



A classification of restrictive polynomial correspondences

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Abstract. In this manuscript, we study a special class of correspondences on $\mathbb{P}^1 \times \mathbb{P}^1$ given by a polynomial relation, say $P(z, w)$. We focus on what we call restrictive polynomial correspondence and characterise that it can be written as $P(z, w) = g_1(w)h_1(z) + \dots + g_\rho(w)h_\rho(z)$, for some appropriate $\rho \in \mathbb{Z}_+$, where g_r and h_r are polynomials. In particular, when $\rho = 2$, we say P is irreducible and observe that the equation $P(z, w) = 0$ can be rewritten as $R(z) = S(w)$, where R and S are rational maps of appropriate degree. Further, we also define an operation that, with the exception of degenerate cases, constructs a new irreducible restrictive polynomial correspondence from any two given irreducible restrictive polynomial correspondences.

1. Introduction

The study of dynamics of correspondences owes its origins to the papers of Bullett and his collaborators, especially [2–5]. Bullett began his investigations by considering what he called quadratic correspondences. A quadratic correspondence is an implicit relation between two variables in such a way that fixing either of them in the correspondence and finding the zero set of the resulting expression yields a set with cardinality 2 (counting multiplicity). A simple example of the same is to consider a bivariate polynomial, say $P \in \mathbb{C}[z, w]$ with degree 2 in both variables.

Starting with an arbitrary point $z_{(1)} \in \mathbb{C}$, suppose that $w_{(1)}$ and $w_{(2)}$ are the solutions of the equation $P(z_{(1)}, w) = 0$. Then, the point $z_{(1)}$ is necessarily one of the solutions of equations $P(z, w_{(1)}) = 0$ and $P(z, w_{(2)}) = 0$. Suppose that the other of the solutions is $z_{(2)}$ and $z_{(3)}$, respectively. Bullett, in [2], defined the polynomial correspondence P to be a map of pairs if for every $z_{(1)} \in \mathbb{C}$, we have $z_{(2)} = z_{(3)}$. He further obtained characterizing conditions for a quadratic polynomial correspondence to be a map of pairs. Though the paper contains far more interesting results concerning the dynamics of such quadratic correspondences, we are currently focused only on the above-mentioned ideas.

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The authors in [9] generalize the case of quadratic correspondences to include polynomial correspondences of any arbitrary degree $d \geq 2$, although the degree being equal in both the variables to define and characterize, what they call maps of d -tuples. A specific hurdle that one encounters while increasing the degree beyond 2 is the absence of involutions, that Bullett makes extensive use of in his proofs of the characterizing results. The authors, in [9], overcome this difficulty by making use of some elementary results from linear algebra.

Following the work in [9], we consider a furthermore generalisation of polynomial correspondences where the degrees in each of the variables need not be equal and complete the analysis of the same. Whenever a polynomial correspondence exhibits a behaviour analogous to the map of pairs under the corresponding situation here, we shall call it a restrictive polynomial correspondence, a precise definition of the same shall appear in Section 2. Moreover, we classify such restrictive polynomial correspondences as reducible and irreducible ones, in Sections 3 and 4, thus exhausting the class of restrictive polynomial correspondences. In Section 5, we define a new multiplication operation between two polynomial correspondences and obtain conditions for two irreducible restrictive polynomial correspondences to result in an irreducible restrictive polynomial correspondence. The paper is strewn with examples illustrating every concept that we introduce, define and characterise.

We conclude this section by stating that this study of restrictive polynomial correspondences is a small drop in the ocean of holomorphic correspondences, a topic of sufficient interest for many mathematicians in the recent decades, for example [1, 6–8]. Towards that end, we start with the definition of a holomorphic correspondence.

Definition 1.1 Consider an implicit relation given by $f(z, w)$, analytic in the variables z and w . Let \mathcal{G}_f denote the graph of f in $\mathbb{P}^1 \times \mathbb{P}^1$ i.e., the biprojective completion of the zero set of f in $\mathbb{P}^1 \times \mathbb{P}^1$. Suppose

1. $\Pi_w (\Pi_z^{-1} (\{z_0\}) \cap \mathcal{G}_f)$ is generically d_z -valued for any $z_0 \in \mathbb{P}^1$;
2. $\Pi_z (\Pi_w^{-1} (\{w_0\}) \cap \mathcal{G}_f)$ is generically d_w -valued for any $w_0 \in \mathbb{P}^1$;

where Π_z and Π_w denote the projection maps defined on \mathcal{G}_f along the coordinates respectively, then such an implicit relation f is called a holomorphic correspondence of bidegree (d_w, d_z) .

Observe that by virtue of Definition 1.1, every complex-valued rational map of degree d defined on \mathbb{P}^1 is now, accorded the status of being a holomorphic correspondence of bidegree $(d, 1)$. We now state a theorem due to Bullett and Penrose, as can be found in [6] that characterises holomorphic correspondences and provides us a motivation to consider polynomial correspondences, as we have done in this paper.

Theorem 1.2 ([6], Theorem 2, Page 121) Every holomorphic correspondence on the Riemann sphere is algebraic, that is to say, [it] can be expressed in the form $P(z, w) = 0$ for some polynomial P in two variables.

2. Restrictive polynomial correspondences

Consider a polynomial correspondence P of bidegree (d_w, d_z) in given by

$$P_{(d_w, d_z)}(z, w) = \sum_{j=0}^{d_w} \sum_{k=0}^{d_z} \Lambda_{(j,k)} z^j w^k. \tag{1}$$

Bearing in mind the conditions mentioned in Definition 1.1, we shall be interested in those polynomial correspondences $P_{(d_w, d_z)}$ given by Equation (1) that satisfy:

- (P1) No linear polynomial in z divides $P_{(d_w, d_z)}(z, w)$;
- (P2) No linear polynomial in w divides $P_{(d_w, d_z)}(z, w)$.

In this paper, we consider polynomial correspondences as described above, where d_w is not necessarily equal to d_z , however with the condition that both d_w and d_z being strictly positive integers with at least one of them being strictly bigger than 1. To every polynomial correspondence $P_{(d_w, d_z)}$, we associate the matrix $M_{P_{(d_w, d_z)}}$ of order $(d_w + 1) \times (d_z + 1)$ given by

$$P_{(d_w, d_z)} \sim M_{P_{(d_w, d_z)}} = \begin{pmatrix} \Lambda_{(d_w, d_z)} & \Lambda_{(d_w, d_z-1)} & \cdots & \Lambda_{(d_w, 0)} \\ \Lambda_{(d_w-1, d_z)} & \Lambda_{(d_w-1, d_z-1)} & \cdots & \Lambda_{(d_w-1, 0)} \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda_{(0, d_z)} & \Lambda_{(0, d_z-1)} & \cdots & \Lambda_{(0, 0)} \end{pmatrix}. \tag{2}$$

For such polynomial correspondences, we now motivate a situation when it could be restrictive and write the formal definition of a restrictive polynomial correspondence later. For any $z_0 \in \mathbb{C}$, consider the list of solutions of the polynomial equation $P_{(d_w, d_z)}(z_0, w) = 0$ given by, say $L_w(z_0) = [w^{(1)}(z_0), \dots, w^{(d_z)}(z_0)]$. Substituting $w = w^{(k)}(z_0)$ for $1 \leq k \leq d_z$ and solving for the polynomial equation $P_{(d_w, d_z)}(z, w^{(k)}(z_0)) = 0$, we obtain a solution list given by

$$L_z(w^{(k)}(z_0)) = [z^{(1)}(w^{(k)}(z_0)), \dots, z^{(d_w)}(w^{(k)}(z_0))].$$

Note that for all $1 \leq k \leq d_z$, at least one of the solutions $z^{(j)}(w^{(k)}(z_0)) \equiv z_0$. The list of solutions in the w -plane naturally depends on the starting point z_0 and the list of solutions in the z -plane depends on our choice of $w^{(k)}(z_0)$. Suppose the latter list is independent of our choice of $w^{(k)}(z_0)$, i.e., the list of points in $L_z(w^{(k)}(z_0))$ remains the same for any $1 \leq k \leq d_z$. This leads us to a situation when a list in the z -plane of cardinality d_w and a list in the w -plane of cardinality d_z are associated to each other *via* the correspondence. In general, this need not be so. In fact, the list of points associated to each other *via* the correspondence $[z^{(j)}(w^{(k)}(z_0)) : 1 \leq j \leq d_w; 1 \leq k \leq d_z]$ has cardinality $d_w d_z$. Our ability to reduce this cardinality naturally prompts a thorough study of the same. Towards that direction, is the following definition.

Definition 2.1 Consider a polynomial correspondence $P_{(d_w, d_z)}(z, w)$ of bidegree (d_w, d_z) , as given in Equation (1). It is said to be a restrictive polynomial correspondence if the list of points

$$P_{(d_w, d_z)} : L_z(w^{(1)}(z_0)) \equiv \dots \equiv L_z(w^{(d_z)}(z_0)) \text{ for any } z_0 \in \mathbb{C}.$$

As a consequence of Definition 2.1, we immediately have the following lemma.

Lemma 2.2 Let $P_{(d_w, d_z)}(z, w)$ be a restrictive polynomial correspondence of bidegree (d_w, d_z) . Then, for any $w_0 \in \mathbb{C}$ as the starting point, we have

$$P_{(d_w, d_z)} : L_w(z^{(1)}(w_0)) \equiv \dots \equiv L_w(z^{(d_w)}(w_0)).$$

In other words, our definition of the correspondence as restrictive starting from one of the variables forces a similar restriction on the other variable, too.

Proof. For some fixed $w_0 \in \mathbb{C}$, let $L_z(w_0) = [z^{(1)}(w_0), \dots, z^{(d_w)}(w_0)]$. Substituting $z = z^{(1)}(w_0)$ and solving for the polynomial equation $P_{(d_w, d_z)}(z^{(1)}(w_0), w) = 0$, we obtain a solution list given by

$$L_w(z^{(1)}(w_0)) = [w^{(1)}(z^{(1)}(w_0)), \dots, w^{(d_z)}(z^{(1)}(w_0))],$$

where $w_0 \in L_w(z^{(1)}(w_0))$ and hence, without loss of generality, we assume $w_0 = w^{(1)}(z^{(1)}(w_0))$. Since $P_{(d_w, d_z)}$ is restrictive, we have

$$L_z(w_0) \equiv L_z(w^{(2)}(z^{(1)}(w_0))) \equiv \dots \equiv L_z(w^{(d_z)}(z^{(1)}(w_0))). \tag{3}$$

Now, fix $1 \leq j \leq d_w$ and consider $P_{(d_w, d_z)}(z^{(j)}(w_0), w) = 0$. By Equation 3, it is clear that, $w^{(k)}(z^{(1)}(w_0))$ is a root of $P_{(d_w, d_z)}(z^{(j)}(w_0), w)$, for all $1 \leq k \leq d_z$. Further, since there are exactly d_z many roots for $P_{(d_w, d_z)}(z^{(j)}(w_0), w)$, these are the only roots. Thus, we have $L_w(z^{(1)}(w_0)) \equiv \dots \equiv L_w(z^{(d_w)}(w_0))$. \square

Remark 2.3 As a consequence of Definition (2.1) and Lemma (2.2), we can find lists of d_w points in the z -plane, say $[z_1, \dots, z_{d_w}]$ and lists of d_z points in the w -plane, say $[w_1, \dots, w_{d_z}]$ that are related to each other via the correspondence $P_{(d_w, d_z)}$ and satisfy $(z_j, w_k) \in \mathcal{G}_{P_{(d_w, d_z)}}$. In other words, any starting point z_j results in the list of w_k 's and vice-versa.

We denote this by $L_z(w_k) = [z_1, \dots, z_{d_w}] \xleftrightarrow{P_{(d_w, d_z)}} [w_1, \dots, w_{d_z}] = L_w(z_j)$. Further, $(z, w_k) \notin \mathcal{G}_{P_{(d_w, d_z)}}$ for $1 \leq k \leq d_z$, unless $z = z_j$ for some $1 \leq j \leq d_w$ and $(z_j, w) \notin \mathcal{G}_{P_{(d_w, d_z)}}$ for $1 \leq j \leq d_w$ unless $w = w_k$ for some $1 \leq k \leq d_z$.

We now explain the above definition with two examples, that illustrate the concept of restrictive polynomial correspondences.

Example 2.4 Consider the polynomial correspondence of bidegree (3, 2) given by

$$P_{(3,2)}(z, w) = iz^3w^2 + 5z^3w - z^2w^2 + z^3 - 2z^2w + 11izw^2 - 6z^2 + 55zw - w^2 + 11z - 2w - 6.$$

Observe that $P_{(3,2)}$ satisfies the required properties, as mentioned in the beginning of this section. Substituting $w \in \left\{0, \frac{28}{1-6i}\right\}$, we obtain $z \in \{1, 2, 3\}$ and vice-versa. In fact, such an association between 3 points in the z -plane and 2 points in the w -plane via the polynomial correspondence $P_{(3,2)}$ can be obtained for any starting point z_0 or w_0 in \mathbb{C} , thus making $P_{(3,2)}$ a restrictive polynomial correspondence.

Example 2.5 The polynomial correspondence $P_{(6,4)} = (P_{(3,2)})^2$ of bidegree (6, 4) where $P_{(3,2)}$ is as stated in Example 2.4 is also a restrictive polynomial correspondence. However, the association between the points is preserved, in this case, with an increase in the multiplicities of each of the points.

3. Reducible restrictive polynomial correspondences

We begin this section by carefully delineating a distinction between what we describe as the concept of reducibility and irreducibility among restrictive polynomial correspondences, as one might have observed from Examples 2.4 and 2.5. We call a restrictive polynomial correspondence $P_{(d_w, d_z)}(z, w)$, as written in Equation (1), *irreducible* if it can not be expressed as a product of finitely many (more than one) restrictive polynomial correspondences and *reducible*, otherwise. Thus, the example stated in 2.4 is an irreducible restrictive polynomial correspondence, while that in 2.5 is a reducible restrictive polynomial correspondence.

Suppose the polynomial correspondence $P_{(d_w, d_z)}$ can be written as a product of two irreducible restrictive polynomial correspondences, say $P_{(m_w, m_z)}^{(1)}$ and $P_{(n_w, n_z)}^{(2)}$ so that $d_w = m_w + n_w$ and $d_z = m_z + n_z$. We then obtain a necessary condition in order that $P_{(d_w, d_z)}$ is a restrictive polynomial correspondence. For $P_{(d_w, d_z)}$ to be a restrictive polynomial correspondence, we need

$$L_z(w_k^{(i)}) = [z_1^{(1)}, \dots, z_{m_w}^{(1)}, z_1^{(2)}, \dots, z_{n_w}^{(2)}] \xleftrightarrow{P_{(d_w, d_z)}} [w_1^{(1)}, \dots, w_{m_z}^{(1)}, w_1^{(2)}, \dots, w_{n_z}^{(2)}] = L_w(z_j^{(i)}).$$

Since $P_{(m_w, m_z)}^{(1)}$ and $P_{(n_w, n_z)}^{(2)}$ are irreducible restrictive polynomial correspondences, by Definition 2.1, we know that

$$P_{(m_w, m_z)}^{(1)} : L_z(w^{(1)}(z_0)) \equiv \dots \equiv L_z(w^{(m_z)}(z_0)) \text{ for any } z_0 \in \mathbb{C}, \text{ and}$$

$$P_{(n_w, n_z)}^{(2)} : L_z(w^{(1)}(z_0)) \equiv \dots \equiv L_z(w^{(n_z)}(z_0)) \text{ for any } z_0 \in \mathbb{C}.$$

Owing to Remark 2.3, this forces the lists $[z_1^{(1)}, \dots, z_{m_w}^{(1)}]$ and $[z_1^{(2)}, \dots, z_{n_w}^{(2)}]$ to be one and the same and further, the lists $[w_1^{(1)}, \dots, w_{m_z}^{(1)}]$ and $[w_1^{(2)}, \dots, w_{n_z}^{(2)}]$ also to be one and the same. This results in the following theorem.

Theorem 3.1 Any restrictive polynomial correspondence $P_{(d_w, d_z)}$, as stated in Equation (1) can be factorised as $[Q_{(d'_w, d'_z)}]^d$, where $Q_{(d'_w, d'_z)}$ is an irreducible restrictive polynomial correspondence and d is a common divisor of d_w and d_z satisfying $dd'_w = d_w$ and $dd'_z = d_z$.

The following example illustrates the fact that the product of two distinct irreducible restrictive polynomial correspondences is not a restrictive polynomial correspondence.

Example 3.2 Consider the following irreducible restrictive polynomial correspondences given by

$$P_{(3,2)}^{(1)}(z, w) = z^3w^2 + z^3 - 6z^2 + w^2 + 11z - 6 \quad \text{and} \quad P_{(2,2)}^{(2)}(z, w) = z^2w^2 + z^2 - 2zw + w^2 - 2z + 1.$$

Note however, that $P_{(5,4)} = (P_{(3,2)}^{(1)} \times P_{(2,2)}^{(2)})$ given by

$$\begin{aligned} P_{(5,4)}(z, w) = & z^5w^4 + 2z^5w^2 - 2z^4w^3 + z^3w^4 - 8z^4w^2 + z^2w^4 + z^5 - 2z^4w + 13z^3w^2 \\ & - 8z^4 + 12z^3w - 11z^2w^2 - 2zw^3 + w^4 + 24z^3 - 22z^2w + 9zw^2 \\ & - 34z^2 + 12zw - 5w^2 + 23z - 6, \end{aligned}$$

is not a restrictive polynomial correspondence. For example, the list of points $L_z = [1, 2, 3]$ correspond to the list of points $L_w = [0, 0]$ via $P_{(3,2)}^{(1)}$, while the list of points $L_z = [1, 1]$ correspond to the list of points $L_w = [0, 1]$ via $P_{(2,2)}^{(2)}$. However, the union of the list of points in L_z and the union of the list of points in L_w for $P_{(3,2)}^{(1)}$ and $P_{(2,2)}^{(2)}$ do not correspond to the list of points in L_z and L_w for $P_{(5,4)}$, i.e., $L_z = [1, 2, 3, 1, 1] \xrightarrow{P_{(5,4)}} [0, 0, 0, 1] = L_w$.

We conclude this section with a result regarding the rank of the matrix pertaining to a reducible restrictive polynomial correspondence. We will prove this theorem, later in Section 6.

Theorem 3.3 Let $P_{(d_w, d_z)}$, as stated in Equation (1) be a reducible restrictive polynomial correspondence that can be factorised as $P_{(d_w, d_z)} = [Q_{(d'_w, d'_z)}]^d$, where $Q_{(d'_w, d'_z)}$ and d are determined as in Theorem 3.1. Then, the rank of the matrix associated to $P_{(d_w, d_z)}$ is $(d + 1)$.

4. Irreducible restrictive polynomial correspondences: A characterisation

We now turn our focus to irreducible restrictive polynomial correspondences. In this section, we prove an important characterising result about irreducible restrictive polynomial correspondences.

Theorem 4.1 Consider the polynomial correspondence $P_{(d_w, d_z)}$, as stated in Equation (1). Then, the following statements are equivalent.

1. $P_{(d_w, d_z)}$ is an irreducible restrictive polynomial correspondence.
2. The equation $P_{(d_w, d_z)}(z, w) = 0$ can be rewritten by separating the variables z and w , as $R(z) = S(w)$, where R and S are rational maps of degrees d_w and d_z respectively.
3. The rank of the matrix associated to $P_{(d_w, d_z)}$ is 2.

Proof. 1 \implies *2*: Assuming $P_{(d_w, d_z)}$ to be an irreducible restrictive polynomial correspondence, we start by rewriting the equation $P_{(d_w, d_z)}(z, w) = 0$ in two different manners as follows:

$$q_0(z)w^{d_z} + q_1(z)w^{d_z-1} + \dots + q_{d_z}(z) = 0 \text{ and } p_0(w)z^{d_w} + p_1(w)z^{d_w-1} + \dots + p_{d_w}(w) = 0, \tag{4}$$

where q_m 's are polynomials in z of degree at most d_w and p_n 's are polynomials in w of degree at most d_z . Further, by Remark 2.3, we know that

$$L_z(w_k) = [z_1, \dots, z_{d_w}] \xrightarrow{P_{(d_w, d_z)}} [w_1, \dots, w_{d_z}] = L_w(z_j).$$

Owing to the hypothesis on $P_{(d_w, d_z)}$, there exists at least one $0 \leq m \leq d_z$ such that the degree of the polynomial q_m is equal to d_w . Suppose the degree of the polynomial q_0 is equal to d_w , then one may choose any non-zero polynomial q_m for $1 \leq m \leq d_z$ and continue with the proof. However, if the degree of q_0 is not equal to d_w , then we choose that $1 \leq m \leq d_w$ for which the degree of the polynomial q_m is equal to d_w and proceed with the proof. Then, Viète's formula (see [12], page 89) asserts that

$$\frac{q_m(z_j)}{q_0(z_j)} = (-1)^m \sum_{1 \leq i_1 < \dots < i_m \leq d_z} w_{i_1} \times \dots \times w_{i_m} \quad \forall z_j \in L_z(w_k).$$

In fact, the precise solutions to the equation of degree d_w , namely,

$$\frac{q_m(z)}{q_0(z)} = (-1)^m \sum_{1 \leq i_1 < \dots < i_m \leq d_z} w_{i_1} \times \dots \times w_{i_m}$$

is given by the d_w points $z_j \in L_z(w_k)$. Similarly, there exists some $1 \leq n \leq d_z$ such that the solutions to the equation of degree d_z given by

$$\frac{p_n(w)}{p_0(w)} = (-1)^n \sum_{1 \leq i_1 < \dots < i_n \leq d_w} z_{i_1} \times \dots \times z_{i_n}$$

is precisely the d_z points $w_k \in L_w(z_j)$. Note that $\frac{q_m}{q_0}$ and $\frac{p_n}{p_0}$ are rational maps in the variables z and w of degree d_w and d_z respectively. Finally, we make use of an automorphism of the Riemann sphere, namely a Möbius map M , and define $R = M \circ \frac{q_m}{q_0}$ and $S = \frac{p_n}{p_0}$ to complete the proof, in this part.

2 \implies *3*: Owing to the hypothesis, we assume that the equation $P_{(d_w, d_z)}(z, w) = 0$ that yields the zero set of the correspondence can be alternately expressed as

$$R(z) = \frac{A_{d_w}z^{d_w} + \dots + A_0}{B_{d_w}z^{d_w} + \dots + B_0} = \frac{C_{d_z}w^{d_z} + \dots + C_0}{D_{d_z}w^{d_z} + \dots + D_0} = S(w), \tag{5}$$

where $A_j, B_j, C_k, D_k \in \mathbb{C}$. Then, the matrix $M_{P_{(d_w, d_z)}}$ of order $(d_w + 1) \times (d_z + 1)$ pertaining to the correspondence is given by

$$\begin{pmatrix} A_{d_w}D_{d_z} - B_{d_w}C_{d_z} & A_{d_w}D_{d_z-1} - B_{d_w}C_{d_z-1} & \dots & A_{d_w}D_0 - B_{d_w}C_0 \\ A_{d_w-1}D_{d_z} - B_{d_w-1}C_{d_z} & A_{d_w-1}D_{d_z-1} - B_{d_w-1}C_{d_z-1} & \dots & A_{d_w-1}D_0 - B_{d_w-1}C_0 \\ \vdots & \vdots & \ddots & \vdots \\ A_0D_{d_z} - B_0C_{d_z} & A_0D_{d_z-1} - B_0C_{d_z-1} & \dots & A_0D_0 - B_0C_0 \end{pmatrix},$$

that can be factorised as the product of two matrices, say \mathcal{R} and \mathcal{S} of sizes $(d_w + 1) \times 2$ and $2 \times (d_z + 1)$ respectively as

$$\mathcal{RS} = \begin{pmatrix} A_{d_w} & B_{d_w} \\ A_{d_w-1} & B_{d_w-1} \\ \vdots & \vdots \\ A_0 & B_0 \end{pmatrix} \begin{pmatrix} D_{d_z} & D_{d_z-1} & \dots & D_0 \\ -C_{d_z} & -C_{d_z-1} & \dots & -C_0 \end{pmatrix}.$$

Since R and S are non-constant rational maps of degree d_w and d_z respectively, the matrices \mathcal{R} and \mathcal{S} are necessarily of rank 2. This forces that the rank of $M_{P_{(d_w, d_z)}}$ is almost 2. It is a simple observation that $\text{Rank} \left(M_{P_{(d_w, d_z)}} \right) \neq 0$, since it is not the zero matrix. Thus, we prove that the $\text{Rank} \left(M_{P_{(d_w, d_z)}} \right) \neq 1$ to complete the proof of this part.

We start by assuming that $\text{Rank} \left(M_{P_{(d_w, d_z)}} \right) = 1$ and end up with a contradiction. Then, one may suppose that there exists one column, say Col_j for $0 \leq j \leq d_w$ such that every other column in $M_{P_{(d_w, d_z)}}$ is a scalar multiple of Col_j . This then gives rise to a factorisation of the equation $P_{(d_w, d_z)} = 0$ as $q_j(z)T(w) = 0$, where q_j is the polynomial, as mentioned in Equation (4) and T is a polynomial in w . This is a contradiction to the properties (P1) and (P2) of the correspondence, as stated in Section 2.

3 \implies 2 By hypothesis, we consider the polynomial correspondence $P_{(d_w, d_z)}$, as stated in Equation (1) such that the rank of the matrix associated to $P_{(d_w, d_z)}$ is 2. The strategy of the proof is to employ the full rank factorisation theorem, as can be found in, say [10, 11], that asserts that there exists two matrices, say \mathcal{R} and \mathcal{S} of orders $(d_w + 1) \times 2$ and $2 \times (d_z + 1)$ each of whose rank is 2, so that $M_{P_{(d_w, d_z)}} = \mathcal{R}\mathcal{S}$ and to construct the rational maps R and S using the matrices \mathcal{R} and \mathcal{S} .

Note that there exists two columns, say Col_j and Col_k , in $M_{P_{(d_w, d_z)}}$ that form a basis for the column space in order that every column of the matrix $M_{P_{(d_w, d_z)}}$ can be written as a linear combination of Col_j and Col_k , i.e., $\text{Col}_m = D_m \text{Col}_j - C_m \text{Col}_k \forall 0 \leq m \leq d_z$. Define

$$\mathcal{R} = (\text{Col}_j \quad \text{Col}_k) \quad \mathcal{S} = \begin{pmatrix} D_{d_z} & \cdots & D_0 \\ -C_{d_z} & \cdots & -C_0 \end{pmatrix},$$

and consider the rational maps pertaining to the matrices \mathcal{R} and \mathcal{S} given by

$$R(z) = \frac{A_{d_w} z^{d_w} + \cdots + A_0}{B_{d_w} z^{d_w} + \cdots + B_0} \quad \text{and} \quad S(w) = \frac{C_{d_z} w^{d_z} + \cdots + C_0}{D_{d_z} w^{d_z} + \cdots + D_0},$$

where $\text{Col}_j = (A_n)_{d_w \geq n \geq 0}$ and $\text{Col}_k = (B_n)_{d_w \geq n \geq 0}$, thereby enabling us to write the equation $P_{(d_w, d_z)}(z, w) = 0$ as $R(z) = S(w)$.

2 \implies 1 Owing to the hypothesis, we first observe that $P_{(d_w, d_z)}$ is a restrictive polynomial correspondence, by definition, meaning $L_z(w_k) = [z_1, \dots, z_{d_w}] \xrightarrow{P_{(d_w, d_z)}} [w_1, \dots, w_{d_z}] = L_w(z_j)$. In order to prove that $P_{(d_w, d_z)}$ represents an irreducible restrictive polynomial correspondence by assuming that the equation $P_{(d_w, d_z)} = 0$ can be rewritten as $R(z) = S(w)$, we prove the contra-positive statement.

On that note, we assume $P_{(d_w, d_z)}$ to be a reducible restrictive polynomial correspondence. Then, by Theorem 3.1 one can factorise $P_{(d_w, d_z)}$ as $[Q_{(d'_w, d'_z)}]^d$, where $Q_{(d'_w, d'_z)}$ is an irreducible restrictive polynomial correspondence and $d > 1$ is a common divisor of d_w and d_z satisfying $dd'_w = d_w$ and $dd'_z = d_z$. Further, from the 1 \implies 2 part of the current theorem, one can write the equation $Q_{(d'_w, d'_z)} = 0$ as $R_1(z) = S_1(w)$, where R_1 and S_1 are rational maps of appropriate degrees. Observing that the equation $[R_1(z) - S_1(w)]^d = 0$ can never be written as $R(z) = S(w)$ for any rational maps R and S , whenever $d > 1$, finally completes the proof. \square

As written in the proof of the last implication, even though $[Q_{(d'_w, d'_z)}]^d = 0$ can never be written as $R(z) = S(w)$ for any rational maps R and S , when $d > 1$, Lemma 6.1 describes the precise alternate expression for such correspondences.

One obtains the results that characterise, what the authors call a map of d -tuples in [9], as a corollary to Theorem 4.1, when $d_z = d_w = d \geq 2$. Moreover, the specific case when $d = 2$ was investigated by Bullett in [2].

Example 4.2 Recall the correspondence $P_{(3,2)}$ in Example (2.4), as stated in Section 2. Then, note that the equation $P_{(3,2)}(z, w) = 0$ can be rewritten as $\frac{z^3 + 11z}{z^2 + 1} = \frac{w^2 + 2w + 6}{iw^2 + 5w + 1}$. Further, the matrix $M_{P_{(3,2)}}$ pertaining to the given correspondence has rank 2.

The next two remarks concern a certain symmetry that one may be interested in polynomial correspondences.

- Remark 4.3**
1. Suppose $P_{(d_w, d_z)}$ is an irreducible restrictive polynomial correspondence that satisfies $P_{(d_w, d_z)}(z, w) = 0$ iff $P_{(d_w, d_z)}(-z, -w) = 0$. Then, any two consecutive columns of the matrix $M_{P_{(d_w, d_z)}}$ pertaining to the correspondence are linearly independent, provided neither of them is the zero vector.
 2. Suppose in Equation (5), we have the quantities A_j, B_j, C_k and D_k for all $0 \leq j \leq d_w$ and for all $0 \leq k \leq d_z$ to be real. Then, $P_{(d_w, d_z)}(z, w) = 0$ iff $P_{(d_w, d_z)}(\bar{z}, \bar{w}) = 0$. The converse is also true, in the sense that suppose the irreducible restrictive polynomial correspondence satisfies $P_{(d_w, d_z)}(z, w) = 0$ iff $P_{(d_w, d_z)}(\bar{z}, \bar{w}) = 0$. Then, $P_{(d_w, d_z)} \in \mathbb{R}[z, w]$, upto a constant multiple.

5. *-product of irreducible restrictive polynomial correspondences

In this section, we consider two irreducible restrictive polynomial correspondences, define an operation between them and investigate when the resulting expression yields something interesting, possibly an irreducible restrictive polynomial correspondence of appropriate bidegree. Towards that end, consider $P_{(m_w, m_z)}$ and $Q_{(n_w, n_z)}$ with $m_z = n_w = d$ given by

$$P_{(m_w, d)}(z, w) = \sum_{j=0}^{m_w} \sum_{k=0}^d \Lambda_{(j,k)} z^j w^k \text{ and } Q_{(d, n_z)}(z, w) = \sum_{j=0}^d \sum_{k=0}^{n_z} \Omega_{(j,k)} z^j w^k.$$

As mentioned in Equation (4), one can then express these correspondences $P_{(m_w, d)}$ and $Q_{(d, n_z)}$ alternatively as

$$P_{(m_w, d)}(z, w) = \sum_{j=0}^{m_w} p_j(w) z^j \text{ and } Q_{(d, n_z)}(z, w) = \sum_{k=0}^{n_z} q_k(z) w^k, \tag{6}$$

where p_j 's and q_k 's are polynomials of degree atmost d in w and z respectively given by $p_j(w) = \sum_{r=0}^d \alpha_r^{(j)} w^r$

for $0 \leq j \leq m_w$ and $q_k(z) = \sum_{r=0}^d \beta_r^{(k)} z^r$ for $0 \leq k \leq n_z$. For such a collection of polynomials consisting of p_j 's and q_k 's, define the *-product of $P_{(m_w, d)}$ and $Q_{(d, n_z)}$ as:

$$(P_{(m_w, d)} * Q_{(d, n_z)})(z, w) = \sum_{j=0}^{m_w} \sum_{k=0}^{n_z} \langle p_j, q_k \rangle z^j w^k \text{ where } \langle p_j, q_k \rangle = \sum_{r=0}^d \alpha_r^{(j)} \beta_r^{(k)}. \tag{7}$$

We first discuss the cases when the *-product between any two irreducible restrictive polynomial correspondences results in an expression that violates the conditions (P1) and (P2), as mentioned in Section 2 for polynomial correspondences and hence, are not desirable. Consider the following examples.

Example 5.1 Given two irreducible restrictive polynomial correspondences,

$$\begin{aligned} P_{(3,4)}(z, w) &= 2z^3w^4 + z^3w^3 - w^4 + 7z^3 - 7w^3 - w^2 - w + 4 \text{ and} \\ Q_{(4,2)}(z, w) &= z^4w^2 - 2z^3w^2 - 53z^4 + 13z^2w^2 + 15z^3 + 13. \end{aligned}$$

We note that $\langle p_j, q_k \rangle = 0$ for all j, k , that yields $(P_{(3,4)} * Q_{(4,2)}) \equiv 0$.

Example 5.2 Let

$$\begin{aligned}
 P_{(3,2)}(z, w) &= 2iz^3w^2 + 3z^3w + z^3 + 5z^2w + 2izw^2 + 3zw + z + 2iw^2 + 28w + 1 \text{ and} \\
 Q_{(2,3)}(z, w) &= z^2w^3 + 1
 \end{aligned}$$

be the two given irreducible restrictive polynomial correspondences. Then, we obtain

$$\begin{aligned}
 (P_{(3,2)} * Q_{(2,3)})(z, w) &= 2iz^3w^3 + z^3 + 2izw^3 + z + 2iw^3 + 1 \\
 &= (z^3 + z + 1)(2iw^3 + 1).
 \end{aligned}$$

In this example, $P_{(3,2)} * Q_{(2,3)}$ is undesirable since it disobeys the conditions (P1) and (P2), as mentioned in Section 2 for polynomial correspondences.

Since the *-product of correspondences have reduced to being uninteresting in the above two examples, we examine the *-product further using alternate expressions of $P_{(m_w,d)}$ and $Q_{(d,n_z)}$. Rewriting the equation $P_{(m_w,d)}(z, w) = 0$, by virtue of Theorem 4.1, we have

$$R_1(z) = \frac{A_{m_w}z^{m_w} + \dots + A_0}{B_{m_w}z^{m_w} + \dots + B_0} = \frac{C_d w^d + \dots + C_0}{D_d w^d + \dots + D_0} = S_1(w). \tag{8}$$

Similarly, the equation $Q_{(d,n_z)}(z, w) = 0$ can be rewritten as

$$R_2(z) = \frac{E_d z^d + \dots + E_0}{F_d z^d + \dots + F_0} = \frac{G_{n_z} w^{n_z} + \dots + G_0}{H_{n_z} w^{n_z} + \dots + H_0} = S_2(w). \tag{9}$$

We now define four bivariate polynomials denoted by f_i for $1 \leq i \leq 4$, given by

$$\begin{aligned}
 f_1(z, w) &= \text{Nr.}(S_1) \text{Dr.}(R_2), & f_2(z, w) &= \text{Dr.}(S_1) \text{Nr.}(R_2), \\
 f_3(z, w) &= \text{Nr.}(S_1) \text{Nr.}(R_2), & f_4(z, w) &= \text{Dr.}(S_1) \text{Dr.}(R_2),
 \end{aligned} \tag{10}$$

where Nr. and Dr. denotes the numerator and the denominator of the appropriate rational function, as written above. Note that each $f_i(z, w)$ represents a bivariate polynomial of bidegree (d, d) and hence, is associated to a matrix of size $(d + 1) \times (d + 1)$, as written in Equation (2). Denoting by $\text{Tr}(f_i)$, the trace of the matrix associated to f_i , we now define

$$S(w) = \frac{[H_{n_z} \text{Tr}(f_3) - G_{n_z} \text{Tr}(f_1)] w^{n_z} + \dots + [H_0 \text{Tr}(f_3) - G_0 \text{Tr}(f_1)]}{[H_{n_z} \text{Tr}(f_2) - G_{n_z} \text{Tr}(f_4)] w^{n_z} + \dots + [H_0 \text{Tr}(f_2) - G_0 \text{Tr}(f_4)]}.$$

We let $R = R_1$ and consider the equation $R(z) = S(w)$ that can be re-written as

$$T(z, w) = 0. \tag{11}$$

We first prove that $(P_{(m_w,d)} * Q_{(d,n_z)}) = T$. In order to do the same, we prove that the coefficient of the $z^j w^k$ in both the sides of the above equation are equal for any $0 \leq j \leq m_w$ and $0 \leq k \leq n_z$. Owing to the representation of $P_{(m_w,d)}$ and $Q_{(d,n_z)}$, as given in Equation (6) and the alternative manner in which the equations $P_{(m_w,d)}(z, w) = 0$ and $Q_{(d,n_z)}(z, w) = 0$ can be written, as mentioned in Equations (8) and (9), we note that for $0 \leq j \leq m_w$ and $0 \leq k \leq n_z$, we have

$$\begin{aligned}
 \langle p_j, q_k \rangle &= A_j (H_k \text{Tr}(f_2) - G_k \text{Tr}(f_4)) - B_j (H_k \text{Tr}(f_3) - G_k \text{Tr}(f_1)) \\
 &= \text{Coefficient of } z^j w^k \text{ in } T(z, w).
 \end{aligned}$$

Making use of this new representation for the *-product, we reexamine Examples 5.1 and 5.2 to understand why they gave rise to undesirable *-products. One can note that in Example 5.1, we obtain $S(w)$ as an

indeterminate, while in Example 5.2, $\text{Nr.}(S) = 0$. Further, we note that we can avoid these undesirabilities by demanding the following conditions:

$$\langle \text{Nr.}(S_1), q_0 \rangle \neq 0 \quad \text{and} \quad \langle \text{Dr.}(S_1), q_{n_z} \rangle \neq 0.$$

In addition to the cases explained above, it is quite possible that $S(w)$ turns out to be a scalar quantity. This results in $\frac{H_k}{G_k} = \alpha$ for all $0 \leq k \leq n_z$, for some $\alpha \in \mathbb{C}$. However, this is not possible, since S_2 is a rational map of degree n_z . The only other possibility by way of which we may end up with $S(w)$ being a scalar quantity is when there exists a $\beta \in \mathbb{C}$ such that $\text{Tr}(f_2) - \beta\text{Tr}(f_3) = 0$ and $\text{Tr}(f_4) - \beta\text{Tr}(f_1) = 0$. However, if $S(w)$ turns out to be a scalar, we will lose one variable in our expression of $T(z, w)$. We provide one such example, here.

Example 5.3 Consider the irreducible restrictive polynomial correspondences

$$\begin{aligned} P_{(3,4)}(z, w) &= 2z^3w^4 + z^3w^3 - w^4 + 7z^3 - 7w^3 - w^2 - w + 4 \quad \text{and} \\ Q_{(4,2)}(z, w) &= z^4w^2 - 2z^3w^2 - 53z^4 + 13z^2w^2 - 15z^3 + 13, \end{aligned}$$

which when equated to 0 can be alternately expressed as

$$z^3 = \frac{w^4 + 7w^3 + w^2 + w - 4}{2w^4 + w^3 + 7} \quad \text{and} \quad \frac{53z^4 + 15z^3 - 13}{z^4 - 2z^3 + 13z^2} = w^2,$$

respectively. We then observe in this case that

$$\text{Tr}(f_1) = 0, \quad \text{Tr}(f_2) = 30, \quad \text{Tr}(f_3) = 210, \quad \text{and} \quad \text{Tr}(f_4) = 0.$$

Choosing $\beta = \frac{1}{7}$, we see that $\text{Tr}(f_2) = \beta\text{Tr}(f_3)$ and $\text{Tr}(f_4) = \beta\text{Tr}(f_1)$, thus, reducing $S(w)$ to be a constant function, a case that we prefer to avoid, since the degree of the w -variable in the polynomial is zero and hence, $T(z, w)$ is not even a correspondence, according to the conditions mentioned in Section 2.

We now focus on the conditions to obtain $T(z, w)$ as an irreducible restrictive polynomial correspondence, by making such demands as appropriate to overcome the above mentioned limitations. In what follows, we consider irreducible restrictive polynomial correspondences $P_{(m_w, d)}$ and $Q_{(d, n_z)}$. The polynomials p_j 's and q_k 's are those given in Equation 6, the rational maps R_1, R_2, S_1 and S_2 are the ones mentioned in Equation 8. We also consider the four bivariate polynomials $f_i(z, w)$, where $i = 1, 2, 3, 4$ as in Equation 10.

Definition 5.4 A pair of irreducible restrictive polynomial correspondences $P_{(m_w, d)}$ and $Q_{(d, n_z)}$ is said to be non-degenerate if it satisfies the following criteria:

1. $\langle \text{Nr.}(S_1), q_0 \rangle \neq 0$ and $\langle \text{Dr.}(S_1), q_{n_z} \rangle \neq 0$;
2. There exists no $\beta \in \mathbb{C}$ that satisfies $\text{Tr}(f_2) = \beta\text{Tr}(f_3)$ and $\text{Tr}(f_4) = \beta\text{Tr}(f_1)$ simultaneously;
3. $\langle p_{m_w}, q_{n_z} \rangle \neq 0$ and $\langle p_0, q_0 \rangle \neq 0$.

Then, we have the following theorem.

Theorem 5.5 Let $P_{(m_w, d)}$ and $Q_{(d, n_z)}$ be a pair of non-degenerate irreducible restrictive polynomial correspondences. Then, $T = P_{(m_w, d)} * Q_{(d, n_z)}$ is an irreducible restrictive polynomial correspondence of bidegree (m_w, n_z) .

Since the genesis of the conditions mentioned in the hypothesis of the theorem have been explained already, an appeal to Theorem 4.1 completes the proof of Theorem 5.5. We also note that instead of fixing $R(z) = R_1(z)$ and defining $S(w)$ using the bivariate polynomials f_i , for $1 \leq i \leq 4$, one may as well fix $S(w) = S_2(w)$ and obtain $R(z)$ using a new set of bivariate polynomials, say g_i for $1 \leq i \leq 4$, by means of the other three rational maps R_1, S_1 and R_2 . We now write an example to illustrate the idea that Theorem 5.5 propounds.

Example 5.6 Consider the following irreducible restrictive polynomial correspondences

$$\begin{aligned}
 P_{(3,4)}(z, w) &= 2z^3w^4 + iz^3w^3 + 3z^2w^4 + (7 + i)z^2w^3 + z^2w^2 - 2iw^4 + 7z^3 + 2z^2w \\
 &\quad + w^3 + 3z^2 - 7i, \\
 Q_{(4,5)}(z, w) &= 2z^4w^4 + z^3w^5 + 5z^3w^4 + 3z^2w^5 + 4z^4w^2 + iz^3w^3 + 13z^2w^4 + zw^5 \\
 &\quad + (4 + 6i)z^3w^2 + 3iz^2w^3 + 3zw^4 - w^5 + z^4 + z^3w \\
 &\quad + (8 + 18i)z^2w^2 + izw^3 + (2i - 3)w^4 + 2z^3 + 3z^2w + 6izw^2 \\
 &\quad - iw^3 + 5z^2 + zw - 2iw^2 + z - w + (i - 1).
 \end{aligned}$$

After due computations, one finds that

$$\begin{aligned}
 (P_{(3,4)} * Q_{(4,5)})(z, w) &= (-7 + i)z^3w^5 + (-17 + 19i)z^3w^4 + (9 + i)z^2w^5 - (1 + 7i)z^3w^3 \\
 &\quad + (51 + 11i)z^2w^4 + (2 - 10i)z^3w^2 + (-1 + 9i)z^2w^3 + (1 + 7i)w^5 \\
 &\quad + (-7 + i)z^3w + (42 + 70i)z^2w^2 + (19 + 17i)w^4 + (-5 + 9i)z^3 \\
 &\quad + (9 + i)z^2w + (-7 + i)w^3 + (21 + 5i)z^2 - (10 + 2i)w^2 \\
 &\quad + (1 + 7i)w + (9 + 5i)
 \end{aligned}$$

is an irreducible restrictive polynomial correspondence, courtesy Theorem 4.1, since the matrix corresponding to $(P_{(3,4)} * Q_{(4,5)})$ has rank 2.

We now state and prove a proposition that provides a partial converse to Theorem 5.5.

Proposition 5.7 Any irreducible restrictive polynomial correspondence $P_{(d_w, d_z)}$ can be written as $P_{(d_w, d_z)}(z, w) = (P_{(d_w, 1)}^{(1)} * P_{(1, d_z)}^{(2)})(z, w)$, where $P_{(d_w, 1)}^{(1)}$ and $P_{(1, d_z)}^{(2)}$ are maps on the appropriate variables, masquerading as irreducible restrictive polynomial correspondences.

Proof. Let $P_{(d_w, d_z)}(z, w) = 0$ be rewritten as in Equation (5). Inspired by the ideas used in Theorem 4.1, we consider the irreducible restrictive polynomial correspondences $P_{(d_w, 1)}^{(1)}$ and $P_{(1, d_z)}^{(2)}$, respectively, obtained from equations

$$\frac{A_{d_w} z^{d_w} + \dots + A_0}{B_{d_w} z^{d_w} + \dots + B_0} = \frac{C_{d_z} w + C_0}{D_{d_z} w + D_0} \quad \text{and} \quad z = \frac{-\frac{J_{(d_z-1, d_z)}}{J_{(0, d_z)}} w^{d_z-1} - \dots - \frac{J_{(d_z-1, 1)}}{J_{(0, d_z)}} w - 1}{w^{d_z} + \frac{J_{(0, d_z-1)}}{J_{(0, d_z)}} w^{d_z-1} + \dots + \frac{J_{(0, 1)}}{J_{(0, d_z)}} w},$$

where $J_{m,n} = D_m C_n - D_n C_m$. One can then verify that $(P_{(d_w, 1)}^{(1)} * P_{(1, d_z)}^{(2)}) = P_{(d_w, d_z)}$. \square

We conclude this section with a remark on two different symmetries that one may explore when dealing with irreducible restrictive polynomial correspondences; with the focus being when $d_w \neq d_z$.

Remark 5.8 Let $P_{(d_w, d_z)}$ be any irreducible restrictive polynomial correspondence. Suppose that the transpose (and the conjugate transpose) of $M_{P_{(d_w, d_z)}}$, given by $M_{P_{(d_w, d_z)}}^T$ (and $\overline{M}_{P_{(d_w, d_z)}}^T$) of order $(d_z + 1) \times (d_w + 1)$ pertains to the polynomial correspondence, denoted by ${}^{\dagger}P_{(d_z, d_w)}$ (and $\overline{{}^{\dagger}P}_{(d_z, d_w)}$), respectively of bidegree (d_z, d_w) . Then,

1. The matrix pertaining to the correspondence $P_{(d_w, d_z)} * {}^{\dagger}P_{(d_z, d_w)}$, i.e., $M_{(P_{(d_w, d_z)} * {}^{\dagger}P_{(d_z, d_w)})}$ is symmetric.
2. The matrix pertaining to the correspondence $P_{(d_w, d_z)} * {}^{\dagger}\overline{P}_{(d_z, d_w)}$, i.e., $M_{(P_{(d_w, d_z)} * {}^{\dagger}\overline{P}_{(d_z, d_w)})}$ is Hermitian.

6. Proof of Theorem 3.3

We begin this section with the statement and the proof of a lemma, that will be useful in the proof of Theorem 3.3.

Lemma 6.1 *Let $P_{(d_w, d_z)}$ be a restrictive polynomial correspondence of bidegree (d_w, d_z) . Then, $\text{Rank} \left(M_{P_{(d_w, d_z)}} \right) = \rho$ iff all the three following statements hold.*

1. $P_{(d_w, d_z)}(z, w)$ can be rewritten as $g_1(w)h_1(z) + \dots + g_\rho(w)h_\rho(z)$, where

$$g_r(w) = \sum_{n=0}^{d_z} \alpha_{(n,r)} w^n \text{ and } h_r(z) = \sum_{m=0}^{d_w} \beta_{(m,r)} z^m,$$

with $\alpha_{(d_z,r)} \neq 0$ for some $1 \leq r \leq \rho$ and $\beta_{(d_w,r)} \neq 0$ for some $1 \leq r \leq \rho$.

2. The collection of vectors $\left\{ (\beta_{(m,r)})_{0 \leq m \leq d_w} \right\}_{1 \leq r \leq \rho}$ is linearly independent.
3. The collection of vectors $\left\{ (\alpha_{(n,r)})_{0 \leq n \leq d_z} \right\}_{1 \leq r \leq \rho}$ is linearly independent.

Proof. At the outset, we urge the readers to observe that the case $\rho = 2$ is the same as the equivalence between the statements 2 and 3 in Theorem 4.1.

We will initially assume that the matrix $M_{P_{(d_w, d_z)}}$ is of rank ρ and rewrite the given correspondence $P_{(d_w, d_z)}(z, w)$ as $\sum_{n=0}^{d_z} q_n(z)w^n$, where $q_{d_z} \neq 0$, each polynomial q_n is of degree at most d_w and there exists at least one polynomial q_n whose degree is equal to d_w . Moreover, the coefficients of the polynomial q_n forms the n -th column of the matrix $M_{P_{(d_w, d_z)}}$, for $0 \leq n \leq d_z$. Then, the matrix has ρ many column vectors, say $\{\text{Col}_{n_r}\}_{1 \leq r \leq \rho}$ that form a basis for the column space, where $0 \leq n_r \leq d_z$. Thus, we have $q_n(z) = \sum_{r=1}^{\rho} \alpha_{(n,r)} q_{n_r}(z)$, thereby the correspondence can be written as

$$P_{(d_w, d_z)}(z, w) = \sum_{n=0}^{d_z} \sum_{r=1}^{\rho} \alpha_{(n,r)} q_{n_r}(z)w^n = \sum_{r=1}^{\rho} q_{n_r}(z) \sum_{n=0}^{d_z} \alpha_{(n,r)} w^n.$$

We now define $g_r(w) = \sum_{n=0}^{d_z} \alpha_{(n,r)} w^n$ and $h_r(z) = q_{n_r}(z) = \sum_{m=0}^{d_w} \beta_{(m,r)} z^m$, say, so that the first statement is obtained. By our choice of the column vectors, $\{\text{Col}_{n_r}\}_{1 \leq r \leq \rho}$, we have the collection $\left\{ (\beta_{(m,r)})_{0 \leq m \leq d_w} \right\}_{1 \leq r \leq \rho}$ to be linearly independent. Further, for the matrix, say N of size $(d_z+1) \times \rho$ consisting of entries $\alpha_{(n,r)}$, observe that for $1 \leq s \leq \rho$, $\alpha_{(n_s, r)} = 1$ whenever $s = r$ and 0 otherwise, since $q_{n_s}(z) = \sum_{r=1}^{\rho} \alpha_{(n_s, r)} q_{n_r}(z)$. Thus there exists a $\rho \times \rho$ submatrix of N with a non-zero determinant, implying that the collection $\left\{ (\alpha_{(n,r)})_{0 \leq n \leq d_z} \right\}_{1 \leq r \leq \rho}$ is linearly independent.

Conversely, suppose all three statements in the Lemma hold, then the matrix associated to the correspon-

dence $P_{(d_w, d_z)}$ is given by

$$\begin{aligned}
 M_{P_{(d_w, d_z)}} &= \begin{pmatrix} \sum_{r=1}^{\rho} \alpha_{(d_z, r)} \beta_{(d_w, r)} & \sum_{r=1}^{\rho} \alpha_{(d_z-1, r)} \beta_{(d_w, r)} & \cdots & \sum_{r=1}^{\rho} \alpha_{(0, r)} \beta_{(d_w, r)} \\ \sum_{r=1}^{\rho} \alpha_{(d_z, r)} \beta_{(d_w-1, r)} & \sum_{r=1}^{\rho} \alpha_{(d_z-1, r)} \beta_{(d_w-1, r)} & \cdots & \sum_{r=1}^{\rho} \alpha_{(0, r)} \beta_{(d_w-1, r)} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{r=1}^{\rho} \alpha_{(d_z, r)} \beta_{(0, r)} & \sum_{r=1}^{\rho} \alpha_{(d_z-1, r)} \beta_{(0, r)} & \cdots & \sum_{r=1}^{\rho} \alpha_{(0, r)} \beta_{(0, r)} \end{pmatrix} \\
 &= \begin{pmatrix} \beta_{(d_w, 1)} & \beta_{(d_w, 2)} & \cdots & \beta_{(d_w, \rho)} \\ \beta_{(d_w-1, 1)} & \beta_{(d_w-1, 2)} & \cdots & \beta_{(d_w-1, \rho)} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{(0, 1)} & \beta_{(0, 2)} & \cdots & \beta_{(0, \rho)} \end{pmatrix} \begin{pmatrix} \alpha_{(d_z, 1)} & \alpha_{(d_z-1, 1)} & \cdots & \alpha_{(0, 1)} \\ \alpha_{(d_z, 2)} & \alpha_{(d_z-1, 2)} & \cdots & \alpha_{(0, 2)} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{(d_z, \rho)} & \alpha_{(d_z-1, \rho)} & \cdots & \alpha_{(0, \rho)} \end{pmatrix}.
 \end{aligned}$$

Since each of the matrices in the above product is of rank ρ , the full rank factorisation theorem, as one may find in [10, 11], then implies that $\text{Rank} \left(M_{P_{(d_w, d_z)}} \right) = \rho$. \square

We now prove Theorem 3.3.

Proof. In order to prove Theorem 3.3, we consider an irreducible restrictive polynomial correspondence $Q_{(d'_w, d'_z)}$ and prove that for any $d > 1$, the matrix pertaining to the reducible restrictive polynomial correspondence $P_{(dd'_w, dd'_z)} = [Q_{(d'_w, d'_z)}]^d$ given by $M_{P_{(dd'_w, dd'_z)}}$ has rank $d + 1$. In fact, owing to Lemma 6.1, it is sufficient to prove that

$$P_{(dd'_w, dd'_z)}(z, w) = g_0(w)h_0(z) + \cdots + g_d(w)h_d(z),$$

where the coefficients of g_r and h_r satisfy the three conditions stated therein.

Since $Q_{(d'_w, d'_z)}$ is an irreducible restrictive polynomial correspondence, using Theorem 4.1 and Lemma 6.1, we have that $Q_{(d'_w, d'_z)}(z, w) = p_1(w)q_1(z) + p_2(w)q_2(z)$, where

$$p_j(w) = \sum_{n=0}^{d'_z} \alpha_{(n, j)} w^n \text{ and } q_j(z) = \sum_{m=0}^{d'_w} \beta_{(m, j)} z^m \text{ for } j = 1, 2;$$

such that

- (1) either p_1 or p_2 has degree d'_z ;
- (2) either q_1 or q_2 has degree d'_w ;
- (3) there exists no $c_1 \in \mathbb{C}$ such that $p_1 = c_1 p_2$ and
- (4) there exists no $c_2 \in \mathbb{C}$ such that $q_1 = c_2 q_2$.

Then,

$$\begin{aligned} P_{(dd'_w, dd'_z)}(z, w) &= [p_1(w)q_1(z) + p_2(w)q_2(z)]^d \\ &= \sum_{r=0}^d \binom{d}{r} (p_1(w))^{d-r} (p_2(w))^r (q_1(z))^{d-r} (q_2(z))^r. \end{aligned}$$

Define $g_r(w) = \binom{d}{r} (p_1(w))^{d-r} (p_2(w))^r$ and $h_r(z) = (q_1(z))^{d-r} (q_2(z))^r$. Owing to conditions (1) and (2) mentioned above, we have that at least one of the polynomials g_r 's has degree dd'_z and that at least one of the polynomials h_r 's has degree dd'_w .

Thus, what remains to be proven is that the collection of $(d + 1)$ vectors, each being a $(dd'_z + 1)$ -tuple determined by the coefficients of the polynomial g_r is linearly independent for $0 \leq r \leq d$ and so is the corresponding collection of $(d + 1)$ vectors, each being a $(dd'_w + 1)$ -tuple determined by h_r . To do the same, we consider the matrix, say \mathcal{G} of size $(dd'_z + 1) \times (d + 1)$, where each column is filled with the coefficients of g_r in order, for $0 \leq r \leq d$ and prove that there exists a submatrix of \mathcal{G} of order $(d + 1)$ whose determinant is non-zero. Observe that the entries in row t of the matrix \mathcal{G} correspond to the coefficients of $w^{dd'_z-t}$ in each of the polynomials g_r 's. Thus, the entry corresponding to row t and column r of the matrix \mathcal{G} is given by

$$\mathcal{G}_{(t,r)} = \binom{d}{r} \sum_{k_1+k_2=dd'_z-t} [\text{Coefficients of } w^{k_1} \text{ in } p_1^{d-r}] [\text{Coefficients of } w^{k_2} \text{ in } p_2^r].$$

Owing to condition (3), as mentioned above, there exists $0 \leq n_1, n_2 \leq d'_z$ such that $\alpha_{(n_1,1)}\alpha_{(n_2,2)} - \alpha_{(n_2,1)}\alpha_{(n_1,2)} \neq 0$. Without loss of generality, we shall assume that $n_1 = d'_z$ and $n_2 = d'_z - 1$. Then, considering the square submatrix of size $(d + 1)$ formed by the first $(d + 1)$ rows of \mathcal{G} , we note that its determinant is equal to

$$\prod_{r=0}^d \binom{d}{r} (\alpha_{(d'_z,1)}\alpha_{(d'_z-1,2)} - \alpha_{(d'_z-1,1)}\alpha_{(d'_z,2)})^{\frac{d(d+1)}{2}} \neq 0.$$

One way to compute this determinant is to factorise the square submatrix as LUP where L is a lower triangular matrix with diagonal entries 1, P is a permutation matrix and U is an upper triangular matrix with diagonal entries $\binom{d}{r} \alpha_{(d'_z,1)}^{d-2r} (\alpha_{(d'_z,1)}\alpha_{(d'_z-1,2)} - \alpha_{(d'_z-1,1)}\alpha_{(d'_z,2)})^r$ for $0 \leq r \leq d$. For any general $0 \leq n_1, n_2 \leq d'_z$, one must choose an appropriate square submatrix of \mathcal{G} of size $(d + 1)$ to conclude that its determinant is non-zero. This establishes that the coefficients of g_r for $0 \leq r \leq d$ are linearly independent.

The proof for the coefficients of h_r for $0 \leq r \leq d$ being linearly independent follows *mutatis mutandis*. \square

Example 6.2 Recall the polynomial correspondence $P_{(6,4)}$, as stated in Example (2.5), that was obtained by squaring the correspondence $P_{(3,2)}$, as given in Example (2.4). The matrix $M_{P_{(6,4)}}$ pertaining to this correspondence has rank 3.

Further, defining

$$\begin{aligned} h_1(z) &= -z^6 - 2iz^5 - 21z^4 - 24iz^3 - 119z^2 - 22iz + 1, \\ h_2(z) &= 10iz^6 - (10 + 4i)z^5 + (4 + 220i)z^4 - (120 + 48i)z^3 + (8 + 1210i)z^2 \\ &\quad - (110 + 44i)z + 4, \\ h_3(z) &= (25 + 2i)z^6 - (22 + 12i)z^5 + (566 + 44i)z^4 - (264 + 144i)z^3 + (3057 + 242i)z^2 \\ &\quad - (242 + 132i)z + 16, \\ g_1(w) &= w^4 + \frac{-15792 + 11872i}{841}w + \frac{509432 + 776720i}{24389}, \\ g_2(w) &= w^3 + \frac{5050 + 3528i}{841}w + \frac{149563 - 83748i}{24389}, \\ g_3(w) &= w^2 + \frac{34 - 56i}{29}w + \frac{-495 - 952i}{841}, \end{aligned}$$

one observes that $P_{(6,4)}$ can be rewritten as $g_1(w)h_1(z) + g_2(w)h_2(z) + g_3(w)h_3(z)$, as asserted by Lemma 6.1.

Finally, we conclude this manuscript with a more general example, that motivated the proof of Theorem 3.3.

Example 6.3 Consider the irreducible restrictive polynomial correspondence

$$P_{(d_w, d_z)}(z, w) = z^{d_w}w^{d_z} + 1,$$

where $d_w, d_z \geq 2$. Then, we see that for any $n \in \mathbb{Z}_+$,

$$P_{(d_w, d_z)}^n(z, w) = z^{nd_w}w^{nd_z} + \binom{n}{1}z^{(n-1)d_w}w^{(n-1)d_z} + \binom{n}{2}z^{(n-2)d_w}w^{(n-2)d_z} + \dots + 1.$$

Then, setting $g_r(w) = w^{d_z(n-r)}$ and $h_r(z) = \binom{n}{r}z^{d_w(n-r)}$ for $0 \leq r \leq n$, we see that $P_{(d_w, d_z)}^n(z, w)$ can be rewritten as $\sum_{r=0}^n g_r(w)h_r(z)$. Also the matrix of size $(nd_w + 1) \times (nd_z + 1)$, namely $M_{P_{(d_w, d_z)}^n}$ whose entries are given by,

$$a_{j,k} = \begin{cases} \binom{n}{n-r} & \text{for } j = d_w(n-r), k = d_z(n-r) \\ 0 & \text{otherwise} \end{cases}$$

with $0 \leq r \leq n$, has rank $(n + 1)$.

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