{\Upsilon}_3 \setminus \widehat{\Upsilon}_2$ and $\widehat{y_2} \in K \cap \widehat{\Upsilon}_2 \setminus \widehat{\Upsilon}_1$ satisfying

$$1 = L_1 \leq \widehat{y_1} \leq L_2 = 1.5 \leq \widehat{y_2} \leq 2 = L_3.$$

Note that the given functions f and g in (4.7) are not globally Lipschitz in y on $\mathbb{R}^+$, since

$$\frac{d}{dy} f(t,y) = \left(\ln\left(\sin^2 100\pi t + 0.01\right)\right)^2 \frac{10}{2\sqrt{10y}},$$
$$\frac{d}{dy} g(y) = -4\left(\ln\left(\left(\sin^2\sqrt{y}\right)+0.001\right)\right)^3 \frac{2\sin\sqrt{y}\cos\sqrt{y}}{2\sqrt{y}\left(\sin^2\sqrt{y}+0.001\right)},$$

are not bounded on $\mathbb{R}^+$. Thus, existence of periodic solution of the equation (4.2) cannot be proved by using [13, Theorem 2.7], since Lipschitz condition of the functions f and g is not fulfilled in y on $\mathbb{R}^+$, since the finding in [13] mainly relies on the fact that all nonlinear functions are globally Lipschitz with respect to the variable y. Hence, in this study, we do not only suggest an alternative technique for discussing positive periodic solutions to generalized problems of equation (1.5) but also improve current literature.

5. Conclusion

This article is devoted to studying the existence of positive periodic solutions for a class of second-order nonlinear neutral differential equations with variable coefficients and mixed delays. Under some sufficient criteria and by virtue of a technique based on the theory of fixed point index in cones together with the Green functions method, as well as some useful functional analysis tools, we establish several interesting existence results of single and multiple positive periodic solutions for the proposed model. In our applications, two simple examples are analyzed to demonstrate the real power of Theorem 1.1 which guarantees the existence of twin positive periodic solutions even when Krasnoselskii's fixed point theorems cannot be used. The proof techniques used in this paper are new and generalize some previous studies and can be extended to investigate other types of second-order differential equations such as, impulsive differential equations, fractional differential equations, etc. We hope some authors can use the methods provided in this article to conduct more in-depth research on various types of second-order differential equations with variable coefficients.

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Conflict of interests.

The authors declare that they do not have any conflict of interest.

Data availability.

The authors declare that no external data were used in the research carried out in this paper.

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