



Bi periodic Jacobsthal polynomial sequence and its various binomial transformations

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Abstract. In this paper, we introduce a bi-periodic Jacobsthal polynomial sequence as a natural polynomial generalization of the bi-periodic Jacobsthal numbers. Explicit forms, generating functions, and Binet-type formulas are derived by exploiting the underlying parity structure of the recurrence relation. We then investigate several classes of binomial-type transformations of this sequence, namely the classical binomial transform, the k -binomial transform, and their rising and falling variants. For each transformation, we establish the recurrence relations, the generating functions, and the explicit formulas. These results provide a unified framework for analyzing binomial transformations of bi-periodic polynomial sequences and extend a number of earlier results obtained for Jacobsthal-type sequences. These results highlight the structural stability of bi-periodic Jacobsthal polynomials under binomial transformations and may serve as a basis for further investigations in combinatorial identities and polynomial sequence theory.

1. Introduction

The Jacobsthal sequence denoted by (j_n) is defined recursively by $j_n = j_{n-1} + 2j_{n-2}$; with starting values $j_0 = 0, j_1 = 1$ for $n \geq 2, n \in \mathbb{N}$. The Binet formula for the Jacobsthal number sequence is given as $j_n = \frac{1}{3}(2^n - (-1)^n)$ in [8]. There are many generalizations of special integer sequences. One of these is the bi-periodic number sequence. Edson and Yayenie first introduced the bi-periodic sequence for the Fibonacci sequence in [5]. Subsequently, many researchers studied different bi-periodic sequences. Bilgici studied the bi-periodic Lucas sequence in [2]. In [3], Coskun and Taskara applied bi-periodic sequences to matrix theory and defined the bi-periodic Fibonacci and Lucas matrix sequences. Uygun and Owusu defined the bi-periodic Jacobsthal and bi-periodic Jacobsthal Lucas numbers and matrix sequences and provided some properties of these sequences in [25, 26, 33–35]. Similarly, Uygun, Karatas, and Akinci provided some properties of the bi-periodic Jacobsthal and Jacobsthal Lucas sequences in [32].

The binomial transformation is an important concept in combinatorics, discrete mathematics, and signal processing. It provides a systematic method for transforming one sequence into another using binomial coefficients. Many combinatorial identities can be derived or simplified using binomial transforms and their inverses. In digital signal processing, the binomial transform acts as a smoothing operator or low-pass filter because the binomial coefficients act as weights that average the nearby terms. It is used in

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interpolation and finite difference methods for converting between sequence forms or evaluating sums involving binomial coefficients.

Many articles have revealed the relationships between binomial transforms and different sequences. Falcon and Plaza [6] defined the binomial transform of k -Fibonacci sequence. In [38], Yilmaz and Taskara studied the binomial transform of Padovan and Perrin matrix sequences. In [1], Bhadouria et al. investigated the binomial transform of Lucas sequences using a method similar to that in [6]. In [36], Yazlik, Yilmaz, and Taskara provided generalized (s, t) -matrix sequences and their binomial transforms. Uygun applied binomial transforms for the generalized (s, t) -Jacobsthal matrix sequence and k -Jacobsthal and k -Jacobsthal-Lucas sequences in [27–29]. In [37], Yilmaz presented the binomial transforms of the balancing and Lucas-balancing polynomials. In [13], Kizilates, Tuglu, and Cekim worked on the binomial transform of quadrapell sequences and quadrapell matrix sequences. In [15], Kwon introduced the binomial transforms of the modified k -Fibonacci-like sequence. In [9], Kaplan and Ozkoc Ozturk obtained the binomial transforms of Horadam quaternion sequences. In [4], Coskun and Taskara investigated the bi-periodic Fibonacci matrix polynomial and its binomial transforms. Soykan [18–24] studied different binomial transforms of sequences such as the generalized Tribonacci, the generalized third order Pell, the generalized Narayana, the generalized Pentanacci, the generalized fourth order Pell, the generalized fifth order Pell, the generalized Jacobsthal-Padovan. Uygun and Haklidir investigated the properties of binomial transforms for the (s, t) -Pell and modified (s, t) -Pell matrices in [30, 31]. In [7], Ghosal and Mandal presented a binomial transform-based fragile watermarking for image authentication. In [10, 11], Kaplan and Ozkoc Ozturk studied binomial transforms for all Horadam-type sequences and binomial transforms of k -Narayana sequences and gave fundamental properties about them. The authors expressed binomial transform for quadra Fibona-Pell sequence in [16]. The binomial transform of the bivariate Fibonacci quaternion polynomials is obtained in [12].

Although bi-periodic Jacobsthal numbers and related matrix sequences have been introduced and studied, a systematic treatment of bi-periodic Jacobsthal polynomial sequences and their behavior under different binomial-type transformations is still missing in the literature. Motivated by the aforementioned studies, we study different binomial transforms of the bi-periodic Jacobsthal polynomial sequence. The remainder of this paper is organized as follows. In Section 2, we define the bi-periodic Jacobsthal polynomial sequence and provide some of its basic properties. In Section 3, we define the binomial transforms of the bi-periodic Jacobsthal polynomial sequence and provide the recurrence relation, the Binet formula and the generating function. In Section 4, we present the k -binomial transforms of the bi-periodic Jacobsthal polynomial sequence. In Section 5, we provide the rising binomial transforms of the bi-periodic Jacobsthal polynomial sequence. In Section 6, we discuss the falling binomial transforms of the bi-periodic Jacobsthal polynomial sequence. By deriving explicit formulas for each transformation, we extend known results for Jacobsthal sequences.

The bi-periodic Jacobsthal sequence $\{J_n\}_{n=0}^{\infty}$ was introduced by $J_n = a^{1-\xi(n)}b^{\xi(n)}J_{n-1} + 2J_{n-2}$, $n \geq 2$, $n \in \mathbb{N}$ with $J_0 = 0$ and $J_1 = 1$ where a and b are nonzero arbitrary real numbers, $\xi(n)$ is the parity function defined as $\xi(n) = \frac{1-(-1)^n}{2}$.

When $a = b = 1$, the classical Jacobsthal sequence is obtained. The characteristic equation of the bi-periodic Jacobsthal sequence is $x^2 - abx - 2ab = 0$. The roots of this equation are given as $\alpha = \frac{ab + \sqrt{a^2b^2 + 8ab}}{2}$ and $\beta = \frac{ab - \sqrt{a^2b^2 + 8ab}}{2}$.

Theorem 1.1. [33] *The generating function $\hat{J}(x)$ of the bi-periodic Jacobsthal sequence is defined as*

$$\hat{J}(x) = \sum_{i=1}^{\infty} J_i x^{i-1} = \frac{x(1 + ax - 2x^2)}{1 - (ab + 4)x^2 + 4x^4}$$

Theorem 1.2. [33] *For the bi-periodic Jacobsthal sequence, the Binet formula is given in the form*

$$J_m = \left(\frac{a^{1-\xi(m)}}{(ab)^{\lfloor \frac{m}{2} \rfloor}} \right) \frac{\alpha^m - \beta^m}{\alpha - \beta}$$

where $\lfloor m \rfloor$ is the floor function of m .

2. The Bi-Periodic Jacobsthal Polynomial Sequence

Definition 2.1. We call a bi-periodic Jacobsthal polynomial sequence $\{d_n(x)\}_{n=0}^\infty$ which has the recurrence relation $d_n = a^{1-\xi(n)}b^{\xi(n)}xd_{n-1} + 2d_{n-2}$, $n \geq 2$ with initial conditions $d_0 = 0$ and $d_1 = 1$ where a and b are nonzero arbitrary real numbers.

The bi-periodic Jacobsthal polynomial sequence has the characteristic equation $\alpha^2 - abx\alpha - 2ab = 0$, with the roots of the equation being $\alpha_1 = \frac{abx + \sqrt{(abx)^2 + 8ab}}{2}$ and $\beta_1 = \frac{abx - \sqrt{(abx)^2 + 8ab}}{2}$.

The roots of the characteristic equation defined above α_1 and α_2 provide the following properties:

- $(\alpha_1x + 2)(\beta_1x + 2) = 4$
- $\alpha_1 + \beta_1 = abx$
- $\alpha_1\beta_1 = -2ab$
- $\alpha_1^2 + \beta_1^2 = (abx)^2 + 4ab$.

Lemma 2.2. The bi-periodic Jacobsthal polynomial sequence $\{d_n\}_{n=0}^\infty$ satisfies the following identities:

- a) $d_{2n} = (abx^2 + 4)d_{2n-2} - 4d_{2n-4}$,
- b) $d_{2n+1} = (abx^2 + 4)d_{2n-1} - 4d_{2n-3}$.

Proof. By the recurrence relation of the bi-periodic Jacobsthal polynomial sequence, we have

$$\begin{aligned} d_{2n} &= axd_{2n-1} + 2d_{2n-2} \\ &= ax(bxd_{2n-2} + 2d_{2n-3}) + 2d_{2n-2} \\ &= (abx^2 + 2)d_{2n-2}(x) + 2(d_{2n-2} - 2d_{2n-4}) \\ &= (abx^2 + 4)d_{2n-2} - 4d_{2n-4} \end{aligned}$$

The other proof is obtained similarly. \square

Theorem 2.3. The generating function of the bi-periodic Jacobsthal polynomial sequence is given as

$$d(t) = \frac{t + ax t^2 - 2t^3}{1 - (abx^2 + 4)t^2 + 4t^4}.$$

Proof. Let's separate the generator function based on whether it contains terms with even or odd indices

$$d(t) = \sum_{i=0}^\infty d_i t^i = \sum_{i=0}^\infty d_{2i} t^{2i} + \sum_{i=0}^\infty d_{2i+1} t^{2i+1} = d_0(t) + d_1(t).$$

Let n be even, we know that $d_{n+2} = axd_{n+1} + 2d_n$ and $d_{n+4} = (abx^2 + 4)d_{n+2} - 4d_n$. Then, we write $d_0(t)$ in a different way in the following:

$$d_0(t) = \sum_{i=0}^\infty d_{2i} t^{2i} = d_0 + d_2 t^2 + \sum_{i=2}^\infty d_{2i} t^{2i} = ax t^2 + \sum_{i=2}^\infty d_{2i} t^{2i}.$$

Multiplying $d_0(t)$ by $-(abx^2 + 4)t^2$ and $4t^4$ respectively, we get

$$\begin{aligned} -(abx^2 + 4)t^2 d_0(t) &= -(abx^2 + 4)t^2 d_0 + \sum_{i=2}^\infty d_{2i-2} t^{2i}, \\ 4t^4 d_0(t) &= 4 \sum_{i=2}^\infty d_{2i-4} t^{2i}. \end{aligned}$$

By combining three sums, we get

$$(1 - (abx^2 + 4)t^2 + 4t^4)d_0(t) = ax^2t^2 + t^4(d_4 - (abx^2 + 4)d_2 + 4d_0) + \dots$$

Therefore, we obtain

$$d_0(t) = \frac{ax^2t^2}{1 - (abx^2 + 4)t^2 + 4t^4}.$$

Let n be odd, we get

$$d_1(t) = \sum_{i=0}^{\infty} d_{2i+1}t^{2i+1} = d_1t + d_3t^3 + \sum_{i=2}^{\infty} d_{2i+1}t^{2i+1}.$$

Multiplying $d_0(t)$ by $-(abx^2 + 4)t^2$ and $4t^4$ respectively, we get

$$\begin{aligned} -(abx^2 + 4)t^2d_1(t) &= d_1(abx^2 + 4)t^3 + \sum_{i=2}^{\infty} d_{2i-1}t^{2i+1}, \\ d_1(t)4t^4 &= 4 \sum_{i=2}^{\infty} d_{2i-3}t^{2i+1}. \end{aligned}$$

By combining three sums, it is obtained that

$$\begin{aligned} d_1(t)(1 - (abx^2 + 4)t^2 + 4t^4) &= d_1t + d_3t^3 - t^3(abx^2 + 4)d_1 \\ &\quad + t^5(d_5 - d_3(abx^2 + 4) + 4d_1) + \dots \end{aligned}$$

Hence, we get

$$d_1(t) = \frac{t + t^3(abx^2 + 2 - abx^2 - 4)}{1 - (abx^2 + 4)t^2 + 4t^4}.$$

From the results found, the generating function of the bi-periodic Jacobsthal polynomial sequence is obtained. \square

Theorem 2.4. For every n that belongs to the set of natural numbers, the Binet formula of the bi-periodic Jacobsthal polynomial sequence is

$$d_n = \frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha_1^n - \beta_1^n}{\alpha_1 - \beta_1} \right).$$

Proof. We use the induction method on n for the proof. The result is obviously satisfied for $n = 0, 1$. Suppose that the result is valid for $k = n \in \mathbb{N}$, namely $d_n = \frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha_1^n - \beta_1^n}{\alpha_1 - \beta_1} \right)$. We shall show that it is also true for $k = n + 1$

$$\begin{aligned} d_{n+1} &= a^{1-\xi(n+1)}b^{\xi(n+1)}xd_n + 2d_{n-1} \\ &= a^{1-\xi(n+1)}b^{\xi(n+1)}x \left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \right) \frac{\alpha_1^n - \beta_1^n}{\alpha_1 - \beta_1} + 2 \left(\frac{a^{1-\xi(n-1)}}{(ab)^{\lfloor \frac{n-1}{2} \rfloor}} \right) \frac{\alpha_1^{n-1} - \beta_1^{n-1}}{\alpha_1 - \beta_1} \\ &= a^{1-\xi(n+1)} \frac{\alpha_1^{n-1}}{\alpha_1 - \beta_1} \left[\frac{ab}{b^{1-\xi(n+1)}a^{\xi(n)}(ab)^{\lfloor \frac{n}{2} \rfloor}} x\alpha_1 + 2 \frac{ab}{(ab)^{\lfloor \frac{n-1}{2} \rfloor}} \right] - \\ &\quad a^{1-\xi(n+1)} \frac{\beta_1^{n-1}}{\alpha_1 - \beta_1} \left[\frac{ab}{b^{1-\xi(n+1)}a^{\xi(n)}(ab)^{\lfloor \frac{n}{2} \rfloor}} x\beta_1 + 2 \frac{ab}{(ab)^{\lfloor \frac{n-1}{2} \rfloor + 1}} \right] \\ &= a^{1-\xi(n+1)} \frac{\alpha_1^{n-1}}{\alpha_1 - \beta_1} \left[\frac{abx\alpha_1 + 2ab}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \right] - a^{1-\xi(n+1)} \frac{\beta_1^{n-1}}{\alpha_1 - \beta_1} \left[\frac{abx\beta_1 + 2ab}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \right] \end{aligned}$$

$$= a^{1-\xi(n+1)} \frac{1}{\alpha_1 - \beta_1} \left[\frac{\alpha_1^{n+1}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \right] - a^{1-\xi(n+1)} \frac{1}{\alpha_1 - \beta_1} \left[\frac{\beta_1^{n+1}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \right]$$

where $abx\alpha_1 + 2ab = \alpha_1^2$. Hence, we get the desired result. \square

3. Binomial Transforms for The Bi-Periodic Jacobsthal Polynomial Sequence

In this section, we mainly focus on the new polynomial sequence obtained by the bi-periodic Jacobsthal sequence to obtain some basic properties. In [7], Knuth introduced the binomial transform. Given a sequence of numbers (a_n) , its binomial transform (b_n) , is demonstrated by the rule $b_n = \sum_{i=0}^n \binom{n}{i} a_i$.

Definition 3.1. The new polynomial sequence $e_n(a, b, x) = e_n$, obtained by the bi-periodic Jacobsthal polynomial sequence, for $n \in \mathbb{N}$, and a and b any non-zero real numbers, is defined as

$$e_n(a, b, x) = \sqrt{xa}^{\frac{\xi(n)}{2}} b^{\frac{1-\xi(n)}{2}} d_n$$

Theorem 3.2. For $\forall n \in \mathbb{N}$, the bi-periodic Jacobsthal polynomial sequence satisfies the following relations:

$$\begin{aligned} e_{2n+2} &= (abx^2 + 4)e_{2n} - 4e_{2n-2}, \\ e_{2n+1} &= (abx^2 + 4)e_{2n-1} - 4e_{2n-3}. \end{aligned}$$

Proof. By the recurrence relation of the bi-periodic Jacobsthal polynomial sequence, we obtain

$$\begin{aligned} e_{2n+2} &= \sqrt{bx}(axd_{2n+1} + 2d_{2n}) \\ &= \sqrt{bx}[ax(bxd_{2n} + 2d_{n-1}) + 2d_{2n}] \\ &= \sqrt{bx}[(abx^2 + 2)d_{2n} + 2axd_{2n-1}] \\ &= \sqrt{bx}[(abx^2 + 2)d_{2n} + 2d_{2n} - 4d_{2n-2}] \\ &= \sqrt{bx}[(abx^2 + 4)d_{2n} - 4d_{2n-2}] \\ &= (abx^2 + 4)e_{2n} - 4e_{2n-2} \end{aligned}$$

The other proof is obtained similarly. \square

Definition 3.3. The binomial transformation of the bi-periodic Jacobsthal polynomial sequence $(\lambda_n(a, b, x))$ is defined as follows:

$$\lambda_n = \lambda_n(a, b, x) = \sum_{i=0}^n \binom{n}{i} e_i.$$

Theorem 3.4. The binomial transformation of the bi-periodic Jacobsthal polynomial sequence provides the following properties:

$$\lambda_{n+1} = \sum_{i=0}^n \binom{n}{i} \sqrt{xa}^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} \left[d_i + a^{\frac{1-2\xi(i)}{2}} b^{\frac{2\xi(i)-1}{2}} d_{i+1} \right].$$

Proof. By the property of binomial expansion, we have

$$\lambda_{n+1} = \sum_{i=0}^{n+1} \binom{n+1}{i} e_i$$

$$= \sqrt{bx}d_0(a, b, x) + \sum_{i=1}^{n+1} \left[\binom{n}{i} + \binom{n}{i-1} \right] \sqrt{xa}^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} d_i.$$

By $\binom{n}{n+1} = 0$, and the change of the index in the second term, it is obtained that

$$\begin{aligned} \lambda_{n+1} &= \sum_{i=0}^n \binom{n}{i} \sqrt{xa}^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} d_i + \sum_{i=0}^n \binom{n}{i} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i+1} \\ &= \sum_{i=0}^n \binom{n}{i} \sqrt{xa}^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} \left[d_i + a^{\frac{1-2\xi(i)}{2}} b^{\frac{2\xi(i)-1}{2}} d_{i+1} \right]. \end{aligned}$$

□

Theorem 3.5. *The recurrence relation of the binomial transformation of the bi-periodic Jacobsthal polynomial sequence is*

$$\lambda_{n+1} = (2 + x\sqrt{ab})\lambda_n - (x\sqrt{ab} - 1)\lambda_{n-1}.$$

Proof. By the definitions of λ_n, e_n and Theorem 3.4, we obtain

$$\begin{aligned} \lambda_{n+1} &= \lambda_n + \sum_{i=0}^n \binom{n}{i} e_{i+1} \\ &= \sum_{i=1}^n \binom{n}{i} \sqrt{xa}^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} \left(d_i + a^{\frac{1-2\xi(i)}{2}} b^{\frac{2\xi(i)-1}{2}} d_{i+1} \right) + \sqrt{bx}d_0 + \sqrt{ax}d_1. \end{aligned}$$

From the definition of d_{i+1} , we have

$$\begin{aligned} \lambda_{n+1} &= (x\sqrt{ab} + 1) \sum_{i=1}^n \binom{n}{i} \sqrt{xa}^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} d_i + 2 \sum_{i=1}^n \binom{n}{i} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} \\ &\quad + \sqrt{bx}d_0 + \sqrt{ax}d_1 \\ &= (x\sqrt{ab} + 1) \sum_{i=0}^n \binom{n}{i} \sqrt{xa}^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} d_i + 2 \sum_{i=1}^n \binom{n}{i} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} + \sqrt{ax}d_1 - bx\sqrt{ax}d_0 \\ &= (x\sqrt{ab} + 1)\lambda_n + 2 \sum_{i=1}^n \sqrt{x} \binom{n}{i} a^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} + \sqrt{ax}d_1 - bx\sqrt{ax}d_0. \end{aligned} \tag{1}$$

If we substitute n for $n - 1$ in (1), then we obtain

$$\begin{aligned} \lambda_n &= (x\sqrt{ab} + 1)\lambda_{n-1} + 2 \sum_{i=1}^{n-1} \binom{n-1}{i} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} + \sqrt{ax}d_1 - bx\sqrt{ax}d_0 \\ &= x\sqrt{ab}\lambda_{n-1} + \sum_{i=1}^n \binom{n-1}{i-1} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} \\ &\quad + 2 \sum_{i=1}^n \binom{n-1}{i} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} + \sqrt{ax}d_1 - bx\sqrt{ax}d_0 \\ &= x\sqrt{ab}\lambda_{n-1} + 2 \sum_{i=1}^n \left[\binom{n-1}{i} + \binom{n-1}{i-1} \right] \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} - \lambda_{n-1} + \sqrt{ax}d_1 - bx\sqrt{ax}d_0 \end{aligned}$$

$$= x\sqrt{ab}\lambda_{n-1} + 2\sum_{i=1}^n \binom{n}{i} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} - \lambda_{n-1} + \sqrt{ax}d_1 - bx\sqrt{ax}d_0. \tag{2}$$

After substituting (1) in (2), we have $\lambda_{n+1} = (x\sqrt{ab} + 1)\lambda_n + \lambda_n - x\sqrt{ab}\lambda_{n-1} + \lambda_{n-1}$.

The characteristic equation from the recurrence relation of the binomial transformation of the bi-periodic Jacobsthal polynomial sequence is $r^2 - (x\sqrt{ab} + 2)r + (x\sqrt{ab} - 1) = 0$ with the roots $r_1 = \frac{x\sqrt{ab} + 2 + \sqrt{abx^2 + 8}}{2}$, $r_2 = \frac{x\sqrt{ab} + 2 - \sqrt{abx^2 + 8}}{2}$. \square

Theorem 3.6. *The Binet formula for the transformation of the bi-periodic Jacobsthal polynomial sequence is given as*

$$\lambda_n = \frac{\sqrt{bx}d_0}{2}(r_1^n + r_2^n) + \frac{2\sqrt{ax}d_1 - bx\sqrt{ax}d_0}{2\sqrt{abx^2 + 8}}(r_1^n - r_2^n).$$

Proof. By the Definition 3.3, we have $\lambda_n = \sum_{i=0}^n \binom{n}{i} e_i$, we will demonstrate $\lambda_n = A(r_1^n + Br_2^n) + B(r_1^n - r_2^n)$. For $n = 0$, we have $\lambda_0 = 2A = \sqrt{bx}d_0$. For $n = 1$, we get $\lambda_1 = A(r_1 + r_2) + B(r_1 - r_2) = \sqrt{bx}d_0 + \sqrt{ax}d_1$. Therefore, we get $B = \frac{2\sqrt{ax}d_1 - bx\sqrt{ax}d_0}{2\sqrt{abx^2 + 8}}$. Substituting for the values into equation, the proof is obtained. \square

Theorem 3.7. *The generating function for $(\lambda_n(a, b, x))$ is found as*

$$\lambda(t) = \sum_{i=0}^{\infty} \lambda_i t^i = \frac{\sqrt{ax}t}{1 - (x\sqrt{ab} + 2)t + (x\sqrt{ab} - 1)t^2}.$$

Proof. The generating function is denoted as $\lambda(t) = \sum_{i=0}^{\infty} \lambda_i t^i = \lambda_0 + \lambda_1 t + \sum_{i=2}^{\infty} \lambda_i t^i$. Multiplying $\lambda(t)$ by $-(x\sqrt{ab} + 2)t$ and $(x\sqrt{ab} - 1)t^2$ respectively, we get

$$\begin{aligned} -(x\sqrt{ab} + 2)t\lambda(t) &= -(x\sqrt{ab} + 2)t\lambda_0 - (x\sqrt{ab} + 2)\sum_{i=2}^{\infty} \lambda_{i-1}t^i \\ + (x\sqrt{ab} - 1)t^2\lambda(t) &= (x\sqrt{ab} - 1)\sum_{i=2}^{\infty} \lambda_{i-2}t^i. \end{aligned}$$

By combining three sums, we get

$$\begin{aligned} \lambda(t) &= \frac{\lambda_0 + \lambda_1 t - (x\sqrt{ab} + 2)t\lambda_0}{1 - (x\sqrt{ab} + 2)t + (x\sqrt{ab} - 1)t^2} \\ &= \frac{(\sqrt{ax}d_1 - (x\sqrt{ab} + 1)\sqrt{bx}d_0)t + \sqrt{bx}d_0}{1 - (x\sqrt{ab} + 1)t + (x\sqrt{ab} - 1)t^2}. \end{aligned}$$

It is seen using the following result proved by Prodinger [17], we can find the generating function for the binomial transformation of the bi-periodic Jacobsthal polynomial sequence

$$\lambda(t) = \frac{1}{1-t} d\left(\frac{t}{1-t}\right).$$

\square

Theorem 3.8. *The exponential generating function for the transformation of the bi-periodic Jacobsthal polynomial sequence is found as*

$$\sum_{n=0}^{\infty} \frac{\lambda_n(a, b, x)t^n}{n!} = \frac{\sqrt{bx}d_0}{2}(e^{r_1 t} + e^{r_2 t}) + \frac{2\sqrt{ax}d_1 - bx\sqrt{ax}d_0}{2\sqrt{abx^2 + 8}}(e^{r_1 t} - e^{r_2 t})$$

Proof.

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\lambda_n t^n}{n!} &= \sum_{n=0}^{\infty} \left[\frac{\sqrt{bx}d_0}{2} (r_1^n + r_2^n) + \frac{2\sqrt{ax}d_1 - bx\sqrt{ax}d_0}{2\sqrt{abx^2 + 8}} (r_1^n - r_2^n) \right] \frac{t^n}{n!} \\ &= \frac{\sqrt{bx}d_0}{2} \sum_{n=0}^{\infty} \left\{ \frac{(r_1 t)^n}{n!} + \frac{(r_2 t)^n}{n!} \right\} + \frac{2\sqrt{ax}d_1 - bx\sqrt{ax}d_0}{2\sqrt{abx^2 + 8}} \sum_{n=0}^{\infty} \left\{ \frac{(r_1 t)^n}{n!} - \frac{(r_2 t)^n}{n!} \right\}. \end{aligned}$$

By the Maclaurin series expansion of e^t , we complete the proof. \square

4. K-Binomial Transforms for The Bi-Periodic Jacobsthal Polynomial Sequence

In this section, we focus on the k -binomial transform of the bi-periodic Jacobsthal polynomial to obtain basic properties.

Definition 4.1. The k -binomial transformation of the bi-periodic Jacobsthal polynomial sequence $\varphi_n(a, b, x) = \varphi_n$ is defined as

$$\varphi_n = \sum_{i=0}^n \binom{n}{i} k^i e_i.$$

Throughout this definition $k = x\sqrt{ab}$.

Theorem 4.2. The k -binomial transformation of the bi-periodic Jacobsthal polynomial sequence $\varphi_n(a, b, x) = \varphi_n$ satisfies the following recurrence property:

$$\varphi_{n+1} = (k^2 + 2k)\varphi_n - (k^3 - k^2)\varphi_{n-1}.$$

Proof. From Theorem 3.5, we have $\lambda_{n+1} = (2 + \sqrt{abx})\lambda_n - (x\sqrt{ab} - 1)\lambda_{n-1}$. By $\varphi_n = k^n \lambda_n$ where $k = x\sqrt{ab}$, we get

$$\begin{aligned} \varphi_{n+1} &= k^{n+1} \left[(2 + \sqrt{abx})\lambda_n - (x\sqrt{ab} - 1)\lambda_{n-1} \right] \\ &= k \left[(2 + \sqrt{abx})\varphi_n - k^2(x\sqrt{ab} - 1)\varphi_{n-1} \right] \\ &= x\sqrt{ab}(2 + \sqrt{abx})\varphi_n - x^2 ab(x\sqrt{ab} - 1)\varphi_{n-1} \\ &= (abx^2 + 2x\sqrt{ab})\varphi_n - (x^3 ab\sqrt{ab} - x^2 ab)\varphi_{n-1}. \end{aligned}$$

The characteristic equation for this relation is given as $t^2 - (abx^2 + 2x\sqrt{ab})t + (x^3 ab\sqrt{ab} - x^2 ab) = 0$. The roots of the characteristic equation are $t_1 = \frac{k^2 + 2k + \sqrt{k^4 + 8k^2}}{2}$ and $t_2 = \frac{k^2 + 2k - \sqrt{k^4 + 8k^2}}{2}$. \square

Theorem 4.3. The k -binomial transformation of the bi-periodic Jacobsthal polynomial sequence φ_n satisfies the following property

$$\varphi_n = \frac{\sqrt{bx}d_0}{2} (t_1^n + t_2^n) + \frac{2ax\sqrt{bx}d_1 - (k^2 + 2k - 2)\sqrt{bx}d_0}{2\sqrt{k^4 + 8k^2}} (t_1^n - t_2^n).$$

Proof. Assume that $\varphi_n = A(t_1^n + t_2^n) + B(t_1^n - t_2^n)$. For $n = 0$, we get $\varphi_0 = 2A = \sqrt{bx}d_0$. For $n = 1$, we get $\varphi_1 = A(t_1 + t_2) + B(t_1 - t_2) = \sqrt{bx}d_0 + ax\sqrt{bx}d_1$. By this equation, we have $B = \frac{2ax\sqrt{bx}d_1 - (k^2 + 2k - 2)\sqrt{bx}d_0}{2\sqrt{k^4 + 8k^2}}$. Substituting the values of A and B , we get the result. \square

Theorem 4.4. The generating function of φ_n is given as

$$\varphi(t) = \sum_{i=0}^{\infty} \varphi_i t^i = \frac{\varphi_0 + \varphi_1 t - (k^2 + 2k)t\varphi_0}{1 - (k^2 + 2k)t + (k^3 - k^2)t^2}.$$

Proof. The generating function is demonstrated as $\varphi(t) = \sum_{i=0}^{\infty} t^i \varphi_i = \varphi_0 + \varphi_1 t + \sum_{i=2}^{\infty} t^i \varphi_i$. By multiplying $\varphi(t)$ by $-(k^2 + 2k)t$, we get

$$-(k^2 + 2k)t\varphi(t) = -(k^2 + 2k)t\varphi_0 - (k^2 + 2k) \sum_{i=2}^{\infty} t^i \varphi_{i-1}.$$

From multiplying $\varphi(t)$ by $(k^3 - k^2)t^2$, we have

$$(k^3 - k^2)t^2 \sum_{i=0}^{\infty} t^i \varphi_i = (k^3 - k^2) \sum_{i=2}^{\infty} t^i \varphi_{i-2}.$$

By combining three sums, we have $\varphi(t) = \frac{\varphi_0 + \varphi_1 t - (k^2 + 2k)t\varphi_0}{1 - (k^2 + 2k)t + (k^3 - k^2)t^2}$. \square

Theorem 4.5. The exponential generating function for φ_n is found as

$$\sum_{n=0}^{\infty} \frac{\varphi_n t^n}{n!} = \frac{\sqrt{bx}d_0}{2} (e^{t_1 t} + e^{t_2 t}) + \frac{2ax \sqrt{bx}d_1 - (k^2 + 2k - 2) \sqrt{bx}d_0}{2 \sqrt{k^4 + 8k^2}} (e^{t_1 t} - e^{t_2 t}).$$

5. Rising Binomial Transforms for The Bi-Periodic Jacobsthal Polynomial Sequence

In this section, we focus on the rising k -binomial transform of the bi-periodic Jacobsthal polynomial to obtain some basic properties.

Definition 5.1. Let $n \in \mathbb{N}$, a and b be non-zero real numbers, the rising k -binomial transformation of the bi-periodic Jacobsthal polynomial sequence $(p_n(a, b, x)) = (p_n)$ is defined as

$$p_n = \sum_{i=0}^n \binom{n}{i} k^i e_i$$

with $p_0 = \sqrt{bx}d_0$ and $p_1 = \sqrt{bx}d_0 + ax \sqrt{bx}d_1$.

Theorem 5.2. For $n \in \mathbb{N}$, the elements of the rising k -binomial transformation polynomial of the bi-periodic Jacobsthal polynomial sequence is expressed as

$$p_n = \frac{a \sqrt{bx}}{\alpha_1 - \beta_1} \left[\left(\frac{\alpha_1^2}{ab} - 1 \right)^n - \left(\frac{\beta_1^2}{ab} - 1 \right)^n \right].$$

Proof.

$$\begin{aligned} p_n &= \sum_{i=0}^n \binom{n}{i} k^i e_i = \sum_{i=0}^n \binom{n}{i} (x \sqrt{ab})^i \sqrt{xa}^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} d_i \\ &= \sum_{i=0}^n \binom{n}{i} (ab)^{\frac{i}{2}} x^{i+\frac{1}{2}} a^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} \left(\frac{a^{1-\xi(i)}}{(ab)^{\lfloor \frac{i}{2} \rfloor}} \right) \frac{\alpha_1^i - \beta_1^i}{\alpha_1 - \beta_1} \\ &= \sqrt{bx} \sum_{i=0}^n \binom{n}{i} (ab)^{\frac{i}{2}} x^i \left(\frac{a}{(ab)^{\frac{\xi(i)}{2}} (ab)^{\lfloor \frac{i}{2} \rfloor}} \right) \frac{\alpha_1^i - \beta_1^i}{\alpha_1 - \beta_1} \end{aligned}$$

$$\begin{aligned}
 &= \frac{a\sqrt{bx}}{\alpha_1 - \beta_1} \sum_{i=0}^n \binom{n}{i} x^i (\alpha_1^i - \beta_1^i) \\
 &= \frac{a\sqrt{bx}}{\alpha_1 - \beta_1} [(\alpha_1 x + 1)^n - (\beta_1 x + 1)^n].
 \end{aligned}$$

□

Theorem 5.3. *The rising k -binomial transformation of the bi-periodic Jacobsthal polynomial sequence provides the following property:*

$$p_{n+1} = p_n + k \sum_{i=0}^n \binom{n}{i} k^i e_{i+1}.$$

Proof. By the property of binomial expansion we have

$$\begin{aligned}
 p_{n+1} &= \sum_{i=0}^{n+1} \binom{n+1}{i} k^i e_i = \sum_{i=0}^{n+1} \binom{n+1}{i} k^i \sqrt{xa}^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} d_i \\
 &= \sqrt{bx}d_0 + \sum_{i=1}^{n+1} \left[\binom{n}{i} + \binom{n}{i-1} \right] k^i \sqrt{xa}^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} d_i.
 \end{aligned}$$

By $\binom{n}{n+1} = 0$ and the change of the index in the second term, it is obtained that

$$\begin{aligned}
 p_{n+1} &= \sum_{i=0}^n \binom{n}{i} k^i \sqrt{xa}^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} d_i + \sum_{i=0}^n \binom{n}{i} k^{i+1} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i+1} \\
 &= \sum_{i=0}^n \binom{n}{i} k^i \sqrt{xa}^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} \left[d_i + ka^{\frac{1-2\xi(i)}{2}} b^{\frac{2\xi(i)-1}{2}} d_{i+1} \right] \\
 &= p_n + \sum_{i=0}^n \binom{n}{i} k^{i+1} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i+1} = p_n + k \sum_{i=0}^n \binom{n}{i} k^i e_{i+1}.
 \end{aligned}$$

□

Theorem 5.4. *The recurrence relation of p_n is*

$$p_{n+1} = (k^2 + 2)p_n - (k - 1)^2 p_{n-1}.$$

Proof. By the definitions of p_n, e_n and Theorem 5.3, we obtain

$$\begin{aligned}
 p_{n+1} &= p_n + k \sum_{i=0}^n \binom{n}{i} k^i e_{i+1} \\
 &= \sum_{i=1}^n \binom{n}{i} k^i \sqrt{xa}^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} \left[d_i + ka^{\frac{1-2\xi(i)}{2}} b^{\frac{2\xi(i)-1}{2}} d_{i+1} \right] + \sqrt{bx}d_0 + \sqrt{ax}d_1.
 \end{aligned}$$

From the definition of d_{i+1} , we have

$$\begin{aligned}
 p_{n+1} &= (k^2 + 1) \sum_{i=1}^n \binom{n}{i} k^i \sqrt{xa}^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} d_i + 2 \sum_{i=1}^n \binom{n}{i} k^i \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} + \sqrt{bx}d_0 + \sqrt{ax}d_1 \\
 &= (k^2 + 1) \sum_{i=0}^n \binom{n}{i} k^i \sqrt{xa}^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} d_i + 2 \sum_{i=1}^n \binom{n}{i} k^i \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} + \sqrt{ax}d_1 - kbx \sqrt{ax}d_0
 \end{aligned}$$

$$= (kx\sqrt{ab} + 1)p_n + 2 \sum_{i=1}^n \binom{n}{i} k^i \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} + \sqrt{ax}d_1 - bx\sqrt{ax}d_0 \tag{3}$$

If we substitute n for $n - 1$, then we have

$$\begin{aligned} p_n &= (k^2 + 1)p_{n-1} + 2 \sum_{i=1}^{n-1} \binom{n-1}{i} k^i \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} + \sqrt{ax}d_1 - bx\sqrt{ax}d_0 \\ &= k^2 p_{n-1} + \sum_{i=1}^n \binom{n-1}{i-1} k^{i-1} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} \\ &\quad + 2 \sum_{i=1}^n \binom{n-1}{i} k^i \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} + \sqrt{ax}d_1 - bx\sqrt{ax}d_0 \\ &= k^2 p_{n-1} + \sum_{i=1}^n \left[2k \binom{n-1}{i} + \binom{n-1}{i-1} \right] k^{i-1} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} \\ &\quad + \sqrt{ax}d_1 - bx\sqrt{ax}d_0 \\ &= k^2 p_{n-1} + 2k \sum_{i=1}^n \binom{n}{i} k^{i-1} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} + \sqrt{ax}d_1 - bx\sqrt{ax}d_0 \\ &\quad + (1-2k) \sum_{i=1}^n \binom{n-1}{i-1} k^{i-1} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} \\ &= k^2 p_{n-1} + 2k \sum_{i=1}^n \binom{n}{i} k^{i-1} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} + \sqrt{ax}d_1 - bx\sqrt{ax}d_0 \\ &\quad + (1-2k) \sum_{i=0}^{n-1} \binom{n-1}{i} k^i \sqrt{xa}^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} d_i \\ &= k^2 p_{n-1} + 2k \sum_{i=1}^n \binom{n}{i} k^{i-1} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} + \sqrt{ax}d_1 - bx\sqrt{ax}d_0 \\ &\quad + (1-2k)p_{n-1} \end{aligned} \tag{4}$$

After substituting (3) in (4), we have

$$p_{n+1} = (k^2 + 2)p_n - (k^2 + 1 - 2k)p_{n-1}.$$

The characteristic equation for this relation is given as $t^2 - (k^2 + 2)t + (k - 1)^2 = 0$. The roots of the characteristic equation are $\tilde{p}_1 = \frac{k^2+2+\sqrt{k^4+8k}}{2}$, $\tilde{p}_2 = \frac{k^2+2-\sqrt{k^4+8k}}{2}$. \square

Theorem 5.5. *The rising k -binomial transformation of the bi-periodic Jacobsthal polynomial sequence p_n satisfies thenfollowing property*

$$p_n = \frac{\sqrt{bx}d_0}{2} (\tilde{p}_1^n + \tilde{p}_2^n) + \frac{2\sqrt{ax}d_1 - k\sqrt{bx}d_0}{2\sqrt{k^2+8}} (\tilde{p}_1^n - \tilde{p}_2^n).$$

Proof. Assume that $p_n = A(\tilde{p}_1^n + \tilde{p}_2^n) + B(\tilde{p}_1^n - \tilde{p}_2^n)$. For $n = 0$, we get $p_0 = 2A = \sqrt{bx}d_0$.

For $n = 1$, we obtain $p_1 = A(\tilde{p}_1 + \tilde{p}_2) + B(\tilde{p}_1 - \tilde{p}_2) = \sqrt{bx}d_0 + k\sqrt{ax}d_1$. By this equation, we have $B = \frac{2k\sqrt{ax}d_1 - k^2\sqrt{bx}d_0}{2\sqrt{k^4+8k^2}} = \frac{2\sqrt{ax}d_1 - k\sqrt{bx}d_0}{2\sqrt{k^2+8}}$.

Substituting the values of A and B , we get the result. \square

Theorem 5.6. The generating function of p_n is given as

$$p(t) = \sum_{i=0}^{\infty} p_i t^i = \frac{p_0 + p_1 t - (k^2 + 2) t p_0}{1 - (k^2 + 2) t + (k^2 + 1 - 2k) t^2}.$$

Proof. The generating function is demonstrated as $p(t) = p_0 + p_1 t + \sum_{i=2}^{\infty} p_i t^i$. From multiplying $p(t)$ by $(k^2 + 2)t$, we get

$$-(k^2 + 2) t p(t) = -(k^2 + 2) t p_0 - (k^2 + 2) \sum_{i=2}^{\infty} t^i p_{i-1}.$$

From multiplying $p(t)$ by $(x^3 ab \sqrt{ab - x^2 ab}) t^2$, we have

$$(k^2 + 1 - 2k) t^2 p(t) = (k^2 + 1 - 2k) \sum_{i=2}^{\infty} t^i p_{i-2}.$$

By these equalities, we obtain $p(t) = \frac{p_0 + p_1 t - (k^2 + 2) t p_0}{1 - (k^2 + 2) t + (k^2 + 1 - 2k) t^2}$. \square

Theorem 5.7. The exponential generating function for the rising k -binomial transformation of the bi-periodic Jacobsthal polynomial sequence is expressed as

$$\sum_{n=0}^{\infty} \frac{p_n t^n}{n!} = \frac{\sqrt{b x d_0}}{2} (e^{\tilde{p}_1 t} + e^{\tilde{p}_2 t}) + \frac{2 \sqrt{a x d_1} - k \sqrt{b x d_0}}{2 \sqrt{k^2 + 8}} (e^{\tilde{p}_1 t} - e^{\tilde{p}_2 t}).$$

6. Falling Binomial Transforms for The Bi-Periodic Jacobsthal Polynomial Sequence

In this section, we focus on the falling k -binomial transform of the bi-periodic Jacobsthal polynomial to obtain some basic properties.

Definition 6.1. Let $n \in \mathbb{N}$, and a, b be non-negative real numbers, the falling k -binomial transformation of the bi-periodic Jacobsthal polynomial sequence $q_n(a, b, x) = q_n$ is defined as

$$q_n(a, b, x) = q_n = \sum_{i=0}^n \binom{n}{i} k^{n-i} e_i.$$

Theorem 6.2. The falling k -binomial transformation of the bi-periodic Jacobsthal polynomial sequence provides the following property:

$$q_{n+1} = k q_n + \sum_{i=0}^n \binom{n}{i} k^{n-i} e_{i+1}.$$

Proof. By the property of the binomial expansion, we have

$$\begin{aligned} q_{n+1} &= \sum_{i=0}^{n+1} \binom{n+1}{i} k^{n+1-i} e_i \\ &= \sum_{i=0}^{n+1} \binom{n+1}{i} k^{n+1-i} \sqrt{x a}^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} d_i \end{aligned}$$

$$= k^{n+1} \sqrt{bx}d_0 + \sum_{i=1}^{n+1} \left[\binom{n}{i} + \binom{n}{i-1} \right] k^{n+1-i} \sqrt{xa}^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} d_i.$$

By $\binom{n}{n+1} = 0$ and the change of the index in the second term, it is obtained that

$$\begin{aligned} q_{n+1} &= \sum_{i=0}^n \binom{n}{i} k^{n+1-i} \sqrt{xa}^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} d_i + \sum_{i=0}^n \binom{n}{i} k^{n-i} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i+1} \\ &= \sum_{i=0}^n \binom{n}{i} k^{n-i} \sqrt{xa}^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} \left[kd_i + a^{\frac{1-2\xi(i)}{2}} b^{\frac{2\xi(i)-1}{2}} d_{i+1} \right] \\ &= kq_n + \sum_{i=0}^n \binom{n}{i} k^{n-i} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i+1} \\ &= kq_n + \sum_{i=0}^n \binom{n}{i} k^{n-i} e_{i+1}. \end{aligned}$$

□

Theorem 6.3. *The recurrence relation of the transformation q_n is given as*

$$q_{n+1} = 3kq_n + (2 - 2k^2)q_{n-1}.$$

Proof. By the definitions of q_n, e_n and Theorem 6.2, we obtain

$$q_{n+1} = \sum_{i=1}^n \binom{n}{i} k^{n-i} \sqrt{xa}^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} \left(kd_i + a^{\frac{1-2\xi(i)}{2}} b^{\frac{2\xi(i)-1}{2}} d_{i+1} \right) + k^{n+1} \sqrt{bx}d_0 + k^n \sqrt{ax}d_1.$$

From the definition of d_{i+1} , we have

$$\begin{aligned} q_{n+1} &= 2 \sum_{i=1}^n \binom{n}{i} k^{n+1-i} \sqrt{xa}^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} d_i + k^{n+1} \sqrt{bx}d_0 + k^n \sqrt{ax}d_1 \\ &\quad + 2 \sum_{i=1}^n \binom{n}{i} k^{n-i} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} \\ &= 2kq_n + 2 \sum_{i=1}^n \binom{n}{i} k^{n-i} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} + k^n \sqrt{ax}d_1 + k^{n+1} \sqrt{bx}d_0 \end{aligned} \tag{5}$$

If we substitute n for $n - 1$, we have

$$\begin{aligned} q_n &= 2kq_{n-1} + 2 \sum_{i=1}^{n-1} \binom{n-1}{i} k^{n-1-i} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} + k^{n-1} \sqrt{ax}d_1 + k^n \sqrt{bx}d_0 \\ &= kq_{n-1} + \sum_{i=0}^{n-1} \binom{n-1}{i} k^{n-i} \sqrt{xa}^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} d_i + k^{n-1} \sqrt{ax}d_1 + k^n \sqrt{bx}d_0 \\ &\quad + 2 \sum_{i=1}^n \binom{n-1}{i} k^{n-1-i} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} \\ &= kq_{n-1} + \sum_{i=1}^n \binom{n-1}{i-1} k^{n+1-i} \sqrt{xa}^{\frac{\xi(i-1)}{2}} b^{\frac{1-\xi(i-1)}{2}} d_{i-1} + k^{n-1} \sqrt{ax}d_1 + k^n \sqrt{bx}d_0 \end{aligned}$$

$$\begin{aligned}
 & +2 \sum_{i=1}^n \binom{n-1}{i} k^{n-1-i} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} \\
 = & kq_{n-1} + \sum_{i=1}^n k^{n-1-i} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} \left[k^2 \binom{n-1}{i-1} + 2 \binom{n-1}{i} \pm 2 \binom{n-1}{i-1} \right] \\
 & + k^{n-1} \sqrt{ax}d_1 + k^n \sqrt{bx}d_0 \\
 = & kq_{n-1} + (k^2 - 2) \sum_{i=1}^n \binom{n-1}{i-1} k^{n-1-i} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} \\
 & + 2 \sum_{i=1}^n \binom{n}{i} k^{n-1-i} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} + k^{n-1} \sqrt{ax}d_1 + k^n \sqrt{bx}d_0 \\
 = & kq_{n-1} + (k^2 - 2) \sum_{i=0}^{n-1} \binom{n-1}{i} k^{n-2-i} \sqrt{xa}^{\frac{\xi(i)}{2}} b^{\frac{1-\xi(i)}{2}} d_i \\
 & + 2 \sum_{i=1}^n \binom{n}{i} k^{n-1-i} \sqrt{xa}^{\frac{1-\xi(i)}{2}} b^{\frac{\xi(i)}{2}} d_{i-1} + k^{n-1} \sqrt{ax}d_1 + k^n \sqrt{bx}d_0 \tag{6}
 \end{aligned}$$

If we substitute (5) in (6), we have

$$q_{n+1} = 3kq_n + (2 - 2k^2)q_{n-1}.$$

□

Theorem 6.4. The Binet formula for the transformation q_n given as

$$q_n = \frac{\sqrt{bx}d_0}{2} (\tilde{q}_1^n + \tilde{q}_2^n) + \frac{2\sqrt{ax}d_1 + (2 - 3x)k\sqrt{bx}d_0}{2\sqrt{9k^2x^2 - 8k^2 + 8}} (\tilde{q}_1^n - \tilde{q}_2^n).$$

Moreover, $\tilde{q}_1 = \frac{3kx + \sqrt{9k^2x^2 - 8k^2 + 8}}{2}$ and $\tilde{q}_2 = \frac{3kx - \sqrt{9k^2x^2 - 8k^2 + 8}}{2}$ are the roots of the characteristic equation $r^2 - 3xkr + 2k^2 - 2 = 0$.

Proof. Assume that $q_n = A(\tilde{q}_1^n + \tilde{q}_2^n) + B(\tilde{q}_1^n - \tilde{q}_2^n)$. For $n = 0$, we get $q_0 = 2A = \sqrt{bx}d_0$.

For $n = 1$, we get $q_1 = A(\tilde{q}_1 + \tilde{q}_2) + B(\tilde{q}_1 - \tilde{q}_2) = k\sqrt{bx}d_0 + \sqrt{ax}d_1$. By this equation, we have $B = \frac{2\sqrt{ax}d_1 + (2 - 3x)k\sqrt{bx}d_0}{2\sqrt{9k^2x^2 - 8k^2 + 8}}$. Substituting the values of A and B we get the result. □

Theorem 6.5. The generating function for the transformation q_n given as

$$q(t) = \sum_{i=0}^{\infty} q_i t^i = \frac{q_0 + q_1 t - 3ktq_0}{1 - 3kt + (2 - 2k^2)t^2}.$$

Proof. The generating function is given as $q(t) = q_0 + q_1 t + \sum_{i=2}^{\infty} q_i t^i$. From multiplying $q(t)$ by $-3kt$ and $(2 - 2k^2)t^2$ respectively, we get

$$\begin{aligned}
 -3ktq(t) &= -3ktq_0 - 3k \sum_{i=2}^{\infty} q_{i-1} t^i, \\
 (2 - 2k^2)t^2 q(t) &= (2 - 2k^2) \sum_{i=2}^{\infty} q_{i-2} t^i.
 \end{aligned}$$

The three equalities provide the proof. □

Theorem 6.6. *The exponential generating function for the transformation q_n is found as*

$$\sum_{n=0}^{\infty} \frac{q_n t^n}{n!} = \frac{\sqrt{bx}d_0}{2} (e^{\tilde{q}_{1t}} + e^{\tilde{q}_{2t}}) + \frac{2\sqrt{ax}d_1 + (2-3x)k\sqrt{bx}d_0}{2\sqrt{9k^2x^2 - 8k^2 + 8}} (e^{\tilde{q}_{1t}} - e^{\tilde{q}_{2t}}).$$

Conclusion

We have developed a systematic study of the bi-periodic Jacobsthal polynomial sequence and its associated binomial-type transformations. By deriving generating functions and Binet-type representations, we clarified the role of parity-dependent coefficients. The analysis of binomial, k -binomial, rising, and falling binomial transformations reveals that the bi-periodic Jacobsthal polynomials preserve their fundamental recurrence structure under these transformations. This observation extends several known results on binomial transforms of classical and some generalized Jacobsthal-type sequences.

The methods presented here can be applied to other bi-periodic polynomial sequences, suggesting potential extensions to higher-order recurrences and matrix-valued analogues. Such directions may lead to further connections with combinatorial identities and discrete transformation theory.

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