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A new lower bound for the doubly metric dimension and related extremal differences

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Abstract. In this paper a new graph invariant $mhs_{\leq}(G)$ based on the minimum hitting set problem is introduced. It is shown that it represents a tight lower bound for the doubly metric dimension of a graph. Exact values of this new invariant for paths, cycles, stars, complete graphs and complete bipartite graphs are obtained. The paper analyzes certain tight bounds for the new invariant in general case. Also, several extremal differences between some related invariants are determined.

1. Introduction

The metric dimension problem was introduced independently by Slater in 1975 [20] and Harary and Melter in 1976 [9]. Given a simple connected undirected graph $G = (V, E)$, $d(u, v)$ denotes the distance between vertices u and v, i.e. the length of a shortest $u - v$ path. A vertex $w \in V(G)$ of graph G is said to resolve two vertices $u, v \in V(G)$ if $d(u, w) \neq d(v, w)$.

A set $S \subseteq V(G)$ is a resolving set of G if any pair of distinct vertices of G are resolved by some vertex from S. A metric basis of G is a resolving set of the minimum cardinality. The metric dimension of G, denoted by $\beta(G)$, is the cardinality of a metric basis of G. The problem of computing the metric dimension of an arbitrary graph is NP-hard [12].

In 2007, Caceres et al. [4] defined the notion of a doubly resolving set as follows. Vertices $x, y \in V(G)$ are said to doubly resolve vertices $u, v \in V(G)$ if $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$. Set $D \subseteq V(G)$ is a doubly resolving set of G if every two distinct vertices of G are doubly resolved by some two vertices of D. The doubly metric dimension of G, denoted by $\psi(G)$, is the minimum cardinality of a doubly resolving set of G. The problem of finding the doubly metric dimension of an arbitrary graph G is also NP-hard [14].

The concepts of edge and mixed metric dimensions were introduced by Kelenc et. al [10, 11]. For the sake of simplicity, edge $e \in E(G)$ with endpoints u and v will be denoted by $e = uv$. The distance between edge $e = uv \in E(G)$ and vertex $w \in V(G)$, denoted by $d(e, w)$, is defined as $d(e, w) = min\{d(u, w), d(v, w)\}$. Vertex w resolves two edges e_1 and e_2 if $d(e_1, w) \neq d(e_2, w)$. Set $N \subseteq V(G)$ is an edge resolving set if for any

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pair of edges from $E(G)$ there is some vertex in N that resolves them. The edge metric dimension of G, denoted by $\beta_E(G)$, is the minimum cardinality of an edge resolving set of G.

Finally, the concept of mixed metric dimension unifies both the metric and the edge metric dimensions in the following way. Vertex w resolves two items $a, b \in V(G) \bigcup E(G)$ if $d(a, w) \neq d(b, w)$. A mixed resolving set $M \subseteq V(G)$ is defined as a set such that for any pair of items from $V(G) \bigcup E(G)$ there is some vertex in M that resolves them. Following earlier definitions, the mixed metric dimension of G, denoted by $\beta_M(G)$, is defined as the minimum cardinality of a mixed resolving set of G. Finding $\beta_E(G)$ and $\beta_M(G)$ are both NP-hard in general case [10, 11].

In the sequel the following definitions will be used. Let $\Delta(G)$ denote the maximum degree of vertices in graph G. Additionally, $N(v) = \{w \in V(G) | vw \in E(G)\}$ and $N[v] = \{v\} \bigcup N(v)$ are open and closed neighbourhood of v, respectively.

Definition 1.1. ([1]) *For an arbitrary edge $e = uv \in E(G)$*

$$W_{uv} = \{w \in V(G) | d(u, w) < d(v, w)\}$$

and

$$W_{vu} = \{w \in V(G) | d(v, w) < d(u, w)\}$$

Set W_{uv} is sometimes in the literature denoted as $\mathcal{N}_1(uv|G)$ (see [8]).

The complements of sets W_{uv} and W_{vu}, denoted by $\overline{W_{uv}}$ and $\overline{W_{vu}}$, are the following sets: $\overline{W_{uv}} = \{w \in V(G) | d(u, w) \geq d(v, w)\}$ and $\overline{W_{vu}} = \{w \in V(G) | d(v, w) \geq d(u, w)\}$.

The set of vertices on equal distances from u and v is denoted in the literature by ${}_uW_v$ ([1]). Note that ${}_uW_v = \overline{W_{uv}} \setminus W_{vu} = \overline{W_{vu}} \setminus W_{uv}$. Also, $\overline{W_{uv}} \bigcup \overline{W_{vu}} = V(G)$.

The following definition introduces the well-known concept of a hitting set [19].

Definition 1.2. *For a given set S and a family $F = \{S_1, ..., S_k\}$, $S_i \subseteq S$, $\bigcup\limits_{i=1}^{k} S_i = S$, a hitting set $H \subseteq S$ of the family F is a set which has a non-empty intersection with each S_i, i.e. $(\forall i \in \{1, ..., k\}) H \bigcap S_i \neq \emptyset$.*

The minimal hitting set problem is to find a hitting set of the minimum cardinality. This problem is equivalent to the set covering problem and is known to be NP-hard [5].

Definition 1.3. ([18]) *The value $mhs_<(G)$ is defined to be the minimum cardinality of a hitting set of the family $\{W_{uv}, W_{vu} | uv \in E(G)\}$.*

It is easy to see that the following remark holds:

Remark 1.4. $mhs_<(G) \geq 2$.

Proof. Let uv be an arbitrary edge of G. Since $W_{uv} \cap W_{vu} = \emptyset$, a hitting set of $\{W_{uv}, W_{vu}\}$ has at least two elements, one from W_{uv} and one from W_{vu}. Therefore, a hitting set of the family $\{W_{uv}, W_{vu} | uv \in E(G)\}$ also has at least two elements. By Definition 1.3, it follows that $mhs_<(G) \geq 2$. □

The following proposition was proved in [18]:

Proposition 1.5. ([18]) *Let G be a connected graph, $uv \in E(G)$ an arbitrary edge and M a mixed resolving set of G. Then*

(i) $W_{uv} \cap M \neq \emptyset$;

(ii) $W_{vu} \cap M \neq \emptyset$.

Corollary 1.6. ([18]) *For every connected graph G it holds $\beta_M(G) \geq mhs_<(G)$.*

In the sequel we will use the following results about $\psi(G)$, $\beta_E(G)$ and $\beta_M(G)$.

Proposition 1.7. ([4]) *Let G be any graph of order n, then $2 \leq \psi(G) \leq n-1$.*

Proposition 1.8. ([4]) *Let K_n be a complete graph of order n, then $\psi(K_n) = max\{2, n-1\}$.*

Proposition 1.9. ([4]) *Let T be any tree with $l(T)$ leaves, then $\psi(T) = l(T)$.*

Corollary 1.10. *For any path P_n it holds $\psi(P_n) = 2$.*

Proposition 1.11. ([4]) *For any cycle C_n, $n \geq 3$, it holds*

$$\psi(C_n) = \begin{cases} 2, & n \text{ is odd} \\ 3, & n \text{ is even} \end{cases}$$

Proposition 1.12. ([10]) *For any cycle C_n, $n \geq 3$, it holds $\beta_M(C_n) = 3$.*

Proposition 1.13. ([11]) *For $n \geq 2$ it holds $\beta_E(P_n) = \beta(P_n) = 1$, $\beta_E(C_n) = \beta(C_n) = 2$, $\beta_E(K_n) = \beta(K_n) = n-1$. Moreover, $\beta_E(G) = 1$ if and only if G is a path.*

Proposition 1.14. ([11]) *Let G be a connected graph. Then $\beta_E(G) \geq \lceil log_2 \Delta(G) \rceil$*

Proposition 1.15. ([10]) *For a connected graph G, it holds $2 \leq \beta_M(G) \leq n$. Moreover, $\beta_M(G) = 2$ if and only if G is a path.*

Proposition 1.16. ([10]) *For any complete bipartite graph $K_{h,n-h}$, $2 \leq h \leq n-h$, it holds*

$$\beta_M(K_{h,n-h}) = \begin{cases} n-1, & h = 2 \\ n-2, & otherwise \end{cases}$$

Definition 1.17. *Let v be a vertex of graph G. A vertex $u \in N(v)$ is called a maximal neighbour of v if $N[v] \subseteq N[u]$. G is a maximal neighbour graph if each vertex $v \in V(G)$ has a maximal neighbour.*

Theorem 1.18. ([10]) *Let G be any graph of order n. Then $\beta_M(G) = n$ if and only if G is a maximal neighbour graph.*

The following definition introduces a new graph invariant which will be analyzed in this paper.

Definition 1.19. *The value $mhs_\leq(G)$ is defined to be the minimum cardinality of a hitting set of the family $\{\overline{W_{uv}}, \overline{W_{vu}} | uv \in E(G)\}$.*

It is interesting to consider extremal differences of some pairs of graph invariants. This topic has been adressed e.g. in [2, 3] for the difference between the determinig number and the metric dimension of graph G. Additionally, in [6] the difference between the locating-domination number and the determining number of G is analysed. The extremal difference between the edge metric dimension and the metric dimension has been considered in [7, 13, 21]. In [15, 18] extremal differences between the mixed and the edge metric dimensions, as well as, between the strong and the mixed metric dimensions are determined.

Formally, the extremal difference of a pair of graph invariants is defined as follows.

Definition 1.20. *Let $\xi_1(G)$ and $\xi_2(G)$ be two graph invariants. Then extremal difference $(\xi_1 - \xi_2)(n)$ is defined as the maximum value of the difference $\xi_1(G) - \xi_2(G)$ for all connected graphs G of order n.*

It should be noted that the minimum value of the difference $\xi_1(G) - \xi_2(G)$ is equal to $-(\xi_2 - \xi_1)(n)$. Also, only non-trivial maximum differences are in cases when $n \geq 3$, since path P_2 is the only connected graph of order 2, so it holds $(\xi_1 - \xi_2)(2) = \xi_1(P_2) - \xi_2(P_2)$.

The paper is organized as follows. In Section 2 it is shown that $mhs_\leq(G)$ is a new lower bound for $\psi(G)$. Exact values of the new invariant are obtained for some special classes of graphs. Also, some tight bounds for $mhs_\leq(G)$ in general case are derived. Section 3 provides the exact values of extremal differences between invariants $\psi(G)$ and $mhs_\leq(G)$, $mhs_<(G)$ and $mhs_\leq(G)$, $\beta_M(G)$ and $mhs_<(G)$, as well as, some upper and lower bounds for the extremal difference between $\psi(G)$ and $\beta_E(G)$. Section 4 gives concluding remarks about obtained results and some directions for future work.

2. A new lower bound for $\psi(G)$

The following theorem and its corollary give a new lower bound for $\psi(G)$.

Theorem 2.1. *Let G be a connected graph, $uv \in E(G)$ an arbitrary edge and S a doubly resolving set of G. Then*

(i) $\overline{W_{uv}} \cap S \neq \emptyset$;

(ii) $\overline{W_{vu}} \cap S \neq \emptyset$.

Proof. Let us suppose that the statement of Theorem 2.1 does not hold, i.e. there exists an edge $uv \in E(G)$ such that
$\overline{W_{uv}} \cap S = \emptyset$ or $\overline{W_{vu}} \cap S = \emptyset$.
In the first case it follows that $d(u,w) < d(v,w)$ for each $w \in S$. Since $uv \in E(G)$ it follows that $d(v,w) = d(u,w)+1$, i.e. $d(v,w) - d(u,w) = 1$ for each $w \in S$. This implies that vertices u and v are not doubly resolved by any pair of vertices from S, which is a contradiction.
In the case when $\overline{W_{vu}} \cap S = \emptyset$ a similar contradiction is derived. □

Theorem 2.1 implies that any doubly resolving set S of G is a hitting set of the family $\{\overline{W_{uv}}, \overline{W_{vu}} | uv \in E(G)\}$. Since $mhs_{\leq}(G)$ is the minimum cardinality of a hitting set of this family, the following corollary of Theorem 2.1 holds.

Corollary 2.2. *For every connected graph G it holds $\psi(G) \geq mhs_{\leq}(G)$*

The next lemma gives tight lower and upper bounds for $mhs_{\leq}(G)$ and $mhs_{<}(G)$ for an arbitrary connected graph G.

Lemma 2.3. *For $n \geq 3$ and graph G of order n it holds:*

(i) $mhs_{\leq}(G) \leq mhs_{<}(G)$

(ii) $2 \leq mhs_{\leq}(G) \leq n-1$

(iii) $mhs_{<}(G) \leq n$

Proof. (i) As $W_{vu} \subseteq \overline{W_{uv}}$ and $W_{uv} \subseteq \overline{W_{vu}}$ for each $uv \in E(G)$ it follows that $mhs_{\leq}(G) \leq mhs_{<}(G)$.
(ii) Let us prove that $mhs_{\leq}(G) > 1$. If we suppose that $mhs_{\leq}(G) = 1$ then there exists a vertex $x \in V(G)$ such that $\{x\}$ is a hitting set of the family $\{\overline{W_{uv}}, \overline{W_{vu}} | uv \in E(G)\}$, i.e. $x \in \overline{W_{uv}}$ and $x \in \overline{W_{vu}}$ for each $uv \in E(G)$. As G is connected there exists vertex $a \in V(G)$ such that $ax \in E(G)$. Since $\overline{W_{ax}} = \{w \in V(G) | d(a,w) \geq d(x,w)\}$ and $\overline{W_{xa}} = \{w \in V(G) | d(x,w) \geq d(a,w)\}$ it follows that $x \in \overline{W_{ax}}$ but $x \notin \overline{W_{xa}}$, which is a contradiction. Therefore, $mhs_{\leq}(G) \geq 2$. From Corollary 2.2 it is known that $mhs_{\leq}(G) \leq \psi(G)$, while by Propostion 1.7 from [4] it is known that $2 \leq \psi(G) \leq n-1$, implying that $mhs_{\leq}(G) \leq n-1$.
(iii) As W_{uv} and W_{vu} are subsets of $V(G)$ for each $uv \in E(G)$, and G is of order n, it follows $mhs_{<}(G) \leq n$. □

As ${}_uW_v = \emptyset$ for some $uv \in E(G)$, implies $\overline{W_{uv}} = W_{vu}$ and $\overline{W_{vu}} = W_{uv}$, then the next remark gives a sufficient condition for $mhs_{\leq}(G) = mhs_{<}(G)$.

Remark 2.4. *If for each $uv \in E(G)$ ${}_uW_v = \emptyset$ then $mhs_{\leq}(G) = mhs_{<}(G)$.*

The sufficient condition from Remark 2.4 is satisfied for some classes of graphs, such as: paths, stars, complete bipartite graphs, hypercubes, etc.

Since $mhs_{<}(G) \geq 2$, it is interesting to show cases when this lower bound is tight. The following theorem gives a complete characterization of the condition $mhs_{<}(G) = 2$.

Theorem 2.5. $mhs_{<}(G) = 2$ *if and only if there exist two vertices $x, y \in V(G)$ such that each edge $e \in E(G)$ belongs to some shortest path between x and y.*

Proof. (⇒) Since $mhs_<(G) = 2$ then there exists a hitting set S of the family $\{W_{uv}, W_{vu} | uv \in E(G)\}$ of cardinality two. Let $S = \{x, y\}$, $l = d(x, y)$ and let us define two functions $f_x, f_y : E(G) \to V(G)$ which for each edge $e \in E(G)$ give its endpoints nearest to vertices x and y, respectively. Formaly, $(\forall e \in E(G))(x \in W_{f_x(e)f_y(e)} \wedge y \in W_{f_y(e)f_x(e)})$, i.e. $(\forall e \in E(G))(d(x, f_x(e)) < d(x, f_y(e)) \wedge d(y, f_y(e)) < d(y, f_x(e)))$. Since $S = \{x, y\}$ is a hitting set of the family $\{W_{uv}, W_{vu} | uv \in E(G)\}$, then $S \bigcap W_{uv} \neq \emptyset$ and $S \bigcap W_{vu} \neq \emptyset$ implying that $d(u, x) \neq d(v, x)$, $d(u, y) \neq d(v, y)$ and $f_x(uv) \neq f_y(uv)$. Therefore, both functions f_x and f_y are well-defined. For each integer $i \geq 0$ let us define sets $E_i = \{e \in E(G) | d(x, f_x(e)) = i\}$. It is obvious that $E_0 = \{e \in E(G) | f_x(e) = x\}$, so E_0 contains all edges incident to x.

In the next two paragraphs, we will prove that $(\forall e \in E_i)(d(x, f_x(e)) = i \wedge d(x, f_y(e)) = i + 1 \wedge d(f_y(e), y) = l - i - 1 \wedge d(f_x(e), y) = l - i)$. It shoud be noted that the first condition holds by definition of E_i, while remaining three conditions will be proved by the mathematical induction.

$k = 0$

Since $E_0 = \{e \in E(G) | f_x(e) = x\}$ it is obvious that for $e \in E_0$ it holds $d(f_x(e), y) = d(x, y) = l$. Because x and $f_y(e)$ are endpoints of edge e, it is clear that $d(x, f_y(e)) = 1$. Since $y \in W_{f_y(e)f_x(e)}$ then $d(f_y(e), y) < d(f_x(e), y)$ implying $d(f_y(e), y) = d(f_x(e), y) - 1 = d(x, y) - 1 = l - 1$.

$k = i$

Let us suppose that the inductive hypothesis is true for all edges in E_i, i.e. $(\forall e' \in E_i)(d(x, f_x(e')) = i \wedge d(x, f_y(e')) = i + 1 \wedge d(f_y(e'), y) = l - i - 1 \wedge d(f_x(e'), y) = l - i)$. Let e be an arbitrary edge from E_{i+1}, i.e. $d(x, f_x(e)) = i + 1$, and let $x - .... - u - f_x(e)$ be the corresponding shortest-path of length $i + 1$. Let $e' = uf_x(e)$. It is obvious that $f_x(e') = u$, $f_y(e') = f_x(e)$ and $d(x, u) = i$ ($e' \in E_i$). Then, $d(x, f_x(e')) = i \wedge d(x, f_y(e')) = i + 1 \wedge d(f_y(e'), y) = l - i - 1 \wedge d(f_x(e'), y) = l - i$ is true by the inductive hypothesis. Since $f_x(e) = f_y(e')$ and $d(x, f_y(e)) > d(x, f_x(e))$ then $d(x, f_y(e)) = d(x, f_y(e')) + 1 = i + 2$ and $d(f_x(e), y) = d(f_y(e'), y) = l - i - 1 = l - (i + 1)$. As $f_x(e) = f_y(e')$ and $f_y(e)$ are endpoints of edge e, it is clear that $d(f_x(e), f_y(e)) = 1$. Since $y \in W_{f_y(e)f_x(e)}$ then $d(f_y(e), y) < d(f_x(e), y) = d(f_y(e'), y) = l - i - 1$ implying $d(f_y(e), y) = l - i - 2 = l - (i + 1) - 1$. This proves the inductive step for $k = i + 1$, i.e. $(\forall e \in E_{i+1})(d(x, f_x(e)) = i + 1 \wedge d(x, f_y(e)) = i + 2 \wedge d(f_y(e), y) = l - i - 2 \wedge d(f_x(e), y) = l - i - 1)$.

Since distances cannot be negative, set E_i is not defined for $i \geq l$ ($E_i = \emptyset$), so $\bigcup\limits_{i=0}^{l-1} E_i = E(G)$. Since $(\forall e \in E_{l-1})(d(x, f_x(e)) = l - 1 \wedge d(x, f_y(e)) = l \wedge d(f_y(e), y) = 0 \wedge d(f_x(e), y) = 1)$ then $f_y(e) = y$, implying $E_{l-1} = \{e | f_y(e) = y\}$. Therefore, E_{l-1} contains all edges incident to y.

Finally, let us consider an arbitrary edge $e \in E(G)$ which belongs to some E_i, $0 \leq i \leq l - 1$. Since $(d(x, f_x(e)) = i \wedge d(x, f_y(e)) = i + 1 \wedge d(f_y(e), y) = l - i - 1 \wedge d(f_x(e), y) = l - i$ then $d(x, f_x(e)) + d(f_y(e), y) = i + l - i - 1 = l - 1 = d(x, y) - 1$. Hence, edge e lies on some shortest path between vertices x and y.

(⇐) Let there exist vertices x and y such that each edge $uv \in E(G)$ is on some shortest path between x and y. We have two possibilities for that shortest x-y path:

Case 1: $x - ... - u - v - ... - y$

It is obvious that in this case $d(x, u) < d(x, v)$ and $d(y, v) < d(y, u)$ so $x \in W_{uv}$ and $y \in W_{vu}$.

Case 2: $x - ... - v - u - ... - y$

In this case $d(x, v) < d(x, u)$ and $d(y, u) < d(y, v)$ so $y \in W_{uv}$ and $x \in W_{vu}$.

In both cases set $\{x, y\}$ has non-empty intersections with W_{uv} and W_{vu}, for each edge $uv \in E(G)$. Therefore, set $\{x, y\}$ is a hitting set of the family $\{W_{uv}, W_{vu} | uv \in E(G)\}$ and hence $mhs_<(G) \leq 2$. Since by Remark 2.3 $mhs_<(G) \geq 2$ it holds $mhs_<(G) = 2$. □

Since $mhs_<(G) \geq mhs_\leq(G) \geq 2$, the next corrolary can be stated.

Corollary 2.6. *If there exist two vertices $x, y \in V(G)$ such that each edge $e \in E(G)$ belongs to some shortest path between x and y then $mhs_\leq(G) = 2$.*

It is obvious that Corrolary 2.6 in other direction does not hold. For example, by Proposition 2.8, for the complete graph K_n it holds $mhs_\leq(K_n) = 2$ but for each pair of vertices x and y there exist exactly $\binom{|V|}{2} - 1$ edges not participating in $x - y$ shortest path, since all shortest paths between vertices have length one.

It is easy to see that neccesary and sufficient conditions of Theorem 2.5 are satisfied for the path P_n, the cycle C_n with even n and the complete bipartite graph $K_{2,n-2}$, $n \geq 4$. The path P_n is defined by

$V(P_n) = \{v_i | 1 \le i \le n\}$, $E(P_n) = \{v_i v_{i+1} | 1 \le i \le n-1\}$. The cycle C_n is defined by $V(C_n) = \{v_i | 1 \le i \le n\}$, $E(C_n) = \{v_1 v_n\} \bigcup \{v_i v_{i+1} | 1 \le i \le n-1\}$. The complete bipartite graph $K_{h,n-h}$ is defined by $V(K_{h,n-h}) = V_1 \bigcup V_2$, $V_1 = \{u_i | 1 \le i \le h\}$, $V_2 = \{v_i | 1 \le i \le n-h\}$ and $E(K_{h,n-h}) = \{u_i v_j | 1 \le i \le h, 1 \le j \le n-h\}$. Then,

- for the path P_n it holds $x = v_1, y = v_n$;
- for the even cycle C_n it holds $x = v_1, y = v_{n/2+1}$;
- for the complete bipartite graph $K_{2,n-2}$ it holds $x = u_1, y = u_2$.

As P_n, C_n with even n and $K_{2,n-2}$ also satisfy the sufficient condition of Remark 2.4, then the following corollary holds.

Corollary 2.7.

(i) For the path P_n, $n \ge 2$, it holds that $mhs_{\le}(P_n) = mhs_{<}(P_n) = 2$;

(ii) For the cycle C_n, for even $n \ge 4$, it holds that $mhs_{\le}(C_n) = mhs_{<}(C_n) = 2$;

(iii) For the complete bipartite graph $K_{2,n-2}$, $n \ge 4$, it holds that $mhs_{<}(K_{2,n-2}) = mhs_{\le}(K_{2,n-2}) = 2$.

In the following Proposition 2.8 exact values of $mhs_{\le}(G)$ and $mhs_{<}(G)$ for some special types of graphs are derived. The obtained values will be used to prove that lower and upper bounds of $mhs_{\le}(G)$ and $mhs_{<}(G)$ from Lemma 2.3 are tight.

Proposition 2.8. *(i) For the cycle C_n, for odd n, $n \ge 3$, it holds that $mhs_{\le}(C_n) = 2$ and $mhs_{<}(C_n) = 3$*

(ii) For the star S_n, $n \ge 3$, it holds that $mhs_{\le}(S_n) = mhs_{<}(S_n) = n-1$;

(iii) For the complete graph K_n, $n \ge 3$, it holds that $mhs_{\le}(K_n) = 2, mhs_{<}(K_n) = n$.

(iv) For the complete bipartite graph $K_{h,n-h}$, $3 \le h \le n-h$, it holds that $mhs_{<}(K_{h,n-h}) = mhs_{\le}(K_{h,n-h}) = min\{h, 4\}$.

Proof. (i) From Corrolary 2.2 it follows $2 \le mhs_{\le}(G) \le \psi(G)$. By Proposition 1.11 for odd n it holds $\psi(C_n) = 2$ implying $mhs_{\le}(C_n) = 2$. Let us consider $mhs_{<}(C_n)$ for odd n. Since for odd n cycle C_n obviously does not satisfy necessary and sufficient conditions of Theorem 2.5 it follows that $mhs_{<}(C_n) \ge 3$. On the other hand, by Corollary 1.6 and Proposition 1.12 it holds $mhs_{<}(C_n) \le \beta_M(C_n) = 3$. Therefore, $mhs_{<}(C_n) = 3$.

(ii) The star S_n is defined by $V(S_n) = \{v_i | 1 \le i \le n\}$, $E(S_n) = \{v_1 v_i | 2 \le i \le n\}$. It is easy to see that $W_{v_i v_1} = \{v_i\}$ and $W_{v_1 v_i} = V(S_n) \setminus \{v_i\}$ for each $i = 2, ..., n$. Therefore, a hitting set H of the family $\{W_{v_i v_1}, W_{v_1 v_i} | 2 \le i \le n\}$ satisfies $\{v_2, ..., v_n\} \subseteq H$. Since $W_{v_1 v_i} \bigcap \{v_2, ..., v_n\} \ne \emptyset$ and $W_{v_i v_1} \bigcap \{v_2, ..., v_n\} \ne \emptyset$ for $i = 2, ..., n$ it follows that $\{v_2, ..., v_n\}$ is a hitting set of the minimum cardinality, i.e. $mhs_{<}(S_n) = n-1$. Since for each $uv \in E(S_n)$ ${}_uW_v = \emptyset$, according to Remark 2.4, $mhs_{\le}(S_n) = mhs_{<}(S_n) = n-1$.

(iii) It is easy to check that $W_{uv} = \{u\}$, $W_{vu} = \{v\}$ for each $uv \in E(K_n)$. Therefore $V(K_n)$ is a hitting set of the family $\{W_{uv}, W_{vu} | uv \in E(K_n)\}$ with the minimum cardinality and $mhs_{<}(K_n) = n$.
Also, $\overline{W_{uv}} = V(K_n) \setminus \{u\}$ and $\overline{W_{vu}} = V(K_n) \setminus \{v\}$, for each $uv \in E(K_n)$. As $u \notin \overline{W_{uv}}$ and $v \notin \overline{W_{vu}}$, a set consisting of one vertex can not be a hitting set of the family $\{\overline{W_{uv}}, \overline{W_{vu}} | uv \in E(K_n)\}$, i.e. $mhs_{\le}(K_n) \ge 2$.
Let $a, b \in V(K_n)$, $a \ne b$, be two arbitrary distinct vertices. Then for each $uv \in E(K_n)$ there are three possible cases:
Case 1: If $u \ne a$ and $v \ne a$ then $a \in \overline{W_{uv}}$ and $a \in \overline{W_{vu}}$;
Case 2: If $u = a$ then $a \notin \overline{W_{av}}$ and $a \in \overline{W_{va}}$;
Case 3: If $v = a$ then $a \in \overline{W_{ua}}$ and $a \notin \overline{W_{au}}$.
However, $b \in \overline{W_{av}}$ and $b \in \overline{W_{au}}$. Therefore, $\{a, b\}$ is a hitting set of the family $\{\overline{W_{uv}}, \overline{W_{vu}} | uv \in E(K_n)\}$ with the minimum cardinality and hence $mhs_{\le}(K_n) = 2$.

Table 1: Intersections of T with members of family F are non-empty

vertex from T	members of F
u'	$W_{u'v}, v \in V_2$
u''	$W_{u''v}, v \in V_2$
v'	$W_{uv''}, u \in V_1 \setminus \{u', u''\}$
v''	$W_{uv'}, u \in V_1 \setminus \{u', u''\}$
v'	$W_{uv}, u \in V_1 \setminus \{u', u''\}, v \in V_2 \setminus \{v', v''\}$
v''	$-\|\|-$
u''	$W_{vu'}, v \in V_2$
u'	$W_{vu''}, v \in V_2$
v''	$W_{v''u}, u \in V_1 \setminus \{u', u''\}$
v'	$W_{v'u}, u \in V_1 \setminus \{u', u''\}$
u'	$W_{vu}, u \in V_1 \setminus \{u', u''\}, v \in V_2 \setminus \{v', v''\}$
u''	$-\|\|-$

Table 2: Intersections of S with elements of family F are empty

S	members of F
$\{u', u'', v'\}$	$W_{uv'}, u \notin \{u', u''\}$
$\{u', u'', v''\}$	$W_{uv''}, u \notin \{u', u''\}$
$\{u', v', v''\}$	$W_{vu'}, v \notin \{v', v''\}$
$\{u'', v', v''\}$	$W_{vu''}, v \notin \{v', v''\}$
$\{u', v'\}$	$W_{uv'}, u \neq u'$
$\{u', v''\}$	$W_{uv''}, u \neq u'$
$\{u'', v'\}$	$W_{uv'}, u \neq u''$
$\{u'', v''\}$	$W_{uv''}, u \neq u''$

(iv) Let us consider family $F = \{W_{u_iv_j}, W_{v_ju_i} | 1 \leq i \leq h, 1 \leq j \leq n-h\}$. As $d(a,b) = 1$ if $a \in V_1$ and $b \in V_2$ and $d(a,b) = 2$ for $a,b \in V_1$ or $a,b \in V_2$ it follows that for each edge $u_iv_j \in E(K_{h,n-h})$ we have $W_{u_iv_j} = \{u_i\} \bigcup (V_2 \setminus \{v_j\})$, $W_{v_ju_i} = \{v_j\} \bigcup (V_1 \setminus \{u_i\})$.

Step 1. First we prove that sets V_1 and V_2 are both hitting sets of the family F. Let $u \in V_1$, $u' \in V_1 \setminus \{u\}$ and $v \in V_2$. Then obviously $u \in W_{uv}$ and $u' \in W_{vu}$. Therefore, $V_1 \bigcap W_{uv} \neq \emptyset$ and $V_1 \bigcap W_{vu} \neq \emptyset$, i.e. V_1 is a hitting set of family F. In the same way, it can be shown that V_2 is a hitting set of the family F. From Remark 1.4 it follows that $mhs_<(K_{h,n-h}) \geq 2$. In the sequel we assume $h \geq 3$ (and consequently $n-h \geq 3$).

Step 2. Next we will prove that any set S such that $S \subset V_1$ or $S \subset V_2$ is not a hitting set of family F. Let us suppose $S \subset V_1$ and $u^* \in V_1 \setminus S$. Then, for each $u \in S$ and $v \in V_2$ it follows that $u \notin W_{u^*v}$, i.e. $S \bigcap W_{u^*v} = \emptyset$, which implies that S is not a hitting set of family F. In the same way, it can be shown that any $S \subset V_2$ is not a hitting set of F.

Step 3. We will prove that set $T = \{u', u'', v', v''\}$ where $u', u'' \in V_1$ and $v', v'' \in V_2$ is a hitting set of family F. It should be proved that $T \bigcap W_{uv} \neq \emptyset$ and $T \bigcap W_{vu} \neq \emptyset$ for $u \in V_1$ and $v \in V_2$ (Table 1).

In each row of Table 1 the first column contains a vertex from the non-empty intersection of T with members of family F which are specified in the second column.

Step 4. Let $S \subset \{u', u'', v', v''\}$, where $|S| \geq 2$, $u', u'' \in V_1$ and $v', v'' \in V_2$. Then S is not a hitting set of family F.

If $S = \{u', u''\}$ or $S = \{v', v''\}$ then according to Step 2 and $h \geq 3$, it folows that S is not a hitting set of F. All other possibilities for the set S and the corresponding sets W_{uv} or W_{vu} which have empty intersections with S are given in Table 2.

Now, directly from previous steps, it follows that $mhs_<(K_{h,n-h}) = min\{h, 4\}$.

Since for each $uv \in E(K_{h,n-h})$ ${}_uW_v = \emptyset$, according to Remark 2.4, $mhs_\leq(K_{h,n-h}) = mhs_<(K_{h,n-h}) =$

$min\{h, 4\}$. □

According to Corollary 1.10, Corollary 2.2 and Corollary 2.7, the new invariant $mhs_{\leq}(G)$ represents a tight lower bound for the doubly metric dimension $\psi(G)$ as $\psi(P_n) = mhs_{\leq}(P_n) = 2$. It should be noted that the lower bound $mhs_{<}(G) \geq 2$ from Remark 1.4 is tight for e.g. path P_n.

Results from Corollary 2.7 and Proposition 2.8 also show that bounds from Lemma 2.3 are tight:

- Bound (i) is reached for the path P_n;
- Lower bound (ii) is reached for the path P_n and the complete graph K_n;
- Upper bound (ii) is reached for the star S_n;
- Upper bound (iii) is reached for the complete graph K_n.

3. Extremal differences

The first task in this section is to find extremal diferences between the doubly metric dimension $\psi(G)$ and the new invariant $mhs_{\leq}(G)$. The complete answer is given in Theorem 3.1.

Theorem 3.1. *For $n \geq 3$ it holds*

(i) $(mhs_{\leq} - \psi)(n) = 0$

(ii) $(\psi - mhs_{\leq})(n) = n - 3$

Proof. (i) For an arbitrary connected graph G, from Corollary 2.2 it follows that $mhs_{\leq}(G) \leq \psi(G)$, i.e. $(mhs_{\leq} - \psi)(n) \leq 0$. Since, by Corrolary 1.10 and Corollary 2.7, $\psi(P_n) = mhs_{\leq}(P_n) = 2$ it follows $(mhs_{\leq} - \psi)(n) = 0$.

(ii) From Proposition 1.7 and Lemma 2.3 it is evident that for every graph G it holds $\psi(G) - mhs_{\leq}(G) \leq n - 3$. By Proposition 1.8 and Proposition 2.8 this upper bound is reached for the complete graph K_n. □

Since $mhs_{\leq}(G))$ is a lower bound for $\psi(G)$ and $mhs_{<}(G)$ represents a lower bound for $\beta_M(G)$ it is interesting to find the extremal differences between these lower bounds. Their exact values are given by Theorem 3.2.

Theorem 3.2. *For $n \geq 3$ it holds*

(i) $(mhs_{\leq} - mhs_{<})(n) = 0$

(ii) $(mhs_{<} - mhs_{\leq})(n) = n - 2$

Proof. (i) By Lemma 2.3 it holds $(mhs_{\leq} - mhs_{<})(n) \leq 0$. By Corollary 2.7 this upper bound is reached for the path P_n, which implies
$(mhs_{\leq} - mhs_{<})(n) = 0$.

(ii) From Lemma 2.3 it follows $mhs_{<}(G) \leq n$ and $mhs_{\leq}(G) \geq 2$, which implies that $mhs_{<}(G) - mhs_{\leq}(G) \leq n - 2$. Acording to Proposition 2.8 this upper bound is reached for the complete graph K_n. □

Since both invarinats $mhs_{\leq}(G)$ and $mhs_{<}(G)$ are based on the minimum hitting set problem and their definitions are similar, it is expected that there exist classes of graphs for which they have the same value. However, Theorem 3.2 shows that values of these invariants can drastically differ.

In [18] it is shown that $mhs_{<}(G)$ is a lower bound of $\beta_M(G)$. Therefore, it is interesting to see what extremal differences of them are. This task is solved in Theorem 3.5, which uses Theorem 3.3 and Corollary 3.4.

Theorem 3.3. *Let G be any connected graph of order n. Then $mhs_{<}(G) = n$ if and only if G is a maximal neighbour graph.*

Proof. (⇒) If $mhs_<(G) = n$ from Corollary 1.6 it follows $\beta_M(G) = n$ which by Theorem 1.18 implies that G is a maximal neighbour graph.
(⇐)Let G be a maximal neighbour graph and $v \in V(G)$ be an arbitrary vertex. Then there exists vertex $u \in N(v)$ such that $N(v) \subseteq N(u)$. Let us prove that $W_{vu} = \{w | d(v,w) < d(u,w)\} = \{v\}$. Let z be an arbitrary vertex of G different from v. If $z \in N(v)$ then $d(v,z) = 1 = d(u,z)$ and hence $z \notin W_{vu}$.
If $z \in V(G) \setminus N(v)$ then $d(v,z) = 1 + d(x,z)$ where $x \in N(v)$.
If $x = u$ then $d(v,z) = 1 + d(u,z)$ and $z \notin W_{vu}$.
If $x \neq u$ then $d(v,z) = 1 + d(x,z) = d(u,z)$ and we again conclude that $z \notin W_{vu}$.
Since $W_{vu} = \{v\}$ for each $v \in V(G)$ the hitting set of the family
$\{W_{uv}, W_{vu} | uv \in E(G)\}$ has to contain all nodes from $V(G)$ and therefore $mhs_<(G) = n$. □

Theorem 1.18 and Theorem 3.3 imply the following corollary.

Corollary 3.4. *Let G be any connected graph of order n. Then $mhs_<(G) = n$ if and only if $\beta_M(G) = n$.*

Theorem 3.5. *For $n \geq 3$ it holds*

(i) $(mhs_< - \beta_M)(n) = 0$

(ii) $(\beta_M - mhs_<)(n) = n - 3$

Proof. (i) For any connected graph G, from Corollary 1.6 it follows that $mhs_<(G) - \beta_M(G) \leq 0$. Since, by Proposition 1.15 and Corollary 2.7, $\beta_M(P_n) = mhs_<(P_n) = 2$ it follows $(mhs_< - \beta_M)(n) = 0$.
(ii) By Proposition 1.15 it holds $\beta_M(G) \leq n$ and by Remark 1.4 it follows $mhs_<(G) \geq 2$, but $(\beta_M - mhs_<)(n) \neq n-2$:
Case 1. $\beta_M(G) = n$.
According to Corollary 3.4 it holds $mhs_<(G) = n$ and consequently it is $\beta_M(G) - mhs_<(G) = 0$.
Case 2. $\beta_M(G) \leq n - 1$.
In this case it holds $\beta_M(G) - mhs_<(G) \leq n - 3$.
Since $n \geq 3$ in both cases it holds $\beta_M(G) - mhs_<(G) \leq n - 3$. This upper bound is reached for the complete bipartite graph $K_{2,n-2}$ since by Proposition 1.16 from [10] it holds $\beta_M(K_{2,n-2}) = n - 1$ and by Corollary 2.7 it holds $mhs_<(K_{2,n-2}) = 2$. □

Finally, it is interesting to find extremal diferences between $\psi(G)$ and $\beta_E(G)$.

For $n = 3$ this task is completely resolved by Remark 3.6. For $n \geq 4$, this task is partially resolved by Theorem 3.7, which provides both upper and lower bounds for the extremal difference between $\psi(G)$ and $\beta_E(G)$.

Remark 3.6. $(\psi - \beta_E)(3) = 1$ *and* $(\beta_E - \psi)(3) = 0$

Proof. There exist only two connected graphs of order 3: the path P_n and the cycle C_n. Since $\beta_E(P_3) = 1$ and $\beta_E(C_3) = \psi(P_3) = \psi(C_3) = 2$ then $(\psi - \beta_E)(3) = 1$ and $(\beta_E - \psi)(3) = 0$. □

Theorem 3.7. *For $n \geq 4$ it holds* $\lfloor \frac{n}{2} \rfloor - 1 \leq (\psi - \beta_E)(n) \leq n - 3$

Proof. Let $m = \lfloor \frac{n}{2} \rfloor$ and let T'_n be a tree with $V(T'_n) = \{v_1, ..., v_n\}$ and $E(T'_n) = \{v_i v_{i+1} | 1 \leq i \leq n-m\} \bigcup \{v_i v_{n-m+i} | 2 \leq i \leq m\}$ [16, 17]. Figure 1 illustrates trees T'_n for odd and even n. From Proposition 1.9 it follows that $\psi(T'_n) = l(T'_n) = m + 1$. It is proved in [16, 17] that $\beta_E(T'_n) = 2$ with edge metric base $\{v_1, v_{n-m+1}\}$. Therefore $\psi(T'_n) - \beta_E(T'_n) = m - 1 = \lfloor \frac{n}{2} \rfloor - 1$ implying $(\psi - \beta_E)(n) \geq \lfloor \frac{n}{2} \rfloor - 1$. Since for any graph G it holds $\psi(G) \leq n - 1$ and $\beta_E(G) \geq 1$ it follows that $(\psi - \beta_E)(n) \leq n - 2$.
Furthermore, this bound can be improved using a similar argument as in [16, 17].
Since G is a connected graph of order $n \geq 3$ it follows that the maximum degree $\Delta(G) \geq 2$. We consider two cases:
Case 1. $\Delta(G) = 2$. The only such graphs of order n, $n \geq 3$ are the path P_n and the cycle C_n. Since by Corollary 1.10 it holds $\psi(P_n) = 2$ and by Proposition 1.13 from [11] $\beta_E(P_n) = 1$ it follows that $\psi(P_n) - \beta_E(P_n) = 1$. Also, by Proposition 1.11 from [4] and Proposition 1.13 from [11] it follows that $\psi(C_n) - \beta_E(C_n) \leq 1$. In both cases

the difference for $n \geq 4$ is less or equal than $n-3$.
Case 2. $\Delta(G) \geq 3$. From Proposition 1.14 ([11]) $\beta_E(G) \geq \lceil log_2 \Delta(G) \rceil \geq \lceil log_2 3 \rceil = 2$. As $\psi(G) \leq n-1$ it follows $\psi(G) - \beta_E(G) \leq n-3$. □

Figure 1: Trees T_8' and T_9'.

4. Conclusions

This paper defines a new graph invariant $mhs_{\leq}(G)$, which is a new lower bound for $\psi(G)$. Exact values of this new invariant are obtained for some special classes of graphs. Next, some tight bounds for the new invariant in general case are derived. Finally, some extremal differences between several related invariants are obtained.

Direction of future work could be focused to find exact values of the new invariant for some other interesting classes of graphs and consider other extremal differences.

References

[1] K. Balakrishnan, M. Changat, I. Peterin, S. Špacapan, P. Šparl, A.R. Subhamathi, *Strongly distance-balanced graphs and graph products*, Eur. J. Combin. **30**(5) (2009), 1048–1053.
[2] D.L. Boutin, *Identifying graph automorphisms using determining sets*, Electron. J. Combin. **13** (2006), Article Number R78.
[3] J. Cáceres, D. Garijo, M.L. Puertas, C. Seara, *On the determining number and the metric dimension of graphs*, Electron. J. Combin. **17** (2010), Article Number 63.
[4] J. Cáceres, C. Hernando, M. Mora, I. Pelayo, M. Puertas, C. Seara, D. Wood, *On the metric dimension of Cartesian products of graphs*, SIAM J. Discrete Math. **21**(2) (2007), 423–441.
[5] M.R. Garey, D.S. Johnson, *Computers and intractability*, vol. 174. Freeman, San Francisco, 1979.
[6] D. Garijo, A. González, A. Márquez, *The difference between the metric dimension and the determining number of a graph*, Appl. Math. Comput. **249** (2014), 487–501.
[7] J. Geneson, S. Kaustav, A. Labelle, *Extremal results for graphs of bounded metric dimension*, Discrete Appl. Math. **309** (2022), 123–129.
[8] I. Gutman, K. Xu, M. Liu, *A Congruence Relation for Wiener and Szeged Indices*, Filomat **29** (2015), 1081–1083.
[9] F. Harary, R. Melter, *On the metric dimension of a graph*, Ars Combin. **2**(1) (1976), 191–195.
[10] A. Kelenc, D. Kuziak, A. Taranenko, I.G. Yero, *Mixed metric dimension of graphs*, Appl. Math. Comput. **314** (2017), 429–438.
[11] A. Kelenc, N. Tratnik, I.G. Yero, *Uniquely identifying the edges of a graph: The edge metric dimension*, Discrete Appl. Math. **251** (2018), 204–220.
[12] S. Khuller, B. Raghavachari, A. Rosenfeld, *Landmarks in graphs* Discrete Appl. Math. **70**(3) (1996), 217–229.
[13] M. Knor, S. Majstorović, A.T.M. Toshi, R. Škrekovski, I.G. Yero, *Graphs with the edge metric dimension smaller than the metric dimension*, Appl. Math. Comput. **401** (2021), Article Number 126076.
[14] J. Kratica, M. Čangalović, V. Kovačević-Vujčić, *Computing minimal doubly resolving sets of graphs*, Comput. Oper. Res. **36**(7) (2009), 2149–2159.
[15] M. Milivojević-Danas, *The mixed metric dimension of flower snarks and wheels*, Open Math. **19**(1) (2021), 629–640.
[16] M. Milivojević-Danas, *Mixed metric dimension of graphs*, Ph.D. thesis, University of Kragujevac, Serbia, 2022. (in Serbian)
[17] M. Milivojević-Danas, *The difference between several metric dimension graph invariants*, Discrete Appl. Math. **332** (2023), 1–6.
[18] M. Milivojević-Danas, J. Kratica, A. Savić, Z. Maksimović, *Some new general lower bounds for mixed metric dimension of graphs*, Filomat **35** (2021), 4275–4285.
[19] D. Simovici, C. Reischer, *Collections of sets and intractable problems in relational databases*, Congr. Numer. **59** (1970), 291.
[20] P.J. Slater, *Leaves of trees*, Congr. Numer **14**(37) (1975), 549–559.
[21] N. Zubrilina, *On the edge dimension of a graph* Discrete Math. **341**(7) (2018), 2083–2088.