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Some log-Minkowski inequalities for L_p-mixed geominimal surface area

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Abstract. In this article, we mainly study log-Minkowski inequalities. We first provide the definition of L_p-mixed geominimal surface area for probability measure and get relevant log-Minkowski inequalities for nonsymmetric convex bodies. In addition, we also establish the functional inequalities, which are more general form than the log-Minkowski inequalities of L_p-mixed geominimal surface area.

1. Introduction and Main Results

Geominimal surface area first introduced by Petty [25] about 40 years ago, is a significant concept in the classical Brunn-Minkowski theory. The classical Brunn-Minkowski theory has a natural extension called the L_p Brunn-Minkowski theory, which was initiated in the early 1960s when Firey introduced the concept of L_p composition of convex bodies (see [8, 23]) and was born in the works of Lutwak [19, 20]. This theory has attracted increasing interest in recent years, see e.g., [3, 4, 11, 12, 21, 22]. A basic concept in the L_p Brunn-Minkowski theory is the L_p geominimal surface area ($p \geq 1$), introduced by Lutwak [20], which extends Petty's geominimal surface area. Many well-known inequalities were presented in [20, 36]. In [31], Ye introduced the L_p geominimal surface area for all $n \neq p < 1$, which generalizes Lutwak's L_p geominimal surface area for $p \geq 1$. Other major contributions to the L_p geominimal surface area can be found in [26, 33]. Later, the L_p geominimal surface area and its related inequalities were further extended to their Orlicz counterparts, see e.g., [32, 34, 35].

In 2015, Zhu et al. [37] give an integral formula of L_p geominimal surface area by the p-Petty body and introduce the concept of L_p mixed geominimal surface area which is a natural extension of L_p geominimal surface area. Recently, Li et al. [16] defined the (p, q)-mixed geominimal surface areas and obtained affine isoperimetric inequalities. For more research about the (p, q)-mixed geominimal surface areas, we may refer the reader to [7].

A convex body is a compact convex subset of $\mathbb{R}^n$ with nonempty interior, we denote by $\mathcal{K}^n$ the set of convex bodies, by $\mathcal{K}^n_o$ the set of convex bodies containing the origin in their interiors. Let $\mathcal{F}^n$ and $\mathcal{F}^n_o$ denote

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the set of all bodies in $\mathcal{K}^n$ and $\mathcal{K}_o^n$, that have a positive continuous curvature function. Denote by $V(K)$ the n-dimensional volume of a body K. The unit sphere in $\mathbb{R}^n$ will be denoted by S^{n-1}.

In [3], Böröczky, Lutwak, Yang and Zhang first proposed the conjectured log-Brunn-Minkowski inequality and log-Minkowski inequality as follows:

Conjecture A (The conjectured log-Brunn-Minkowski inequality). *If K and L are origin-symmetric convex bodies, then for $\lambda \in [0,1]$,*

$$V((1-\lambda)\cdot K +_0 \lambda \cdot L) \geq V(K)^{1-\lambda} V(L)^{\lambda},$$

where $(1-\lambda)\cdot K +_0 \lambda \cdot L$ denotes the log-Minkowski combination of K and L.

Conjecture B (The conjectured log-Minkowski inequality). *If K and L are origin-symmetric convex bodies, then the following inequality holds*

$$\int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right) d\overline{V}_L \geq \frac{1}{n} \ln\left(\frac{V(K)}{V(L)}\right),$$

where $dV_L = \frac{1}{n} h_L(u) dS(L,u)$ denotes the cone-volume measure of L for $u \in S^{n-1}$ and $d\overline{V}_L = \frac{1}{V(L)} dV_L$ is its normalization.

Meanwhile, for origin-symmetric convex bodies, Böröczky et al. [3] established a log-Brunn–Minkowski inequality and a corresponding log-Minkowski inequality that are both stronger than their classical counterparts; they also proved the equivalence of these two inequalities in the plane. Later, Stancu [24] validated the conjectured log-Minkowski inequality and proved a modified log-Minkowski inequality for n-dimensional convex bodies as follows:

Theorem 1.A. *If $K, L \in \mathcal{K}_o^n$, then*

$$\int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right) d\overline{V}_1(L,K) \geq \ln\left(\frac{V_1(L,K)}{V(L)}\right) \geq \frac{1}{n} \ln\left(\frac{V(K)}{V(L)}\right),$$

where $dv_1(L,K)$ denotes the mixed cone-volume measure of L and K, and $d\overline{V}_1(L,K) = \frac{dv_1(L,K)}{V_1(L,K)}$ denotes its normalization. Equality holds if and only if K is homothetic to L.

Böröczky's work inspired much subsequent research on this topic. In [30], Xi and Leng gave a positive answer to Dar's conjecture for all planar convex bodies and gained a general form of the log-Brunn-Minkowski inequality. In 2017, Wang and Feng [27] proved log-Minkowski inequality for mixed quermassintegrals which is more general than Stancu's results. Moreover, Wang, Xu and Zhou [29] proposed p-mixed cone-volume measure and proved the log-Minkowski inequality for L_p-mixed volumes. After, Li and Wang [17] obtained the log-Minkowski inequality for the L_p-mixed quermassintegrals. In the very recent article [18], the authors got some log-Minkowski inequalities for L_p-mixed affine surface area. Research of log-Minkowski inequality has achieved great developments, see e.g.,[1, 2, 4, 6, 13, 14, 28].

A convex body $K \in \mathcal{K}_o^n$ has L_p curvature function $f_p(K,\cdot) : S^{n-1} \mapsto \mathbb{R}$, if the L_p surface area measure is absolutely continuous with respect to spherical Lebesgue measure, i.e.,

$$\frac{dS_p(K,\cdot)}{dS(\cdot)} = f_p(K,\cdot). \tag{1.1}$$

When $p = 1$ and $u \in S^{n-1}$ in (1.1), $f_1(K,u) = f(K,u)$ is just the curvature function of K at u. In this paper, we write $f_p(K,u) = f_{p,K}$ and $f(K,u) = f_K$. And the Radon-Nikodym derivative is [19]

$$dS_p(K,\cdot) = h^{1-p}(K,\cdot) dS(K,\cdot), \tag{1.2}$$

where the surface area measure $S(K,\cdot)$ of K is absolutely continuous with respect to spherical Lebesgue measure S. It follows from (1.2) that the measure $S_1(K,\cdot)$ is just the classical surface area measure $S(K,\cdot)$ of K.

By (1.1) and (1.2), we know that

$$f_p(K,u) = h^{1-p}(K,u) f(K,u),$$

for a convex body K in $\mathbb{R}^n$ and $u \in S^{n-1}$.

In [37], Zhu et al. defined the L_p geominimal surface area by the p-Petty body: For each $K \in \mathcal{F}_o^n$ and $p \geq 1$, there exists a unique convex body $T = T_pK \in \mathcal{T}^n$ with

$$G_p(K) = \int_{S^{n-1}} h_T^p(u) f_p(K,u) dS(u).$$

Let $g_p(K,u) = h_T^p(u) f_p(K,u)$. Then, the $G_p(K)$ can be written as follows:

$$G_p(K) = \int_{S^{n-1}} g_p(K,u) dS(u). \tag{1.3}$$

Definition of the p-Petty body of K given by Lutwak in [20]: For each $K \in \mathcal{K}_o^n$ and $p \geq 1$, there exists a unique body $T_pK \in \mathcal{T}^n$ with $G_p(K) = nV_p(K, T_pK)$.

Furthermore, Zhu et al. [37] introduced the concept of L_p mixed geominimal surface area: For $p \geq 1$ and $i \in \mathbb{R}$, the L_p mixed geominimal surface area, $G_{p,i}(K,L)$, of $K, L \in \mathcal{F}_o^n$ is defined by

$$G_{p,i}(K,L) = \int_{S^{n-1}} g_p(K,u)^{\frac{n-i}{n}} g_p(L,u)^{\frac{i}{n}} dS(u). \tag{1.4}$$

Here $G_{p,i}(K,L) = G_p(\underbrace{K,\cdots,K}_{n-i},\underbrace{L,\cdots,L}_{i})$. For $p = 1$, (1.4) is the classical counterparts.

Now, we begin to define the L_p-mixed geominimal surface area probability measure.

Definition 1.1. *For $K, L \in \mathcal{F}_o^n$, $p \geq 1$ and $i \in \mathbb{R}$, we define the L_p-mixed geominimal surface area measure, $d\phi_{p,i}(L,K)$, by*

$$d\phi_{p,i}(L,K) = g_p(L,u)^{\frac{n-i}{n}} g_p(K,u)^{\frac{i}{n}} dS(u). \tag{1.5}$$

Based on this, the L_p-mixed geominimal surface area probability measure is written by

$$d\overline{G}_{p,i}(L,K) = \frac{1}{G_{p,i}(L,K)} d\phi_{p,i}(L,K). \tag{1.6}$$

When taking $i = 0$ in (1.5) and (1.6), then we obtain the L_p-geominimal surface area measure as follows:

$$d\phi_p(L) = g_p(L,u) dS(u). \tag{1.7}$$

Its normalization is the L_p-geominimal surface area probability measure:

$$d\overline{G}_p(L) = \frac{1}{G_p(L)} d\phi_p(L). \tag{1.8}$$

If we take $p = 1$ in (1.5) and (1.6), then $d\phi_i(L,K) = g(L,u)^{\frac{n-i}{n}} g(K,u)^{\frac{i}{n}} dS(u)$ is the mixed geominimal surface area measure, $d\overline{G}_i(L,K) = \frac{1}{G_i(L,K)} d\phi_i(L,K)$ denotes its normalization; if $p = 1$ in (1.7) and (1.8), then $d\phi(L) = g(L,u)dS(u)$ is the geominimal surface area measure, $d\overline{G}(L) = \frac{1}{G(L)} d\phi(L)$ denotes its normalization.

Our main results are as follows for the L_p-mixed geominimal surface area probability measure. The following result is a generalization of the log-Minkowski inequalities.

Theorem 1.1. *Let $K, L \in \mathcal{F}_o^n$ and $n \neq p \geq 1$. If $i > n$, then*

$$\int_{S^{n-1}} \ln\left(\frac{g_{p,K}}{g_{p,L}}\right) d\overline{G}_{p,i}(L,K) \geq \frac{n}{i} \ln\left(\frac{G_{p,i}(L,K)}{G_p(L)}\right) \geq \ln\left(\frac{G_p(K)}{G_p(L)}\right); \tag{1.9}$$

if $i < 0$, then

$$\int_{S^{n-1}} \ln\left(\frac{g_{p,K}}{g_{p,L}}\right) d\overline{G}_{p,i}(L,K) \leq \frac{n}{i} \ln\left(\frac{G_{p,i}(L,K)}{G_p(L)}\right) \leq \ln\left(\frac{G_p(K)}{G_p(L)}\right). \tag{1.10}$$

Each inequality holds as an equality if $g_{p,K}/g_{p,L}$ is a constant.

The following theorem is an improved version of (1.9).

Theorem 1.2. *Let $K, L \in \mathcal{F}_o^n$ and $n \neq p \geq 1$. If $i > 0$, then*

$$\int_{S^{n-1}} \ln\left(\frac{g_{p,K}}{g_{p,L}}\right) d\overline{G}_p(L) \leq \frac{n}{i} \ln\left(\frac{G_{p,i}(L,K)}{G_p(L)}\right) \leq \int_{S^{n-1}} \ln\left(\frac{g_{p,K}}{g_{p,L}}\right) d\overline{G}_{p,i}(L,K), \tag{1.11}$$

in each case, equality holds if and only if $g_{p,K}/g_{p,L}$ is a constant.

Next, for convenience, we will denote

$$\left(\frac{g_{p,K}}{g_{p,L}}\right)_{\text{p-average}} = \frac{\int_{S^{n-1}} (\frac{g_{p,K}}{g_{p,L}})^{\frac{i}{n}} d\phi_p(L)}{\int_{S^{n-1}} d\phi_p(L)},$$

$$\left(\frac{g_{p,K}}{g_{p,L}}\right)_{\max} = \max_{u \in \operatorname{supp}\phi_p(L)} \frac{g_{p,K}}{g_{p,L}},$$

$$\left(\frac{g_{p,K}}{g_{p,L}}\right)_{\min} = \min_{u \in \operatorname{supp}\phi_p(K)} \frac{g_{p,K}}{g_{p,L}},$$

here $\operatorname{supp}\phi_p(L)$ and $\operatorname{supp}\phi_p(K)$ denotes the support of the L_p-geominimal surface area measure of $\phi_p(L)$ and $\phi_p(K)$. Our result can be stated as below:

Theorem 1.3. *Let $K, L \in \mathcal{F}_o^n$ with $g_{p,L} \leq g_{p,K}$, $n \neq p \geq 1$ and $i > n$, then*

$$\int_{S^{n-1}} \ln\left(\frac{g_{p,K}}{g_{p,L}}\right) d\overline{G}_p(L) \geq \frac{\left(\frac{g_{p,K}}{g_{p,L}}\right)_{\text{p-average}}}{\left(\frac{g_{p,K}}{g_{p,L}}\right)_{\max}^{\frac{i}{n}}} \ln\left(\frac{G_p(K)}{G_p(L)}\right), \tag{1.12}$$

with equality holds if and only if $g_{p,K} = g_{p,L}$.

In general, if $K, L \in \mathcal{F}_o^n$, then

$$\int_{S^{n-1}} \ln\left(\frac{g_{p,K}}{g_{p,L}}\right) d\overline{G}_p(L) \geq \frac{\left(\frac{g_{p,K}}{g_{p,L}}\right)_{\text{p-average}}}{\left(\frac{g_{p,K}}{g_{p,L}}\right)_{\max}^{\frac{i}{n}}} \ln\left(\frac{G_p(K)}{G_p(L)}\right) + \ln\left[\left(\frac{g_{p,K}}{g_{p,L}}\right)_{\min}\right]\left[1 - \frac{\left(\frac{g_{p,K}}{g_{p,L}}\right)_{\text{p-average}}}{\left(\frac{g_{p,K}}{g_{p,L}}\right)_{\max}^{\frac{i}{n}}}\right], \tag{1.13}$$

equality holds if and only if $g_{p,K}/g_{p,L}$ is a constant.

The rest of this paper is organized as follows: The next section is devoted to prove main results, we also obtain some results about the log-Minkowski inequality for geominimal surface area. Along with the study in Section 2, some functional inequalities for L_p-mixed geominimal surface area are provided in the last Section.

2. Proofs of Theorems 1.1-1.3.

This section is dedicated to give the proofs of Theorems 1.1-1.3. To prove Theorem 1.1 we require the following lemma:

Lemma 2.1([37]). *If $K, L \in \mathcal{F}_o^n$, $n \neq p \geq 1$, $i \in \mathbb{R}$, then for $i < 0$ and $i > n$,*

$$G_{p,i}(L,K) \geq G_p(L)^{\frac{n-i}{n}} G_p(K)^{\frac{i}{n}}; \tag{2.1}$$

for $0 < i < n$,

$$G_{p,i}(L,K) \leq G_p(L)^{\frac{n-i}{n}} G_p(K)^{\frac{i}{n}}. \tag{2.2}$$

Each inequality holds as an equality if $g_{p,K}/g_{p,L}$ is a constant. For $i=0$ or $i=n$, (2.1) or (2.2) is identical.

Proof of Theorem 1.1. For $K,L\in\mathcal{F}_o^n$, $n\neq p\geq 1$ and for all $u\in S^{n-1}$, it follows from (1.5) and (1.7) that

$$\int_{S^{n-1}}\left(\frac{g_{p,K}}{g_{p,L}}\right)^{\frac{i}{n}}\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)d\phi_p(L)=\int_{S^{n-1}}\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)d\phi_{p,i}(L,K). \tag{2.3}$$

By Lebesgue's dominated convergence theorem, and combined with formula (1.4), (1.7) and (2.3), as $t\mapsto\infty$,

$$\int_{S^{n-1}}\left(\frac{g_{p,K}}{g_{p,L}}\right)^{\frac{i}{n}\cdot\frac{t}{t+n}}d\phi_p(L)\mapsto G_{p,i}(L,K), \tag{2.4}$$

and

$$\int_{S^{n-1}}\left(\frac{g_{p,K}}{g_{p,L}}\right)^{\frac{i}{n}\cdot\frac{t}{t+n}}\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)d\phi_p(L)\mapsto\int_{S^{n-1}}\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)d\phi_{p,i}(L,K). \tag{2.5}$$

Considering the function $M(K,L;t):[1,\infty)\mapsto\mathbb{R}$ defined by

$$M(K,L;t)=\frac{1}{G_{p,i}(L,K)}\int_{S^{n-1}}\left(\frac{g_{p,K}}{g_{p,L}}\right)^{\frac{i}{n}\cdot\frac{t}{t+n}}d\phi_p(L), \tag{2.6}$$

using L'Hôpital's rule, together (2.4) with (2.5), give that

$$\begin{aligned}\lim_{t\mapsto\infty}\ln M(K,L;t)^{t+n}&=\lim_{t\mapsto\infty}\frac{\ln M(K,L;t)}{\frac{1}{t+n}}=\lim_{t\mapsto\infty}\frac{M'(K,L;t)}{-\frac{M(K,L;t)}{(t+n)^2}}\\
&=\lim_{t\mapsto\infty}\frac{\frac{1}{G_{p,i}(L,K)}\cdot\frac{i}{(t+n)^2}\int_{S^{n-1}}(\frac{g_{p,K}}{g_{p,L}})^{\frac{i}{n}\cdot\frac{t}{t+n}}\ln(\frac{g_{p,K}}{g_{p,L}})d\phi_p(L)}{-\frac{M(K,L;t)}{(t+n)^2}}\\
&=\lim_{t\mapsto\infty}-\frac{\frac{i}{G_{p,i}(L,K)}\int_{S^{n-1}}(\frac{g_{p,K}}{g_{p,L}})^{\frac{i}{n}\cdot\frac{t}{t+n}}\ln(\frac{g_{p,K}}{g_{p,L}})d\phi_p(L)}{\frac{1}{G_{p,i}(L,K)}\int_{S^{n-1}}(\frac{g_{p,K}}{g_{p,L}})^{\frac{i}{n}\cdot\frac{t}{t+n}}d\phi_p(L)}\\
&=-\frac{i}{G_{p,i}(L,K)}\int_{S^{n-1}}\left(\frac{g_{p,K}}{g_{p,L}}\right)^{\frac{i}{n}}\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)d\phi_p(L).\end{aligned} \tag{2.7}$$

It follows from (2.6) and (2.7),

$$\begin{aligned}\exp&\left[-\frac{i}{G_{p,i}(L,K)}\int_{S^{n-1}}\left(\frac{g_{p,K}}{g_{p,L}}\right)^{\frac{i}{n}}\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)d\phi_p(L)\right]\\
&=\lim_{t\mapsto\infty}M(K,L;t)^{t+n}\\
&=\lim_{t\mapsto\infty}\left[\frac{1}{G_{p,i}(L,K)}\int_{S^{n-1}}\left(\frac{g_{p,K}}{g_{p,L}}\right)^{\frac{i}{n}\cdot\frac{t}{t+n}}d\phi_p(L)\right]^{t+n}.\end{aligned} \tag{2.8}$$

By Hölder's inequality ([10]), (1.3), (1.4) and (1.7), we have

$$\left[\int_{S^{n-1}}\left(\frac{g_{p,K}}{g_{p,L}}\right)^{\frac{i}{n}\cdot\frac{t}{t+n}}d\phi_p(L)\right]^{\frac{t+n}{t}}\cdot\left[\int_{S^{n-1}}d\phi_p(L)\right]^{-\frac{n}{t}}\leq\int_{S^{n-1}}\left(\frac{g_{p,K}}{g_{p,L}}\right)^{\frac{i}{n}}d\phi_p(L)=G_{p,i}(L,K).$$

Thus,

$$\left[\frac{1}{G_{p,i}(L,K)}\int_{S^{n-1}}\left(\frac{g_{p,K}}{g_{p,L}}\right)^{\frac{i}{n}\cdot\frac{t}{t+n}}d\phi_p(L)\right]^{t+n}\leq\left(\frac{G_p(L)}{G_{p,i}(L,K)}\right)^n. \tag{2.9}$$

The equality condition of the Hölder's inequality [10] implies that equality holds in (2.9) if and only if $g_{p,K}/g_{p,L}$ is a constant.

Moreover, together (2.8) with (2.9), it follows that

$$\exp\left[\frac{-i}{G_{p,i}(L,K)}\int_{S^{n-1}}\left(\frac{g_{p,K}}{g_{p,L}}\right)^{\frac{i}{n}}\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)d\phi_p(L)\right]\leq\left(\frac{G_p(L)}{G_{p,i}(L,K)}\right)^n,$$

i.e.,

$$\frac{i}{G_{p,i}(L,K)}\int_{S^{n-1}}\left(\frac{g_{p,K}}{g_{p,L}}\right)^{\frac{i}{n}}\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)d\phi_p(L)\geq n\ln\left(\frac{G_{p,i}(L,K)}{G_p(L)}\right). \tag{2.10}$$

Therefore,

(i) For $i > n$, by (2.10), (1.5), (1.6), (1.7) and (2.1), we obtain

$$\int_{S^{n-1}}\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)d\overline{G}_{p,i}(L,K)\geq\frac{n}{i}\ln\left(\frac{G_{p,i}(L,K)}{G_p(L)}\right)\geq\frac{n}{i}\ln\left(\frac{G_p(L)^{\frac{n-i}{n}}G_p(K)^{\frac{i}{n}}}{G_p(L)}\right)=\ln\left(\frac{G_p(K)}{G_p(L)}\right).$$

This yields inequality (1.9).

(ii) For $i < 0$, by (2.10), (1.5), (1.6), (1.7) and (2.1), we have

$$\int_{S^{n-1}}\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)d\overline{G}_{p,i}(L,K)\leq\frac{n}{i}\ln\left(\frac{G_{p,i}(L,K)}{G_p(L)}\right)\leq\frac{n}{i}\ln\left(\frac{G_p(L)^{\frac{n-i}{n}}G_p(K)^{\frac{i}{n}}}{G_p(L)}\right)=\ln\left(\frac{G_p(K)}{G_p(L)}\right),$$

which is exactly the required inequality (1.10).

The equality condition of (1.9) and (1.10) directly follows from the equality condition of inequality (2.1). □

Using (1.9) of Theorem 1.1, we have the following result.

Corollary 2.1. *If $K, L \in \mathcal{F}_o^n$ with $g_{p,L}(u) \leq g_{p,K}(u)$, $n \neq p \geq 1$ and $i > n$, then*

$$\int_{S^{n-1}}\left(\frac{g_{p,K}}{g_{p,L}}\right)^{\frac{i}{n}}\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)d\overline{G}_p(L)\geq\left(\frac{G_p(K)}{G_p(L)}\right)^{\frac{i}{n}}\ln\left(\frac{G_p(K)}{G_p(L)}\right),$$

with equality if and only if $g_{p,K}/g_{p,L}$ is a constant.

Proof. From (1.5), (1.6), (1.7), (1.8), (1.9) and (2.1), for all $u \in S^{n-1}$,

$$\begin{aligned}\int_{S^{n-1}}\left(\frac{g_{p,K}}{g_{p,L}}\right)^{\frac{i}{n}}\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)d\overline{G}_p(L)&=\frac{1}{G_p(L)}\int_{S^{n-1}}\left(\frac{g_{p,K}}{g_{p,L}}\right)^{\frac{i}{n}}\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)d\phi_p(L)\\&=\frac{G_{p,i}(L,K)}{G_p(L)}\int_{S^{n-1}}\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)d\overline{G}_{p,i}(L,K)\\&\geq\frac{G_{p,i}(L,K)}{G_p(L)}\cdot\frac{n}{i}\ln\left(\frac{G_{p,i}(L,K)}{G_p(L)}\right)\\&\geq\left(\frac{G_p(K)}{G_p(L)}\right)^{\frac{i}{n}}\ln\left(\frac{G_p(K)}{G_p(L)}\right).\end{aligned}$$

The equality conditions for the above inequalities can be derived from the equality conditions in (1.9) and (2.1). □

Setting $p = 1$ in Theorem 1.1 yields the log-Minkowski inequality for mixed geominimal surface area as follows:

Corollary 2.2. *Let $K, L \in \mathcal{F}_o^n$, if $i > n$, then*

$$\int_{S^{n-1}} \ln\left(\frac{g_K}{g_L}\right) d\overline{G}_i(L,K) \geq \frac{n}{i} \ln\left(\frac{G_i(L,K)}{G(L)}\right) \geq \ln\left(\frac{G(K)}{G(L)}\right);$$

if $i < 0$, then

$$\int_{S^{n-1}} \ln\left(\frac{g_K}{g_L}\right) d\overline{G}_i(L,K) \leq \frac{n}{i} \ln\left(\frac{G_i(L,K)}{G(L)}\right) \leq \ln\left(\frac{G(K)}{G(L)}\right),$$

in each case, equality holds if and only if $g_{p,K}/g_{p,L}$ is a constant.

Lemma 2.2 ([5, 15]). *Let $f(x)$ and $r(x)$ be the probability density functions on a measure space (X, v), for v-almost all $x \in X$, if $\int_X f(x)dv(x) = 1$, $\int_X r(x)dv(x) = 1$, then*

$$\int_X f(x) \ln f(x) dv(x) \geq \int_X f(x) \ln r(x) dv(x),$$

with equality if and only if $f(x) = r(x)$.

Proof of Theorem 1.2. For $K, L \in \mathcal{F}_o^n$, $n \neq p \geq 1$, $i > 0$ and for all $u \in S^{n-1}$, let

$$f(u) = \frac{1}{G_p(L)}\left(\frac{g_{p,L}}{g_{p,K}}\right)^{\frac{i}{n}}, \quad r(u) = \frac{1}{G_{p,i}(L,K)}, \quad dv(u) = d\phi_{p,i}(L,K),$$

then

$$\int_{S^{n-1}} f(u)dv(u) = 1, \quad \int_{S^{n-1}} r(u)dv(u) = 1.$$

This together with Lemma 2.2 yields that

$$\int_{S^{n-1}} \frac{1}{G_p(L)}\left(\frac{g_{p,L}}{g_{p,K}}\right)^{\frac{i}{n}} \ln\left[\frac{1}{G_p(L)}\left(\frac{g_{p,L}}{g_{p,K}}\right)^{\frac{i}{n}}\right] d\phi_{p,i}(L,K) \geq \int_{S^{n-1}} \frac{1}{G_p(L)}\left(\frac{g_{p,L}}{g_{p,K}}\right)^{\frac{i}{n}} \ln\left(\frac{1}{G_{p,i}(L,K)}\right) d\phi_{p,i}(L,K).$$

Thus

$$\int_{S^{n-1}} \frac{1}{G_p(L)}\left(\frac{g_{p,L}}{g_{p,K}}\right)^{\frac{i}{n}} \ln\left(\frac{g_{p,L}}{g_{p,K}}\right)^{\frac{i}{n}} d\phi_{p,i}(L,K) \geq \ln\left(\frac{G_p(L)}{G_{p,i}(L,K)}\right) \int_{S^{n-1}} \frac{1}{G_p(L)}\left(\frac{g_{p,L}}{g_{p,K}}\right)^{\frac{i}{n}} d\phi_{p,i}(L,K). \tag{2.11}$$

By $\int_{S^{n-1}} \frac{1}{G_p(L)}\left(\frac{g_{p,L}}{g_{p,K}}\right)^{\frac{i}{n}} d\phi_{p,i}(L,K) = 1$, (1.5), (1.7) and (1.8), we get

$$\int_{S^{n-1}} \ln\left(\frac{g_{p,K}}{g_{p,L}}\right) d\overline{G}_p(L) \leq \frac{n}{i} \ln\left(\frac{G_{p,i}(L,K)}{G_p(L)}\right). \tag{2.12}$$

The equality condition of inequality (2.12) follows from the equality condition of the Lemma 2.2, i.e., $g_{p,K}/g_{p,L}$ is a constant.

In the same way, taking

$$f(u) = \frac{1}{G_{p,i}(L,K)}, \quad r(u) = \frac{1}{G_p(L)}\left(\frac{g_{p,L}}{g_{p,K}}\right)^{\frac{i}{n}}, \quad dv(u) = d\phi_{p,i}(L,K),$$

combined Lemma 2.2, we deduce

$$\frac{n}{i} \ln\left(\frac{G_{p,i}(L,K)}{G_p(L)}\right) \leq \int_{S^{n-1}} \ln\left(\frac{g_{p,K}}{g_{p,L}}\right) d\overline{G}_{p,i}(L,K), \tag{2.13}$$

with equality if and only if $g_{p,K}/g_{p,L}$ is a constant.

Thus by (2.12) and (2.13), we have desired result (1.11). □

Taking $p = 1$ in Theorem 1.2, for $i > 0$, the following is a direct result which is stronger than the Corollary 2.2.

Corollary 2.3. *Let $K, L \in \mathcal{F}_o^n$, if $i > 0$, then*

$$\int_{S^{n-1}} \ln\left(\frac{g_K}{g_L}\right) d\overline{G}(L) \leq \frac{n}{i} \ln\left(\frac{G_i(L,K)}{G(L)}\right) \leq \int_{S^{n-1}} \ln\left(\frac{g_K}{g_L}\right) d\overline{G}_i(L,K),$$

in each case, with equality if and only if $g_{p,K}/g_{p,L}$ is a constant.

Lemma 2.3 (Hadamard type inequality [9]). *Let f be a positive, log-convex function on $[a,b]$, then*

$$\frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(b)-f(a)}{\ln f(b) - \ln f(a)}.$$

Proof of Theorem 1.3. Consider the function

$$B(q) = \int_{S^{n-1}} \left(\frac{g_{p,K}}{g_{p,L}}\right)^{q\cdot \frac{i}{n}} \ln\left(\frac{g_{p,K}}{g_{p,L}}\right) d\phi_p(L), \quad q \in \mathbb{R}. \tag{2.14}$$

For $g_{p,K} \geq g_{p,L}$, we obtain $\left(\frac{g_{p,K}}{g_{p,L}}\right)^{q\cdot\frac{i}{n}} \ln\left(\frac{g_{p,K}}{g_{p,L}}\right) \geq 0$, thus $B(q)$ is non-negative. If $u \to \ln\left(\frac{g_{p,K}}{g_{p,L}}\right)(u)$ is zero on the support of the L_p-geominimal surface area measure of L, then B is identically zero. If B is not identically zero, we know $B(1) \geq B(0) > 0$ by (2.14). If $B(1) = B(0)$, then $g_{p,K}$ must be equal to $g_{p,L}$. So we assume $B(1) > B(0)$.

Next, we verify that $B(q)$ is a log-convex function. In fact, by (2.14) and Hölder's inequality [10], it follows that for $\lambda \in (0,1)$ and $\mu, \nu \in \mathbb{R}$,

$$\begin{aligned}
B((1-\lambda)\mu + \lambda\nu) &= \int_{S^{n-1}} \left(\frac{g_{p,K}}{g_{p,L}}\right)^{((1-\lambda)\mu+\lambda\nu)\cdot\frac{i}{n}} \ln\left(\frac{g_{p,K}}{g_{p,L}}\right) d\phi_p(L) \\
&= \int_{S^{n-1}} \left(\frac{g_{p,K}}{g_{p,L}}\right)^{\mu(1-\lambda)\cdot\frac{i}{n}} \left(\frac{g_{p,K}}{g_{p,L}}\right)^{\lambda\nu\cdot\frac{i}{n}} \ln\left(\frac{g_{p,K}}{g_{p,L}}\right) d\phi_p(L) \\
&\leq \left[\int_{S^{n-1}} \left(\frac{g_{p,K}}{g_{p,L}}\right)^{\mu\cdot\frac{i}{n}} \ln\left(\frac{g_{p,K}}{g_{p,L}}\right) d\phi_p(L)\right]^{(1-\lambda)} \times \left[\int_{S^{n-1}} \left(\frac{g_{p,K}}{g_{p,L}}\right)^{\nu\cdot\frac{i}{n}} \ln\left(\frac{g_{p,K}}{g_{p,L}}\right) d\phi_p(L)\right]^{\lambda} \\
&= B(\mu)^{1-\lambda} B(\nu)^{\lambda}.
\end{aligned}$$

It follows from (2.14) and Hadamard type inequality that

$$\frac{B(1)-B(0)}{\ln B(1) - \ln B(0)} \geq \int_0^1 \left[\int_{S^{n-1}} \left(\left(\frac{g_{p,K}}{g_{p,L}}\right)^{\frac{i}{n}}\right)^q \ln\left(\frac{g_{p,K}}{g_{p,L}}\right) d\phi_p(L)\right] dq. \tag{2.15}$$

We see that $B(1) > B(0)$, together (2.15) with Fubini-Toneli's theorem, we get

$$\begin{aligned}
B(0) &\geq B(1) \cdot \exp\left[-\frac{B(1)-B(0)}{\int_0^1 \int_{S^{n-1}} (\frac{g_{p,K}}{g_{p,L}})^{\frac{i}{n}\cdot q} \ln(\frac{g_{p,K}}{g_{p,L}}) d\phi_p(L) dq}\right] \\
&= B(1) \cdot \exp\left[-\frac{B(1)-B(0)}{\int_{S^{n-1}} [\int_0^1 (\frac{g_{p,K}}{g_{p,L}})^{\frac{i}{n}\cdot q} \ln(\frac{g_{p,K}}{g_{p,L}}) dq] d\phi_p(L)}\right]
\end{aligned}$$

$$= B(1)\cdot\exp\left[-\frac{i}{n}\cdot\frac{B(1)-B(0)}{\int_{S^{n-1}}[(\frac{g_{p,K}}{g_{p,L}})^{\frac{i}{n}}-1]d\phi_p(L)}\right]. \tag{2.16}$$

In (2.16), note that

$$\frac{B(1)-B(0)}{\int_{S^{n-1}}[(\frac{g_{p,K}}{g_{p,L}})^{\frac{i}{n}}-1]d\phi_p(L)}=\frac{\int_{S^{n-1}}\ln(\frac{g_{p,K}}{g_{p,L}})[(\frac{g_{p,K}}{g_{p,L}})^{\frac{i}{n}}-1]d\phi_p(L)}{\int_{S^{n-1}}[(\frac{g_{p,K}}{g_{p,L}})^{\frac{i}{n}}-1]d\phi_p(L)}\le\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)_{\max}.$$

This together with (2.16) gives

$$\int_{S^{n-1}}\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)d\phi_p(L)\ge\int_{S^{n-1}}\left(\frac{g_{p,K}}{g_{p,L}}\right)^{\frac{i}{n}}\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)d\phi_p(L)\cdot\exp\left[\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)_{\max}^{-\frac{i}{n}}\right].$$

Thus,

$$\frac{1}{G_p(L)}\int_{S^{n-1}}\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)d\phi_p(L)\ge\frac{G_{p,i}(L,K)}{G_p(L)}\frac{1}{G_{p,i}(L,K)}\left(\frac{g_{p,K}}{g_{p,L}}\right)_{\max}^{-\frac{i}{n}}\int_{S^{n-1}}\left(\frac{g_{p,K}}{g_{p,L}}\right)^{\frac{i}{n}}\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)d\phi_p(L).$$

Then, by (1.5), (1.6), (1.7), (1.8) and (1.9), we obtain

$$\begin{aligned}\int_{S^{n-1}}\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)d\overline{G}_p(L)&\ge\frac{G_{p,i}(L,K)}{G_p(L)}\cdot\left(\frac{g_{p,K}}{g_{p,L}}\right)_{\max}^{-\frac{i}{n}}\cdot\int_{S^{n-1}}\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)d\overline{G}_{p,i}(L,K)\\&=\frac{\left(\frac{g_{p,K}}{g_{p,L}}\right)_{\text{p-average}}}{\left(\frac{g_{p,K}}{g_{p,L}}\right)_{\max}^{\frac{i}{n}}}\int_{S^{n-1}}\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)d\overline{G}_{p,i}(L,K)\\&\ge\frac{\left(\frac{g_{p,K}}{g_{p,L}}\right)_{\text{p-average}}}{\left(\frac{g_{p,K}}{g_{p,L}}\right)_{\max}^{\frac{i}{n}}}\ln\left(\frac{G_p(K)}{G_p(L)}\right).\end{aligned}$$

From this, we get the inequality (1.12).

Assume that $B(q)$ is identically zero, then $f_p(K,u)=f_p(L,u)$ for almost all u in the support of the L_p-geominimal surface area measure of L, its means $G_{p,i}(L,K)=G_p(L)$. From (2.1), we know that $G_p(K)^{\frac{i}{n}}G_p(L)^{\frac{n-i}{n}}\le G_{p,i}(L,K)=G_p(L)$, i.e., $G_p(K)\le G_p(L)$; since $g_{p,L}\le g_{p,K}$, by (1.3), we know that $G_p(K)\ge G_p(L)$. Thus, from the equality condition of inequality (2.1), we see that equality holds in (1.12) if and only if $g_{p,K}=g_{p,L}$.

Suppose that K and L are arbitrary convex bodies. If L is not included in K, there exists a λ and $0<\lambda<1$, such that $\widetilde{L}:\lambda L\subseteq K$, i.e., $g_{p,\widetilde{L}}=\lambda g_{p,L}\le g_{p,K}$. From this, by (1.3) and (1.12), we get

$$\int_{S^{n-1}}\ln\left(\frac{g_{p,K}}{\lambda g_{p,L}}\right)d\overline{G}_p(L)\ge\frac{\left(\frac{g_{p,K}}{\lambda g_{p,L}}\right)_{\text{p-average}}}{\left(\frac{g_{p,K}}{\lambda g_{p,L}}\right)_{\max}^{\frac{i}{n}}}\ln\left(\frac{G_p(K)}{\lambda G_p(L)}\right).$$

That is,

$$\int_{S^{n-1}}\ln\left(\frac{g_{p,K}}{\lambda g_{p,L}}\right)d\overline{G}_p(L)\ge\frac{\left(\frac{g_{p,K}}{g_{p,L}}\right)_{\text{p-average}}}{\left(\frac{g_{p,K}}{g_{p,L}}\right)_{\max}^{\frac{i}{n}}}\ln\left(\frac{G_p(K)}{\lambda G_p(L)}\right).$$

Thus,

$$\int_{S^{n-1}} \ln\left(\frac{g_{p,K}}{g_{p,L}}\right) d\overline{G}_p(L) \geq \frac{\left(\frac{g_{p,K}}{g_{p,L}}\right)_{\text{p-average}}}{\left(\frac{g_{p,K}}{g_{p,L}}\right)_{\max}^{\frac{i}{n}}} \ln\left(\frac{G_p(K)}{G_p(L)}\right) + \ln\lambda \cdot \left[1 - \frac{\left(\frac{g_{p,K}}{g_{p,L}}\right)_{\text{p-average}}}{\left(\frac{g_{p,K}}{g_{p,L}}\right)_{\max}^{\frac{i}{n}}}\right]. \tag{2.17}$$

Noting that taking $\lambda = \left(\frac{g_{p,K}}{g_{p,L}}\right)_{\min}$ in (2.17), which is the desired result (1.13). According to $\lambda g_{p,L} \leq g_{p,K}$ and the equality condition of inequality (1.12), the equality condition of the above inequality if and only if $g_{p,K}/g_{p,L}$ is a constant. This complete the proof. □

3. Functional inequalities

Having the log-Minkowski inequalities of Lp-mixed geominimal surface area, we can prove more general form—functional inequalities. The main tool of prove is Lemma 3.1.

Theorem 3.1. *Let $K, L \in \mathcal{F}_o^n$ and $\varphi(x) : (0,\infty) \to (0,\infty)$ be a monotonous convex function. If $n \neq p \geq 1$ and $0 < i < n$, then*

$$\varphi\left[\int_{S^{n-1}} \varphi^{-1}\left(\left(\frac{g_{p,K}}{g_{p,L}}\right)^{\frac{i}{n}}\right) d\overline{G}_p(L)\right] \leq \left(\frac{G_p(K)}{G_p(L)}\right)^{\frac{i}{n}}, \tag{3.1}$$

with equality if and only if $g_{p,K}/g_{p,L}$ is a constant.

Lemma 3.1(Jessen's inequality [10]). *Suppose that μ is a probability measure on a space X and $g : X \to I \subset \mathbb{R}$ is a μ-integrable function, where I is a possibly infinites interval. If $\varphi : I \subset \mathbb{R}$ is a convex function, then*

$$\int_X \varphi(g(x)) d\mu(x) \geq \varphi\left(\int_X g(x) d\mu(x)\right),$$

if φ is strictly convex, with equality if and only if $g(x)$ is a constant for μ-almost all $x \in X$.

Proof of Theorem 3.1. Since $K, L \in \mathcal{F}_o^n$ and $\varphi(x) : (0,\infty) \to (0,\infty)$ is a monotonous convex function. It follows from (1.4), (1.7), (1.8) and Lemma 3.1 that

$$\begin{aligned}
G_{p,i}(L,K) &= \int_{S^{n-1}} g_p(L,u)^{\frac{n-i}{n}} g_p(K,u)^{\frac{i}{n}} dS(u) \\
&= \int_{S^{n-1}} g_p(L,u) \left(\frac{g_p(K,u)}{g_p(L,u)}\right)^{\frac{i}{n}} dS(u) \\
&= G_p(L) \int_{S^{n-1}} \left(\frac{g_p(K,u)}{g_p(L,u)}\right)^{\frac{i}{n}} d\overline{G}_p(L) \\
&= G_p(L) \int_{S^{n-1}} \varphi\left[\varphi^{-1}\left(\left(\frac{g_p(K,u)}{g_p(L,u)}\right)^{\frac{i}{n}}\right)\right] d\overline{G}_p(L) \\
&\geq G_p(L) \varphi\left[\int_{S^{n-1}} \varphi^{-1}\left(\left(\frac{g_p(K,u)}{g_p(L,u)}\right)^{\frac{i}{n}}\right) d\overline{G}_p(L)\right].
\end{aligned}$$

This implies

$$\frac{G_{p,i}(L,K)}{G_p(L)} \geq \varphi\left[\int_{S^{n-1}} \varphi^{-1}\left(\left(\frac{g_p(K,u)}{g_p(L,u)}\right)^{\frac{i}{n}}\right) d\overline{G}_p(L)\right]. \tag{3.2}$$

The equality condition of the Lemma 3.1 yields that equality in (3.2) holds if and only if $\varphi^{-1}\left(\left(\frac{g_p(K,u)}{g_p(L,u)}\right)^{\frac{i}{n}}\right)$ is a constant, i.e., $g_{p,K}/g_{p,L}$ is a constant.

Combining inequalities (2.2) and (3.2), it follows that

$$\varphi\left[\int_{S^{n-1}}\varphi^{-1}\left(\left(\frac{g_p(K,u)}{g_p(L,u)}\right)^{\frac{i}{n}}\right)d\overline{G}_p(L)\right]\le\frac{G_{p,i}(L,K)}{G_p(L)}\le\left(\frac{G_p(K)}{G_p(L)}\right)^{\frac{i}{n}}.$$

The equality condition of (3.1) directly follows from the equality condition of (2.2) and (3.2), with equality if and only if $g_{p,K}/g_{p,L}$ is a constant. This completes the proof. □

Theorem 3.2. *Let $K,L\in\mathcal{F}_o^n$ and $\varphi(x):(0,\infty)\to(0,\infty)$ be a monotonous convex function. If $n\neq p\ge 1$ and $0<i<n$, then*

$$\varphi\left[\int_{S^{n-1}}\varphi^{-1}\left(\left(\frac{g_{p,L}}{g_{p,K}}\right)^{\frac{n-i}{n}}\right)d\overline{G}_p(K)\right]\le\left(\frac{G_p(L)}{G_p(K)}\right)^{\frac{n-i}{n}},\tag{3.3}$$

with equality if and only if $g_{p,K}/g_{p,L}$ is a constant.

Proof of Theorem 3.2. From (1.4), (1.7), (1.8) and Lemma 3.1, we see that

$$\begin{aligned}G_{p,i}(L,K)&=\int_{S^{n-1}}g_p(L,u)^{\frac{n-i}{n}}g_p(K,u)^{\frac{i}{n}}dS(u)\\&=\int_{S^{n-1}}g_p(K,u)\left(\frac{g_p(L,u)}{g_p(K,u)}\right)^{\frac{n-i}{n}}dS(u)\\&=G_p(K)\int_{S^{n-1}}\left(\frac{g_p(L,u)}{g_p(K,u)}\right)^{\frac{n-i}{n}}d\overline{G}_p(K)\\&=G_p(K)\int_{S^{n-1}}\varphi\left[\varphi^{-1}\left(\left(\frac{g_p(L,u)}{g_p(K,u)}\right)^{\frac{n-i}{n}}\right)\right]d\overline{G}_p(K)\\&\ge G_p(K)\varphi\left[\int_{S^{n-1}}\varphi^{-1}\left(\left(\frac{g_p(L,u)}{g_p(K,u)}\right)^{\frac{n-i}{n}}\right)d\overline{G}_p(K)\right],\end{aligned}$$

i.e.,

$$\frac{G_{p,i}(L,K)}{G_p(K)}\ge\varphi\left[\int_{S^{n-1}}\varphi^{-1}\left(\left(\frac{g_p(L,u)}{g_p(K,u)}\right)^{\frac{n-i}{n}}\right)d\overline{G}_p(K)\right].\tag{3.4}$$

The equality condition of the Lemma 3.1 yields that equality in (3.4) holds if and only if $\varphi^{-1}\left(\left(\frac{g_p(L,u)}{g_p(K,u)}\right)^{\frac{n-i}{n}}\right)$ is a constant, i.e., $g_{p,K}/g_{p,L}$ is a constant.

Together (2.2) with (3.4), we have

$$\varphi\left[\int_{S^{n-1}}\varphi^{-1}\left(\left(\frac{g_p(L,u)}{g_p(K,u)}\right)^{\frac{n-i}{n}}\right)d\overline{G}_p(K)\right]\le\frac{G_{p,i}(L,K)}{G_p(K)}\le\left(\frac{G_p(L)}{G_p(K)}\right)^{\frac{n-i}{n}}.$$

This is the desired result (3.3).

By the equality condition of inequality (2.2) and (3.4), there exists equality in (3.3) if and only if $g_{p,K}/g_{p,L}$ is a constant. □

Particularly, for $0<i<n$, let $\varphi(x)=\exp(x)$ in the previous two theorems, we immediately have the following log-Minkowski inequalities.

Corollary 3.1. *For $K,L\in\mathcal{F}_o^n$ and $p\ge 1$, then*

$$\int_{S^{n-1}}\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)d\overline{G}_p(L)\le\ln\left(\frac{G_p(K)}{G_p(L)}\right)\le\int_{S^{n-1}}\ln\left(\frac{g_{p,K}}{g_{p,L}}\right)d\overline{G}_p(K),$$

in each case, with equality if and only if $g_{p,K}/g_{p,L}$ is a constant.

As a direct consequence of Corollary 3.1, when $p = 1$ we can obtain the following log-Minkowski inequality.

Corollary 3.2. *For $K, L \in \mathcal{F}_o^n$ then*

$$\int_{S^{n-1}} \ln\left(\frac{g_K}{g_L}\right) d\overline{G}(L) \leq \ln\left(\frac{G(K)}{G(L)}\right) \leq \int_{S^{n-1}} \ln\left(\frac{g_K}{g_L}\right) d\overline{G}(K),$$

in each case, with equality if and only if g_K/g_L is a constant.

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