Filomat 40:9 (2026), 3471–3480
https://doi.org/10.2298/FIL2609471N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: `http://www.pmf.ni.ac.rs/filomat`

Refinements of numerical radius estimates via the Moore–Penrose inverse

Mehdi Naimi[a,*], Mohammed Benharrat[b]

[a]*Université Abdelhamid Ibn Badis de Mostaganem, Mostaganem 27000, Algérie*
[b]*Ecole Nationale Polytechnique d'Oran-Maurice Audin (Ex. ENSET d'Oran), BP 1523 Oran-El M'naouar, 31000 Oran, Algérie*

Abstract. We establish new bounds for the numerical radius and generalized numerical radius of bounded operators on Hilbert spaces. By combining the Moore–Penrose inverse with elementary operator inequalities, we derive refined upper estimates that sharpen several existing results in the literature. Our approach yields improvements for single operators as well as for sums of operators with closed range. Concrete examples are provided to illustrate the sharpness and effectiveness of the obtained bounds.

1. Introduction

Throughout the paper, let $B(\mathcal{H})$ denote the C^*-algebra of all bounded operators on a complex separable Hilbert space, endowed with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. For $T \in B(\mathcal{H})$, the numerical range of T is the convex subset of $\mathbb{C}$ defined by

$$W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H},\ \|x\| = 1\}.$$

The numerical radius of T, denoted by $w(T)$, is then given by $w(T) = \sup\{|\lambda| : \lambda \in W(T)\}$. This notion plays a fundamental role, since $w(\cdot)$ defines a norm on $B(\mathcal{H})$ equivalent to the usual operator norm. A remarkable characterization of $w(T)$ was obtained by Yamazaki in [34], who proved that for any operator $T \in B(\mathcal{H})$,

$$w(T) = \sup_{\theta \in \mathbb{R}} \left\| \mathfrak{R}(e^{i\theta} T) \right\|, \tag{1}$$

where $\mathfrak{R}(T) = \frac{1}{2}(T + T^*)$ and $\mathfrak{I}(T) = \frac{1}{2i}(T - T^*)$ are the real and imaginary parts of T, respectively.

The study of the numerical range and numerical radius has long been a central theme in operator theory and matrix analysis. These concepts not only provide sharp estimates for the spectral radius, but also serve as powerful tools in deriving inequalities, analyzing perturbations, and developing methods in numerical

---

2020 *Mathematics Subject Classification*. Primary 47A12; Secondary 47A30.
Keywords. Numerical radius, Generalized numerical radius, Moore–Penrose inverse.
Received: 18 November 2025; Accepted: 13 January 2026
Communicated by Dijana Mosić
Research supported by the Laboratory of Fundamental and Applicable Mathematics of Oran (LMFAO) and the Algerian research project: PRFU, No: C00L03ES310120220003 (D.G.R.S.D.T).

* Corresponding author: Mehdi Naimi
Email addresses: `mehdi.naimi@univ-mosta.dz` (Mehdi Naimi), `mohammed.benharrat@enp-oran.dz` (Mohammed Benharrat)
ORCID iDs: `https://orcid.org/0000-0001-8405-2251` (Mehdi Naimi), `https://orcid.org/0000-0001-5766-6717` (Mohammed Benharrat)

linear algebra (see, for instance, [4, 5, 8, 14, 15, 19–21, 24, 29, 30]). Over the last decade, a wealth of research has advanced these ideas, and in recent times, particular attention has been directed toward the use of the Moore–Penrose inverse as a means of obtaining sharper and more flexible inequalities. In our previous works (see, e.g., [25, 26]), we studied related inequalities for the numerical radius. Here, we focus instead on approaches involving the Moore–Penrose inverse.

The Moore–Penrose inverse of $T \in B(\mathcal{H})$ [13, 27] is the operator $X : \mathcal{R}(T) \oplus \mathcal{R}(T)^{\perp} \to \mathcal{H}$ satisfying

$$TXT = T, \qquad XTX = X, \qquad XT = P_{\mathcal{N}(T)^{\perp}}, \qquad AX = P_{\overline{\mathcal{R}(T)}}\Big|_{\mathcal{R}(T)\oplus\mathcal{R}(T)^{\perp}}.$$

Here, $\mathcal{N}(T)$ stands for the null space of T, while P_L denotes the orthogonal projection onto a closed subspace L. It is worth emphasizing that the Moore–Penrose inverse of an operator T is unique and denoted by $T^{\dagger}$, moreover, it is bounded if and only if the range of T is closed. We denote by $\mathcal{CR}(\mathcal{H})$ the set of bounded operators with closed range. Using this tool, Sababheh et al. proved in [28, Theorem 2.2] that for any $T \in \mathcal{CR}(\mathcal{H})$,

$$w(T) \leq \frac{1}{2}\left\| |T|^2 + TT^{\dagger} \right\|. \tag{2}$$

Building on this, several further inequalities have been obtained in [11]. For example, for two operators $T, S \in \mathcal{CR}(\mathcal{H})$, it was shown that

$$w(T+S) \leq \frac{1}{2}\||T|^2 + |S|^2 + TT^{\dagger} + SS^{\dagger}\|. \tag{3}$$

The aim of the second section of this paper is to continue along this line of research, by establishing new bounds and refinements for the numerical radius of operators, and by illustrating the significance of our inequality through simple concrete examples.

One of the most striking generalizations of the numerical radius was introduced in [1]. Motivated by the Yamazaki identity (1), Abu-Omar et al. defined the generalized numerical radius as

$$w_N(T) = \sup_{\theta\in\mathbb{R}} N\Big(\mathfrak{R}(e^{i\theta}T)\Big),$$

where $N(\cdot)$ is an arbitrary norm on $B(\mathcal{H})$. This elegant abstraction not only encompasses the classical numerical radius -as the special case when N is the usual operator norm- but also paves the way for a wealth of new inequalities, especially when N is chosen to be a Schatten p-norm (see, for instance, [1, 6, 12, 23, 31]).

In the third section, we establish new bounds for the generalized numerical radius. For any operator $T \in B(\mathcal{H})$ we prove

$$w_N(T) \ \geq \ N(T) \ - \ \min\{N(\mathfrak{R}(T)), N(\mathfrak{I}(T))\}.$$

When $N(\cdot)$ is an algebra norm, that is, a norm satisfying

$$N(AB) \leq N(A)N(B) \quad \text{for all } A, B \in B(\mathcal{H}),$$

the multiplicative structure allows for stronger refinements. In particular, we prove the following sharper quadratic inequality:

$$w_N^2(T) \ \geq \ \frac{N\Big(|T|^2 + |T^*|^2\Big)}{2} \ - \ \min\{N(\mathfrak{R}^2(T)), N(\mathfrak{I}^2(T))\}.$$

These inequalities generalize and refine the recent results of Ghris and Mansour [17], who used the Cartesian decomposition and Maligranda's inequality to derive numerical radius bounds for classical radius $w(\cdot)$.

2. Numerical radius inequalities via the Moore–Penrose inverse

The Moore–Penrose inverse is a fundamental tool in operator theory, especially for operators with closed range. Recently, several authors have explored its role in deriving bounds for the numerical radius (see, for instance, [11, 28]). In this section we refine and generalize these results. Our approach combines techniques from convexity, the McCarthy inequality, and projection properties of the Moore–Penrose inverse. We also illustrate the sharpness of our results with concrete examples. We begin with the following sequence of lemmas.

Lemma 2.1. *[28, Theorem 2.1] Let $T \in \mathcal{CR}(\mathcal{H})$. Then for all $x, y \in \mathcal{H}$,*

$$|\langle Tx, y\rangle| \leq \sqrt{\langle |T|^2x, x\rangle\langle TT^{\dagger}y, y\rangle}. \tag{4}$$

The following result can be viewed as an extension of the well-known Buzano inequality.

Lemma 2.2. *[3, Lemma 2.5] Let $x, y, e \in \mathcal{H}$ with $\|e\| = 1$. Then,*

$$|\langle x, e\rangle\langle e, y\rangle|^r \leq \frac{1+\alpha}{2}\|x\|^r\|y\|^r + \frac{1-\alpha}{2}|\langle x, y\rangle|^r,$$

for every $0 \leq \alpha \leq 1$ and $r \geq 1$.

Lemma 2.3. *(McCarthy inequality)([32, p.20]) Let T be a positive operator in $B(\mathcal{H})$ and let x be in $\mathcal{H}$ such that $\|x\| = 1$. Then for all $r \geq 1$,*

$$\langle Tx, x\rangle^r \leq \langle T^r x, x\rangle. \tag{5}$$

The following lemma is a direct consequence of the convexity of the function $f(t) = t^r$ for all $r \geq 1$.

Lemma 2.4. *Let $a_1, \ldots, a_n$ be positive real numbers. Then, for every $r \geq 1$,*

$$\left(\sum_{i=1}^{n} a_i\right)^r \leq n^{r-1}\sum_{i=1}^{n} a_i^r.$$

We begin our results with the following theorem, which provides a new upper bound for the numerical radius of an operator. This bound can be viewed as a refinement of inequality (2).

Theorem 2.5. *Let $T \in \mathcal{CR}(\mathcal{H})$ and $r \geq 1$. Then*

$$w^{2r}(T) \leq \min_{0\leq\alpha\leq1}\left\|\frac{\alpha}{2}\left(|T|^{4r} + TT^{\dagger}\right) + (1-\alpha)|T|^{2r}\right\| \tag{6}$$

and

$$w^{2r}(T) \leq \min_{0\leq\alpha\leq1}\left\|\frac{\alpha}{2}\left(|T|^{4r} + TT^{\dagger}\right) + (1-\alpha)|T^*|^{2r}\right\|. \tag{7}$$

Proof. Let x be a unit vector in $\mathcal{H}$ and $\alpha \in [0, 1]$. Then

$$|\langle Tx, x\rangle| = \alpha|\langle Tx, x\rangle| + (1-\alpha)|\langle Tx, x\rangle|.$$

By the convexity of the function $f(t) = t^{2r}$ we get

$$\begin{aligned}
|\langle Tx, x\rangle|^{2r} &= \alpha|\langle Tx, x\rangle|^{2r} + (1-\alpha)|\langle Tx, x\rangle|^{2r}\\
&\leq \alpha\left(\langle |T|^2x, x\rangle^r\langle TT^{\dagger}x, x\rangle^r\right) + (1-\alpha)|\langle Tx, x\rangle|^{2r} \quad \text{(by Lemma 2.1)}\\
&\leq \frac{\alpha}{2}\left(\langle |T|^2x, x\rangle^{2r} + \langle TT^{\dagger}x, x\rangle^{2r}\right) + (1-\alpha)|\langle Tx, x\rangle|^{2r} \text{ (by AM-GM inequality)}\\
&\leq \frac{\alpha}{2}\left(\langle |T|^{4r}x, x\rangle + \langle [TT^{\dagger}]^{2r}x, x\rangle\right) + (1-\alpha)|\langle Tx, x\rangle|^{2r} \quad \text{(by Lemma 2.3).}
\end{aligned}$$

Since $T \in \mathcal{CR}(\mathcal{H})$, the operator $TT^{\dagger} = P_{\mathcal{R}(T)}$ is an orthogonal projection onto $\mathcal{R}(T)$, hence $(TT^{\dagger})^2 = TT^{\dagger}$, therefore $(TT^{\dagger})^{2r} = TT^{\dagger}$ for every $r \geq 1$. It follows that

$$\begin{aligned}
|\langle Tx, x\rangle|^{2r} &\leq \frac{\alpha}{2}\left(\langle |T|^{4r}x, x\rangle + \langle TT^{\dagger}x, x\rangle\right) + (1-\alpha)\|Tx\|^{2r} \\
&\leq \frac{\alpha}{2}\left(\langle |T|^{4r}x, x\rangle + \langle TT^{\dagger}x, x\rangle\right) + (1-\alpha)\langle |T|^{2r}x, x\rangle \quad \text{(by Lemma 2.3)} \\
&\leq \left\langle \left(\frac{\alpha}{2}\left(|T|^{4r} + TT^{\dagger}\right) + (1-\alpha)|T|^{2r}\right)x, x\right\rangle \\
&\leq \left\|\frac{\alpha}{2}\left(|T|^{4r} + TT^{\dagger}\right) + (1-\alpha)|T|^{2r}\right\|.
\end{aligned}$$

The last inequality follows by taking the supremum over all unit vectors $x \in \mathcal{H}$. Hence, we obtain the first inequality (6). The second inequality (7) can be derived in a similar manner. □

We illustrate the sharpness of our estimation with the following example:

Example 2.6. *Let*

$$T = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}. \textit{ Then } \quad T^{\dagger} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix},$$

$$|T|^4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 16 \end{pmatrix}, \quad TT^{\dagger} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad |T^*|^2 = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, we take $r = 1$ *in* (7), *so we get*

$$\begin{aligned}
&\min_{0\leq\alpha\leq 1}\left\|\frac{\alpha}{2}\left(|T|^4 + TT^{\dagger}\right) + (1-\alpha)|T^*|^2\right\| \\
&\leq \left\|\frac{1}{20}\left(|T|^4 + TT^{\dagger}\right) + \frac{9}{10}|T^*|^2\right\| \qquad (\textit{for } \alpha = \frac{1}{10}) \\
&= 8.15.
\end{aligned}$$

For comparison, we have

$$\frac{1}{4}\left\||T|^2 + TT^{\dagger}\right\|^2 = 25.$$

Therefore, our result yields a sharper bound in this example than that provided by (2).

Theorem 2.7. *Let* $T \in \mathcal{CR}(\mathcal{H})$ *and* $r \geq 1$. *Then*

$$w^{2r}(T) \leq \min_{0\leq\alpha\leq 1}\left[\frac{1-\alpha}{2}w^r\left(TT^{\dagger}|T|^2\right) + \frac{1+\alpha}{4}\left\||T|^{4r} + TT^{\dagger}\right\|\right]. \tag{8}$$

Proof. Let x be a unit vector in $\mathcal{H}$ and $\alpha \in [0,1]$. Then

$$\begin{aligned}
|\langle Tx, x\rangle|^{2r} &\leq \langle |T|^2x, x\rangle^r \langle TT^{\dagger}x, x\rangle^r \quad \text{(by Lemma 2.1)} \\
&\leq \frac{1-\alpha}{2}|\langle |T|^2x, TT^{\dagger}x\rangle^r| + \frac{1+\alpha}{2}\||T|^2x\|^r\|TT^{\dagger}x\|^r \quad \text{(by Lemma 2.2)} \\
&\leq \frac{1-\alpha}{2}|\langle |T|^2x, TT^{\dagger}x\rangle^r| + \frac{1+\alpha}{2}\langle |T|^4x, x\rangle^{\frac{r}{2}}\langle TT^{\dagger}x, x\rangle^{\frac{r}{2}} \\
&\leq \frac{1-\alpha}{2}|\langle TT^{\dagger}|T|^2x, x\rangle^r| + \frac{1+\alpha}{4}\left(\langle |T|^4x, x\rangle^r + \langle TT^{\dagger}x, x\rangle^r\right) \quad \text{(by AM-GM inequality)} \\
&\leq \frac{1-\alpha}{2}|\langle TT^{\dagger}|T|^2x, x\rangle^r| + \frac{1+\alpha}{4}\left(\langle |T|^{4r}x, x\rangle + \langle TT^{\dagger}x, x\rangle\right) \\
&\leq \frac{1-\alpha}{2}w^r\left(TT^{\dagger}|T|^2\right) + \frac{1+\alpha}{4}\left\||T|^{4r} + TT^{\dagger}\right\|.
\end{aligned}$$

By taking the supremum over all unit vectors x in $\mathcal{H}$, we obtain the desired inequality. □

Theorem 2.8. *Let $T \in \mathcal{CR}(\mathcal{H})$ and $r \geq 1$. Then*

$$2w^{2r}(T) \leq \min_{0\leq\alpha\leq1} \left\| \alpha\left(|T|^{4r} + TT^{\dagger}\right) + (1-\alpha)\left(|T^*|^{4r} + T^{\dagger}T\right)\right\|. \tag{9}$$

Proof. Let x be a unit vector in $\mathcal{H}$ and $\alpha \in [0,1]$. Then

$$|\langle Tx, x\rangle| = \alpha|\langle Tx, x\rangle| + (1-\alpha)|\langle T^*x, x\rangle|.$$

by the convexity of the function $f(t) = t^{2r}$ we get

$$\begin{aligned}
|\langle Tx, x\rangle|^{2r} &= \alpha|\langle Tx, x\rangle|^{2r} + (1-\alpha)|\langle T^*x, x\rangle|^{2r} \\
&\leq \alpha\left(\langle |T|^2x, x\rangle^r \langle TT^{\dagger}x, x\rangle^r\right) + (1-\alpha)\left(\langle |T^*|^2x, x\rangle^r \langle T^{\dagger}Tx, x\rangle^r\right) \quad \text{(by Lemma 2.1)} \\
&\leq \frac{1}{2}\Big[\alpha\left(\langle |T|^2x, x\rangle^{2r} + \langle TT^{\dagger}x, x\rangle^{2r}\right) + (1-\alpha)\left(\langle |T^*|^2x, x\rangle^{2r} + \langle T^{\dagger}Tx, x\rangle^{2r}\right)\Big] \\
&\leq \frac{1}{2}\left\langle \alpha\left(|T|^{4r} + TT^{\dagger}\right) + (1-\alpha)\left(|T^*|^{4r} + T^{\dagger}T\right)x, x\right\rangle \quad \text{(by Lemma 2.3)} \\
&\leq \frac{1}{2}\left\| \alpha\left(|T|^{4r} + TT^{\dagger}\right) + (1-\alpha)\left(|T^*|^{4r} + T^{\dagger}T\right)\right\|.
\end{aligned}$$

□

We consider the following example to show that our estimation is proper.

Example 2.9.

$$T = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

A direct calculation yields,

$$T^{\dagger} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}, \quad TT^{\dagger} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T^{\dagger}T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$|T|^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & 81 & 0 \\ 0 & 0 & 0 & 16 \end{pmatrix} \quad \text{and} \quad |T^*|^4 = \begin{pmatrix} 16 & 0 & 0 & 0 \\ 0 & 81 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Consider the expression appearing in (9) *with $r = 1$, we have*

$$\begin{aligned}
&\min_{0\leq\alpha\leq1} \frac{1}{2}\left\| \alpha\left(|T|^4 + TT^{\dagger}\right) + (1-\alpha)\left(|T^*|^4 + T^{\dagger}T\right)\right\| \\
&\leq \frac{1}{4}\left\| |T|^4 + TT^{\dagger} + |T^*|^4 + T^{\dagger}T\right\| \qquad (\text{for } \alpha = \frac{1}{2}) \\
&= \frac{99}{4} = 24.75.
\end{aligned}$$

For comparison,

$$\frac{1}{4}\left\| |T|^2 + TT^{\dagger}\right\|^2 = 25.$$

so the new bound is strictly sharper for this example than (2).

In the following, we provide an upper bound for the numerical radius of the sum of n operators in $\mathcal{CR}(\mathcal{H})$, This result generalizes the upper bound given in [11, Theorem 2.6].

Theorem 2.10. *Let $T_i \in \mathcal{CR}(\mathcal{H})$ for all $i \in \{1, ..., n\}$ and $r \geq 1$. Then*

$$w^r\left(\sum_{i=1}^{n} T_i\right) \leq \frac{n^{r-1}}{2}\left\|\sum_{i=1}^{n}(|T_i|^{2r} + T_iT_i^{\dagger})\right\|.$$

Proof. Let x be a unit vector in $\mathcal{H}$. Then

$$\begin{aligned}
\left|\left\langle\sum_{i=1}^{n} T_i x, x\right\rangle\right|^r &\leq \left(\sum_{i=1}^{n} |\langle T_i x, x\rangle|\right)^r \\
&\leq n^{r-1}\left(\sum_{i=1}^{n} |\langle T_i x, x\rangle|^r\right) \qquad \text{(by Lemma 2.4)} \\
&\leq n^{r-1}\left(\sum_{i=1}^{n} \left\langle|T_i|^2 x, x\right\rangle^{r/2} \left\langle T_iT_i^{\dagger} x, x\right\rangle^{r/2}\right) \qquad \text{(by Lemma 2.1)}
\end{aligned}$$

Applying the AM-GM inequality to the product and McCarthy inequality, gives

$$\begin{aligned}
\left|\left\langle\sum_{i=1}^{n} T_i x, x\right\rangle\right|^r &\leq n^{r-1}\left(\sum_{i=1}^{n} \left\langle|T_i|^{2r} x, x\right\rangle^{1/2} \left\langle[T_iT_i^{\dagger}]^r x, x\right\rangle^{1/2}\right) \qquad \text{(by Lemma 2.3)} \\
&\leq \frac{n^{r-1}}{2}\left(\sum_{i=1}^{n} \left(\left\langle|T_i|^{2r} x, x\right\rangle + \left\langle T_iT_i^{\dagger} x, x\right\rangle\right)\right) \\
&\leq \frac{n^{r-1}}{2}\left\|\sum_{i=1}^{n} \left(|T_i|^{2r} + T_iT_i^{\dagger}\right)\right\|.
\end{aligned}$$

Taking the supremum over all unit vectors x yields the operator norm on the right-hand side and gives the desired inequality. □

3. Bounds for the generalized numerical radius

In this section, we study the generalized numerical radius associated with a general operator norm. Our work is inspired by the approach of Ghris and Mansour in [17], where their arguments rely on an application of Maligranda's inequality [22]. In contrast, we provide a more general framework and obtain our results through a simpler method that does not require this Maligranda's inequality.

Theorem 3.1. *Let $T \in B(\mathcal{H})$ and $N(\cdot)$ be a norm. Then*

$$w_N(T) \geq N(T) - \min\{N(\Re(T)), N(\Im(T))\}. \tag{10}$$

Proof. By the definition of the generalized numerical radius, we have

$$w_N(T) = \sup_{\theta \in \mathbb{R}} N(\Re(e^{i\theta}T)).$$

Taking $\theta = 0$ and $\theta = -\frac{\pi}{2}$ yields

$$w_N(T) \geq N(\Re(T)), \qquad w_N(T) \geq N(\Im(T)).$$

Hence

$$\begin{aligned} 2w_N(T) &\geq 2\max\{N(\mathfrak{R}(T)), N(\mathfrak{I}(T))\} \\ &= N(\mathfrak{R}(T)) + N(\mathfrak{I}(T)) + |N(\mathfrak{R}(T)) - N(\mathfrak{I}(T))| \\ &\geq N(\mathfrak{R}(T) + i\mathfrak{I}(T)) + |N(\mathfrak{R}(T)) - N(\mathfrak{I}(T))| \\ &= N(T) + |N(\mathfrak{R}(T)) - N(\mathfrak{I}(T))|. \end{aligned}$$

On the other hand, using the triangle inequality,

$$\begin{aligned} 2\min\{N(\mathfrak{R}(T)), N(\mathfrak{I}(T))\} &= N(\mathfrak{R}(T)) + N(\mathfrak{I}(T)) - |N(\mathfrak{R}(T)) - N(\mathfrak{I}(T))| \\ &\geq N(\mathfrak{R}(T) + i\mathfrak{I}(T)) - |N(\mathfrak{R}(T)) - N(\mathfrak{I}(T))| \\ &= N(T) - |N(\mathfrak{R}(T)) - N(\mathfrak{I}(T))|. \end{aligned}$$

Combining the above inequalities, we obtain

$$\begin{aligned} 2w_N(T) &\geq N(T) + |N(\mathfrak{R}(T) - N(\mathfrak{I}(T)| \\ &\geq 2N(T) - 2\min\{N(\mathfrak{R}(T)), N(\mathfrak{I}(T))\}. \end{aligned}$$

as required. □

The next theorem gives a lower bound for $w_N^2(T)$ under the algebra norm assumption.

Theorem 3.2. *Let $T \in B(\mathcal{H})$ and $N(\cdot)$ be an algebra norm. Then*

$$w_N^2(T) \geq \frac{N(|T|^2 + |T^*|^2)}{2} - \min\{N(\mathfrak{R}^2(T)), N(\mathfrak{I}^2(T))\}. \tag{11}$$

Proof. By definition of the generalized numerical radius,

$$w_N^2(T) = \sup_{\theta \in \mathbb{R}} N^2\Big(\mathfrak{R}(e^{i\theta}T)\Big).$$

In particular, taking $\theta = 0$ and $\theta = -\frac{\pi}{2}$ yields

$$2w_N^2(T) \ \geq \ 2\max\{N^2(\mathfrak{R}(T)),\ N^2(\mathfrak{I}(T))\}.$$

By the submultiplicativity of $N(\cdot)$, we obtain

$$2w_N^2(T) \ \geq \ 2\max\{N(\mathfrak{R}^2(T)),\ N(\mathfrak{I}^2(T))\}.$$

Therefore

$$\begin{aligned} 2w_N^2(T) &\geq N(\mathfrak{R}^2(T)) + N(\mathfrak{I}^2(T)) + |N(\mathfrak{R}^2(T)) - N(\mathfrak{I}^2(T))| \\ &\geq N(\mathfrak{R}^2(T) + \mathfrak{I}^2(T)) + |N(\mathfrak{R}^2(T)) - N(\mathfrak{I}^2(T))|. \\ &\geq \tfrac{1}{2}N(|T|^2 + |T^*|^2) + |N(\mathfrak{R}^2(T)) - N(\mathfrak{I}^2(T))|. \end{aligned}$$

Now observe that

$$\begin{aligned} 2\min\{N(\mathfrak{R}^2(T)), N(\mathfrak{I}^2(T))\} &= N(\mathfrak{R}^2(T)) + N(\mathfrak{I}^2(T)) - |N(\mathfrak{R}^2(T)) - N(\mathfrak{I}^2(T))| \\ &\geq \tfrac{1}{2}N(|T|^2 + |T^*|^2) - |N(\mathfrak{R}^2(T)) - N(\mathfrak{I}^2(T))|. \end{aligned}$$

Combining the above inequalities, we conclude that

$$\begin{aligned} 2w_N^2(T) &\geq \tfrac{1}{2}N(|T|^2 + |T^*|^2) + |N(\mathfrak{R}^2(T)) - N(\mathfrak{I}^2(T))| \\ &\geq N(|T|^2 + |T^*|^2) - 2\min\{N(\mathfrak{R}^2(T)), N(\mathfrak{I}^2(T))\}. \end{aligned}$$

This completes the proof. □

The following corollary is an immediate consequence of Theorem 3.5 by taking $N(\cdot) = \|\cdot\|$, and was proved recently in [17].

Corollary 3.3. *Let $T \in B(\mathcal{H})$. Then*

$$w^2(T) \geq \frac{\||T|^2 + |T^*|^2\|}{2} - \min\{\|\mathfrak{R}(T)\|^2, \|\mathfrak{I}(T)\|^2\}. \tag{12}$$

To show the sharpness of inequalities in Corollary 3.3 and Theorem 3.2, we propose the following example

Example 3.4. *Let*

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

A short calculation gives

$$|T|^2 + |T^*|^2 = T^*T + TT^* = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix},$$

hence $\Big\| |T|^2 + |T^*|^2 \Big\| = 5$ and therefore

$$\tfrac{1}{2}\Big\| |T|^2 + |T^*|^2 \Big\| = \tfrac{5}{2}.$$

Moreover,

$$\|\mathfrak{R}(T)\| = \tfrac{3}{2}, \qquad \|\mathfrak{I}(T)\| = \tfrac{1}{2},$$

so that

$$\min\{\|\mathfrak{R}(T)\|^2, \|\mathfrak{I}(T)\|^2\} = \tfrac{1}{4}.$$

Thus, the right-hand side of (12) *equals*

$$\tfrac{5}{2} - \tfrac{1}{4} = \tfrac{9}{4}.$$

On the other hand, the numerical range of T is the disc centered at 1 *with radius* 1/2, *hence*

$$w(T) = 1 + \tfrac{1}{2} = \tfrac{3}{2}, \qquad w^2(T) = \left(\tfrac{3}{2}\right)^2 = \tfrac{9}{4}.$$

Therefore, equality holds, showing that our inequalities are sharp for this operator T.

In [10], it is shown that the numerical radius of a bounded linear operator T satisfies

$$w(T) = \max_{\lambda \in \mathbb{T}} \Big\|\mathfrak{R}_\lambda(T)\Big\|,$$

where

$$\mathfrak{R}_\lambda(T) = \frac{T + \lambda T^*}{2}, \qquad \mathfrak{I}_\lambda(T) = \frac{T - \lambda T^*}{2i}$$

are called the generalized real and generalized imaginary parts of T, respectively. Here

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}.$$

Thus, T admits the generalized Cartesian decomposition

$$T = \mathfrak{R}_\lambda(T) + i\,\mathfrak{I}_\lambda(T).$$

It is also straightforward to see that the generalized numerical radius satisfies

$$w_N(T) = \max_{\lambda \in \mathbb{T}} N(\mathfrak{R}_\lambda(T)).$$

Theorem 3.5. *Let $T \in B(\mathcal{H})$ and let $N(\cdot)$ be a norm. Then, for every $\lambda, \mu \in \mathbb{T}$,*

$$w_N(T) \ \geq \ N\left(T + \frac{\lambda + \mu}{2} T^*\right) \ - \ \frac{1}{2} \min\{ N(T + \lambda T^*), \, N(T + \mu T^*)\}.$$

Proof. For any $\lambda, \mu \in \mathbb{T}$ we have

$$2w_N(T) \ \geq \ N(T + \lambda T^*), \qquad 2w_N(T) \ \geq \ N(T + \mu T^*).$$

Hence,

$$\begin{aligned} 4w_N(T) &\geq 2\max\{N(T + \lambda T^*), \, N(T + \mu T^*)\} \\ &= N(T + \lambda T^*) + N(T + \mu T^*) + \left| N(T + \lambda T^*) - N(T + \mu T^*) \right| \\ &\geq N(2T + (\lambda + \mu)T^*) + \left| N(T + \lambda T^*) - N(T + \mu T^*) \right|. \end{aligned}$$

On the other hand,

$$\begin{aligned} 2\min\{N(T + \lambda T^*), \, N(T + \mu T^*)\} &= N(T + \lambda T^*) + N(T + \mu T^*) - \left| N(T + \lambda T^*) - N(T + \mu T^*) \right| \\ &\geq N(2T + (\lambda + \mu)T^*) - \left| N(T + \lambda T^*) - N(T + \mu T^*) \right|. \end{aligned}$$

Combining the two estimates yields

$$4w_N(T) \ \geq \ 2N(2T + (\lambda + \mu)T^*) - 2\min\{N(T + \lambda T^*), \, N(T + \mu T^*)\},$$

which gives the desired inequality after dividing by 4. □

Conflict of interest The authors declare that they have no conflict of interest.

References

[1] A. Abu-Omar and F. Kittaneh, A generalization of the numerical radius, *Linear Algebra Appl.* **569** (2019), 323–334.

[2] S. Abu Sammour, F. Kittaneh and M. Sababheh, A geometric approach to numerical radius inequalities, *Linear Algebra Appl.* **652** (2022), 1–17.

[3] M. Al–dolat and F. Kittaneh, Upper bounds for the numerical radii of powers of Hilbert space operators, *Quaest. Math.* **47** (2024), no. 2, 341–352.

[4] W. Audeh, H. R. Moradi, and M. Sababheh, Numerical radius bounds for certain operator matrices, *Filomat* **39** (2025), no. 13, 4341–4355.

[5] O. Axelsson, H. Lu and B. Polman, On the numerical radius of matrices and its application to iterative solution methods, *Linear Multilinear Algebra* **37** (1994), no. 1–3, 225–238.

[6] M. Bakherad, M. Hajmohamadi, R. Lashkaripour and M. Sababheh, Some estimates for the generalized numerical radius, *J. Anal.* **31** (2023), 2163–2172.

[7] Y. Bedrani, F. Kittaneh and M. Sababheh, Numerical radii of accretive matrices, *Linear Multilinear Algebra* **69** (2020), no. 5, 957–970.

[8] M. Benzi, Some uses of the field of values in numerical analysis, *Boll. Unione Mat. Ital.* **14** (2021), no. 1, 159–177.

[9] P. Bhunia, K. Paul and A. Sen, Numerical radius inequalities of sectorial matrices, *Ann. Funct. Anal.* **14** (2023), no. 3, 66.

[10] P. Bhunia, A. Sen and K. Paul, Generalized Cartesian decomposition and numerical radius inequalities, *Rend. Circ. Mat. Palermo (2)* **73** (2024), no. 3, 887–897.

[11] P. Bhunia, F. Kittaneh and S. Sahoo, Improved numerical radius bounds using the Moore–Penrose inverse, *Linear Algebra Appl.* **711** (2025), 1–16.

[12] T. Bottazzi and C. Conde, Generalized numerical radius and related inequalities, *Oper. Matrices* **15** (2021), no. 4, 1289–1308.

[13] F. Bouchelaghem and M. Benharrat, The Moore–Penrose inverse of accretive operators with application to quadratic operator pencils, *Filomat* **36** (2022), no. 7, 2475–2491.

[14] G. Cassier, M. Naimi and M. Benharrat, Harnack parts for 4-by-4 truncated shift, *Adv. Oper. Theory* **9** (2023), no. 1, 11.

[15] Y. Chen and Y. Wei, Numerical radius for the asymptotic stability of delay differential equations, *Linear Multilinear Algebra* **65** (2017), no. 11, 2306–2315.

[16] C. W. Groetsch, *Generalized Inverses of Linear Operators: Representation and Approximation*, Marcel Dekker, New York, 1977.

[17] A. Ghris and A. Mansour, Improved bounds for the numerical radius via Maligranda inequality, *Gulf J. Math.* **19** (2025), no. 1, 208–216.

[18] O. Hirzallah, F. Kittaneh and K. Shebrawi, Numerical radius inequalities for certain 2×2 operator matrices, *Integral Equations Oper. Theory* **71** (2011), no. 1, 129–147.

[19] F. Kittaneh and A. Zamani, Bounds for A-numerical radius based on an extension of A-Buzano inequality, *J. Comput. Appl. Math.* **426** (2023), 115070.

[20] F. Kittaneh, A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix, *Studia Math.* **158** (2003), no. 1, 11–17.

[21] F. Kittaneh, Numerical radius inequalities for Hilbert space operators, *Studia Math.* **168** (2005), no. 1, 73–80.

[22] L. Maligranda, Simple norm inequalities, *Amer. Math. Monthly* **113** (2006), no. 3, 256–260.

[23] B. Mayyas and M. Sababheh, Hilbert–Schmidt-Type Radii of Operator Pairs, *J. Funct. Spaces* **2025** (2025), Art. 2696198, 16 pp.

[24] H. R. Moradi and M. Sababheh, New estimates for the numerical radius, *Filomat* **35** (2021), no. 14, 4957–4962.

[25] M. Naimi and M. Benharrat, On the numerical radius of an upper triangular operator matrix, *Tamkang J. Math.* **55** (2023), no. 2, 195–201.

[26] M. Naimi and M. Benharrat, Further inequalities for the numerical radius of off-diagonal part of 2×2 operator matrices, *Tamkang J. Math.* **56** (2025), no. 3, 321–333.

[27] M. Z. Nashed, Generalized Inverses and Applications, Academic Press, New York, 1976.

[28] M. Sababheh, D. S. Djordjević and H. R. Moradi, Numerical radius and norm bounds via the Moore–Penrose inverse, *Complex Anal. Oper. Theory* **18** (2024), no. 5, 117.

[29] M. Sababheh, H. R. Moradi, and M. Krnić, Upper bounds for the norm of the sum and Kronecker product of Hilbert space operators, *Linear Multilinear Algebra* **73** (2025), no. 2, 351–367.

[30] M. Sababheh, H. R. Moradi, and S. Sahoo, Numerical radii of operator sums and operator matrices, *Complex Anal. Oper. Theory* **19** (2025), no. 7, Art. 163, 20 pp.

[31] S. Sahoo and M. Sababheh, Hilbert-Schmidt numerical radius of block operators, *Filomat* **35** (2021), no. 8, 2663–2678.

[32] B. Simon, *Trace Ideals and Their Applications*, Cambridge Univ. Press, Cambridge, 1979.

[33] H. Wielandt, On eigenvalues of sums of normal matrices, *Pacific J. Math.* **5** (1955), no. 4, 633–638.

[34] T. Yamazaki, On upper and lower bounds of the numerical radius and an equality condition, *Stud. Math.* **178** (2007), no. 1, 83–89.