



Weighted composition operators from Bloch to Zygmund spaces: An alternative characterization

Lien Vuong Lam^a, Nguyen Van Dai^b, Thai Thuan Quang^{c,*}

^aDepartment of Mathematics, Pham Van Dong University, Vietnam

^bDepartment of Education, Quy Nhon University, Vietnam

^cDepartment of Mathematics and Statistics, Quy Nhon University, Vietnam

Abstract. Let ψ be a holomorphic function and φ a holomorphic self-map of the unit ball $\mathbb{B} \subset \mathbb{C}^N$. Denote by \mathcal{B}_ν and \mathcal{Z}_μ the weighted Bloch-type and Zygmund-type spaces with normal weights ν and μ . The weighted composition operator

$$W_{\psi,\varphi} : \mathcal{B}_\nu \text{ (or } \mathcal{B}_{\nu,0}) \rightarrow \mathcal{Z}_\mu \text{ (or } \mathcal{Z}_{\mu,0}), \quad f \mapsto \psi \cdot (f \circ \varphi),$$

is studied in terms of boundedness, compactness, and asymptotic norm behavior. The obtained characterizations are expressed via analytic properties of the symbols ψ and φ , particularly a component function φ_k of φ .

1. Introduction and Preliminaries

We fix a positive integer N . Let $\mathbb{B} := \{z \in \mathbb{C}^N : |z| = \sqrt{\langle z, z \rangle} < 1\}$ be the open unit ball in the Euclidean space \mathbb{C}^N . When $N = 1$, the ball \mathbb{B} is the unit disk \mathbb{D} .

Let $H(\mathbb{B})$ be the class of all holomorphic functions on \mathbb{B} and $S(\mathbb{B})$ be the set of holomorphic self-maps of \mathbb{B} .

For Banach spaces E, F of holomorphic functions on \mathbb{B} , the weighted composition operator with given symbols $\psi \in H(\mathbb{B})$ and $\varphi \in S(\mathbb{B})$ is the linear operator $W_{\psi,\varphi} : E \rightarrow F$ defined by

$$W_{\psi,\varphi}(f) := \psi \cdot (f \circ \varphi), \quad f \in E.$$

The component operators are the multiplication operator $M_\psi(f) := \psi \cdot f$ and the composition operator $C_\varphi(f) := f \circ \varphi$, which correspond to the cases when the composition symbol is the identity function on \mathbb{B} and the multiplication symbol is the constant function 1, respectively.

2020 *Mathematics Subject Classification*. Primary 47B38; Secondary 30H30, 47B33, 47B91.

Keywords. Weighted composition operator, Bloch spaces, Zygmund spaces, boundedness, compactness.

Received: 16 November 2025; Accepted: 29 January 2026

Communicated by Dragan S. Djordjević

This research received no external funding.

* Corresponding author: Thai Thuan Quang

Email addresses: lv1lam@pdu.edu.vn (Lien Vuong Lam), nguyenvandai@qnu.edu.vn (Nguyen Van Dai),

thai thuanquang@qnu.edu.vn (Thai Thuan Quang)

ORCID iDs: <https://orcid.org/0000-0003-0634-8019> (Lien Vuong Lam), <https://orcid.org/0000-0001-6128-5335> (Nguyen Van Dai), <https://orcid.org/0000-0003-3029-9157> (Thai Thuan Quang)

In recent years, weighted composition operators and related operators have attracted considerable attention in various contexts. Their boundedness and compactness have been extensively studied on classical Banach spaces of analytic functions such as Hardy, Bergman, BMOA, and Bloch-type spaces (see, e.g., [7, 11, 12, 22]). For more recent developments in the unit ball and Bloch-type settings, see [1, 2, 4–6, 8, 9, 13, 14] and references therein.

Previous research has primarily focused on the operator-theoretic properties of C_φ , M_ψ , and $W_{\psi,\varphi}$ in relation to the analytic behavior of the symbols ψ and φ as whole functions. However, little attention has been given to the role of a particular component function φ_k of φ . A natural question arises: can the boundedness and compactness of $W_{\psi,\varphi}$ be characterized solely in terms of the function-theoretic properties of ψ and one component φ_k of φ ? Equivalently, do these properties depend on the intersection of the image $\varphi(\mathbb{B})$ with a complex line through the origin?

Recently, Quang, Dai, and Lam [9] addressed this question for the case of weighted composition operators between Bloch spaces. The novelty of our approach lies in expressing boundedness and compactness entirely in terms of a single component function φ_k of φ , particularly when the image $\varphi(\mathbb{B})$ contains at least one coordinate disk $\mathbb{D}_k = \{\lambda e_k : |\lambda| < 1\}$, $k = 1, \dots, N$.

Motivated by this line of research, the present paper extends the study to weighted composition operators between Bloch-type and Zygmund-type spaces. Furthermore, we show that our results hold for a much broader class of holomorphic self-maps than those considered previously, namely, those whose range contains the origin and whose image diameter coincides with that of some component function φ_k of φ .

A positive continuous function ω on the interval $[0, 1)$ is called *normal* if there are three constants $0 \leq \delta < 1$ and $0 < a < b < \infty$ such that

$$\frac{\omega(t)}{(1-t)^a} \text{ is decreasing on } [\delta, 1), \quad \lim_{t \rightarrow 1} \frac{\omega(t)}{(1-t)^a} = 0, \quad (W_1)$$

$$\frac{\omega(t)}{(1-t)^b} \text{ is increasing on } [\delta, 1), \quad \lim_{t \rightarrow 1} \frac{\omega(t)}{(1-t)^b} = \infty. \quad (W_2)$$

If we say that a function $\omega : \mathbb{B} \rightarrow [0, \infty)$ is normal, we also assume that it is radial, that is, $\omega(z) = \omega(|z|)$ for every $z \in \mathbb{B}$.

Strictly positive continuous functions on \mathbb{B} are called *weights*.

For a normal weight ω on \mathbb{B} , we use there certain quantities, which will be used in this work:

$$I_\omega(z) := \int_0^{|z|} \frac{dt}{\omega(t)}, \quad \omega^*(z) := \frac{1}{\omega(z)(1-|z|^2)}, \quad z \in \mathbb{B}.$$

Remark 1.1. (see (2.1) in [8]). *Since ω is positive, continuous, $m_{\omega,\delta} := \min_{t \in [0,\delta]} \omega(t) > 0$. Moreover, it follows from (W_1) that ω is strictly decreasing on $[\delta, 1)$ and $\omega(t) \rightarrow 0$ as $t \rightarrow 1$ (see [4]). It is clear that $\max_{t \in (0,1)} \omega(t) =: M_\omega < \infty$. Then, it is easy to see that $\omega(z)I_\omega(z) < \infty$ for every $z \in \mathbb{B}$.*

We define the first-order and second-order radial derivatives of $f \in H(\mathbb{B})$ as follows:

$$Rf(z) = \langle \nabla f(z), \bar{z} \rangle, \quad R^{(2)}f(z) = R(Rf)(z), \quad z \in \mathbb{B},$$

where $\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_N}(z) \right)$ is the (holomorphic) gradient of f at z .

We now introduce the (little) Bloch-type and (little) Zygmund-type spaces of holomorphic functions on \mathbb{B} . Let ω be a normal weight on \mathbb{B} . The *Bloch-type space* \mathcal{B}_ω and its little subspace $\mathcal{B}_{\omega,0}$ are defined by

$$\mathcal{B}_\omega := \left\{ f \in H(\mathbb{B}) : \|f\|_{\mathcal{B}_\omega} := \sup_{z \in \mathbb{B}} \omega(z) |\nabla f(z)| < \infty \right\},$$

$$\mathcal{B}_{\omega,0} := \left\{ f \in \mathcal{B}_\omega : \lim_{|z| \rightarrow 1} \omega(z) |\nabla f(z)| = 0 \right\}.$$

Similarly, the Zygmund-type space \mathcal{Z}_ω and its little subspace $\mathcal{Z}_{\omega,0}$ are defined by

$$\mathcal{Z}_\omega := \left\{ f \in H(\mathbb{B}) : \|f\|_{s\mathcal{Z}_\omega} := \sup_{z \in \mathbb{B}} \omega(z) |\nabla(Rf)(z)| < \infty \right\},$$

$$\mathcal{Z}_{\omega,0} := \left\{ f \in \mathcal{Z}_\omega : \lim_{|z| \rightarrow 1} \omega(z) |\nabla(Rf)(z)| = 0 \right\}.$$

It is easy to check that $\mathcal{B}_{\omega,0}, \mathcal{B}_\omega$ and $\mathcal{Z}_{\omega,0}, \mathcal{Z}_\omega$ are Banach spaces, respectively, under the norms

$$\|f\|_{\mathcal{B}_\omega} = |f(0)| + \|f\|_{s\mathcal{B}_\omega}, \quad \|f\|_{\mathcal{Z}_\omega} = |f(0)| + \|f\|_{s\mathcal{Z}_\omega}.$$

The α -Bloch space \mathcal{B}^α is obtained for $\omega(z) = (1 - |z|^2)^\alpha$, $\alpha \in (0, \infty)$ (see, e.g., [13, 14]). For $\alpha = 1$ the space $\mathcal{B}^1 = \mathcal{B}$ is the classical Bloch space. For $0 < \alpha < 1$, it is known that the space \mathcal{B}^α coincides with the Lipschitz space $Lip_{1-\alpha}$. The classical Bloch space \mathcal{B} of analytic functions on the open unit disk \mathbb{D} of \mathbb{C} plays an important role in geometric function theory and it has been studied by many authors. Timoney [17, 18] extended the notion of Bloch function by considering bounded homogeneous domains in \mathbb{C}^N , such as the open unit ball \mathbb{B} and the polydisk \mathbb{D}^N .

When $\omega(r) = 1 - r$, from [23, p. 261] we see that $f \in \mathcal{Z}_{1-r} := \mathcal{Z}$ if and only if $f \in A(\mathbb{B})$, the ball algebra on \mathbb{B} , and there exists a constant $C > 0$ such that $|f(\zeta + h) + f(\zeta - h) - 2f(\zeta)| < C|h|$, for all $\zeta \in \partial\mathbb{B}$ and $\zeta \pm h \in \partial\mathbb{B}$. The space \mathcal{Z}_ω can be considered as a generalization of the classical Zygmund space which was introduced in [14].

In the study of several complex variables, challenges frequently arise when exploring and applying the properties of the Bloch-type and Zygmund-type spaces, particularly in the context of products involving composition, multiplication, and radial derivative operators. To overcome these obstacles, we need various techniques. However, the situation is much more complex than in the case of the open unit disk because there are multiple types of derivatives to choose from on the open unit ball.

Bloch-type spaces and Zygmund-type spaces are usually defined by using radial derivative of f instead of its gradient. We recall the following results:

Proposition 1.2 ([16], Theorem 2.1). *For every normal weight ω on \mathbb{B} , we have*

- (a) $\mathcal{B}_\omega = \mathcal{B}_\omega^R := \left\{ f \in H(\mathbb{B}) : \|f\|_{\mathcal{B}_\omega^R} := |f(0)| + \sup_{z \in \mathbb{B}} \omega(z) |Rf(z)| < \infty \right\}$, moreover, $\|\cdot\|_{\mathcal{B}_\omega} \cong \|\cdot\|_{\mathcal{B}_\omega^R}$.
- (b) $f \in \mathcal{B}_{\omega,0}$ if and only if $\lim_{|z| \rightarrow 1} \omega(z) |Rf(z)| = 0$.
- (c) There exists $C > 0$ such that for every $f \in \mathcal{B}_\omega$ and for every $z \in \mathbb{B}$, we have

$$|f(z)| \leq C(1 + I_\omega(z)) \|f\|_{\mathcal{B}_\omega}. \tag{1}$$

Remark 1.3. *In fact, the estimate (1) can be written as follows:*

$$|f(z)| \leq C(|f(0)| + I_\omega(z)) \|f\|_{s\mathcal{B}_\omega}.$$

Proposition 1.4 ([21], Lemma 2.2, [19], Lemma 2.1). *For every normal weight ω on \mathbb{B} , we have*

$$\begin{aligned} \mathcal{Z}_\omega &= \mathcal{Z}_\omega^R := \left\{ f \in H(\mathbb{B}) : \|f\|_{\mathcal{Z}_\omega^R} := |f(0)| + \sup_{z \in \mathbb{B}} \omega(z) |R^{(2)}f(z)| < \infty \right\} \\ &= \mathcal{Z}_\omega^\nabla := \left\{ f \in H(\mathbb{B}) : \|f\|_{\mathcal{Z}_\omega^\nabla} := |f(0)| + \sup_{z \in \mathbb{B}} \omega(z) |\nabla^2 f(z)| < \infty \right\}, \end{aligned}$$

moreover, $\|\cdot\|_{\mathcal{Z}_\omega^R} \cong \|\cdot\|_{\mathcal{Z}_\omega^\nabla} \cong \|\cdot\|_{\mathcal{Z}_\omega^\mu}$.

Lemma 1.5 ([10], Lemma 3.4(2), [15], Lemma 2.4). Let ω be a normal weight on \mathbb{B} . Then there exists $C > 0$ such that

$$|R^{(2)}f(z)| \leq C\omega^*(z)\|f\|_{\mathcal{B}_\omega}, \quad |\nabla(Rf(z))| \leq C\omega^*(z)\|f\|_{\mathcal{B}_\omega} \quad (2)$$

for every $f \in \mathcal{B}_\omega$, for every $z \in \mathbb{B}$, and

$$|Rf(z)| \leq CI_\omega(z)\|f\|_{\mathcal{Z}_\omega}, \quad |\nabla f(z)| \leq C(1 + I_\omega(z))\|f\|_{\mathcal{Z}_\omega}, \quad (3)$$

$$|f(z)| \leq |f(0)| + C\left(\int_0^{|z|} I_\omega(t)dt\right)\|f\|_{\mathcal{Z}_\omega} \leq \left(1 + C\left(\int_0^{|z|} I_\omega(t)dt\right)\right)\|f\|_{\mathcal{Z}_\omega} \quad (4)$$

for every $f \in \mathcal{Z}_\omega$ and for every $z \in \mathbb{B}$.

Depending on a specific situation, it is sometimes more appropriate to use one of the definitions of the Bloch-type spaces and the Zygmund-type spaces. In this paper, we will employ the notation \mathcal{B}_ω and \mathcal{Z}_ω in place of $(\mathcal{B}_\omega, \|\cdot\|_{\mathcal{B}_\omega^R})$ and $(\mathcal{Z}_\omega, \|\cdot\|_{\mathcal{Z}_\omega^R})$, respectively.

Throughout this paper, we use the notions $a \lesssim b$ and $a \asymp b$ for non-negative quantities a and b to mean $a \leq Cb$ and, respectively, $b/C \leq a \leq Cb$ for some inessential constant $C > 0$.

2. Set $\widetilde{S}_k(\mathbb{B})$ and Condition $\mathcal{Z}_{+,0}$

In this section, we introduce several concepts, conditions, and related results that are required in the hypotheses of the main theorems.

2.1. The set $\widetilde{S}_k(\mathbb{B})$ and some auxiliary results

We begin this section by revisiting the sets $S^*(\mathbb{B})$ and $S_k^*(\mathbb{B})$, $1 \leq k \leq N$, introduced in [8], and by introducing a new set $\widetilde{S}_k(\mathbb{B})$ which extends both of them:

$$\begin{aligned} S^*(\mathbb{B}) &= \left\{ \varphi \in S(\mathbb{B}) : \varphi(\mathbb{B}) \supseteq \bigcup_{j=1}^N \mathbb{D}_j := \bigcup_{j=1}^N \{\lambda e_j : \lambda \in \mathbb{C}, |\lambda| < 1\} \right\}, \\ S_k^*(\mathbb{B}) &= \left\{ \varphi \in S(\mathbb{B}) : \varphi(\mathbb{B}) \supseteq \mathbb{D}_k := \{\lambda e_k : \lambda \in \mathbb{C}, |\lambda| < 1\} \right\}, \\ \widetilde{S}_k(\mathbb{B}) &= \left\{ \varphi \in S(\mathbb{B}) : 0 \in \varphi(\mathbb{B}) \cap \mathbb{D}_k, \forall z \in \mathbb{B}, \exists z' \in \mathbb{B} : |\varphi_k(z')| = |\varphi(z)| \right\}. \end{aligned}$$

The set $\widetilde{S}_k(\mathbb{B})$ will serve as an essential assumption in the formulation of our main results. Roughly speaking, while $S^*(\mathbb{B})$ and $S_k^*(\mathbb{B})$ consist of holomorphic self-maps of \mathbb{B} satisfying certain geometric conditions related to the unit disk in the k -th coordinate direction, the set $\widetilde{S}_k(\mathbb{B})$ is defined more generally so that it contains all such mappings whose ranges preserve, in an appropriate sense, the modulus structure of at least one component function of φ :

$$\widetilde{S}_p(\mathbb{B}) = \left\{ \varphi \in S(\mathbb{B}) : 0 \in \varphi(\mathbb{B}), \sup_{x \in \varphi_p(\mathbb{B})} |x| = \sup_{z \in \varphi(\mathbb{B})} |z| \right\}.$$

This generalization allows us to handle a much broader class of holomorphic self-maps and to establish stronger results on the boundedness and compactness of the weighted composition operators considered in this paper.

It is easy to see that, for $N = 2$, the function $\varphi : \mathbb{B} \rightarrow \mathbb{B}$ given by $\varphi(z, w) = (z/2, w^2)$ belongs to $S_2^*(\mathbb{B})$. This example also shows that $S_k^*(\mathbb{B}) \supsetneq S^*(\mathbb{B})$.

It is obvious that if $\varphi \in S(\mathbb{B})$ is surjective then $\varphi \in S^*(\mathbb{B})$. Now let us give an example of non-surjective holomorphic self-map which belongs to $S^*(\mathbb{B})$.

Example 2.1. Consider the case $N = 2$ and the function $\varphi : \mathbb{B} \rightarrow \mathbb{B}$ given by $\varphi(z, w) = (z, w^2)$ for every $(z, w) \in \mathbb{B}$. It is clear that $\varphi \in S^*(\mathbb{B})$. It remains to check that φ is not surjective. Indeed, let $(\frac{1}{\sqrt{2}}, \frac{1}{2}) \in \mathbb{B}$. Assume that there exists $(z_0, w_0) \in \mathbb{B}$ such that $\varphi(z_0, w_0) = (\frac{1}{\sqrt{2}}, \frac{1}{2})$. Then $z_0 = \frac{1}{\sqrt{2}}$ and $w_0 = \pm \frac{1}{\sqrt{2}}$, which contradicts with $(z_0, w_0) \in \mathbb{B}$, because $|z_0|^2 + |w_0|^2 = 1$.

The following examples show that the class $\widetilde{S}_k(\mathbb{B})$ is broader than the class $S_k^*(\mathbb{B})$.

Example 2.2. In \mathbb{C}^3 , consider $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in S(\mathbb{B})$ given by

$$\varphi(z_1, z_2, z_3) = \frac{1}{6}(2z_1, 2z_2^2 + i, z_3).$$

Obviously, $\varphi \in S(\mathbb{B})$ and $0 \in \varphi(\mathbb{B})$. For every $z = (z_1, z_2, z_3) \in \mathbb{B}$, consider $y \in \mathbb{D}$ such that $|y| = |\varphi(z)|$. Then, it is easy to check that for $z' = (z'_1, \sqrt{\frac{6y-i}{2}}, z'_3) \in \mathbb{B}$ we have $\varphi_2(z') = y$, consequently, $|\varphi_2(z')| = |\varphi(z)|$. Thus, $\varphi \in \widetilde{S}_2(\mathbb{B})$. However, it is easy to check that the point $-\frac{i}{2} \in \mathbb{D}_2 \setminus \text{Pr}_{\mathbb{D}_2}\varphi(\mathbb{B})$. This means $\varphi \notin S_2^*(\mathbb{B})$.

Example 2.3. Fix $\alpha_1, \dots, \alpha_N \in \mathbb{R}$ and $a = (a_1, \dots, a_N) \in \mathbb{B} \subset \mathbb{C}^N$ such that $|a_p| = \max_{1 \leq k \leq N} |a_k|$, $p \in \{1, \dots, N\}$. Consider $\varphi = (\varphi_1, \dots, \varphi_N) \in S(\mathbb{B})$ given by

$$\varphi(z) = (e^{i\alpha_1} a_1 z_1, \dots, e^{i\alpha_N} a_N z_N), \quad z = (z_1, \dots, z_N).$$

It is easy to see that $\varphi(\mathbb{B}) \subset \mathbb{B}$ and $\varphi(0) = 0$. We have

$$|\varphi(z)| \leq \max_{1 \leq k \leq N} |a_k| |z| = |a_p| |z| < |a_p| = |e^{i\alpha_p} a_p|, \quad z \in \mathbb{B}.$$

On the other hand, $\varphi_p(\mathbb{B}) = \{x \in \mathbb{C} : |x| \leq |a_p|\}$. This means $\sup_{x \in \varphi_p(\mathbb{B})} |x| \geq \sup_{z \in \varphi(\mathbb{B})} |z|$. Therefore, $\varphi \in \widetilde{S}_p(\mathbb{B})$. While, it is clear that $\varphi \notin S_p^*(\mathbb{B})$ because $\{x \in \mathbb{C} : |x| > |a_p|\} = \mathbb{D}_N \setminus \text{Pr}_{\mathbb{D}_N}\varphi(\mathbb{B}) \neq \emptyset$.

Lemma 2.4 (cf. [8], Lemma 4.1). Let $\varphi = (\varphi_1, \dots, \varphi_N) \in S_k^*(\mathbb{B})$, for some $k \in \{1, \dots, N\}$, $\varphi(0) = 0$, ν be a normal weight and h be a positive, real-valued bounded function defined on \mathbb{B} such that $\lim_{|z| \rightarrow 1} h(z) > 0$. Assume that

$$\mathbb{M}_k := \sup_{w \in \mathbb{B}} h(w) I_\nu(\varphi_k(w)) < \infty.$$

Then there exists constant $C_4 > 0$ such that

$$\sup_{z \in \mathbb{B}} h(z) I_\nu(\varphi(z)) \leq C_4 \mathbb{M}_k. \tag{5}$$

This represents an improvement over Lemma 4.1 in [8] where the estimate $h(z) I_\nu(\varphi(z)) \leq C_4(1 + \mathbb{M}_k)$ is replaced by (5).

The proof of this lemma repeats word for word the proof of Lemma 3.4 in [9], and is therefore omitted.

Remark 2.5. Since $\mathbb{M}_k < \infty$, we can find $C_4^* > 0$ such that

$$\sup_{w \in \mathbb{B}} h(w) \int_0^{\delta_0} \frac{dt}{m_{\nu, \delta_0}} \leq C_4^* \sup_{|\varphi_k(w)| > \delta_0} h(w) I_\nu(\varphi_k(w)).$$

Thus, the estimate (5) can be written as follows

$$\sup_{|\varphi(z)| > \delta_0} h(z) I_\nu(\varphi(z)) \lesssim \sup_{|\varphi_k(w)| > \delta_0} h(w) I_\nu(\varphi_k(w)).$$

For $\alpha \in \mathbb{B} \setminus \{0\}$, let γ_α be an involutive automorphism of \mathbb{B} given by

$$\gamma_\alpha(z) = \frac{\alpha - P_\alpha(z) - s_\alpha Q_\alpha(z)}{1 - \langle z, \alpha \rangle}, \tag{6}$$

where $s_\alpha = \sqrt{1 - |\alpha|^2}$, $P_\alpha(z) = \frac{\langle z, \alpha \rangle}{|\alpha|^2} \alpha$, $Q_\alpha(z) = z - \frac{\langle z, \alpha \rangle}{|\alpha|^2} \alpha$.

The following result can be obtained directly from Lemma 3.6 in [9], but here we present a shorter proof.

Lemma 2.6. *Let ν be a normal weight on \mathbb{B} , $\alpha \in \mathbb{B} \setminus \{0\}$, $\gamma = \gamma_\alpha = (\gamma_1, \dots, \gamma_N)$ defined as in (6) and h be a non-negative, real-valued bounded function defined on \mathbb{B} . Then,*

(a) *For every $k \in \{1, \dots, N\}$ satisfying $\mathbb{M}_k^* := \sup_{w \in \mathbb{B}} h(w)I_\nu(w_k) < \infty$, there exists $C(\alpha, k) > 0$ such that*

$$h(z)I_\nu(\gamma_k(z)) \leq C(\alpha, k) + \mathbb{M}_k^* \quad \text{for every } z \in \mathbb{B};$$

(b) *For every $k \in \{1, \dots, N\}$, we have*

$$\sup_{z \in \mathbb{B}} \frac{\nu(z)}{\nu(\gamma(z))} < \infty, \quad \sup_{z \in \mathbb{B}} \frac{\nu(z_k)}{\nu(\gamma_k(z))} < \infty.$$

Proof. (a) First, recall from [23] that P_α is the orthogonal projection from \mathbb{C}^N onto the one dimensional subspace $[\alpha]$ generated by α and Q_α is the orthogonal projection from \mathbb{C}^N onto $\mathbb{C}^N \ominus [\alpha]$. Note that $\gamma(\alpha) = 0$ and $(\gamma \circ \gamma)(z) = z$ for every $z \in \mathbb{B}$.

Let $k \in \{1, \dots, N\}$ be fixed, with $\mathbb{M}_k^* < \infty$. Since γ is surjective, for every $z \in \mathbb{B}$, there exists $z' \in \mathbb{B}$ such that

$$\gamma(z') = (0, \dots, \gamma_k(z), 0, \dots, 0).$$

It is clear that $\gamma_j(z') = 0$ for $j \in \{1, \dots, N\} \setminus \{k\}$ and

$$\gamma_k(z) = \gamma_k(z') = \frac{\alpha_k - P_{\alpha,k}(z') - s_\alpha Q_{\alpha,k}(z')}{1 - \langle z', \alpha \rangle},$$

where $P_{\alpha,k}(z') := \frac{\langle z', \alpha \rangle}{|\alpha|^2} \alpha_k$, and $Q_{\alpha,k}(z') = z'_k - \frac{\langle z', \alpha \rangle}{|\alpha|^2} \alpha_k$.

Since $\alpha - P_\alpha(z)$ and $Q_\alpha(z)$ are perpendicular in \mathbb{C}^N , we have

$$\begin{aligned} |\alpha_k - P_{\alpha,k}(z') - s_\alpha Q_{\alpha,k}(z')|^2 &= |\alpha_k - P_{\alpha,k}(z')|^2 + (1 - |\alpha|^2)(|z'_k|^2 - |P_{\alpha,k}(z')|^2) \\ &= |\alpha_k|^2 \left(1 - 2\operatorname{Re} \frac{\langle z', \alpha \rangle}{|\alpha|^2} + \frac{|\langle z', \alpha \rangle|^2}{|\alpha|^4} \right) - \frac{|\langle z', \alpha \rangle|^2}{|\alpha|^4} |\alpha_k|^2 + \frac{|\langle z', \alpha \rangle|^2}{|\alpha|^2} |\alpha_k|^2 + (1 - |\alpha|^2)|z'_k|^2 \\ &= \frac{|\alpha_k|^2}{|\alpha|^2} \left(|\alpha|^2 - 1 + 1 - 2\operatorname{Re} \langle z', \alpha \rangle + |\langle z', \alpha \rangle|^2 \right) + (1 - |\alpha|^2)|z'_k|^2 \\ &= \frac{|\alpha_k|^2}{|\alpha|^2} (|\alpha|^2 - 1) + \frac{|\alpha_k|^2}{|\alpha|^2} |1 - \langle z', \alpha \rangle|^2 + (1 - |\alpha|^2)|z'_k|^2. \end{aligned}$$

This yields that

$$\begin{aligned} |\gamma_k(z)|^2 &= |\gamma(z')|^2 = |\gamma_k(z')|^2 = \frac{|\alpha_k|^2}{|\alpha|^2} - \frac{(1 - |\alpha|^2) \left(\frac{|\alpha_k|^2}{|\alpha|^2} - |z'_k|^2 \right)}{|1 - \langle z', \alpha \rangle|^2} \\ &\leq \frac{|\alpha_k|^2}{|\alpha|^2} - \frac{(1 - |\alpha|) \left(\frac{|\alpha_k|^2}{|\alpha|^2} - |z'_k|^2 \right)}{1 + |\alpha|} = \frac{2|\alpha_k|^2}{|\alpha|(1 + |\alpha|)} + \frac{1 - |\alpha|}{1 + |\alpha|} |z'_k|^2 \\ &= A_k^2 + A^2 |z'_k|^2 < A_k^2 + A^2, \end{aligned} \tag{7}$$

where $A_k^2 := \frac{2|\alpha_k|^2}{|\alpha|(1+|\alpha|)}$ and $A^2 := \frac{1-|\alpha|}{1+|\alpha|}$. It is easy to check that $A_k^2 + A^2 < 1$. Then, for all $|z'_k| \geq |\alpha|$, we obtain the following estimate

$$\begin{aligned} I_v(\gamma'_k(z)) &\leq \int_0^{|z'_k|} \frac{dt}{v(t)} + \int_{A|z'_k|}^{\sqrt{A_k^2 + A^2}|z'_k|^2} \frac{dt}{v(t)} \leq \int_0^{|z'_k|} \frac{dt}{v(t)} + \int_{A|\alpha|}^{\sqrt{A_k^2 + A^2}} \frac{dt}{v(t)} \\ &= C_{\alpha,k} + I_v(z'_k), \end{aligned}$$

where $C_{\alpha,k} = \int_{A|\alpha|}^{\sqrt{A_k^2 + A^2}} \frac{dt}{v(t)} < \infty$. Obviously, in the case $|z'_k| \leq |\alpha|$, we have

$$I_v(\gamma'_k(z)) \leq \int_0^{A|\alpha|} \frac{dt}{v(t)} < \infty.$$

Consequently, assertion (a) of the lemma holds, and the lemma is thus proved.

(b) For every $r \in (|\alpha|, 1)$, the continuity of γ ensures that the set $\{\gamma(z) : |z| \leq r\}$ is compact in \mathbb{B} . Since v is positive and continuous, $\inf_{|z| \leq r} v(\gamma(z)) > 0$, it implies that

$$\sup_{|z| \leq r} \frac{v(z)}{v(\gamma(z))} < \infty.$$

On the other hand, for every $z \in \mathbb{B}$, $|z| > r$, by (7), we have

$$\frac{(1 - |z|)^a}{(1 - |\gamma_\alpha(z)|)^b} \leq \frac{(1 - r)^a}{\left(1 - \frac{2r^2}{|\alpha|(1+|\alpha|)} - \frac{1-r}{1+r}|z|^2\right)^b} \rightarrow \frac{(1 - r)^a}{\left(1 - \frac{2r^2}{|\alpha|(1+|\alpha|)} - \frac{1-r}{1+r}\right)^b} < \infty$$

as $|z| \rightarrow 1$ because it is easy to check that $1 - \frac{2r^2}{|\alpha|(1+|\alpha|)} - \frac{1-r}{1+r} > 0$. Therefore,

$$\sup_{|z| > r} \frac{(1 - |z|)^a}{(1 - |\gamma_\alpha(z)|)^b} < \infty$$

for $r \in (|\alpha|, 1)$ sufficiently large. Then, by (W_1) and (W_2) ,

$$\lim_{|z| \rightarrow 1} \frac{v(z)}{v(\gamma(z))} = \lim_{|z| \rightarrow 1} \frac{v(z)}{(1 - |z|)^a} \frac{(1 - |\gamma(z)|)^b}{v(\gamma(z))} \frac{(1 - |z|)^a}{(1 - |\gamma(z)|)^b} = 0. \tag{8}$$

Then, we obtain the first inequality.

Now, it is obvious that $\sup_{|\gamma_k(z)| \leq r} \frac{v(z_k)}{v(\gamma_k(z))} < \infty$ with $r \in (\delta, 1)$. In the case $|\gamma_k(z)| > r$, since $|\gamma_k(z)| \leq |\gamma(z)|$, by an estimate as (8), we obtain the second inequality in (b) of the lemma. \square

Lemma 2.7 ([9], Lemma 3.7). *Let v be a normal weight on \mathbb{B} , $\alpha \in \mathbb{B} \setminus \{0\}$ and $\gamma \in \text{Aut}(\mathbb{B})$ be defined by (6). Then, the composition operator $C_\gamma : \mathcal{B}_v \rightarrow \mathcal{B}_v$, $f \mapsto f \circ \gamma$, is a homeomorphism.*

2.2. The condition $\mathcal{Z}_{+,0}$

In the hypotheses of the main theorems, we make use of the following condition. We say that the pair of functions (ψ, φ_k) satisfies the condition $\mathcal{Z}_{+,0}$ and write $(\psi, \varphi_k) \in \mathcal{Z}_{+,0}$, if

$$\psi \in \mathcal{Z}_{\mu,+r} \quad \psi \cdot \varphi_k, \psi \cdot \varphi_k^2 \in \mathcal{Z}_{\mu,0}. \tag{\mathcal{Z}_{+,0}}$$

In this subsection, we provide examples to demonstrate the existence of the stated conditions.

Example 2.8. For $\alpha \in (0, 1)$, consider the weight $\mu(z) \in H(\mathbb{B})$, $\varphi \in S_k^*(\mathbb{B})$ with

$$\mu(z) = (1 - |z|^2)^\alpha, \quad \varphi_k(z) = \langle z, e_k \rangle.$$

We construct the function $\psi \in H(\mathbb{B})$ as follows: Let $q > 1$ be a large positive integer to be determined, define

$$\widetilde{\psi}(z) = \sum_{j=0}^{\infty} a_j \langle z, e_k \rangle^{n_j} = \sum_{j=0}^{\infty} a_j z_k^{n_j}, \tag{9}$$

where $a_j = q^{j(\alpha-1)+\frac{\alpha}{2}}$, $n_j = q^j$. Because $\widetilde{\psi}$ is a lacunary power series with

$$a_j n_j^{1-\alpha} = q^{j(\alpha-1)+\frac{\alpha}{2}} q^{j(1-\alpha)} = q^{\frac{\alpha}{2}},$$

using Theorem 1 (1) in [20] we have $\widetilde{\psi} \in \mathcal{B}_\alpha$ and since $\alpha \in (0, 1)$, it is easy to check that $\widetilde{\psi} \in H^\infty(\mathbb{B})$.

By modifying an argument in the proof of Theorem 6 in [3], next, we will show

$$|R\widetilde{\psi}(z)| \gtrsim \frac{1}{(1 - |z|)^\alpha} \tag{10}$$

for all $z \in \mathbb{B}$ sufficiently close to the boundary.

We write

$$\begin{aligned} |R\widetilde{\psi}(z)| &= \left| \sum_{i=0}^{\infty} q^{i(\alpha-1)+\frac{\alpha}{2}+i} z_k^{q^i} \right| \\ &\geq q^{k(\alpha-1)+\frac{\alpha}{2}+k} |z_k|^{q^k+1} - \sum_{i=0}^{k-1} q^{i(\alpha-1)+\frac{\alpha}{2}+i} |z_k|^{q^i} - \sum_{i=j+1}^{\infty} q^{i(\alpha-1)+\frac{\alpha}{2}+i} |z_k|^{q^i} \\ &\geq q^{j(\alpha-1)+\frac{\alpha}{2}+j} |z_k|^{q^k+1} - \sum_{i=0}^{j-1} q^{i(\alpha-1)+\frac{\alpha}{2}+i} |z|^{q^i} - \sum_{i=j+1}^{\infty} q^{i(\alpha-1)+\frac{\alpha}{2}+i} |z|^{q^i} \\ &:= Q_1 - Q_2 - Q_3. \end{aligned}$$

For z satisfying

$$1 - \frac{1}{q^j} \leq |z_k| \leq |z| \leq 1 - \frac{1}{q^{j+\frac{1}{2}}} \tag{11}$$

we have

$$|z_k|^{q^j+1} \geq \left(1 - \frac{1}{q^j}\right)^{q^j+1} \geq \frac{1}{3} \tag{12}$$

if q is large enough. Then (12) gives

$$\begin{aligned} Q_1 &\geq \frac{1}{3} q^{(j+\frac{1}{2})\alpha}, \\ Q_2 &\leq \sum_{i=0}^{j-1} q^{i(\alpha-1)+\frac{\alpha}{2}+i} = \sum_{i=0}^{j-1} q^{i\alpha+\frac{\alpha}{2}} = q^{\frac{\alpha}{2}} \frac{q^{j\alpha}}{q^\alpha - 1} = \frac{q^{(j+\frac{1}{2})\alpha}}{q^\alpha - 1}. \end{aligned}$$

Applying (11) again, we have $|z_k|^{q^j} \leq \left(\frac{1}{2}\right)^{q^{-\frac{1}{2}}}$ and

$$\begin{aligned} Q_3 &= \sum_{i=j+1}^{\infty} q^{i\alpha+\frac{\alpha}{2}} |z|^{q^i} = q^{\frac{\alpha}{2}} \sum_{i=j+1}^{\infty} q^{i\alpha} |z|^{q^i} \leq q^{\frac{\alpha}{2}} q^{\alpha(j+1)} |z|^{q^{j+1}} \sum_{i=0}^{\infty} \left(q^\alpha |z|^{q^{i+2}-q^{j+1}}\right)^i = q^{\alpha(j+1)+\frac{\alpha}{2}} \frac{|z|^{j+1}}{1 - q^\alpha |z|^{q^{j+2}-q^{j+1}}} \\ &= q^{\alpha(j+\frac{1}{2})} \frac{q^\alpha |z|^{j+1}}{1 - q^\alpha |z|^{q^{j+2}-q^{j+1}}} \leq q^{(j+\frac{1}{2})\alpha} \frac{q^\alpha \left(\frac{1}{2}\right)^{\frac{1}{2}}}{1 - q^\alpha \left(\frac{1}{2}\right)^{\frac{3}{2}-q^{\frac{1}{2}}}}. \end{aligned}$$

From (11) we have $q^{j+\frac{1}{2}} \geq \frac{1}{1-|z|}$. Combining (12) with the estimates for Q_1, Q_2 , and Q_3 , we get

$$|R\tilde{\psi}(z)| \geq \frac{1}{4}q^{(j+\frac{1}{2})\alpha} \geq \frac{1}{4} \frac{1}{(1-|z|)^\alpha}$$

for z satisfying (11) and q sufficiently large, hence, (10) is proved. This implies that

$$\lim_{|z| \rightarrow 1} (1-|z|^2)^\alpha |R\tilde{\psi}(z)| > 0.$$

Now we put

$$\psi(z) := \int_0^{\langle z, e_k \rangle} \int_0^{\omega_k^{(1)}} \tilde{\psi}(t) dt dt_1, \quad z \in \mathbb{B}.$$

It is easy to verify that $R^{(2)}\psi(z) = R\tilde{\psi}(z)z$. Consequently,

$$\lim_{|z| \rightarrow 1} (1-|z|^2)^\alpha |R^{(2)}\psi(z)| \geq \lim_{|z| \rightarrow 1} (1-|z|^2)^\alpha |R\tilde{\psi}(z)| > 0.$$

That means $\psi \in \mathcal{Z}_{\mu,+}$.

On the other hand, since $\tilde{\psi} \in H^\infty(\mathbb{B})$, there exists $M > 0$ such that $\sup_{z \in \mathbb{B}} |R\psi(z)| \leq M$, hence, $\sup_{z \in \mathbb{B}} |R^{(n)}\psi(z)| < M$ for every $n = 0, 1, 2$. Then,

$$\begin{aligned} \lim_{|z| \rightarrow 1} (1-|z|^2)^\alpha |R^{(2)}(\psi \cdot \varphi_k^j)(z)| &\leq \sum_{i=0}^2 \binom{2}{i} \lim_{|z| \rightarrow 1} (1-|z|^2)^\alpha |R^{(2-i)}\psi(z)R^{(i)}\varphi_k^{j_0}(z)| \\ &\leq \sum_{i=0}^2 \binom{2}{i} \lim_{|z| \rightarrow 1} (1-|z|^2)^\alpha M \frac{j_0!}{(j_0-i)!} |z_k^{j-i}| = 0 \end{aligned}$$

for $j = 1, 2$. Therefore, $\psi \cdot \varphi_k^j \in \mathcal{Z}_{\mu,0}$ for $j = 1, 2$.

Example 2.9. Consider the weight $\mu(z) \in H(\mathbb{B})$, $\varphi \in S_k^*(\mathbb{B})$ with

$$\mu(z) = (1-|z|^2)^\alpha, \quad \varphi_k(z) = \frac{z_k - a_k}{1 - z_k \bar{a}_k},$$

where $\alpha \in (0, 1)$, $a \in \mathbb{B}$ and $\psi = \tilde{\psi}$ which is defined by (9).

First, we check that $\psi \in \mathcal{Z}_{\mu,+}$.

It is clear that

$$R^{(2)}\psi(z) = \sum_{i=0}^{\infty} q^{i(\alpha-1) + \frac{\alpha}{2} + 2i} z_p^{q^i}.$$

Then, using a similar calculation as in the above example, we have

$$\begin{aligned} |R^{(2)}\psi(z)| &= \left| \sum_{i=0}^{\infty} q^{i(\alpha-1) + \frac{\alpha}{2} + 2i} z_p^{q^i} \right| \\ &\geq q^{j(\alpha-1) + \frac{\alpha}{2} + 2j} |z_k|^{q^{j+1}} - \sum_{i=0}^{j-1} q^{i(\alpha-1) + \frac{\alpha}{2} + 2i} |z_p|^{q^i} - \sum_{i=j+1}^{\infty} q^{i(\alpha-1) + \frac{\alpha}{2} + 2i} |z_p|^{q^i} \\ &:= Q'_1 - Q'_2 - Q'_3. \end{aligned}$$

For z satisfying

$$1 - \frac{1}{q^j} \leq |z_k| \leq |z| \leq 1 - \frac{1}{q^{j+\frac{3}{2}}}, \tag{13}$$

as in the above, we also have (12) and

$$Q'_1 \geq \frac{1}{3}q^{(j+1)+(j+\frac{3}{2})\alpha}, \quad Q'_2 \leq \frac{q^{(j+1)+(j+\frac{3}{2})\alpha}}{q^{2+\alpha-1}-1}, \quad Q'_3 \leq q^{(j+1)+(j+\frac{3}{2})\alpha} \frac{q^{2+\alpha-1} \left(\frac{1}{2}\right)^{\frac{1}{2}}}{1 - q^{2+\alpha-1} \left(\frac{1}{2}\right)^{\frac{3}{2}-q^{\frac{1}{2}}}}.$$

From (13) we have $q^{j+\frac{3}{2}} \geq \frac{1}{1-|z|}$. Combining (12) with the estimates for $Q'_1, Q'_2,$ and $Q'_3,$ we get

$$|R^{(2)}\psi(z)| \geq \frac{1}{4}q^{(j+1)+(j+\frac{3}{2})\alpha} \geq \frac{1}{4} \frac{1}{(1-|z|)^\alpha}$$

for z satisfying (13) and q sufficiently large. This implies that

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\alpha |R^{(2)}\psi(z)| > 0. \tag{14}$$

Thus, $\psi \in \mathcal{Z}_{\mu,+}$.

Finally, by similar arguments and estimates as in Example 1, we can also easily prove that $\psi \cdot \varphi_k^j \in \mathcal{Z}_{\mu,0}$ for $j = 1, 2$.

3. The boundedness

In this section, let $\psi \in H(\mathbb{B}), \varphi = (\varphi_1, \dots, \varphi_N) \in S(\mathbb{B}),$ and μ, ν be normal weights on $\mathbb{B}.$ We present here some notations that will be used the main results:

$$\mathcal{B}_{\nu,+} := \left\{ f \in \mathcal{B}_\nu : \lim_{|z| \rightarrow 1} \nu(z) |Rf(z)| > 0 \right\},$$

$$\mathcal{Z}_{\mu,+} := \left\{ f \in \mathcal{Z}_\mu : \lim_{|z| \rightarrow 1} \mu(z) |R^{(2)}f(z)| > 0 \right\},$$

and for $\varphi_\circ = \varphi$ or $\varphi_\circ = \varphi_k,$

$$\begin{aligned} A_{\psi, \varphi_\circ}(z) &:= 2R\psi(z)R\varphi_\circ(z) + \psi(z)R^{(2)}\varphi_\circ(z), & B_{\psi, \varphi_k}(z) &:= \psi(z)(R\varphi_k(z))^2, \\ B_{\psi, \varphi}(z) &:= \psi(z)(R\varphi_1(z))^2, \dots, (R\varphi_N(z))^2, & M_{\mu, \psi}^{I, \varphi_\circ} &:= \sup_{z \in \mathbb{B}} M_{\mu, \psi}^{I, \varphi_\circ}(z), \\ M_{\mu, \psi}^{I, \varphi_\circ}(z) &:= \mu(z) |R^{(2)}\psi(z)| I_\nu(\varphi_\circ(z)), & M_{\mu, \nu}^{\Sigma A} &:= \sup_{z \in \mathbb{B}} M_{\mu, \nu}^{\Sigma A}(z), \\ M_{\mu, \nu}^{\Sigma A}(z) &:= \mu(z) \frac{\sum_{k=1}^N |A_{\psi, \varphi_k}(z)|}{\nu(\varphi(z))}, & M_{\mu, \nu}^{A, \varphi_k} &:= \sup_{z \in \mathbb{B}} M_{\mu, \nu}^{A, \varphi_k}(z), \\ M_{\mu, \nu}^{A, \varphi_k}(z) &:= \mu(z) \frac{|A_{\psi, \varphi_k}(z)|}{\nu(\varphi_k(z))}, & M_{\mu, \nu^*}^{\Sigma B} &:= \sup_{z \in \mathbb{B}} M_{\mu, \nu^*}^{\Sigma B}(z), \\ M_{\mu, \nu^*}^{\Sigma B}(z) &:= \mu(z) \sum_{k=1}^N |B_{\psi, \varphi_k}(z)| \nu^*(\varphi(z)), & M_{\mu, \nu^*}^{B, \varphi_k} &:= \sup_{z \in \mathbb{B}} M_{\mu, \nu^*}^{B, \varphi_k}(z), \\ M_{\mu, \nu^*}^{B, \varphi_k}(z) &:= \mu(z) |B_{\psi, \varphi_k}(z)| \nu^*(\varphi_k(z)), \end{aligned}$$

By Newton-Leibniz formula (see (2.4) in [10]) and the higher-order radial derivative formula of a composite function (see (2.5) in [10]) we have

$$\begin{aligned} &R^{(2)}(\psi \cdot (f \circ \varphi))(z) \\ &= R^{(2)}\psi(z)f(\varphi(z)) + \sum_{k=1}^N \left(2R\psi(z)R\varphi_k(z) + \psi(z)R^{(2)}\varphi_k(z) \right) \frac{\partial f(\varphi(z))}{\partial z_k} + \psi(z) \sum_{j,k=1}^N \frac{\partial^2 f(\varphi(z))}{\partial z_k \partial z_j} R\varphi_k(z)R\varphi_j(z) \\ &= R^{(2)}\psi(z)f(\varphi(z)) + \langle A_{\psi,\varphi}(z), \overline{\nabla f(\varphi(z))} \rangle + \psi(z) \langle R\varphi(z), \overline{\nabla \langle R\varphi(z), \overline{\nabla f(\varphi(z))} \rangle} \rangle \end{aligned}$$

for every $f \in H(\mathbb{B})$. Consequently,

$$|R^{(2)}(W_{\psi,\varphi}f)(z)| \leq |R^{(2)}\psi(z)||f(\varphi(z))| + |A_{\psi,\varphi}(z)||\nabla f(\varphi(z))| + 2|B_{\psi,\varphi}(z)||\nabla^2 f(\varphi(z))|. \tag{15}$$

Theorem 3.1. *The following are equivalent:*

- (a) $W_{\psi,\varphi} : \mathcal{B}_v \rightarrow \mathcal{Z}_\mu$ is bounded;
- (b) $W_{\psi,\varphi} : \mathcal{B}_{v,0} \rightarrow \mathcal{Z}_\mu$ is bounded;
- (c) $\psi, \psi \cdot \varphi_j, \psi \cdot \varphi_j^2 \in \mathcal{Z}_\mu$ for every $j \in \{1, \dots, N\}$ and

$$M_{\mu,\psi}^{L,\varphi} < \infty, \quad M_{\mu,\nu}^{\Sigma A} < \infty, \quad M_{\mu,\nu^*}^{\Sigma B} < \infty. \tag{16}$$

In this case, the following asymptotic relation holds:

$$\|W_{\psi,\varphi}\| \asymp |\psi(0)|I_v(\varphi(0)) + M_{\mu,\psi}^{L,\varphi} + M_{\mu,\nu}^{\Sigma A} + M_{\mu,\nu^*}^{\Sigma B}. \tag{17}$$

Moreover, under the additional conditions that $\varphi \in \widetilde{S}_k(\mathbb{B})$ for some $k \in \{1, \dots, N\}$ and $(\psi, \varphi_k) \in \mathcal{Z}_{+,0}$, the assertions (a)-(c) and the following are equivalent:

- (d) $\psi, \psi \cdot \varphi_j, \psi \cdot \varphi_j^2 \in \mathcal{Z}_\mu$ for every $j \in \{1, \dots, N\}$ and

$$M_{\mu,\psi}^{L,\varphi_k} < \infty, \quad M_{\mu,\nu}^{A,\varphi_k} < \infty, \quad M_{\mu,\nu^*}^{B,\varphi_k} < \infty. \tag{18}$$

Proof. The equivalences (a) \Leftrightarrow (b) \Leftrightarrow (c) represent a special case of Theorem 4.2 in [10], when $n = m = 1$. The implication (c) \Rightarrow (d) is obvious.

Now we prove the implication (d) \Rightarrow (a).

By a direct calculation, we have

$$\begin{aligned} R^{(2)}[\psi(z)\varphi_j(z)] &= R^{(2)}\psi(z)\varphi_j(z) + A_{\psi,\varphi_j}(z), \\ R^{(2)}[\psi(z)\varphi_j^2(z)] &= \varphi_j(z) \left[2R^{(2)}[\psi(z)\varphi_j(z)] - R^{(2)}\psi(z) \right] + 2B_{\psi,\varphi_j}(z) \end{aligned} \tag{19}$$

for every $z \in \mathbb{B}$. Since $\psi, \psi \cdot \varphi_j, \psi \cdot \varphi_j^2 \in \mathcal{Z}_\mu$ for every $j \in \{1, \dots, N\}$ we have

$$\begin{aligned} \sup_{z \in \mathbb{B}} \mu(z)|A_{\psi,\varphi_j}(z)| &\leq \|\psi \cdot \varphi_j\|_{\mathcal{Z}_\mu} + \|\psi\|_{\mathcal{Z}_\mu} < \infty, \\ \sup_{z \in \mathbb{B}} \mu(z)|B_{\psi,\varphi_j}(z)| &\leq \|\psi \cdot \varphi_j^2\|_{\mathcal{Z}_\mu} + 2\|\psi \cdot \varphi_j\|_{\mathcal{Z}_\mu} + \|\psi\|_{\mathcal{Z}_\mu} < \infty \end{aligned} \tag{20}$$

for every $j = 1, \dots, N$. Therefore,

$$\sup_{z \in \mathbb{B}} \mu(z)|A_{\psi,\varphi}(z)| < \infty, \quad \sup_{z \in \mathbb{B}} \mu(z)|B_{\psi,\varphi}(z)| < \infty. \tag{21}$$

- First, we consider the case $\varphi(0) = 0$.

Since $\psi \in \mathcal{Z}_{\mu,+}$, by applying Lemma 2.4 to $h(z) = \mu(z)|R^{(2)}\psi(z)|$, it follows from (1) that there exists $D_1 > 0$ such that

$$\mu(z)|R^{(2)}\psi(z)||f(\varphi(z))| \leq D_1(1 + M_{\mu,\psi}^{A,\varphi_k}(z))\|f\|_{\mathcal{B}_v}, \quad f \in \mathcal{B}_v. \tag{22}$$

We will prove that there exists $\lambda \in (\delta, 1)$ such that

$$\inf_{|\varphi_k(z)| > \lambda} \mu(z)|A_{\psi,\varphi_k}(z)| > 0, \tag{23}$$

$$\inf_{|\varphi_k(z)| > \lambda} \mu(z)|B_{\psi,\varphi_k}(z)| > 0. \tag{24}$$

Since $(\psi, \varphi_k) \in \mathcal{Z}_{+,0}$, and $|\varphi_k(z)| \leq |z|$, we have

$$\begin{aligned} \lim_{|\varphi_k(z)| \rightarrow 1} \mu(z)|A_{\psi,\varphi_k}(z)| &= \lim_{|\varphi_k(z)| \rightarrow 1} |\mu(z)R^{(2)}(\psi \cdot \varphi_k)(z) - \mu(z)R^{(2)}\psi(z)\varphi_k(z)| \\ &= \lim_{|z| \rightarrow 1} |\psi(z)\mu(z)R^{(2)}\psi(z)| > 0, \end{aligned}$$

hence, there exists $\lambda \in (\delta, 1)$ such that (23) holds.

It follows from (19) that

$$R^{(2)}(\psi \cdot \varphi_k^2)(z) = \varphi_k(z)R^{(2)}(\psi \cdot \varphi_k)(z) + A_{\psi,\varphi_k}(z) + 2B_{\psi,\varphi_k}(z).$$

Then, by $\psi \cdot \varphi_k, \psi \cdot \varphi_k^2 \in \mathcal{Z}_{\mu,0}$ and (23), we conclude that (24) holds for some $\lambda \in (0, 1)$.

Thus, by (21),

$$D_2 := \frac{\sup_{z \in \mathbb{B}} \mu(z)|A_{\psi,\varphi}(z)|}{\inf_{|\varphi_k(w)| > \lambda} \mu(w)|A_{\psi,\varphi_k}(w)|} < \infty, \quad D_3 := \frac{\sup_{z \in \mathbb{B}} \mu(z)|B_{\psi,\varphi}(z)|}{\inf_{|\varphi_k(w)| > \lambda} \mu(w)|B_{\psi,\varphi_k}(w)|} < \infty.$$

Then, since $\varphi \in \widetilde{\mathcal{S}}_k(\mathbb{B})$, for each $z \in \mathbb{B}$ where $|\varphi(z)| > \lambda$ (hence, $|z| > \lambda$), there exists $z' \in \mathbb{B}$, such that $|\varphi(z)| = |\varphi_k(z')|$. Therefore,

$$\begin{aligned} \mu(z)|A_{\psi,\varphi}(z)||\nabla f(\varphi(z))| &\leq \frac{\sup_{z \in \mathbb{B}} \mu(z)|A_{\psi,\varphi}(z)|}{\inf_{|\varphi_k(w)| > \lambda} \mu(w)|A_{\psi,\varphi_k}(w)|} \frac{\mu(z')|A_{\psi,\varphi_k}(z')|}{\nu(\varphi_k(z'))} \nu(\varphi(z))|\nabla f(\varphi(z))| \\ &\leq D_2 \sup_{|\varphi(z)| > \lambda} M_{\mu,\nu}^{A,\varphi_k} \|f\|_{\mathcal{B}_v} \leq D_2 M_{\mu,\nu}^{A,\varphi_k} \|f\|_{\mathcal{B}_v}. \end{aligned} \tag{25}$$

Similar, by applying (2), we obtain

$$\mu(z)|B_{\psi,\varphi}(z)||\nabla^2 f(\varphi(z))| \leq D_3 M_{\mu,\nu^*}^{B,\varphi_k}(z) \|f\|_{\mathcal{B}_v}. \tag{26}$$

On the other hand, since $m_{\nu,\lambda} := \min_{t \in [0,\lambda]} \nu(t) > 0$, $\sup_{|\varphi(z)| \leq \lambda} M_{\mu,\nu}^{A,\varphi_k} < \infty$, $\sup_{|\varphi(z)| \leq \lambda} M_{\mu,\nu^*}^{B,\varphi_k}(z) < \infty$. Choose $D_4 > 0$ such that

$$\sup_{|\varphi(z)| \leq \lambda} \mu(z)|A_{\psi,\varphi}(z)| \leq D_4 \sup_{|\varphi_k(z)| \leq \lambda} M_{\mu,\nu}^{A,\varphi_k}(z), \tag{27}$$

$$\sup_{|\varphi(z)| \leq \lambda} \mu(z)|B_{\psi,\varphi}(z)| \leq D_4 \sup_{|\varphi_k(z)| \leq \lambda} M_{\mu,\nu^*}^{B,\varphi_k}(z). \tag{28}$$

Thus, it follows from (25), (27) and (26), (28) that

$$\sup_{z \in \mathbb{B}} \mu(z)|A_{\psi,\varphi}(z)||\nabla f(\varphi(z))| \lesssim M_{\mu,\nu}^{A,\varphi_k} \|f\|_{\mathcal{B}_v}, \tag{29}$$

$$\sup_{z \in \mathbb{B}} \mu(z) |B_{\psi, \varphi}(z)| |\nabla^2 f(\varphi(z))| \lesssim M_{\mu, \nu}^{B_k} \|f\|_{\mathcal{B}_\nu}. \tag{30}$$

Then, using (15), Proposition 1.4, we combine (22), (29) and (30) to yield

$$\mu(z) |R^{(2)}(W_{\psi, \varphi} f)(z)| \lesssim \left[1 + M_{\mu, \psi}^{L, \varphi_k} + M_{\mu, \nu}^{A_k} + M_{\mu, \nu^*}^{B_k} \right] \|f\|_{\mathcal{B}_\nu}. \tag{31}$$

Consequently,

$$\|W_{\psi, \varphi} f(z)\|_{\mathcal{Z}_\mu} \lesssim \left[1 + M_{\mu, \psi}^{L, \varphi_k} + M_{\mu, \nu}^{A_k} + M_{\mu, \nu^*}^{B_k} \right] \|f\|_{\mathcal{B}_\nu},$$

hence, $W_{\psi, \varphi} : \mathcal{B}_\nu \rightarrow \mathcal{Z}_\mu$ is bounded.

• Next, we consider the case $\varphi(0) = \alpha \in \mathbb{B} \setminus \{0\}$.

Let $\gamma \in \text{Aut}(\mathbb{B})$ given by (6). Then $\eta := \gamma \circ \varphi$ satisfies $\eta(0) = 0$. We will prove that $(\psi, \eta_k) \in \mathcal{Z}_{+,0}$ if $(\psi, \varphi_k) \in \mathcal{Z}_{+,0}$ and

$$M_{\mu, \psi}^{L, \eta_k} < \infty, \quad M_{\mu, \nu}^{A, \eta_k} < \infty, \quad M_{\mu, \nu^*}^{B, \eta_k} < \infty. \tag{32}$$

Since $\varphi \in \widetilde{S}_k(\mathbb{B})$, for each $z \in \mathbb{B}$, there exists $z' \in \mathbb{B}$ such that $\eta_k(z) = \varphi_k(z')$. We will show that $|z'| \rightarrow 1$ if $|z| \rightarrow 1$. Indeed, assume that there is a sequence $\{z_m\}_{m \geq 1} \subset \mathbb{B}$, $|z_m| \rightarrow 1$ but $z'_m \rightarrow a$ with $|a| < 1$. By the continuity of φ_k , we have $\varphi_k(z'_m) \rightarrow \varphi_k(a)$ with $|\varphi_k(a)| < 1$. It follows from $\gamma \in \text{Aut}(\mathbb{B})$ that $\varphi(z_m) = \gamma_k^{-1}(\gamma_k(\varphi(z_m))) \rightarrow \gamma_k^{-1}(\gamma_k(a))$. On the other hand, since φ is injective, we get

$$|z_m| \rightarrow |\varphi^{-1}(\gamma_k^{-1}(\gamma_k(a)))| = |(\gamma_k \circ \varphi)^{-1}(\varphi_k(a))| = |\eta_k^{-1}(\varphi_k(a))| < 1.$$

We thus arrive at a contradiction.

In the case $\psi \cdot \varphi_k, \psi \cdot \varphi_k^2 \in \mathcal{Z}_{\mu,0}$, we have

$$\begin{aligned} \lim_{|z| \rightarrow 1} \mu(z) |R^{(2)}(\psi \cdot \eta_k)(z)| &= \lim_{|z| \rightarrow 1} \mu(z) \left| R^{(2)}\psi(z)\eta_k(z) + A_{\psi, \eta_k}(z) \right| \\ &= \lim_{|z| \rightarrow 1} \frac{\mu(z)}{\mu(z')} \left| \frac{R^{(2)}\psi(z)}{R^{(2)}\psi(z')} \mu(z') R^{(2)}\psi(z') \varphi_k(z') + \mu(z') A_{\psi, \varphi_k}(z') \right| \\ &= \lim_{|z| \rightarrow 1} \mu(z') |R^{(2)}(\psi \cdot \varphi_k)(z')| = 0, \\ \lim_{|z| \rightarrow 1} \mu(z) |R(\psi \cdot \eta_k^2)(z)| &= \lim_{|z| \rightarrow 1} \mu(z) \left| R^{(2)}\psi(z)\eta_k^2(z) + 2A_{\psi, \eta_k}(z) + B_{\psi, \eta_k}(z) \right| \\ &= \lim_{|z| \rightarrow 1} \frac{\mu(z)}{\mu(z')} \left| \frac{R^{(2)}\psi(z)}{R^{(2)}\psi(z')} R^{(2)}\psi(z') \varphi_k^2(z') + 2A_{\psi, \varphi_k}(z') + B_{\psi, \varphi_k}(z') \right| \\ &= \lim_{|z| \rightarrow 1} \mu(z') \left| R^{(2)}\psi(z') \varphi_k(z') + 2A_{\psi, \varphi_k}(z') + B_{\psi, \varphi_k}(z') \right| \\ &= \lim_{|z| \rightarrow 1} \mu(z') |R(\psi \cdot \varphi_k^2)(z')| = 0. \end{aligned}$$

Thus $\psi \cdot \eta_k, \psi \cdot \eta_k^2 \in \mathcal{Z}_{\mu,0}$, and, hence, $(\psi, \eta_k) \in \mathcal{Z}_{+,0}$.

Now, we check (32).

Due to the continuity of γ , the set $\{\eta(z) = \gamma(\varphi(z)) : |\varphi(z)| \leq \lambda\}$ is compact in \mathbb{B} for $\lambda \in [|\alpha|, 1)$. Then,

$$\sup_{|\varphi_k(z)| \leq \lambda} M_{\mu, \psi}^{L, \eta_k}(z) \leq \sup_{|\varphi_k(z)| \leq \lambda} M_{\mu, \psi}^{L, \eta}(z) < \infty. \tag{33}$$

It remains to consider the case $|\varphi_k(z)| \geq \lambda \geq |\alpha|$.

By Lemma 2.6(a), there exists $C(\alpha, k) > 0$ such that

$$\begin{aligned} M_{\mu,\psi}^{I,\eta_k}(z) &\leq \sup_{|\varphi_k(z)| \leq \lambda} M_{\mu,\psi}^{I,\eta}(z) + \sup_{|\varphi_k(z)| > \lambda} M_{\mu,\psi}^{I,\eta_k}(z) \\ &\leq \sup_{|\varphi_k(z)| \leq \lambda} M_{\mu,\psi}^{I,\eta}(z) + C(\alpha, k) \|\psi\|_{Z_\mu} + \sup_{|\varphi_k(z)| > \lambda} M_{\mu,\psi}^{I,\varphi_k}(z) < \infty. \end{aligned}$$

Now, we recall that there exists $\lambda \in (0, 1)$ such that (23) and (24) hold. Denote

$$K_{A,\lambda} = \sup_{|\eta_k(z)| \leq \lambda} M_{\mu,\nu}^{A,\eta_k}(z) < \infty, \quad K_{B,\lambda} = \sup_{|\eta_k(z)| \leq \lambda} M_{\mu,\nu}^{B,\eta_k}(z) < \infty.$$

By an argument analogous to (21), we also obtain $\sup_{z \in \mathbb{B}} \mu(z)|A_{\psi,\eta_k}(z)| < \infty$, $\sup_{z \in \mathbb{B}} \mu(z)|B_{\psi,\eta_k}(z)| < \infty$. Then, from (2.6) and the second inequality in Lemma 2.6(b), it follows that

$$\begin{aligned} M_{\mu,\nu}^{A,\eta_k} &\leq K_{A,\lambda} + \sup_{|\varphi_k(z)| > \lambda} M_{\mu,\nu}^{A,\eta_k}(z) \\ &\leq K_{A,\lambda} + \frac{\sup_{z \in \mathbb{B}} \mu(z)|A_{\psi,\eta_k}(z)|}{\inf_{|\varphi_k(w)| > \lambda} \mu(w)|A_{\psi,\varphi_k}(w)|} \sup_{|\varphi_k(w)| > \lambda} \frac{\mu(w)|A_{\psi,\varphi_k}(w)| v(\varphi_k(z))}{v(\varphi_k(z)) v(\eta_k(z))} \\ &\leq K_{A,\lambda} + D_{2,A} \sup_{z \in \mathbb{B}} \frac{v(\varphi_k(z))}{v(\eta_k(z))} \sup_{|\varphi_k(z)| > \lambda} M_{\mu,\nu}^{A,\varphi_k}(z) < \infty, \\ M_{\mu,\nu^*}^{B,\eta_k} &\leq K_{B,\lambda} + \sup_{|\varphi_k(z)| > \lambda} M_{\mu,\nu^*}^{B,\eta_k}(z) \\ &\leq K_{B,\lambda} + \frac{\sup_{z \in \mathbb{B}} \mu(z)|B_{\psi,\eta_k}(z)|}{\inf_{|\varphi_k(w)| > \lambda} \mu(w)|B_{\psi,\varphi_k}(w)|} \sup_{|\varphi_k(w)| > \lambda} \mu(w)|B_{\psi,\varphi_k}(w)| v^*(\varphi_k(w)) \frac{v^*(\eta_k(w))}{v^*(\varphi_k(w))} \\ &\leq K_{A,\lambda} + D_{2,B} \sup_{z \in \mathbb{B}} \frac{v(\varphi_k(z))}{v(\eta_k(z))} \sup_{|\varphi_k(z)| > \lambda} M_{\mu,\nu^*}^{B,\varphi_k}(z) < \infty, \end{aligned}$$

where $D_{2,A} = \frac{\sup_{z \in \mathbb{B}} \mu(z)|A_{\psi,\eta_k}(z)|}{\inf_{|\varphi_k(w)| > \lambda} \mu(w)|A_{\psi,\varphi_k}(w)|}$, $D_{2,B} = \frac{\sup_{z \in \mathbb{B}} \mu(z)|B_{\psi,\eta_k}(z)|}{\inf_{|\varphi_k(w)| > \lambda} \mu(w)|B_{\psi,\varphi_k}(w)|}$.

Thus, (32) is proved.

Combining (32) with the first case, we conclude that $W_{\psi,\eta}$ is bounded.

On the other hand, it is straightforward to verify that $W_{\psi,\eta} = W_{\psi,\varphi} \circ C_\gamma$. Then, by Lemma 2.7, we deduce that $W_{\psi,\varphi}$ is bounded. Hence, (d) \Rightarrow (a) is proved.

Finally, combining (2) with the above estimates, we get (17). The proof of Theorem is completed. \square

To prepare for the proof of the next result, we recall the test function constructed by Hu and Wang [6] and some of its basic properties. We consider the holomorphic function

$$g(z) = 1 + \sum_{i>i_0} 2^i z^{n_i}, \tag{34}$$

where $i_0 = \lceil \log_2 \frac{1}{v(\varrho)} \rceil$, $n_i = \lceil \frac{1}{1-r_i} \rceil$ with $r_i = v^{-1}(1/2^i)$ for every $i \geq 1$. Here the symbol $\lceil x \rceil$ means the greatest integer not bigger than x . By [6, Theorem 2.3], $g(t)$ is increasing on $[0, 1)$ and

$$\begin{aligned} |g(z)| &\leq g(|z|), \quad z \in \mathbb{D}, \\ 0 < C_1 &:= \inf_{t \in [0,1)} v(t)g(t) \leq \sup_{t \in [0,1)} v(t)g(t) =: C_2 < \infty. \end{aligned} \tag{35}$$

Theorem 3.2. *The following are equivalent:*

- (a) $W_{\psi,\varphi} : \mathcal{B}_{\nu,0} \rightarrow \mathcal{Z}_{\mu,0}$ is bounded;
- (b) $\psi, \psi \cdot \varphi_j, \psi \cdot \varphi_j^2 \in \mathcal{Z}_{\mu,0}$ for every $j \in \{1, \dots, N\}$ and (16) holds.

In this case, the following asymptotic relation (17) holds.

Additionally, when supplemented with the conditions that $\varphi \in \widetilde{S}_k(\mathbb{B})$ for a given $k \in \{1, \dots, N\}$, the assertions (a)-(c) and the following are equivalent:

- (c) $\psi, \psi \cdot \varphi_j, \psi \cdot \varphi_j^2 \in \mathcal{Z}_{\mu,0}$ for every $j \in \{1, \dots, N\}$ and (18) holds.

Note that the condition $(\psi, \varphi_k) \in \mathcal{Z}_{+,0}$ is not included in the assumptions of this theorem because an operator $W_{\psi,\varphi}$ taking values in $\mathcal{Z}_{\mu,0}$ would imply that $\psi \in \mathcal{Z}_{\mu,0}$, and therefore it cannot belong to $\mathcal{Z}_{\mu,+}$.

Proof. The implication (a) \Rightarrow (b) is a special case of Theorem 4.2 in [10] when $n = m = 1$. The implication (b) \Rightarrow (c) is straightforward.

Now, we prove (c) \Rightarrow (a).

As above, it suffices to prove the case where $\varphi \in \widetilde{S}_k(\mathbb{B})$, $\varphi(0) = 0$, otherwise, the proof proceeds in a manner analogous to that of Theorem 3.1.

Assume that $\psi, \psi \cdot \varphi_j, \psi \cdot \varphi_j^2 \in \mathcal{Z}_{\mu,0}$ for every $j \in \{1, \dots, N\}$ and (18) holds. It follows from the estimates (20) that

$$\lim_{|z| \rightarrow 1} \mu(z) |A_{\psi,\varphi_j}(z)| = 0, \quad \lim_{|z| \rightarrow 1} \mu(z) |B_{\psi,\varphi_j}(z)| = 0. \tag{36}$$

First, we prove that there exists $C > 0$ such that

$$M_{\mu,\psi}^{I,\varphi} \leq CM_{\mu,\psi}^{I,\varphi_k} =: \widehat{M}_{\mu,\psi}^{I,\varphi_k}, \quad M_{\mu,\nu}^{A,\varphi} \leq CM_{\mu,\nu}^{A,\varphi_k} =: \widehat{M}_{\mu,\nu}^{A,\varphi_k}, \quad M_{\mu,\nu^*}^{B,\varphi} \leq CM_{\mu,\nu^*}^{B,\varphi_k} =: \widehat{M}_{\mu,\nu^*}^{B,\varphi_k}. \tag{37}$$

By $\varphi \in \widetilde{S}_k(\mathbb{B})$, for each $z \in \mathbb{B}$, there exists $z' \in \mathbb{B}$ such that $|\varphi_k(z')| = |\varphi(z)|$. We consider the functions $h_{\varphi_k(z')}, k_{j,z'}, l_{j,z'}, j = 1, \dots, N$, given by

$$\begin{aligned} h_{\varphi_k(z')}(w) &= \mu(w)R^{(2)}\psi(w) \int_0^{|\tau w| |\varphi_k(z')|} g(t)dt, \\ k_{j,z'}(w) &= \mu(w)A_{\psi,\varphi_j}(w)g(|\varphi_k(z')|), \\ l_{j,z'}(w) &= \mu(w)B_{\psi,\varphi_j}(w) \frac{g(|\varphi_k(z')|)}{1 - |\varphi_k(z')|^2}, \end{aligned}$$

for every $w \in \mathbb{B}$, where g is defined by (34).

It follows from $\psi \in \mathcal{Z}_{\mu,0}$ and (36) that $h_{\varphi_k(z')}(w) \rightarrow 0, k_{j,z'}(w) \rightarrow 0, l_{j,z'}(w) \rightarrow 0$ as $|\tau w| \rightarrow 1$. Therefore, for

$$\varepsilon_{z'}^I := M_{\mu,\psi}^{I,\varphi_k}(z') > 0, \quad \varepsilon_{z'}^A := M_{\mu,\nu}^{A,\varphi_k}(z') > 0, \quad \varepsilon_{z'}^B := M_{\mu,\nu^*}^{B,\varphi_k}(z') > 0,$$

there exists $\delta_z \geq \delta$, where δ is in the definition of the weight ν , such that

$$|h_{\varphi_k(z')}(w)| < \varepsilon_{z'}^I, \quad |k_{j,z'}(w)| < \varepsilon_{z'}^A, \quad |l_{j,z'}(w)| < \varepsilon_{z'}^B \tag{38}$$

for $|w| > \delta_z$ and for every $j \in \{1, \dots, N\}$. Put

$$\delta' = \inf_{|z| \geq \delta} \delta_z \geq \delta.$$

In the case $0 < |z| \leq \delta'$, we have $\{\varphi(z) : |z| \leq \delta'\}$ is compact in \mathbb{B} . Then, since $M_{\mu,\psi}^{I,\varphi_k} < \infty$, $M_{\mu,\nu}^{A,\varphi_k} < \infty$, $M_{\mu,\nu^*}^{B,\varphi_k} < \infty$, we have

$$\sup_{|z| \leq \delta'} M_{\mu,\psi}^{I,\varphi}(z) \lesssim M_{\mu,\psi}^{I,\varphi_k}, \quad \sup_{|z| \leq \delta'} M_{\mu,\nu}^{A,\varphi} \lesssim M_{\mu,\nu}^{A,\varphi_k}, \quad \sup_{|z| \leq \delta'} M_{\mu,\nu^*}^{B,\varphi}(z) \lesssim M_{\mu,\nu^*}^{B,\varphi_k}. \tag{39}$$

If $|z| > \delta'$ we can find $y \in \mathbb{B}$ with $|y| > \delta$ such that $\delta' < \delta_y < |z|$. Then, by (38),

$$|h_{\varphi_k(z')}(z)| < \varepsilon_{y'}^I, \quad |k_{j,z'}(z)| < \varepsilon_{y'}^A, \quad |l_{j,z'}(z)| < \varepsilon_{y'}^B,$$

hence, for every $j \in \{1, \dots, N\}$,

$$\sup_{|z| > \delta'} |h_{\varphi_k(z')}(z)| \lesssim M_{\mu,\psi}^{I,\varphi_k}, \quad \sup_{|z| > \delta'} |k_{j,z'}(z)| \lesssim M_{\mu,\nu}^{A,\varphi_k}, \quad \sup_{|z| > \delta'} |l_{j,z'}(z)| \lesssim M_{\mu,\nu^*}^{B,\varphi_k}. \tag{40}$$

Then, from (35), (39) and (40), we obtain (37).

Now, we assume that $W_{\psi,\varphi}(f) \in \mathcal{Z}_{\mu,0}$ for every $f \in \mathcal{B}_{\nu,0}$. Then, by (37) and an argument similar to the proof (d) \Rightarrow (a) for the case $\varphi(0) = 0$ of Theorem 3.1, we can conclude that $W_{\psi,\varphi} : \mathcal{B}_{\nu,0} \rightarrow \mathcal{Z}_{\mu,0}$ is bounded.

Finally, to finish the proof of (c) \Rightarrow (a), we show that $W_{\psi,\varphi}(f) \in \mathcal{Z}_{\mu,0}$ for every $f \in \mathcal{B}_{\nu,0}$.

Take $f \in \mathcal{B}_{\nu,0}$. By [2, Theorem 2.3], $Rf \in \mathcal{B}_{\nu_1,0}(\mathbb{B})$ where $\nu_1(z) = (1 - |z|)\nu(z)$ for every $z \in \mathbb{B}$, and hence, $Rf \in \mathcal{B}_{1/\nu^*,0}(\mathbb{B})$.

These facts together with $\psi \in \mathcal{Z}_{\mu,0}$ imply that, for $\varepsilon > 0$ given, we can pick some $r_0 > 0$ such that, whenever $r_0 < |z| < 1$, we have

$$\mu(z)|R^{(2)}\psi(z)| < \frac{\varepsilon}{4K}, \quad \nu(z)|\nabla f(z)| < \frac{\varepsilon}{4K}, \quad (\nu^*(z))^{-1}|\nabla^2 f(z)| < \frac{\varepsilon}{4K},$$

where $K = (1 + \widehat{M}_{\mu,\psi}^{I,\varphi_k})(1 + \widehat{M}_{\mu,\nu}^{A,\varphi_k})(1 + \widehat{M}_{\mu,\nu^*}^{B,\varphi_k})(1 + |f(0)|)$.

Put $Q = (1 + L)(1 + N)(1 + P)$ where $L := \sup_{|z| \leq r_0} |f(z)|$, $N := \sup_{|z| \leq r_0} |\nabla f(z)|$, and $P := \sup_{|z| \leq r_0} |\nabla^2 f(z)|$. From $\psi \in \mathcal{Z}_{\mu,0}$ and (36), it follows that there exists $s \in (0, 1)$ such that

$$\mu(z)|R^{(2)}\psi(z)| < \frac{\varepsilon}{3Q}, \quad \mu(z)|A_{\psi,\varphi_j}(z)| < \frac{\varepsilon}{3Q}, \quad \mu(z)|B_{\psi,\varphi_j}(z)| < \frac{\varepsilon}{3Q}$$

whenever $s < |z| < 1$.

Now, for $s < |z| < 1$, we consider two following cases:

- The case $|\varphi(z)| > r_0$: We have

$$|f(\varphi(z)) - f(0)| = \left| \int_0^1 \langle \nabla f(t\varphi(z)), \overline{\varphi(z)} \rangle dt \right| \leq \int_0^1 |\nabla f(t\varphi(z))| dt.$$

Then, as the above, by Propositions 1.2, 1.4, (15), (37) and $f \in \mathcal{B}_{\nu,0}$, we obtain

$$\begin{aligned} \mu(z)|R^{(2)}(\psi(f \circ \varphi)(z))| &\leq \mu(z)|R^{(2)}\psi(z)||f(\varphi(z))| + \mu(z)|A_{\psi,\varphi}(z)||\nabla f(\varphi(z))| + \mu(z)|B_{\psi,\varphi}(z)||\nabla^2 f(\varphi(z))| \\ &\leq \mu(z)|R^{(2)}\psi(z)| \int_0^1 |\nabla f(t\varphi(z))| dt + \mu(z)|R^{(2)}\psi(w)||f(0)| \\ &\quad + M_{\mu,\nu}^{A,\varphi}(z)\nu(\varphi(z))|\nabla f(\varphi(z))| + M_{\mu,\nu^*}^{B,\varphi}(z)(\nu^*(\varphi(z)))^{-1}|\nabla^2 f(\varphi(z))| \\ &\lesssim M_{\mu,\psi}^{I,\varphi}(z) \sup_{|\varphi(w)| > r_0} \nu(\varphi(w))|\nabla f(\varphi(w))| + \mu(w)|R^{(2)}\psi(w)||f(0)| \\ &\quad + \widehat{M}_{\mu,\nu}^{A,\varphi_k} \sup_{|\varphi(z)| > r_0} \nu(\varphi(z))|\nabla f(\varphi(z))| + \widehat{M}_{\mu,\nu^*}^{B,\varphi_k} \sup_{|\varphi(z)| > r_0} (\nu^*(\varphi(z)))^{-1}|\nabla^2 f(\varphi(z))| \\ &\lesssim \widehat{M}_{\mu,\psi}^{I,\varphi_k} \frac{\varepsilon}{4K} + |f(0)| \frac{\varepsilon}{4K} + \widehat{M}_{\mu,\nu}^{A,\varphi_k} \frac{\varepsilon}{4K} + \widehat{M}_{\mu,\nu^*}^{B,\varphi_k} \frac{\varepsilon}{4K} < \varepsilon. \end{aligned}$$

- The case $|\varphi(z)| \leq r_0$: Since $s < |z| < 1$ and by (2.4), (15) and (36), we have

$$\begin{aligned} \mu(z)|R^{(2)}(\psi(f \circ \varphi)(z))| &\leq \mu(z)|R^{(2)}\psi(z)||f(\varphi(z))| + \mu(z)|A_{\psi,\varphi}(z)||\nabla f(\varphi(z))| + \mu(z)|B_{\psi,\varphi}(z)||\nabla^2 f(\varphi(z))| \\ &\leq \frac{\varepsilon}{3Q} \sup_{|\varphi(z)| \leq r_0} |f(\varphi(z))| + \frac{\varepsilon}{3Q} \sup_{|\varphi(z)| \leq r_0} |\nabla f(\varphi(z))| + \frac{\varepsilon}{3Q} \sup_{|\varphi(z)| \leq r_0} |\nabla^2 f(\varphi(z))| < \varepsilon. \end{aligned}$$

This means that $W_{\psi,\varphi}(f) \in \mathcal{Z}_{\mu,0}$ for every $f \in \mathcal{B}_{\nu,0}$.

Finally, as in Theorem 3.1, (17) holds for $\|W_{\psi,\varphi}\|$. \square

4. The compactness

In this section, we also consider $\psi \in H(\mathbb{B})$, $\varphi = (\varphi_1, \dots, \varphi_N) \in S(\mathbb{B})$ and μ, ν are normal weights on \mathbb{B} . In order to study the compactness of the operators $W_{\psi,\varphi}$, we need the following:

Lemma 4.1 ([10], Lemma 3.6). *Let $\mathcal{E}_1, \mathcal{E}_2$ be two Banach spaces of holomorphic functions on \mathbb{B} such that the closed unit ball $B_{\mathcal{E}_1}$ of \mathcal{E}_1 is compact in the compact open topology τ_{co} , and the identity map $I : (\mathcal{E}_2, \|\cdot\|_{\mathcal{E}_2}) \rightarrow (\mathcal{E}_2, \tau_{co})$ is continuous. Then, $W_{\psi,\varphi} : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ is compact if and only if $W_{\psi,\varphi}$ is bounded and for any bounded sequence $\{f_m\}_{m \geq 1}$ in \mathcal{E}_1 which converges to zero uniformly on compact subsets of \mathbb{B} , we have $\|W_{\psi,\varphi}(f_m)\|_{\mathcal{E}_2} \rightarrow 0$ as $m \rightarrow \infty$.*

Theorem 4.2. *Assume that $\int_0^1 \frac{dt}{v(t)} = \infty$. Then, the following are equivalent:*

- (a) $W_{\psi,\varphi} : \mathcal{B}_{\nu} \rightarrow \mathcal{Z}_{\mu}$ is compact;
- (b) $W_{\psi,\varphi} : \mathcal{B}_{\nu,0} \rightarrow \mathcal{Z}_{\mu}$ is compact;
- (c) $\psi, \psi \cdot \varphi_j, \psi \cdot \varphi_j^2 \in \mathcal{Z}_{\mu}$ for every $j \in \{1, \dots, N\}$ and

$$\lim_{|\varphi(z)| \rightarrow 1} M_{\mu,\psi}^{I,\varphi}(z) = 0, \quad \lim_{|\varphi(z)| \rightarrow 1} M_{\mu,\nu}^{\Sigma A}(z) = 0, \quad \lim_{|\varphi(z)| \rightarrow 1} M_{\mu,\nu^*}^{\Sigma B}(z) = 0. \tag{41}$$

Moreover, under the additional conditions that $\varphi \in \widetilde{S}_k(\mathbb{B})$ for some $k \in \{1, \dots, N\}$ and $(\psi, \varphi_k) \in \mathcal{Z}_{+,0}$, the assertions (a)-(c) and the following are equivalent:

- (d) $\psi, \psi \cdot \varphi_j, \psi \cdot \varphi_j^2 \in \mathcal{Z}_{\mu}$ for every $j \in \{1, \dots, N\}$ and

$$\lim_{|\varphi_k(z)| \rightarrow 1} M_{\mu,\psi}^{I,\varphi_k}(z) = 0, \quad \lim_{|\varphi_k(z)| \rightarrow 1} M_{\mu,\nu}^{A,\varphi_k}(z) = 0, \quad \lim_{|\varphi_k(z)| \rightarrow 1} M_{\mu,\nu^*}^{B,\varphi_k}(z) = 0. \tag{42}$$

Proof. The statement (a) \Leftrightarrow (b) \Leftrightarrow (c) is directly derived from Theorem 5.3 in [10] when $n = m = 1$. The implication (c) \Rightarrow (d) is obvious.

Now we prove the implication (d) \Rightarrow (a).

As in Theorem 3.1, it suffices to prove for the case where $\varphi \in \widetilde{S}_k(\mathbb{B})$ and $\varphi(0) = 0$.

It follows from the assumption (d) and Theorem 3.1 that $W_{\psi,\varphi} : \mathcal{B}_{\nu} \rightarrow \mathcal{Z}_{\mu}$ is bounded.

Note first that, by Remarks 1.3 and 2.5, in fact, with an argument analogous to the estimate (31), we can find $D_1, D_2 > 0$ such that for $|\varphi(z)| > \lambda$,

$$\begin{aligned} \mu(z)|R^{(2)}(W_{\psi,\varphi}f)(z)| &\leq D_1 f(\varphi(0)) + D_2 \left[\sup_{|\varphi_k(w)| > \lambda} M_{\mu,\psi}^{I,\varphi_k}(w) + \sup_{|\varphi_k(w)| > \lambda} M_{\mu,\nu}^{A,\varphi_k}(w) + \sup_{|\varphi_k(w)| > \lambda} M_{\mu,\nu^*}^{B,\varphi_k}(w) \right] \|f\|_{\mathcal{B}_{\nu}}. \end{aligned} \tag{43}$$

Let $\{f_m\}_{m \geq 1}$ be a bounded sequence in \mathcal{B}_{ν} converging to 0 uniformly on compact subsets of \mathbb{B} , and fix an $\varepsilon > 0$. By the hypothesis, we can choose $m_0 \in \mathbb{N}$ and $\lambda > 0$ such that for every $m \geq m_0$ and $\lambda < |\varphi_k(z)| < 1$,

$$|f_m(\varphi(0))| < \frac{\varepsilon}{4D_1L}, \quad M_{\mu,\psi}^{I,\varphi_k}(z) < \frac{\varepsilon}{4D_2K}, \quad M_{\mu,\nu}^{A,\varphi_k}(z) < \frac{\varepsilon}{4D_2K}, \quad M_{\mu,\nu^*}^{B,\varphi_k}(z) < \frac{\varepsilon}{4D_2K},$$

where $L = \sup_{m \geq 1} |f_m(\varphi(0))|$, $K = \sup_{m \geq 1} \|f_m\|_{\mathcal{B}_v} < \infty$.

As in the proof of Theorem 3.1, by (1), (2), (43), for every $m \geq m_0$ and $|\varphi(z)| > \lambda$, we can find $D_1, D_2 > 0$ such that

$$\begin{aligned} & \mu(z) |R^{(2)}[\psi(z)f_m(\varphi(z))]| \\ & \leq D_1 |f_m(\varphi(0))| + D_2 \left[\sup_{|\varphi_k(w)| > \lambda} M_{\mu, \psi}^{I, \varphi_k}(w) + \sup_{|\varphi_k(w)| > \lambda} M_{\mu, \nu}^{A, \varphi_k}(w) + \sup_{|\varphi_k(w)| > \lambda} M_{\mu, \nu^*}^{B, \varphi_k}(w) \right] \|f_m\|_{\mathcal{B}_v} \\ & \leq \frac{\varepsilon}{4} + \frac{3\varepsilon}{4K} \|f_m\|_{\mathcal{B}_v} = \varepsilon. \end{aligned} \tag{44}$$

Note that, since $\{f_m\}_{m \geq 1}$ converges to 0 uniformly on compact subsets of \mathbb{B} and by Cauchy integral formula, it is clear that

$$\sup_{|\varphi(z)| \leq \lambda} |\nabla f_m(\varphi(z))| \rightarrow 0, \quad \sup_{|\varphi(z)| \leq \lambda} |\nabla^2 f_m(\varphi(z))| \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Then, with the estimate as the above and by (21), we have

$$\begin{aligned} & \sup_{|\varphi(z)| \leq \lambda} \mu(z) |R^{(2)}(W_{\psi, \varphi} f_m)(z)| \\ & \leq \|\psi\|_{\mathcal{Z}_\mu} \sup_{|\varphi(z)| \leq \lambda} |f_m(\varphi(0))| + \sup_{z \in \mathbb{B}} \mu(z) |A_{\psi, \varphi}(z)| \sup_{|\varphi(z)| \leq \lambda} |\nabla f_m(\varphi(z))| \\ & \quad + \sup_{z \in \mathbb{B}} \mu(z) |B_{\psi, \varphi}(z)| \sup_{|\varphi(z)| \leq \lambda} |\nabla^2 f_m(\varphi(z))| \rightarrow 0 \quad \text{as } m \rightarrow \infty. \end{aligned} \tag{45}$$

Therefore, it follows from Lemma 4.1 and (44), (45) that $W_{\psi, \varphi}$ is compact. \square

By $\psi \in \mathcal{Z}_\mu$, (21) and Lemma 5.1 in [10], we obtain easily the following from Theorem 4.2:

Theorem 4.3. Assume that $\int_0^1 \frac{dt}{v(t)} < \infty$. Then, the following are equivalent:

- (a) $W_{\psi, \varphi} : \mathcal{B}_v \rightarrow \mathcal{Z}_\mu$ is compact;
- (b) $W_{\psi, \varphi} : \mathcal{B}_{v,0} \rightarrow \mathcal{Z}_\mu$ is compact;
- (c) $\psi, \psi \cdot \varphi_j, \psi \cdot \varphi_j^2 \in \mathcal{Z}_\mu$ for every $j \in \{1, \dots, N\}$ and

$$\lim_{|\varphi(z)| \rightarrow 1} M_{\mu, \nu}^{\Sigma A}(z) = 0, \quad \lim_{|\varphi(z)| \rightarrow 1} M_{\mu, \nu^*}^{\Sigma B}(z) = 0. \tag{46}$$

Moreover, under the additional conditions that $\varphi \in \widetilde{S}_k(\mathbb{B})$ for some $k \in \{1, \dots, N\}$ and $(\psi, \varphi_k) \in \mathcal{Z}_{+,0}$, the assertions (a)-(c) and the following are equivalent:

- (d) $\psi, \psi \cdot \varphi_j, \psi \cdot \varphi_j^2 \in \mathcal{Z}_\mu$ for every $j \in \{1, \dots, N\}$ and

$$\lim_{|\varphi_k(z)| \rightarrow 1} M_{\mu, \nu}^{A, \varphi_k}(z) = 0, \quad \lim_{|\varphi_k(z)| \rightarrow 1} M_{\mu, \nu^*}^{B, \varphi_k}(z) = 0. \tag{47}$$

Now, the following theorems are derived by combining Theorems 3.2, 4.2 and 4.3. Their proofs are omitted.

Theorem 4.4. Assume that $\int_0^1 \frac{dt}{v(t)} = \infty$. Then, the following are equivalent:

- (a) $W_{\psi, \varphi} : \mathcal{B}_{v,0} \rightarrow \mathcal{Z}_{\mu,0}$ is compact;
- (b) $\psi, \psi \cdot \varphi_j, \psi \cdot \varphi_j^2 \in \mathcal{Z}_{\mu,0}$ for every $j \in \{1, \dots, N\}$ and (41) holds.

Additionally, when supplemented with the conditions that $\varphi \in \widetilde{S}_k(\mathbb{B})$ for a given $k \in \{1, \dots, N\}$, the assertions (a)-(b) and the following are equivalent:

(c) $\psi, \psi \cdot \varphi_j, \psi \cdot \varphi_j^2 \in \mathcal{Z}_{\mu,0}$ for every $j \in \{1, \dots, N\}$ and (42) holds.

Theorem 4.5. Assume that $\int_0^1 \frac{dt}{v(t)} < \infty$. Then, the following are equivalent:

- (a) $W_{\psi,\varphi} : \mathcal{B}_{v,0} \rightarrow \mathcal{Z}_{\mu,0}$ is compact;
 (b) $\psi, \psi \cdot \varphi_j, \psi \cdot \varphi_j^2 \in \mathcal{Z}_{\mu,0}$ for every $j \in \{1, \dots, N\}$ and (46) holds.

Furthermore, with the added conditions that $\varphi \in \widetilde{S}_k(\mathbb{B})$ for some $k \in \{1, \dots, N\}$, the statements (a)-(b) and the following are equivalent:

(c) $\psi, \psi \cdot \varphi_j, \psi \cdot \varphi_j^2 \in \mathcal{Z}_{\mu,0}$ for every $j \in \{1, \dots, N\}$ and (47) holds.

Note that the equivalents (a) \Leftrightarrow (b) \Leftrightarrow (c) of Theorem 4.3 and (a) \Leftrightarrow (b) of Theorems 4.4 and 4.5 are also special cases of Theorem 5.3 in [10].

Declarations

- Conflict of interest: The authors have no conflict of interests related to this work.
- Funding: The authors did not receive support from any organization for this work.

References

- [1] E. Abbasi, X. Zhu, *Product-type operators from the Bloch space into Zygmund-type spaces*, Bull. Iran. Math. Soc., 48 (2022), 385-400.
- [2] J. Chen, M. Wang, *Integration operators between Bloch-type spaces and growth spaces on the unit ball of a complex Banach space*, Annales Polon. Math., 127 (2021), 177-199.
- [3] D. Girela, *On Bloch functions and gap series*, Publ. Matemàtiques, 35 (1991), 403-427.
- [4] H. Hamada, *Bloch-type spaces and extended Cesàro operators in the unit ball of a complex Banach space*, Sci. China Math., 62 (2019), 617-628.
- [5] Q. Hu, *Generalized weighted composition operators from Bloch-type spaces into Zygmund-type spaces*, Bull. Malays. Math. Sci. Soc., 42 (5) (2019), 2381-2394.
- [6] Z.J. Hu, S.S. Wang, *Composition operators on Bloch-type spaces*, Proc. Roy. Soc. Edinburgh Sect. A, 135 (2005), 1229-1239.
- [7] Huang, C-S., Jiang, Z-J.: *On a sum of more complex product-type operators from Bloch-type spaces to the weighted-type spaces*, Axioms 12 (2023), 566, <https://doi.org/10.3390/axioms12060566>
- [8] L. V. Lam, T. T. Quang, *On the boundedness and compactness of extended Cesàro composition operators between weighted Bloch-type spaces*, Hacet. J. Math. Stat., 53 (4) (2024), 897-914.
- [9] L. V. Lam, N. V. Dai, T. T. Quang, *Some new characterizations of the weighted composition operators between Bloch-type spaces*, European J. Math., 11(25), (2025). <https://doi.org/10.1007/s40879-025-00817-w>.
- [10] T. T. Quang, *Weighted composition operators on weighted-type high-order growth spaces on the unit ball*, J. Math. Anal. Appl., 547 (2025), 129266, <https://doi.org/10.1016/j.jmaa.2025.129266>
- [11] S. Stević, C. S. Huang, Z. J. Jiang, *Sum of some product-type operators from Hardy spaces to weighted-type spaces on the unit ball*, Math. Meth. Appl. Sci., 45 (17) (2022), 1-20.
- [12] S. Stević, Z. J. Jiang, *Weighted iterated radial composition operators from weighted Bergman-Orlicz spaces to weighted-type spaces on the unit ball*, Math. Meth. Appl. Sci., 44 (2021), 8684-8696.
- [13] S. Stević, *On a product-type operator from Bloch spaces to weighted-type spaces on the unit ball*, Appl. Math. Comput., 217 (12) (2011), 5930-5935.
- [14] S. Stević, *On α -Bloch spaces with Hadamard gaps*, Abstr. Appl. Anal., 2007 (2007), Article ID 39176, 7 pp.
- [15] S. Stević, *On an integral-type operator from Zygmund-type spaces to mixed-norm spaces on the unit ball*, Abstract and Applied Analysis, 2010 (2010), Article ID 198608, 7 pages.
- [16] X. Tang, *Extended Cesàro operators between Bloch-type spaces in the unit ball of \mathbb{C}^n* , J. Math. Anal. Appl., 326 (2007), 1199-1211.
- [17] R.M. Timoney, *Bloch functions in several complex variables I*, Bull. Lond. Math. Soc., 12 (1980), 241-267.
- [18] R.M. Timoney, *Bloch functions in several complex variables II*, J. Reine Angew. Math., 319 (1980), 1-22.
- [19] S. Xu, X. Zhang, S. Li, *Composition Cesàro operator on the normal weight Zygmund space in high dimensions*, Chin. Ann. Math. Ser. B, 42(1) (2021), 69-84.
- [20] S. Yamashita, *Gap series and α -Bloch functions*, Yokohama Math. J., 28 (1980), 31-36.
- [21] X. Zhang, S. Xu, *Weighted differentiation composition operators between normal weight Zygmund spaces and Bloch spaces in the unit ball of \mathbb{C}^n for $n > 1$* , Complex Anal. Oper. Theory, 13 (2019), 859-878.
- [22] R. Zhao, *Essential norms of composition operators between Bloch type spaces*, Proc. Am. Math. Soc., 138 (2010), 2537-2546.
- [23] K. Zhu, *Spaces of holomorphic functions in the unit ball*, Graduate Texts in Mathematics, vol. 226, Springer-Verlag, New York (2005).