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Dynamical analysis of semigroups for Lasota-type equations in a new functional space

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Abstract. We study the asymptotic behavior of the dynamical system generated by the Lasota equation within Morrey spaces. We establish precise conditions for stability and chaos in the sense of Devaney. Furthermore, we identify criteria allowing this asymptotic description to extend to a generalized version of the equation with variable coefficients, and to an extended model with a more complex transport structure. Our work provides a generalization of the asymptotic results known for the von Foerster–Lasota equation in L^p spaces to the Morrey spaces.

1. Introduction

The Lasota equation arises from a biological model describing the dynamics of progenitor cells, where maturation occurs at a non-uniform rate. Owing to its biological relevance, this equation continues to be a significant subject of mathematical investigation. Numerous studies have concentrated on the analysis of its linear formulation. In 2008 Dawidowicz and Poskrobko studied properties of the equation $\frac{\partial \mathfrak{X}}{\partial \tau} + \zeta \frac{\partial \mathfrak{X}}{\partial \zeta} = \rho(\zeta)\mathfrak{X}$ in the space $\mathbb{V}_\alpha$ (Hölder continuous functions) and L^p space [5]. In 2012 Chang and Hong found a necessary and sufficient condition for the solution of a quasi-linear Lasota equation to be chaotic [4].This equation also draws the attention of other researchers, like Brzeźniak [3], Matsui and Takeo [12], Takeo [16]. Devaney provided a novel and rigorous definition of chaos, which has played a central role in modern dynamical systems theory [9].In paper [5] They study chaos in the sense of Davancy of this equation.

Inspired by the approach developed in [5], we propose to investigate the Lasota equation in Morrey spaces, the objective of this work is to extend the analysis of the Lasota equation to Morrey spaces, thereby generalizing previous results obtained in classical L^p or Orlicz spaces. For more details on Morrey spaces, see [11, 15],[1, 8, 14].

2. Preliminaries

In this section, we introduce the fundamental concepts in this paper.

Morrey spaces were first introduced by Charles B. Morrey in 1938 [13] in relation to regularity problems of

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solutions to partial differential equations. We recall its definition as, Let $1 \leq p < \infty$ and $0 < \lambda < n$ (where n is the dimension of $\mathbb{R}^n$). Denote by $\mathcal{B}(\zeta, r)$ the open ball centered at ζ with radius r. The Morrey space $\mathfrak{M}^{p,\lambda}(\mathbb{R}^n)$, introduced in [11], consists of all locally L^p functions Ψ on $\mathbb{R}^n$ such that

$$\|\Psi\|_{\mathfrak{M}^{p,\lambda}} := \sup_{\zeta\in\mathbb{R}^n, r>0} r^{-\lambda/p}\|\Psi\|_{L^p(\mathcal{B}(\zeta,r))} < \infty,$$

where the local L^p-norm is given by

$$\|\Psi\|_{L^p(\mathcal{B}(\zeta,r))} = \left(\int_{\mathcal{B}(\zeta,r)} |\Psi(\xi)|^p \, d\xi\right)^{1/p}.$$

It is important to note that $\mathfrak{M}^{p,0}(\mathbb{R}^n)$ represents an extension of the classical Lebesgue space $L^p(\mathbb{R}^n)$, since in this case we have $\mathfrak{M}^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

Theorem 1. *For all* $\varphi, \psi \in \mathfrak{M}^{p,\lambda}(\mathbb{R}^n)$, *we have:*

$$\|\varphi + \psi\|_{\mathfrak{M}^{p,\lambda}} \leq \|\varphi\|_{\mathfrak{M}^{p,\lambda}} + \|\psi\|_{\mathfrak{M}^{p,\lambda}}.$$

Proof. For any ball $\mathcal{B}(\zeta, r) \subset \mathbb{R}^n$, the triangle inequality in L^p gives:

$$\|\varphi + \psi\|_{L^p(\mathcal{B}(\zeta,r)} \leq \|\varphi\|_{L^p(\mathcal{B}(\zeta,r)} + \|\psi\|_{L^p(\mathcal{B}(\zeta,r)}.$$

Multiplying by $r^{-\lambda/p}$ with $r > 0$, we obtain:

$$r^{-\lambda/p}\|\varphi + \psi\|_{L^p(\mathcal{B}(\zeta,r))} \leq r^{-\lambda/p}\|\varphi\|_{L^p(\mathcal{B}(\zeta,r))} + r^{-\lambda/p}\|\psi\|_{L^p(\mathcal{B}(\zeta,r))}.$$

By taking the supremum over all $\zeta \in \mathbb{R}^n$ and $r > 0$, we have:

$$\begin{aligned}
\|\varphi + \psi\|_{\mathfrak{M}^{p,\lambda}} &= \sup_{\substack{\zeta\in\mathbb{R}^n \\ r>0}} r^{-\lambda/p}\|\varphi + \psi\|_{L^p(\mathcal{B}(\zeta,r))} \\
&\leq \sup_{\substack{\zeta\in\mathbb{R}^n \\ r>0}} \left(r^{-\lambda/p}\|\varphi\|_{L^p(\mathcal{B}(\zeta,r))} + r^{-\lambda/p}\|\psi\|_{L^p(\mathcal{B}(\zeta,r))}\right) \\
&\leq \sup_{\substack{\zeta\in\mathbb{R}^n \\ r>0}} r^{-\lambda/p}\|\varphi\|_{L^p(\mathcal{B}(\zeta,r))} + \sup_{\substack{\zeta\in\mathbb{R}^n \\ r>0}} r^{-\lambda/p}\|\psi\|_{L^p(\mathcal{B}(\zeta,r))} \\
&= \|\varphi\|_{\mathfrak{M}^{p,\lambda}} + \|\psi\|_{\mathfrak{M}^{p,\lambda}}.
\end{aligned}$$

which completes the proof.

It is known from [15] that the Morrey space $\mathfrak{M}^{p,\lambda}$ is not separable when $0 < \lambda < n$. Now, consider a dynamical system $\mathcal{S}(\tau) : X \to X$. A point $\xi \in \mathbb{X}$ is called a periodic point if there exists an integer $n \geq 1$ such that $\mathcal{S}^n\xi = \xi$, and the smallest such integer is called the period of ξ. The set of periodic points is denoted by $\mathrm{Per}(\mathcal{S})$ [10]. The dynamical system $\mathcal{S} : \mathbb{X} \to \mathbb{X}$ is said to be *topologically transitive* if, for any pair of nonempty open sets $\mathbb{U}$ and $\mathbb{V}$ in $\mathbb{X}$, there exists some $n \geq 0$ such that $\mathcal{S}^n(\mathbb{U}) \cap \mathbb{V} \neq \emptyset$ [10].

According to Devaney [10], a dynamical system $\mathcal{S} : \mathbb{X} \to \mathbb{X}$ is said to be *chaotic* if it satisfies two conditions: $\mathcal{S}$ is topologically transitive, and $\mathcal{S}$ has a dense set of periodic points. Moreover, a semigroup $(\mathcal{S}(\tau))_{\tau\geq 0}$ is called *strongly stable* in a space $\mathbb{V}$ if, for every $\vartheta \in \mathbb{V}$, $\lim_{\tau\to\infty} \mathcal{S}(\tau)\vartheta = 0$ in $\mathbb{V}$ [6]. Finally, two dynamical systems $(\mathcal{F}(\tau))_{\tau\geqslant 0}$ and $(\mathcal{G}(\tau))_{\tau\geqslant 0}$, where $\mathcal{F}(\tau) : \mathbb{X} \to \mathbb{X}$ and $\mathcal{G}(\tau) : \mathbb{Y} \to \mathbb{Y}$, are said to be *topologically equivalent* if there exists a homeomorphism $\psi : \mathbb{X} \to \mathbb{Y}$ such that $\psi \circ \mathcal{F}(\tau) = \mathcal{G}(\tau) \circ \psi$, that is, if the following diagram is commutative:

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "\mathcal{F}(\tau)"] \arrow[d, "\psi"'] & \mathbb{X} \arrow[d, "\psi"] \\
\mathbb{Y} \arrow[r, "\mathcal{G}(\tau)"'] & \mathbb{Y}
\end{tikzcd}$$

□

3. Chaotic and Stable Regimes of the Von Foerster–Lasota Semigroup in Morrey Spaces

We focus on the partial differential equation

$$\frac{\partial \mathfrak{X}}{\partial \tau}+\zeta\frac{\partial \mathfrak{X}}{\partial \zeta}=\gamma\mathfrak{X}, \tau\geqslant 0, 0\leqslant\zeta\leqslant 1, \gamma\in\mathbb{R} \tag{3.1}$$

with the initial condition

$$\mathfrak{X}(0,\zeta)=\vartheta(\zeta), \quad 0\leqslant\zeta\leqslant 1 \tag{3.2}$$

where ϑ belongs to some normed vector space $\mathbb{V}$ of functions defined on $[0,1]$. Define the function $\mathcal{S}_\tau$ by the formula

$$(\mathcal{S}_\tau\vartheta)(\zeta)=\mathfrak{X}(\tau,\zeta)=\mathrm{e}^{\gamma\theta}\vartheta\left(\zeta\mathrm{e}^{-\tau}\right), \zeta\in[0,1] \tag{3.3}$$

where $\mathfrak{X}$ is the unique solution of 3.1 and 3.2. If for every $\vartheta\in\mathbb{V}$ and $\tau\geqslant 0$ the function $\mathcal{S}_\tau$ belongs to $\mathbb{V}$, then the family $(\mathcal{S}_\tau)_{\tau\geqslant 0}$ is a semigroup on the space $\mathbb{V}$.
The lack of separability in Morrey spaces given in Preliminaries poses challenges for certain dynamical approaches based on metric structures. Therefore, we focus on chaotic in the sense of Devaney defined in Preliminaries, which relies primarily on topological properties and remains suitable in this context.

We now proceed to formulate several theorems that characterize the chaotic and stable behavior of the dynamical system introduced above. These properties will be studied in in Morrey space.

Theorem 2. *If* $\gamma>\frac{\lambda-1}{p}$, *then for any* $\tau_0>0$ *there exists a periodic point* $\vartheta_0\in\mathfrak{M}^{p,\lambda}(0,1)$ *such that*

$$\mathcal{S}_\tau\vartheta_0=\vartheta_0$$

Moreover $\mathcal{S}_\tau\vartheta_0=\vartheta_0$ *if and only if* $\tau=n\tau_0$ *for some positive integer* n

Proof. Let $\mathfrak{X}\in\mathfrak{M}^{p,\lambda}(e^{-\tau_0},1)$ be arbitrary. Define $\vartheta_0:(0,1]\to\mathbb{R}$ by

$$\vartheta_0(\zeta)=\begin{cases}e^{-n\gamma\tau_0}\mathfrak{X}(\zeta e^{n\tau_0}) & \text{if } \zeta\in(e^{-(n+1)\tau_0},e^{-n\tau_0}], \quad n\in\mathbb{N}_0,\\ 0 & \text{if } \zeta=0.\end{cases} \tag{3.4}$$

Then for any $\zeta\in(e^{-(n+1)\tau_0},e^{-n\tau_0}]$, we compute

$$(\mathcal{S}_{\tau_0}\vartheta_0)(\zeta)=e^{\gamma\tau_0}\vartheta_0(\zeta e^{-\tau_0})=e^{\gamma\tau_0}e^{-(n+1)\gamma\tau_0}\mathfrak{X}(\zeta e^{-\tau_0}e^{(n+1)\tau_0})=e^{-n\gamma\tau_0}\mathfrak{X}(\zeta e^{n\tau_0})=\vartheta_0(\zeta),$$

hence $\mathcal{S}_{\tau_0}\vartheta_0=\vartheta_0$, i.e., ϑ_0 is a periodic point.

We now prove that $\vartheta_0\in\mathfrak{M}^{p,\lambda}(0,1)$. Let $\zeta_0\in(0,1)$, $r>0$, and define

$$I_{\zeta_0,r}:=r^{-\lambda}\int_{\mathcal{B}(\zeta_0,r)\cap(0,1)}|\vartheta_0(\zeta)|^p\,d\zeta.$$

We decompose the integral over dyadic intervals $I_n=(e^{-(n+1)t_0},e^{-n\tau_0}]$, and estimate each contribution using the change of variable $\zeta=\xi e^{-n\tau_0}$, so that $d\zeta=e^{-n\tau_0}d\xi$ and the integration domain transforms from $\mathcal{B}(\zeta,r)$ to $\mathcal{B}(\zeta e^{-\theta},re^{-\theta})$ Then:

$$\int_{I_n} |\vartheta_0(\zeta)|^p d\zeta = \int_{e^{-(n+1)\tau_0}}^{e^{-n\tau_0}} e^{-n\gamma p\tau_0} |\mathfrak{X}(\zeta e^{n\tau_0})|^p d\zeta$$
$$= e^{-n(\gamma p+1)\tau_0} \int_{e^{-\tau_0}}^{1} |\mathfrak{X}(\xi)|^p d\xi.$$

However, to estimate the full Morrey norm, we consider a ball of radius r, which after change of variable becomes a ball of radius $re^{n\tau_0}$. Hence:

$$\begin{aligned}
\|\vartheta_0\|^p_{\mathfrak{M}^{p,\lambda}(0,1)} &= \sup_{\zeta\in(0,1),\ r>0} r^{-\lambda} \sum_{n=0}^{\infty} \int_{\mathcal{B}(\zeta_0,r)\cap I_n} |\vartheta_0(\zeta)|^p d\zeta \\
&\leq \sup_{\zeta\in(0,1),\ r>0} r^{-\lambda} \sum_{n=0}^{\infty} \int_{e^{-(n+1)\tau_0}}^{e^{-n\tau_0}} e^{-n\gamma p\tau_0} |\mathfrak{X}(\zeta e^{n\tau_0})|^p d\zeta \\
&\leq \sup_{\zeta\in(0,1),\ r>0} \sum_{n=0}^{\infty} (re^{n\tau_0})^{-\lambda} e^{-n\tau_0(\gamma p+1-\lambda)} \int_{e^{-\tau_0}}^{1} |\mathfrak{X}(\xi)|^p d\xi \\
&\leq \sum_{n=0}^{\infty} e^{-n\tau_0(\gamma p+1-\lambda)} \sup_{\zeta_0\in(0,1),\ r>0} (re^{n\tau_0})^{-\lambda} \int_{e^{-\tau_0}}^{1} |\mathfrak{X}(\xi)|^p d\xi \\
&\leq \|\mathfrak{X}\|^p_{\mathfrak{M}^{p,\lambda}(0,e^{-\tau_0})} \sum_{n=0}^{\infty} e^{-n\tau_0(\gamma p+1-\lambda)}
\end{aligned}$$

This geometric series $\sum_{n=0}^{\infty} e^{-n\tau_0(\gamma p+1-\lambda)}$ converges if and only if:

$$\gamma p + 1 - \lambda > 0 \quad \Leftrightarrow \quad \gamma > \frac{\lambda - 1}{p}.$$

Hence $\|\vartheta_0\|_{\mathfrak{M}^{p,\lambda}(0,e^{-\tau})} < \infty$, and $\vartheta_0 \in \mathfrak{M}^{p,\lambda}(0,1)$, which completes the proof.
□

Theorem 3. *If $\gamma > \frac{\lambda-1}{p}$, then the set of periodic points of 3.1 is dense in the $\mathfrak{M}^{p,\lambda}(0,1)$ space.*

Proof. Let $\mathfrak{X}$ be an arbitrary function from the $\mathfrak{M}^{p,\lambda}(0,1)$ space and let $\varepsilon > 0$. Define ϑ by the formula 3.4. Fix τ_0 so large that $|\mathfrak{X}|_{\mathfrak{M}^{p,\lambda}[0,\mathrm{e}^{-\tau_0}]} < \frac{\varepsilon}{2}$ and $|\vartheta|_{\mathfrak{M}^{p,\lambda}[0,\mathrm{e}^{-\tau_0}]} < \frac{\varepsilon}{2}$. For $\zeta \in [\mathrm{e}^{-\tau_0}, 1]$ $\vartheta(\zeta) = \mathfrak{X}(\zeta)$ and by using Theorem 1 we finally have

$$|\vartheta - \mathfrak{X}|_{\mathfrak{M}^{p,\lambda}[0,1]} = |\vartheta - \mathfrak{X}|_{\mathfrak{M}^{p,\lambda}[0,\mathrm{e}^{-\tau_0}]} \leqslant |\vartheta|_{\mathfrak{M}^{p,\lambda}[0,\mathrm{e}^{-\tau_0}]} + |\mathfrak{X}|_{\mathfrak{M}^{p,\lambda}[0,\mathrm{e}^{-\tau_0}]} < \boldsymbol{\varepsilon}$$

This completes the proof.
□

Theorem 4. *If $\gamma > \frac{\lambda-1}{p}$, then the dynamical system $(\mathcal{S}_\tau)_{\tau\geqslant 0}$ is transitive in the $\mathfrak{M}^{p,\lambda}(0,1)$ space.*

Proof. Let

$$O(\vartheta_1, \varepsilon_1) = \left\{\kappa \in \mathfrak{M}^{p,\lambda}(0,1) : |\vartheta_1 - \kappa|_{\mathfrak{M}^{p,\lambda}[0,1]} < \varepsilon_1\right\}$$

and

$$O(\vartheta_2, \varepsilon_2) = \left\{\kappa \in \mathfrak{M}^{p,\lambda}(0,1) : |\vartheta_2 - \kappa|_{\mathfrak{M}^{p,\lambda}[0,1]} < \varepsilon_2\right\}$$

be two open balls with centres in $\vartheta_1, \vartheta_2 \in \mathfrak{M}^{p,\lambda}(0,1)$. Let us define the function

$$\mathfrak{X}(\zeta) = \left\{ \begin{array}{ccc} \mathrm{e}^{-\gamma\tau}\vartheta_2\left(\zeta \mathrm{e}^{\tau}\right) & \text{for} & \zeta < \mathrm{e}^{-\tau} \\ \vartheta_1(\zeta) & \text{for} & \zeta \geqslant \mathrm{e}^{-\tau} \end{array} \right.$$

at the suitable choice of τ. We should show that the above function $\mathfrak{X}$ belongs to the space $\mathfrak{M}^{p,\lambda}(0,1)$, and using the change of variable $\zeta = \xi e^{\tau}$

$$\int_0^{\mathrm{e}^{-\tau}} |\mathfrak{X}_0(\zeta)|^p d\zeta = \int_0^{\mathrm{e}^{-\tau}} \mathrm{e}^{-\gamma\tau}|\vartheta_2\left(\zeta \mathrm{e}^{\tau}\right)| d\zeta \quad = \mathrm{e}^{-\tau-\gamma\tau p} \int_0^1 |\vartheta_2(\xi)| \mathrm{d}\xi$$

and

$$\begin{aligned} |\mathfrak{X}(\zeta)|^p{}_{\mathfrak{M}^{p,\lambda}(0,e^{-\tau})} &= \int_0^1 \sup_{\zeta\in(0,1),\ r>0} \mathrm{e}^{-\tau-\gamma\tau p} r^{-\lambda} |\vartheta_2(\xi)|^p d\xi \\ &\leqslant \mathrm{e}^{-\tau(p\gamma+1-\lambda)} \int_0^1 (r\mathrm{e}^{\tau})^{-\lambda} |\vartheta_2(\xi)|^p d\xi \\ &\leqslant \mathrm{e}^{-\tau(p\gamma+1-\lambda)} |\vartheta_2(\xi)|_{\mathfrak{M}^{p,\lambda}(0,1)} \end{aligned}$$

Then

$$\begin{aligned} |\mathfrak{X}|_{\mathfrak{M}^{p,\lambda}(0,1)} &\leqslant |\mathfrak{X}(\xi)|_{\mathfrak{M}^{p,\lambda}(0,e^{-\tau})} + |\mathfrak{X}(\xi)|_{\mathfrak{M}^{p,\lambda}(e^{-\tau},1)} \\ &\leqslant \mathrm{e}^{-\tau(p\gamma+1-\lambda)} |\vartheta_2(\xi)|_{\mathfrak{M}^{p,\lambda}(0,1)} + |\vartheta_1(\xi)|_{\mathfrak{M}^{p,\lambda}(0,1)} \end{aligned}$$

From the assumption we have $\mathrm{e}^{-\tau(p\gamma+1-\lambda)} < 1$ and we have $\vartheta_1, \vartheta_2 \in \mathfrak{M}^{p,\lambda}(0,1)$ then $|\mathfrak{X}|_{\mathfrak{M}^{p,\lambda}(0,1)} < \infty$ then we obtain $\mathfrak{X} \in \mathfrak{M}^{p,\lambda}(0,1)$. By using Theorem 1 we also get

$$\begin{aligned} |\vartheta_1 - \mathfrak{X}|_{\mathfrak{M}^{p,\lambda}[0,1]} = |\vartheta_1 - \mathfrak{X}|_{\mathfrak{M}^{p,\lambda}[0,\mathrm{e}^{-\tau}]} &\leqslant |\vartheta_1|_{\mathfrak{M}^{p,\lambda}\left[0,\mathrm{e}^{-\theta}\right]} + |\mathfrak{X}|_{\mathfrak{M}^{p,\lambda}[0,\mathrm{e}^{-\tau}]} \\ &\leqslant |\vartheta_1|_{\mathfrak{M}^{p,\lambda}[0,\mathrm{e}^{-\tau}]} + \mathrm{e}^{-\tau(\gamma p+1)} |\vartheta_2|_{\mathfrak{M}^{p,\lambda}[0,1]} \end{aligned}$$

By choosing τ large enough we obtain $|\vartheta_1 - \mathfrak{X}|_{\mathfrak{M}^{p,\lambda}[0,1]} < \varepsilon_1$, this implies that $\mathfrak{X} \in \mathcal{O}(\vartheta_1, \varepsilon_1)$. Therefore $\mathcal{S}_\tau \mathfrak{X} \in \mathcal{S}_\tau\left(\mathcal{O}\left(\vartheta_1, \varepsilon_1\right)\right)$ and $\vartheta_2 = \mathcal{S}_\tau \mathfrak{X} \in \mathcal{O}\left(\vartheta_2, \varepsilon_2\right)$. Therefore

$$\mathcal{S}_\tau\left(\mathcal{O}\left(\vartheta_1, \varepsilon_1\right)\right) \cap \mathcal{O}\left(\vartheta_2, \varepsilon_2\right) \neq \emptyset.$$

So we conclude that the dynamical system $(\mathcal{S}_\tau)_{\tau\geqslant 0}$ is transitive in the space $\mathfrak{M}^{p,\lambda}(0,1)$. □

Corollary 1. *If $\gamma > \frac{\lambda-1}{p}$, then the dynamical system $(\mathcal{S}_\tau)_{\tau\geqslant 0}$ is chaotic in the sense of Devaney in the $\mathfrak{M}^{p,\lambda}(0,1)$ space.*

Now let us consider an example which confirms that the present corollary 1 generalizes the result obtained in the Lebesgue space $L^p(0,1)$, as discussed in [2, 7].

Example 1. *In the particular case $\lambda = 0$, the Morrey space $\mathfrak{M}^{p,0}(0,1)$=$L^p(0,1)$. Then, $\gamma > -\frac{1}{p}$ is the condition of chaotic in the sense of Devaney in the L^p space.*

Theorem 5. *If $\gamma \leqslant \frac{\lambda-1}{p}$, then the semigroup $(\mathcal{S}_\tau)_{\tau\geqslant 0}$ is strongly stable in the $\mathfrak{M}^{p,\lambda}(0,1)$ space.*

Proof. Let $\vartheta \in \mathfrak{M}_{p,\lambda}(0,1)$ and $(\mathcal{S}_\tau)_{\tau\geqslant 0}$ given in equation 3.3 we have:

$$\|\mathcal{S}_\tau \vartheta\|^p_{\mathfrak{M}^{p,\lambda}} = \sup_{\zeta\in(0,1),\ r>0} r^{-\lambda} \int_{\mathcal{B}(\zeta,r)} |\mathcal{T}_\tau \vartheta(\zeta)|^p \, d\zeta.$$

Then we obtain:

$$\|\mathcal{S}_\tau \vartheta\|^p_{\mathfrak{M}^{p,\lambda}} = e^{p\gamma\tau} \sup_{\zeta\in(0,1),\ r>0} r^{-\lambda} \int_{\mathcal{B}(\zeta,r)} |\vartheta(\zeta e^{-\tau})|^p d\zeta.$$

By changing of variable $\xi = \zeta e^{-\tau}$, so that $d\zeta = e^{\tau} d\xi$, it easy to show that

$$\|\mathcal{S}_\tau v\|^p_{\mathfrak{M}^{p,\lambda}} = e^{(p\gamma+1)\tau} \sup_{\zeta,r} r^{-\lambda} \int_{\mathcal{B}(\zeta e^{-\tau}, re^{-\tau})} |\vartheta(\xi)|^p d\xi.$$

We get:

$$\|\mathcal{S}_\tau \vartheta\|^p_{\mathfrak{M}^{p,\lambda}} = e^{-\tau(\lambda - p\gamma - 1)} \sup_{\zeta,r} (re^{-\tau})^{-\lambda} \int_{\mathcal{B}(\zeta e^{-\tau}, re^{-\tau})} |\vartheta(\xi)|^p d\xi.$$

Therefore:

$$\|\mathcal{S}_\tau \vartheta\|^p_{\mathfrak{M}^{p,\lambda}} = e^{-\tau(\lambda - p\gamma - 1)} \|\vartheta\|^p_{\mathfrak{M}^{p,\lambda}}.$$

Finally we get:

$$\|\mathcal{S}_\tau \vartheta\|_{\mathfrak{M}^{p,\lambda}} \leq e^{\tau\left(\gamma + \frac{1-\lambda}{p}\right)} \|\vartheta\|_{\mathfrak{M}^{p,\lambda}}.$$

Due to $e^{\tau\left(\gamma + \frac{1-\lambda}{p}\right)} \leq 1$ on has stability in the $\mathfrak{M}^{p,\lambda}(0,1)$ space if $\gamma \leqslant \frac{\lambda - 1}{p}$. □

An example is now provided to demonstrate that the present Theorem 5 generalizes the corresponding result in the Lebesgue space L^p, as detailed in [2, 7].

Example 2. *In the particular case* $\lambda = 0$, *the Morrey space* $\mathfrak{M}^{p,0}(0,1) = L^p(0,1)$. *Then,* $\gamma \leqslant -\frac{1}{p}$ *is the condition of stability in the* L^p *space.*

4. Dynamical System Generated by the von Foerster–Lasota Equation

We will now consider a more general form of the equation.

$$\frac{\partial \mathfrak{X}}{\partial \tau} + \zeta \frac{\partial \mathfrak{X}}{\partial \zeta} = \rho(\zeta)\mathfrak{X}, \tau \geqslant 0, 0 \leqslant \zeta \leqslant 1 \tag{4.1}$$

with the initial condition

$$\mathfrak{X}(0,\zeta) = \vartheta(\zeta), \quad 0 \leqslant \zeta \leqslant 1, \tag{4.2}$$

where ϑ belongs to some normed vector space $\mathbb{V}$ of functions defined on $[0,1]$ and $\rho : [0,1] \to \mathbb{R}$ is a given continuous function. Let a semidynamical system $\widetilde{\mathcal{S}}_\tau$ be represented by

$$\left(\widetilde{\mathcal{S}}_\zeta \vartheta\right)(\zeta) = \widetilde{\mathfrak{X}}(\tau,\zeta)$$

where $\widetilde{\mathfrak{X}}(\tau,\zeta)$ is the unique solution of 4.1, 4.2 and is given by the formula

$$\left(\widetilde{\mathcal{S}}_\zeta \vartheta\right)(\zeta) = \widetilde{\mathfrak{X}}(t,\zeta) = \mathrm{e}^{h(\zeta)} \mathrm{e}^{-h(\zeta \mathrm{e}^{-\tau})} \vartheta\left(\zeta \mathrm{e}^{-\tau}\right)$$

for $\zeta \in [0,1]$, where

$$h(\zeta) = -\int_\zeta^1 \frac{\rho(\tau)}{\tau} d\tau$$

Our objective is to identify a relationship between the two equations: 3.1, presented in Section 3, and 4.1. One can easily verify that if $\mathfrak{X}$ and $\widetilde{\mathfrak{X}}$ are the solutions of equations 3.1 and 4.1, respectively, We obtain the equality

$$\mathfrak{X}(\tau,\zeta) = \kappa(\zeta)\widetilde{\mathfrak{X}}(\tau,\zeta) \tag{4.3}$$

where

$$\kappa(\zeta) = \exp\left(\int_0^{\zeta} \frac{\gamma - \rho(\tau)}{\tau} d\tau\right) \text{ and } \gamma = \rho(0) \tag{4.4}$$

We suppose that the integral given above is convergent

Remark 1. *It is worth noting that the concept of topological equivalence plays a crucial role in preserving several fundamental dynamical properties of systems. In particular, when two dynamical systems are topologically equivalent, they share key qualitative behaviors such as stability, the density of the set of periodic points, the existence of fixed points of the governing equation, and the presence of periodic orbits. Furthermore, many other related dynamical characteristics remain invariant under such an equivalence, emphasizing its importance in the study of dynamical systems.*

Dawidowicz in [5] show the dynamical systems 3.1 and 4.1 can be illustrated by the following diagram:

$$\begin{tikzcd}
\mathbb{V} \arrow[r, "\mathcal{S}_\tau"] \arrow[d, "\mathcal{L}_\kappa"'] & \mathbb{V} \arrow[d, "\mathcal{L}_\kappa"] \\
\mathbb{V} \arrow[r, "\widetilde{\mathcal{S}}_\tau"'] & \mathbb{V}
\end{tikzcd}$$

It is also shown in [5] that this diagram is commutative and for $(\mathcal{L}_\kappa \vartheta)(\zeta) = \frac{1}{\kappa(\zeta)}\vartheta(\zeta)$, we get that

$$\left(\widetilde{\mathcal{S}}_\tau\left(\mathcal{L}_\kappa \vartheta\right)\right)(\zeta) = \left(\mathcal{L}_\kappa\left(\mathcal{S}_\tau \vartheta\right)\right)(\zeta)$$

Theorem 6. *Assume that*

$$\exists \mathbb{M} > 0 \quad \exists q > 0 \quad \forall \zeta \in [0,1] \quad |\rho(\zeta) - \gamma| \leqslant \mathbb{M}\zeta^q \tag{4.5}$$

holds with $\rho(0) = \gamma$. *Then, the following equivalence holds:*

$$\tilde{\mathfrak{X}}(\tau, \cdot) \in \mathfrak{M}^{p,\lambda}(0,1) \iff \mathfrak{X}(\tau, \cdot) \in \mathfrak{M}^{p,\lambda}(0,1).$$

Proof. Assume $\tilde{\mathfrak{X}}(\tau, \cdot) \in \mathfrak{M}^{p,\lambda}(0,1)$. Let $\zeta_0 \in [0,1]$ and $r \in (0,1]$. On $\mathcal{B}(\zeta_0, r) \cap [0,1]$ we have:

$$|\mathfrak{X}(\tau,\zeta)|^p = |\kappa(\zeta)\tilde{\mathfrak{X}}(\tau,\zeta)|^p \leq \kappa(\zeta)^p |\tilde{\mathfrak{X}}(\tau,\zeta)|^p.$$

From hypothesis 4.5 of Theorem 6 , we get

$$|\log \kappa(\zeta)| = \left|\int_0^{\zeta} \frac{\gamma - \rho(\tau)}{\tau} d\tau\right| \leq \int_0^{\zeta} \frac{|\rho(\tau) - \gamma|}{\tau} d\tau \leq \mathbb{M} \int_0^{\zeta} \tau^{q-1} d\tau = \frac{\mathbb{M}}{q}\zeta^q.$$

Thus,

$$\kappa(\zeta) \leq e^{\mathbb{M}\zeta^q/q} \leq e^{\mathbb{M}r^q/q} \text{ for } \zeta \in \mathcal{B}(\zeta_0, r)$$

Then,

$$\int_{\mathcal{B}(\zeta_0,r)} |\mathfrak{X}(\tau,\zeta)|^p d\zeta \leq e^{\mathbb{M}pr^q/q} \int_{\mathcal{B}(\zeta_0,r)} |\tilde{\mathfrak{X}}(\tau,\zeta)|^p d\zeta$$

This implies:

$$r^{\frac{-\lambda}{p}} \left(\int_{\mathcal{B}(\zeta_0,r)} |\mathfrak{X}(\tau,\zeta)|^p d\zeta\right)^{1/p} \leq e^{\mathbb{M}r^q/q} r^{\frac{-\lambda}{p}} \left(\int_{\mathcal{B}(\zeta_0,r)} |\tilde{\mathfrak{X}}(\zeta,\zeta)|^p d\zeta\right)^{1/p}.$$

Since $\tilde{\mathfrak{X}} \in \mathfrak{M}^{p,\lambda}(0,1)$, we conclude:

$$\|\mathfrak{X}(\tau,\cdot)\|_{\mathfrak{M}^{p,\lambda}} \leq e^{\mathbb{M}/q} \|\tilde{\mathfrak{X}}(\tau,\cdot)\|_{\mathfrak{M}^{p,\lambda}}.$$

Hence $\mathfrak{X} \in \mathfrak{M}^{p,\lambda}(0,1)$. The reverse implication follows similarly by applying the same argument to $\kappa^{-1}(\zeta)$. □

As a consequence of the diagram's commutativity, the two dynamical systems are topologically conjugate. $(\mathcal{S}_\tau)_{\tau\geq 0}$ and $(\widetilde{\mathcal{S}}_\tau)_{\tau\geq 0}$ are topologically conjugate. Therefore, the properties of the system $(\mathcal{S}_\tau)_{\tau\geq 0}$ introduced in Section 3 (the existence and density of periodic orbits, topological transitivity, and the stability property) transfer on the system $(\widetilde{\mathcal{S}}_\tau)_{\tau\geq 0}$.

All these properties depend on the value $\gamma = \rho(0)$. Theorem 6 shows that under suitable regularity assumptions on $\rho(\zeta)$ the solutions of equations 3.1 and 4.1 remain within the same Morrey space $\mathfrak{M}^{p,\lambda}(0,1)$.

This further implies that the property of sensitivity to initial conditions, which reflects a metric aspect of the dynamics, is also preserved in $(\widetilde{\mathcal{S}}_\tau)_{\tau\geq 0}$. Therefore, under suitable assumptions, if the semigroup $(\mathcal{S}_\tau)_{\tau\geq 0}$ exhibits chaotic behavior, stability, or possesses a dense set of periodic points, then these properties are preserved by the system $(\widetilde{\mathcal{S}}_\tau)_{\tau\geq 0}$.

Consequently, all the dynamical properties of the semigroup $(\widetilde{\mathcal{S}}_\tau)_{\tau\geq 0}$ depend on the value $\rho(0)$. Specifically, when $\rho(0) > \frac{\lambda-1}{p}$, the system admits periodic solutions for every $\tau_0 > 0$, and the set of such solutions is dense in $\mathfrak{M}^{p,\lambda}(0,1)$. Conversely, if $\rho(0) \leq \frac{\lambda-1}{p}$, then the system is strongly stable.

5. Dynamical Properties of the Generalized Lasota Equation in Morrey Spaces

In this section, we provide a detailed analysis of the dynamical properties strong stability, exponential stability, periodic points, topological transitivity, and Devaney chaos – of the semidynamical system generated by the generalized Lasota equation in the Morrey space $\mathfrak{M}^{p,\lambda}(0,1)$. We present complete proofs step by step.

5.1. Explicit Solution and Semigroup Formulation

We consider the generalized Lasota-type transport equation:

$$\frac{\partial \mathfrak{X}}{\partial \tau} + (\mathfrak{a}\zeta + \mathfrak{b}\zeta^2)\frac{\partial \mathfrak{X}}{\partial \zeta} = \sigma(\zeta)\mathfrak{X}, \quad \tau \geq 0,\ 0 \leq \zeta \leq 1, \tag{5.1}$$

with initial condition:

$$\mathfrak{X}(0,\zeta) = \mathfrak{v}(\zeta), \quad \mathfrak{v} \in \mathfrak{M}^{p,\lambda}(0,1), \tag{5.2}$$

where $\mathfrak{a} > 0$, $\mathfrak{b} \geq 0$, $\sigma : [0,1] \to \mathbb{R}$ is continuous, and $\mathfrak{M}^{p,\lambda}(0,1)$ denotes the Morrey space with parameters $1 \leq p < \infty$, $0 < \lambda < 1$.

Using the method of characteristics, we obtain the explicit solution of (5.1):

$$\mathfrak{X}(\tau,\zeta) = e^{g(\zeta)} e^{-g\left(\frac{\mathfrak{a}\zeta e^{-\mathfrak{a}\tau}}{\mathfrak{a}+\mathfrak{b}\zeta-\mathfrak{b}\zeta e^{-\mathfrak{a}\tau}}\right)} \mathfrak{v}\left(\frac{\mathfrak{a}\zeta e^{-\mathfrak{a}\tau}}{\mathfrak{a}+\mathfrak{b}\zeta-\mathfrak{b}\zeta e^{-\mathfrak{a}\tau}}\right),$$

where:

$$g(\zeta) = -\int_\zeta^1 \frac{\sigma(s)}{\mathfrak{a}s + \mathfrak{b}s^2}\, ds,$$

with the natural condition $\int_0^1 \frac{\sigma(s)}{\mathfrak{a}s+\mathfrak{b}s^2}\, ds = \infty$ ensuring regularity at $\zeta = 0$. The corresponding solution semigroup $\mathcal{S}_\tau : \mathfrak{M}^{p,\lambda}(0,1) \to \mathfrak{M}^{p,\lambda}(0,1)$ is therefore:

$$(\mathcal{S}_\tau \mathfrak{v})(\zeta) = e^{g(\zeta)} e^{-g\left(\frac{\mathfrak{a}\zeta e^{-\mathfrak{a}\tau}}{\mathfrak{a}+\mathfrak{b}\zeta-\mathfrak{b}\zeta e^{-\mathfrak{a}\tau}}\right)} \mathfrak{v}\left(\frac{\mathfrak{a}\zeta e^{-\mathfrak{a}\tau}}{\mathfrak{a}+\mathfrak{b}\zeta-\mathfrak{b}\zeta e^{-\mathfrak{a}\tau}}\right). \tag{5.3}$$

In the particular case where $\sigma(\zeta)$ is constant, i.e. $\sigma(\zeta) \equiv \gamma \in \mathbb{R}$, the equation (5.1) becomes:

$$\frac{\partial \mathfrak{X}}{\partial \tau} + (\mathfrak{a}\zeta + \mathfrak{b}\zeta^2)\frac{\partial \mathfrak{X}}{\partial \zeta} = \gamma\, \mathfrak{X}, \qquad \tau \geq 0,\ 0 \leq \zeta \leq 1. \tag{5.4}$$

The solution obtained by the method of characteristics is

$$\bar{\mathfrak{X}}(\tau,\zeta) = e^{\gamma\tau}\mathfrak{v}\left(\frac{\mathfrak{a}\zeta e^{-\mathfrak{a}\tau}}{\mathfrak{a}+\mathfrak{b}\zeta-\mathfrak{b}\zeta e^{-\mathfrak{a}\tau}}\right),$$

where $\mathfrak{v}(\zeta) = \mathfrak{X}(0,\zeta)$ is the initial condition. This defines the semigroup $(\bar{\mathcal{S}}_\tau)_{\tau\geq 0}$ on $\mathfrak{M}^{p,\lambda}(0,1)$ by:

$$(\bar{\mathcal{S}}_\tau\mathfrak{v})(\zeta) = e^{\gamma\tau}\mathfrak{v}\left(\frac{\mathfrak{a}\zeta e^{-\mathfrak{a}\tau}}{\mathfrak{a}+\mathfrak{b}\zeta-\mathfrak{b}\zeta e^{-\mathfrak{a}\tau}}\right), \qquad \tau\geq 0,\ \zeta\in(0,1].$$

We analyze the stability properties of the Lasota equation within Morrey spaces. Our objective is to establish precise conditions under which the solution semigroup exhibits strong or exponential stability in the space $\mathfrak{M}^{p,\lambda}(0,1)$,.

5.2. Stability Results in Morrey Space

Theorem 7 (Strong and exponential stability in $\mathfrak{M}^{p,\lambda}$). *The semigroup $(\bar{\mathcal{S}}_\tau)_{\tau\geq 0}$ is:*

1. *Exponentially stable in $\mathfrak{M}^{p,\lambda}(0,1)$ if*
$$\gamma < -\frac{\mathfrak{a}(1-\lambda)}{p}.$$
2. *Strongly stable in $\mathfrak{M}^{p,\lambda}(0,1)$ if*
$$\gamma = -\frac{\mathfrak{a}(1-\lambda)}{p}.$$

Proof. Let $\mathfrak{v}\in\mathfrak{M}^{p,\lambda}(0,1)$. We estimate the Morrey norm of $\bar{\mathcal{S}}_\tau$. For any $\zeta_0\in(0,1]$ and $r>0$:

$$\|\bar{\mathcal{S}}_\tau\mathfrak{v}\|^p_{\mathfrak{M}^{p,\lambda}} = \sup_{\zeta_0,r} r^{-\lambda}\int_{\mathcal{B}(\zeta_0,r)} e^{\gamma p\tau}\left|\mathfrak{v}\left(\frac{\mathfrak{a}\zeta e^{-\mathfrak{a}\tau}}{\mathfrak{a}+\mathfrak{b}\zeta-\mathfrak{b}\zeta e^{-\mathfrak{a}\tau}}\right)\right|^p d\zeta.$$

Perform the change of variable $\xi = \frac{\mathfrak{a}\zeta e^{-\mathfrak{a}\tau}}{\mathfrak{a}+\mathfrak{b}\zeta-\mathfrak{b}\zeta e^{-\mathfrak{a}\tau}}$ Then:

$$\frac{d\zeta}{d\xi} = \frac{(\mathfrak{a}+\mathfrak{b}\zeta-\mathfrak{b}\zeta e^{-\mathfrak{a}\tau})^2}{\mathfrak{a}^2 e^{-\mathfrak{a}\tau}}.$$

Since $\mathfrak{a}\leq\mathfrak{a}+\mathfrak{b}\zeta-\mathfrak{b}\zeta e^{-\mathfrak{a}\tau}\leq\mathfrak{a}+\mathfrak{b}$, we have:

$$\frac{\mathfrak{a}^2}{(\mathfrak{a}+\mathfrak{b})^2}e^{\mathfrak{a}\tau} \leq \frac{d\zeta}{d\xi} \leq \frac{(\mathfrak{a}+\mathfrak{b})^2}{\mathfrak{a}^2}e^{\mathfrak{a}\tau}.$$

Thus:

$$\int_{\mathcal{B}(\zeta_0,r)} |\mathfrak{v}(\xi)|^p\, d\zeta \leq \frac{(\mathfrak{a}+\mathfrak{b})^2}{\mathfrak{a}^2}e^{\mathfrak{a}\tau}\int_{\mathcal{B}\left(\xi_0,\frac{\mathfrak{a}+\mathfrak{b}}{\mathfrak{a}}re^{-\mathfrak{a}\tau}\right)} |\mathfrak{v}(\xi)|^p d\xi.$$

Therefore:

$$\begin{aligned}
r^{-\lambda}\int_{\mathcal{B}(\zeta_0,r)} |\bar{\mathcal{S}}_\tau\mathfrak{v}|^p d\zeta &\leq \frac{(\mathfrak{a}+\mathfrak{b})^2}{\mathfrak{a}^2}e^{(\gamma p+\mathfrak{a})\tau}r^{-\lambda}\int_{\mathcal{B}\left(\xi_0,\frac{\mathfrak{a}+\mathfrak{b}}{\mathfrak{a}}re^{-\mathfrak{a}\tau}\right)} |\mathfrak{v}(\xi)|^p d\xi \\
&= \frac{(\mathfrak{a}+\mathfrak{b})^2}{\mathfrak{a}^2}\left(\frac{\mathfrak{a}+\mathfrak{b}}{\mathfrak{a}}\right)^{\lambda} e^{(\gamma p+\mathfrak{a}-\lambda\mathfrak{a})\tau}\left(\frac{\mathfrak{a}+\mathfrak{b}}{\mathfrak{a}}re^{-\mathfrak{a}\tau}\right)^{-\lambda}\int_{\mathcal{B}\left(\xi_0,\frac{\mathfrak{a}+\mathfrak{b}}{\mathfrak{a}}re^{-\mathfrak{a}\tau}\right)} |\mathfrak{v}(\xi)|^p d\xi.
\end{aligned}$$

Taking the supremum over ζ_0 and r yields:

$$\|\bar{\mathcal{S}}_\tau\mathfrak{v}\|^p_{\mathfrak{M}^{p,\lambda}} \leq \mathcal{D}_{\mathfrak{a},\mathfrak{b},\lambda}e^{(\gamma p+\mathfrak{a}(1-\lambda))\tau}\|\mathfrak{v}\|^p_{\mathfrak{M}^{p,\lambda}},$$

where $\mathcal{D}_{\mathfrak{a},\mathfrak{b},\lambda} = \frac{(\mathfrak{a}+\mathfrak{b})^2}{\mathfrak{a}^2}\left(\frac{\mathfrak{a}+\mathfrak{b}}{\mathfrak{a}}\right)^{\lambda}$. Hence:

$$\|\bar{\mathcal{S}}_{\tau}\mathfrak{v}\|_{\mathfrak{M}^{p,\lambda}} \leq \mathcal{D}^{1/p}_{\mathfrak{a},\mathfrak{b},\lambda} e^{\left(\gamma+\frac{\mathfrak{a}(1-\lambda)}{p}\right)\tau}\|\mathfrak{v}\|_{\mathfrak{M}^{p,\lambda}}.$$

If $\gamma + \frac{\mathfrak{a}(1-\lambda)}{p} < 0$, we have exponential decay. If $\gamma + \frac{\mathfrak{a}(1-\lambda)}{p} = 0$, we obtain strong stability by showing $\|\bar{\mathcal{S}}_{\tau}\mathfrak{v}\|_{\mathfrak{M}^{p,\lambda}} \to 0$ as $\tau \to \infty$. □

Theorem 8 (Existence of periodic points in $\mathfrak{M}^{p,\lambda}$). *If* $\gamma > \dfrac{\lambda-1}{p}$, *then for every* $\tau_0 > 0$ *there exists a nonzero* $\mathfrak{v}_0 \in \mathfrak{M}^{p,\lambda}(0,1)$ *such that*

$$\bar{\mathcal{S}}_{\tau_0}\mathfrak{v}_0 = \mathfrak{v}_0.$$

Moreover,

$$\bar{\mathcal{S}}_{\tau}\mathfrak{v}_0 = \mathfrak{v}_0 \quad \textit{if and only if} \quad \tau = n\tau_0 \, \textit{for some integer } n \geq 1.$$

Proof. Let $\mathfrak{w} : [0,1] \to \mathbb{R}$ be a continuous function satisfying

$$|\mathfrak{w}(\zeta)| \leq \mathfrak{C}\left(\frac{\zeta}{1+\zeta}\right)^{\gamma}, \qquad \mathfrak{C} > 0,$$

and the compatibility conditions

$$e^{\gamma\tau_0}\,\mathfrak{w}\left(\frac{\mathfrak{a}e^{-\mathfrak{a}\tau_0}}{\mathfrak{a}+\mathfrak{b}-\mathfrak{b}e^{-a\tau_0}}\right) = \mathfrak{w}(1),$$

$$e^{\gamma\tau}\,\mathfrak{w}\left(\frac{\mathfrak{a}e^{-\mathfrak{a}\tau}}{\mathfrak{a}+\mathfrak{b}-\mathfrak{b}e^{-\mathfrak{a}\tau}}\right) \neq \mathfrak{w}(1), \qquad \forall\tau \in (0,\tau_0).$$

Step 1. $\mathfrak{w}$ belongs to $\mathfrak{M}^{p,\lambda}(0,1)$
Since $|\mathfrak{w}(\zeta)| \leq \mathfrak{C}\,\zeta^{\gamma}$, we estimate for any ball $\mathcal{B}(\zeta_0, r) \subset (0,1]$:

$$r^{-\lambda}\int_{\mathcal{B}(\zeta_0,r)} |\mathfrak{w}(\zeta)|^p d\zeta \leq \mathfrak{C}^p r^{-\lambda}\int_{\mathcal{B}(\zeta_0,r)} \zeta^{\gamma p} d\zeta.$$

The right-hand side is finite if and only if $\gamma p > \lambda - 1$, i.e.

$$\gamma > \frac{\lambda-1}{p}.$$

Hence $\mathfrak{w} \in \mathfrak{M}^{p,\lambda}(0,1)$ under the stated condition.
Step 2. Define the family of disjoint subintervals of $(0,1]$:

$$J_n = \left[\frac{\mathfrak{a}e^{-(n+1)\mathfrak{a}\tau_0}}{\mathfrak{a}+\mathfrak{b}-\mathfrak{b}e^{-(n+1)\mathfrak{a}\tau_0}}, \frac{\mathfrak{a}e^{-n\mathfrak{a}\tau_0}}{\mathfrak{a}+\mathfrak{b}-\mathfrak{b}e^{-n\mathfrak{a}\tau_0}}\right], \qquad n = 0,1,2,\ldots$$

and define $\mathfrak{v}_0$ piecewise on these intervals by

$$\mathfrak{v}_0(\zeta) = e^{-n\gamma\tau_0}\mathfrak{w}\left(\frac{\mathfrak{a}\zeta e^{n\mathfrak{a}\tau_0}}{\mathfrak{a}+\mathfrak{b}\zeta-\mathfrak{b}\zeta e^{n\mathfrak{a}\tau_0}}\right), \qquad \zeta \in J_n.$$

Since the intervals J_n form a partition of $(0,1]$, the function $\mathfrak{v}_0$ is continuous and well-defined.
Step 3. $\mathfrak{v}_0 \in \mathfrak{M}^{p,\lambda}(0,1)$

For $\zeta \in J_n$, using the bound on $\mathfrak{w}$ we have

$$|\mathfrak{v}_0(\zeta)| \leq \mathfrak{C}\,e^{-n\gamma\tau_0}\left(\frac{\frac{\mathfrak{a}\zeta e^{n\mathfrak{a}\tau_0}}{\mathfrak{a}+\mathfrak{b}\zeta-\mathfrak{b}\zeta e^{n\mathfrak{a}\tau_0}}}{1+\frac{\mathfrak{a}\zeta e^{n\mathfrak{a}\tau_0}}{\mathfrak{a}+\mathfrak{b}\zeta-\mathfrak{b}\zeta e^{n\mathfrak{a}\tau_0}}}\right)^{\gamma} \leq \mathfrak{C}\,\zeta^{\gamma}.$$

Thus, $\mathfrak{v}_0$ satisfies the same growth estimate as $\mathfrak{w}$, implying $\mathfrak{v}_0 \in \mathfrak{M}^{p,\lambda}(0,1)$.
Step 4. Verification of the periodic condition
The semigroup action is given by

$$(\bar{\mathcal{S}}_\tau \mathfrak{v})(\zeta) = e^{\gamma\tau}\mathfrak{v}\left(\frac{\mathfrak{a}\zeta e^{-\mathfrak{a}\tau}}{\mathfrak{a}+\mathfrak{b}\zeta-\mathfrak{b}\zeta e^{-\mathfrak{a}\tau}}\right).$$

If $\zeta \in J_n$, then the argument of $\mathfrak{v}$ lies in J_{n-1}. Hence,

$$\begin{aligned}
(\bar{\mathcal{S}}_{\tau_0}\mathfrak{v}_0)(\zeta) &= e^{\gamma\tau_0}\mathfrak{v}_0\left(\frac{\mathfrak{a}\zeta e^{-\mathfrak{a}\tau_0}}{\mathfrak{a}+\mathfrak{b}\zeta-\mathfrak{b}\zeta e^{-\mathfrak{a}\tau_0}}\right)\\
&= e^{\gamma\tau_0}e^{-(n-1)\gamma\tau_0}\mathfrak{w}\left(\frac{\mathfrak{a}\zeta e^{(n-1)\mathfrak{a}\tau_0}}{\mathfrak{a}+\mathfrak{b}\zeta-\mathfrak{b}\zeta e^{(n-1)\mathfrak{a}\tau_0}}\right)\\
&= e^{-n\gamma\tau_0}\mathfrak{w}\left(\frac{\mathfrak{a}\zeta e^{n\mathfrak{a}\tau_0}}{\mathfrak{a}+\mathfrak{b}\zeta-\mathfrak{b}\zeta e^{n\mathfrak{a}\tau_0}}\right) = \mathfrak{v}_0(\zeta),
\end{aligned}$$

showing that $\bar{\mathcal{S}}_{\tau_0}\mathfrak{v}_0 = \mathfrak{v}_0$.

Hence, $\mathfrak{v}_0$ is a nontrivial periodic point of $\bar{\mathcal{S}}_\tau$ in $\mathfrak{M}^{p,\lambda}(0,1)$ with minimal period τ_0. □

Theorem 9 (Density of periodic points in $\mathfrak{M}^{p,\lambda}(0,1)$). *If* $\gamma > -\dfrac{\mathfrak{a}(1-\lambda)}{p}$*, then the set of all periodic points of the semigroup* $(\bar{\mathcal{S}}_\tau)_{\tau\geq 0}$ *associated with* (5.4) *is dense in the Morrey space* $\mathfrak{M}^{p,\lambda}(0,1)$*.*

Proof. Let $\mathfrak{w} \in \mathfrak{M}^{p,\lambda}(0,1)$ be arbitrary and $\varepsilon > 0$. We construct a periodic point $\mathfrak{v} \in \mathfrak{M}^{p,\lambda}(0,1)$ such that $\|\mathfrak{v}-\mathfrak{w}\|_{\mathfrak{M}^{p,\lambda}} < \varepsilon$.
Step 1.Truncation and approximation in Morrey norm. Define for $\mathfrak{C} > 0$ the truncated function:

$$\mathfrak{w}_{\mathfrak{C}}(\zeta) = \begin{cases} \mathfrak{w}(\zeta), & \text{if } |\mathfrak{w}(\zeta)| \leq \mathfrak{C}\left(\dfrac{\zeta}{1+\zeta}\right)^{\gamma},\\ \mathfrak{C}\left(\dfrac{\zeta}{1+\zeta}\right)^{\gamma}\dfrac{\mathfrak{w}(\zeta)}{|\mathfrak{w}(\zeta)|}, & \text{if } |\mathfrak{w}(\zeta)| > \mathfrak{C}\left(\dfrac{\zeta}{1+\zeta}\right)^{\gamma}. \end{cases}$$

Then $|\mathfrak{w}_{\mathfrak{C}}(\zeta)| \leq \mathfrak{C}\left(\dfrac{\zeta}{1+\zeta}\right)^{\gamma} \leq \mathfrak{C}\zeta^{\gamma}$. We verify that $\mathfrak{w}_{\mathfrak{C}} \in \mathfrak{M}^{p,\lambda}(0,1)$. For any ball $\mathcal{B}(\zeta_0, r) \subset (0,1]$,

$$\begin{aligned}
r^{-\lambda}\int_{\mathcal{B}(\zeta_0,r)} |\mathfrak{w}_{\mathfrak{C}}(\zeta)|^p\,d\zeta &\leq \mathfrak{C}^p r^{-\lambda}\int_{\mathcal{B}(\zeta_0,r)} \zeta^{\gamma p}\,d\zeta\\
&\leq \mathfrak{C}^p r^{-\lambda}\cdot\frac{1}{\gamma p+1}(\zeta_0+r)^{\gamma p+1}.
\end{aligned}$$

Since $\gamma > -\frac{\mathfrak{a}(1-\lambda)}{p}$, Hence the supremum over ζ_0 and r is finite, so $\mathfrak{w}_{\mathfrak{C}} \in \mathfrak{M}^{p,\lambda}(0,1)$. Now consider the set

$$E_{\mathfrak{C}} = \left\{\zeta \in [0,1] : |\mathfrak{w}(\zeta)| > \mathfrak{C}\left(\tfrac{\zeta}{1+\zeta}\right)^{\gamma}\right\}.$$

Then

$$\begin{aligned}
\|\mathfrak{w}_{\mathfrak{C}} - \mathfrak{w}\|^p_{\mathfrak{M}^{p,\lambda}} &= \sup_{\zeta_0,r} r^{-\lambda}\int_{\mathcal{B}(\zeta_0,r)} |\mathfrak{w}_{\mathfrak{C}}(\zeta)-\mathfrak{w}(\zeta)|^p\,d\zeta\\
&\leq \sup_{\zeta_0,r} r^{-\lambda}\int_{\mathcal{B}(\zeta_0,r)\cap E_{\mathfrak{C}}} |\mathfrak{w}(\zeta)|^p\,d\zeta.
\end{aligned}$$

For any $\alpha > 0$,

$$\begin{aligned}\mu(E_{\mathfrak{C}}) &= \mu\Big(\Big\{\zeta \in [0,1] : |\mathfrak{w}(\zeta)| > \mathfrak{C}\left(\tfrac{\zeta}{1+\zeta}\right)^{\gamma}\Big\}\Big)\\ &\leq \alpha + \mu\Big(\Big\{\zeta \in [\alpha,1] : |\mathfrak{w}(\zeta)| > \mathfrak{C}\left(\tfrac{\alpha}{1+\alpha}\right)^{\gamma}\Big\}\Big)\\ &\leq \alpha + \frac{1}{\mathfrak{C}^p\left(\frac{\alpha}{1+\alpha}\right)^{\gamma p}}\int_0^1 |\mathfrak{w}(\zeta)|^p\, d\zeta.\end{aligned}$$

As $\mathfrak{C} \to \infty$, $\mu(E_{\mathfrak{C}}) \to 0$, and by absolute continuity,

$$\|\mathfrak{w}_{\mathfrak{C}} - \mathfrak{w}\|_{\mathfrak{M}^{p,\lambda}} \to 0 \quad , \quad \mathfrak{C} \to \infty.$$

Choose $\mathfrak{C}$ large enough so that

$$\|\mathfrak{w}_{\mathfrak{C}} - \mathfrak{w}\|_{\mathfrak{M}^{p,\lambda}} < \frac{\varepsilon}{4}.$$

Step 2. Construction of the periodic function.
Let $\tau_0 > 0$ be fixed temporarily. Define the intervals

$$J_n = \left[\frac{\mathfrak{a}e^{-(n+1)\mathfrak{a}\tau_0}}{\mathfrak{a} + \mathfrak{b} - \mathfrak{b}e^{-(n+1)\mathfrak{a}\tau_0}}, \frac{\mathfrak{a}e^{-n\mathfrak{a}\tau_0}}{\mathfrak{a} + \mathfrak{b} - \mathfrak{b}e^{-n\mathfrak{a}\tau_0}}\right], \quad n = 0, 1, 2, \ldots$$

Observe that $(0,1] = \bigcup_{n=0}^{\infty} J_n$. Define $\mathfrak{v}$ on each J_n by

$$\mathfrak{v}(\zeta) = e^{-n\gamma\tau_0}\, \mathfrak{w}_{\mathfrak{C}}\left(\frac{\mathfrak{a}\zeta e^{n\mathfrak{a}\tau_0}}{\mathfrak{a} + \mathfrak{b}\zeta - \mathfrak{b}\zeta e^{n\mathfrak{a}\tau_0}}\right), \quad \zeta \in J_n.$$

Since $|\mathfrak{w}_{\mathfrak{C}}(\zeta)| \leq \mathfrak{C}\left(\frac{\zeta}{1+\zeta}\right)^{\gamma}$, a direct computation gives

$$|\mathfrak{v}(\zeta)| \leq \mathfrak{C}\left(\frac{\zeta}{1+\zeta}\right)^{\gamma},$$

hence $\mathfrak{v} \in \mathfrak{M}^{p,\lambda}(0,1)$.
Step 3. Verification of periodicity.
For $\zeta \in J_n$, we have

$$\frac{\mathfrak{a}\zeta e^{-\mathfrak{a}\tau_0}}{\mathfrak{a} + \mathfrak{b}\zeta - \mathfrak{b}\zeta e^{-\mathfrak{a}\tau_0}} \in J_{n-1}.$$

Thus

$$\begin{aligned}(\bar{\mathcal{S}}_{\tau_0}\mathfrak{v})(\zeta) &= e^{\gamma\tau_0}\, \mathfrak{v}\left(\frac{\mathfrak{a}\zeta e^{-\mathfrak{a}\tau_0}}{\mathfrak{a} + \mathfrak{b}\zeta - \mathfrak{b}\zeta e^{-\mathfrak{a}\tau_0}}\right)\\ &= e^{\gamma\tau_0}\, e^{-(n-1)\gamma\tau_0}\, \mathfrak{w}_{\mathfrak{C}}\left(\frac{\mathfrak{a}\zeta e^{(n-1)\mathfrak{b}\tau_0}}{\mathfrak{a} + \mathfrak{b}\zeta - \mathfrak{b}\zeta e^{(n-1)\mathfrak{a}\tau_0}}\right)\\ &= e^{-n\gamma\tau_0}\, \mathfrak{w}_{\mathfrak{C}}\left(\frac{\mathfrak{a}\zeta e^{n\mathfrak{a}\tau_0}}{\mathfrak{a} + \mathfrak{b}\zeta - \mathfrak{b}\zeta e^{n\mathfrak{a}\tau_0}}\right)\\ &= \mathfrak{v}(\zeta).\end{aligned}$$

Therefore $\bar{\mathcal{S}}_{\tau_0}\mathfrak{v} = \mathfrak{v}$, i.e., $\mathfrak{v}$ is periodic with period τ_0.
Step 4. Control of the Morrey norm on the initial interval.
Let

$$\delta(\tau_0) = \frac{\mathfrak{a}e^{-\mathfrak{a}\tau_0}}{\mathfrak{a} + \mathfrak{b} - \mathfrak{b}e^{-\mathfrak{a}\tau_0}}.$$

Since $\mathfrak{v}, \mathfrak{w} \in \mathfrak{M}^{p,\lambda}(0,1)$, we can choose τ_0 sufficiently large so that

$$\|\mathfrak{v}\|_{\mathfrak{M}^{p,\lambda}([0,\delta(\tau_0)))} < \frac{\varepsilon}{4}, \qquad \|\mathfrak{w}\|_{\mathfrak{M}^{p,\lambda}([0,\delta(\tau_0)))} < \frac{\varepsilon}{4}.$$

For $\zeta \in J_0 = [\delta(\tau_0), 1]$, we have $n = 0$, hence $\mathfrak{v}(\zeta) = \mathfrak{w}_{\mathfrak{C}}(\zeta)$. Consequently,

$$\|\mathfrak{v} - \mathfrak{w}_{\mathfrak{C}}\|_{\mathfrak{M}^{p,\lambda}([\delta(\tau_0),1])} = 0.$$

Step 5. Final approximation estimate.

Now combine the estimates:

$$\begin{aligned}
\|\mathfrak{v} - \mathfrak{w}\|_{\mathfrak{M}^{p,\lambda}} &\le \|\mathfrak{v} - \mathfrak{w}\|_{\mathfrak{M}^{p,\lambda}([0,\delta(\tau_0)))} + \|\mathfrak{v} - \mathfrak{w}\|_{\mathfrak{M}^{p,\lambda}([\delta(\tau_0),1])} \\
&\le \|\mathfrak{v}\|_{\mathfrak{M}^{p,\lambda}([0,\delta(\tau_0)))} + \|\mathfrak{w}\|_{\mathfrak{M}^{p,\lambda}([0,\delta(\tau_0)))} \\
&\quad + \|\mathfrak{v} - \mathfrak{w}_{\mathfrak{C}}\|_{\mathfrak{M}^{p,\lambda}([\delta(\tau_0),1])} + \|\mathfrak{w}_{\mathfrak{C}} - \mathfrak{w}\|_{\mathfrak{M}^{p,\lambda}([\delta(\tau_0),1])} \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + 0 + \frac{\varepsilon}{4} \\
&= \frac{3\varepsilon}{4} < \varepsilon.
\end{aligned}$$

Thus we have constructed a periodic point $\mathfrak{v}$ satisfying $\|\mathfrak{v} - \mathfrak{w}\|_{\mathfrak{M}^{p,\lambda}} < \varepsilon$, proving that the set of periodic points is dense in $\mathfrak{M}^{p,\lambda}(0,1)$. □

Theorem 10 (Transitivity in Morrey space). *If* $\gamma > \dfrac{\lambda - 1}{p}$, *then the semidynamical system* $(\bar{\mathcal{S}}_\tau)_{\tau\ge 0}$ *is transitive in* $\mathfrak{M}^{p,\lambda}(0,1)$.

Proof. Let

$$\mathcal{P}(\mathfrak{v}_1, \epsilon_1) = \{\vartheta \in \mathfrak{M}^{p,\lambda} : \|\mathfrak{v}_1 - \vartheta\|_{\mathfrak{M}^{p,\lambda}} < \epsilon_1\},$$

$$\mathcal{P}(\mathfrak{v}_2, \epsilon_2) = \{\vartheta \in \mathfrak{M}^{p,\lambda} : \|\mathfrak{v}_2 - \vartheta\|_{\mathfrak{M}^{p,\lambda}} < \epsilon_2\},$$

be two nonempty open balls. Define

$$\mathfrak{w}(\zeta) = \begin{cases} e^{-\gamma\tau}\mathfrak{v}_2\left(\dfrac{\mathfrak{a}\zeta e^{-\mathfrak{a}\tau}}{\mathfrak{a} + \mathfrak{b}\zeta - \mathfrak{b}\zeta e^{-\mathfrak{a}\tau}}\right), & \zeta < \dfrac{\mathfrak{a}e^{-\mathfrak{a}\tau}}{\mathfrak{a} + \mathfrak{b} - \mathfrak{b}e^{-\mathfrak{a}\tau}}, \\ \mathfrak{v}_1(\zeta), & \zeta \ge \dfrac{\mathfrak{a}e^{-\mathfrak{a}\tau}}{\mathfrak{a} + \mathfrak{b} - \mathfrak{b}e^{-\mathfrak{a}\tau}}, \end{cases}$$

for a suitable choice of τ. Then

$$\|\mathfrak{w}\|_{\mathfrak{M}^{p,\lambda}([0,\delta(\tau)])} \le \mathfrak{C}_{\mathfrak{a},\mathfrak{b},\lambda}^{1/p} e^{-\tau(\gamma + \frac{\mathfrak{a}(1-\lambda)}{p})}\|\mathfrak{v}_2\|_{\mathfrak{M}^{p,\lambda}},$$

where $\delta(\tau) = \frac{\mathfrak{a}e^{-\mathfrak{a}\tau}}{\mathfrak{a}+\mathfrak{b}-\mathfrak{b}e^{-\mathfrak{a}\tau}}$ and $\mathfrak{C}_{\mathfrak{a},\mathfrak{b},\lambda} = \frac{(\mathfrak{a}+\mathfrak{b})^2}{\mathfrak{a}^2}\left(\frac{\mathfrak{a}+\mathfrak{b}}{\mathfrak{a}}\right)^{\lambda}$.

Since $\gamma > \frac{\lambda-1}{p}$, we have $\gamma + \frac{\mathfrak{a}(1-\lambda)}{p} > 0$ for $\mathfrak{a} = 1$, and the factor $e^{-\tau(\gamma + \frac{\mathfrak{a}(1-\lambda)}{p})} \to 0$ as $\tau \to \infty$. Hence for large τ,

$$\|\mathfrak{v}_1 - \mathfrak{w}\|_{\mathfrak{M}^{p,\lambda}} \le \|\mathfrak{v}_1\|_{\mathfrak{M}^{p,\lambda}([0,\delta(\tau)])} + e^{-\tau(\gamma + \frac{\mathfrak{a}(1-\lambda)}{p})}\|\mathfrak{v}_2\|_{\mathfrak{M}^{p,\lambda}} < \epsilon_1,$$

so $\mathfrak{w} \in \mathcal{P}(\mathfrak{v}_1, \epsilon_1)$. Moreover, $\bar{\mathcal{S}}_\tau \mathfrak{w} = \mathfrak{v}_2 \in \mathcal{P}(\mathfrak{v}_2, \epsilon_2)$. Thus $\bar{\mathcal{S}}_\tau(\mathcal{P}(\mathfrak{v}_1, \epsilon_1)) \cap \mathcal{P}(\mathfrak{v}_2, \epsilon_2) \ne \emptyset$. □

5.3. *The dynamical system $(\bar{\mathcal{S}}_\tau)_{\tau\geq 0}$ in Morrey space*

Theorem 11 (Equivalence in Morrey space)**.** *Assume there exist positive constants $\mathfrak{C}$ and q such that for all $\zeta \in [0,1]$,*

$$\frac{|\sigma(0)-\sigma(\zeta)|}{1+\zeta} \leq \mathfrak{C}\,\zeta^q. \tag{5.5}$$

Then $\mathfrak{X}(\tau,\cdot) \in \mathfrak{M}^{p,\lambda}(0,1)$ if and only if $\bar{\mathfrak{X}}(\tau,\cdot) \in \mathfrak{M}^{p,\lambda}(0,1)$, where $\bar{\mathfrak{X}}$ is the solution of the constant-coefficient problem with $\gamma = \sigma(0)$.

Proof. Suppose first that $\mathfrak{X} \in \mathfrak{M}^{p,\lambda}(0,1)$. From the transformation

$$h(\zeta) = \exp\left(\int_0^\zeta \frac{\sigma(0)-\sigma(s)}{\mathfrak{a}s+\mathfrak{b}s^2}\,ds\right), \qquad \bar{\mathfrak{X}} = h\cdot\mathfrak{X},$$

we estimate the exponential factor using assumption (5.5):

$$\left|\int_0^\zeta \frac{\sigma(0)-\sigma(s)}{\mathfrak{a}s+\mathfrak{b}s^2}\,ds\right| \leq \frac{\mathfrak{C}}{\mathfrak{a}}\int_0^\zeta s^{q-1}\,ds = \frac{\mathfrak{C}}{\mathfrak{a}q}\zeta^q.$$

Hence

$$|h(\zeta)| \leq e^{\frac{\mathfrak{C}}{\mathfrak{a}q}\zeta^q} \leq e^{\frac{\mathfrak{C}}{\mathfrak{a}q}}, \quad |h^{-1}(\zeta)| \leq e^{\frac{\mathfrak{C}}{\mathfrak{a}q}}.$$

For any ball $\mathcal{B}(\zeta_0,r) \subset (0,1)$,

$$\int_{\mathcal{B}(\zeta_0,r)} |\bar{\mathfrak{X}}(\tau,\zeta)|^p\,d\zeta = \int_{\mathcal{B}(\zeta_0,r)} |h(\zeta)\mathfrak{X}(\tau,\zeta)|^p\,d\zeta \leq e^{p\mathfrak{C}/(\mathfrak{a}q)}\int_{\mathcal{B}(\zeta_0,r)} |\mathfrak{X}(\tau,\zeta)|^p\,d\zeta.$$

Multiplying by $r^{-\lambda}$ and taking the supremum yields

$$\|\bar{\mathfrak{X}}(\tau,\cdot)\|_{\mathfrak{M}^{p,\lambda}} \leq e^{\mathfrak{C}/(\mathfrak{a}q)}\,\|\mathfrak{X}(\tau,\cdot)\|_{\mathfrak{M}^{p,\lambda}}.$$

Thus $\bar{\mathfrak{X}} \in \mathfrak{M}^{p,\lambda}(0,1)$ whenever $\mathfrak{X} \in \mathfrak{M}^{p,\lambda}(0,1)$.

The converse implication follows by applying the same argument to h^{-1}, since the inverse satisfies the same exponential bound. □

Conclusion

This study provides a complete characterization of the long-term dynamical behavior of the Lasota-type equation within the framework of Morrey spaces. We have demonstrated that the critical value $\gamma_c = \frac{\lambda-1}{p}$ determines a sharp phase transition between two opposing asymptotic regimes for the classical model. For $\gamma > \gamma_c$, the associated semigroup exhibits Devaney chaos, characterized by extreme sensitivity to initial conditions, topological transitivity, and a dense set of periodic orbits in $\mathfrak{M}^{p,\lambda}(0,1)$. Conversely, for $\gamma \leq \gamma_c$, the system is strongly stable, with all solutions converging uniformly to zero.

Then these results were successfully extended to a more complex second model, the generalized Lasota equation with the transport term $(\mathfrak{a}\zeta+\mathfrak{b}\zeta^2)\frac{\partial}{\partial\zeta}$. For this model, we established analogous criteria: chaos occurs when $\gamma > -\frac{\mathfrak{a}(1-\lambda)}{p}$ and strong stability when $\gamma \leq -\frac{\mathfrak{a}(1-\lambda)}{p}$. The analysis further confirmed that the density of periodic points and topological transitivity are preserved in this generalized setting.

Our work thus provides a significant generalization of classical theories developed in L^p spaces, demonstrating that the rich structure of Morrey spaces effectively captures and classifies the complex dynamics of both the classical and extended Lasota equations, which are of considerable biological and mathematical importance.

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