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Lipschitz estimates for the commutators of fractional Hardy and Hardy-Littlewood-Pólya operators on grand variable Herz spaces

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Abstract. In this article, we aim to prove the boundedness for commutators of fractional Hardy and Hardy-Littlewood-Pólya operators on grand variable Herz spaces, where the symbols of the commutators belong to Lipschitz spaces.

1. Introduction and main results

In the n-dimensional Euclidean space $\mathbb{R}^n$, for any $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, where $x_i \in \mathbb{R}$ $(i = 1, ..., n)$. The euclidean norm is denoted by $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. In addition, the ball in the Euclidean space is defined by $B_r(a) = \{x \in \mathbb{R}^n : |x| \leq 2^r\}$, the corresponding Euclidean space spherical shell $S_r(a) = B_r(a) \setminus B_{r-1}(a)$. When $a = 0$, we define $B_r(a) = B_r, S_r(a) = S_r$. Moreover, some methods on Euclidean space can be widely applied to physics and biology [11, 15, 16, 18].

Let f be a non-negative integrable function on $\mathbb{R}^+$, the classical Hardy operators can be defined by

$$Hf(x) = \frac{1}{x}\int_0^x f(y)\,dy, \quad x > 0.$$

The Hardy operators gradually attracted extensive attention, see [1, 8, 19]. [5] defined the form of the following n-dimensional integral inequalities parallel to the 1-dimensional results.

Let $1 < p < \infty$, $f \in L_{\text{loc}}(\mathbb{R}^n)$, n-dimensional Hardy operators can be defined by

$$\mathcal{H}f(x) = \frac{1}{|x|^n}\int_{|y|<|x|} f(y)\,dy, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

As the research on Hardy operator gradually deepens, the commutators generated by the Hardy operator with functions such as Lipschitz or BMO functions have also been widely studied by researchers. The

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commutator of the Hardy operator is defined by

$$[b,\mathcal{H}]f = b\mathcal{H}f - \mathcal{H}(bf),$$

where b is a locally integrable function defined on $\mathbb{R}^n$.

Let $0 < \alpha < n$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then the fractional Hardy operators can be defined by [12]

$$\mathcal{H}_\alpha f(x) = \frac{1}{|x|^{n-\alpha}} \int_{|t|\leq|x|} f(t)dt, \quad \mathcal{H}^*_\alpha f(x) = \int_{|t|>|x|} \frac{f(t)}{|t|^{n-\alpha}}dt, \ x \in \mathbb{R}^n\backslash\{0\}.$$

Let g be a non-negative integrable function on $\mathbb{R}^n$, it is easy to see that $\mathcal{H}_\alpha$ and $\mathcal{H}^*_\alpha$ are satisfied

$$\int_{\mathbb{R}^n} g(x)\mathcal{H}_\alpha f(x)dx = \int_{\mathbb{R}^n} f(x)\mathcal{H}_\alpha g(x)dx.$$

If we take $\alpha = 0$, then $\mathcal{H}_0 = \mathcal{H}$ and $\mathcal{H}^*_\alpha = \mathcal{H}^*$.

Let b be a locally integrable function on $\mathbb{R}^n$, the commutators of fractional Hardy operators can be defined by

$$\begin{aligned} &[b,\mathcal{H}_\alpha]f(x) = \frac{1}{|x|^{n-\alpha}} \int_{|t|\leq|x|} (b(x) - b(t))f(t)dt, \\ &[b,\mathcal{H}^*_\alpha]f(x) = \int_{|t|>|x|} \frac{f(t)}{|t|^{n-\alpha}}(b(x) - b(t))dt, \ x \in \mathbb{R}^n\backslash\{0\}. \end{aligned} \tag{1.1}$$

When $\alpha = 0$, then $[b,\mathcal{H}_\alpha] = [b,\mathcal{H}]$ and $[b,\mathcal{H}^*_\alpha] = [b,\mathcal{H}^*]$.

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, the Hardy-Littlewood-Pólya operator can be defined by [3]

$$Tf(x) = \int_{\mathbb{R}\backslash\{0\}} \frac{f(t)}{\max\{|x|,|t|\}}dt, x \in \mathbb{R}.$$

Let $b : \mathbb{R} \to \mathbb{R}$ be a locally integrable function, the commutator of the Hardy-Littlewood-Pólya operator can be defined by

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x). \tag{1.2}$$

In 2017, Liu et.al [12] considered the boundedness of the commutators of the bilinear fractional Hardy operators. In 2019, Zhou et.al [9] obtained the endpoint estimates of the commutators generated by fractional Hardy operators and Lipschitz fuctions on the Lipschitz spaces. Nafis et.al [14] introduced the boundedness of the sublinear operators on the grand variable Herz spaces. In 2018, Zhao and Zhou [25] proved that the boundedness of the higher order commutators $[b^m, T_{\Omega,\mu}]$ generated by the fractional integral operators $T_{\Omega,\mu}$ with variable kernel and Lipschitz functionon b in variable exponent Herz-Morrey spaces. The boundedness of commutators of rough Hardy operators on grand variable Herz spaces was given by Sultan et.al [20] in 2024. Xue [23] investigated the boundedness of commutators generated by θ-type Calderón-Zygmund operators and Lipschitz functions on homogeneous Herz spaces with three variable exponents. The boundedness of rough generalized commutators with Lipschitz functions on homogeneous variable exponent Herz type spaces have been obtained in [10].

Inspired by the above literatures, the purpose of this paper is to prove the boundedness for commutators of fractional Hardy operator and Hardy-Littlewood-Pólya operator on grand variable Herz spaces, where the symbols of the commutators belong to Lipschitz space. In this paper, the letter C will be adopted to signify constants that may vary across different expressions and define the characteristic function $\mathcal{X}_k := \mathcal{X}_{S_k} = \mathcal{X}_{B_k\backslash B_{k-1}}$.

Theorem 1.1. *Let* $1 \le u < \infty, 0 < \beta < 1, 0 < \alpha < \alpha + \beta < n$, *for all* $\eta(\cdot), q_1(\cdot), q_2(\cdot) \in C^{log}(\mathbb{R}^n)$ *with* $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n), (q_1)_+ < n/(\alpha+\beta)$, $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha+\beta)/n$, $1/q_1(\cdot) + 1/q_1^{'}(\cdot) = 1$ *and* $1/q_1(\infty) + 1/q_1^{'}(\infty) = 1$. *If* $b \in \Lambda_\beta$ *and* $\eta(\cdot)$ *satisfies the following conditions*

(1)$\alpha + \beta - \frac{n}{q_1(0)} < \eta(0) < \frac{n}{q_1'(0)}$;

(2) $\alpha + \beta - \frac{n}{q_1(\infty)} < \eta(\infty) < \frac{n}{q_1'(\infty)}$.

Then the commutator $[b, \mathcal{H}_\alpha]$ *is bounded from* $K_{q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R}^n)$ *to* $K_{q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R}^n)$.

Corollary 1.2. *Let* $1 \le u < \infty, 0 < \beta < 1$, *for all* $\eta(\cdot), q_1(\cdot), q_2(\cdot) \in C^{log}(\mathbb{R}^n)$, *and* $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n), (q_1)_+ < n/\beta$, $1/q_2(\cdot) = 1/q_1(\cdot) - \beta/n$, $1/q_1(\cdot) + 1/q_1^{'}(\cdot) = 1$ *and* $1/q_1(\infty) + 1/q_1^{'}(\infty) = 1$. *If* $b \in \Lambda_\beta$ *and* $\eta(\cdot)$ *satisfies the following conditions*

(1) $\beta - \frac{n}{q_1(0)} < \eta(0) < \frac{n}{q_1'(0)}$;

(2) $\beta - \frac{n}{q_1(\infty)} < \eta(\infty) < \frac{n}{q_1'(\infty)}$.

Then the commutator $[b, \mathcal{H}]$ *is bounded from* $K_{q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R}^n)$ *to* $K_{q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R}^n)$.

Theorem 1.3. *Let* $1 \le u < \infty, 0 < \beta < 1, 0 < \alpha < \alpha + \beta < n$, *for all* $\eta(\cdot), q_1(\cdot), q_2(\cdot) \in C^{log}(\mathbb{R}^n)$ *with* $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n), (q_1)_+ < n/(\alpha+\beta)$, $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha+\beta)/n$, $1/q_1(\cdot) + 1/q_1^{'}(\cdot) = 1$ *and* $1/q_1(\infty) + 1/q_1^{'}(\infty) = 1$. *If* $b \in \Lambda_\beta$ *and* $\eta(\cdot)$ *satisfies the following conditions*

(1) $\alpha + \beta - \frac{n}{q_1(0)} < \eta(0) < \frac{n}{q_1'(0)}$;

(2) $\alpha + \beta - \frac{n}{q_1(\infty)} < \eta(\infty) < \frac{n}{q_1'(\infty)}$.

Then the commutator $[b, \mathcal{H}_\alpha^*]$ *is bounded from* $K_{q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R}^n)$ *to* $K_{q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R}^n)$.

Corollary 1.4. *Let* $1 \le u < \infty, 0 < \beta < 1$, *for all* $\eta(\cdot), q_1(\cdot), q_2(\cdot) \in C^{log}(\mathbb{R}^n)$, *and* $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n), (q_1)_+ < n/\beta$, $1/q_2(\cdot) = 1/q_1(\cdot) - \beta/n$, $1/q_1(\cdot) + 1/q_1^{'}(\cdot) = 1$ *and* $1/q_1(\infty) + 1/q_1^{'}(\infty) = 1$. *If* $b \in \Lambda_\beta$ *and* $\eta(\cdot)$ *satisfies the following conditions*

(1) $\beta - \frac{n}{q_1(0)} < \eta(0) < \frac{n}{q_1'(0)}$;

(2) $\beta - \frac{n}{q_1(\infty)} < \eta(\infty) < \frac{n}{q_1'(\infty)}$.

Then the commutator $[b, \mathcal{H}^*]$ *is bounded from* $K_{q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R}^n)$ *to* $K_{q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R}^n)$.

Theorem 1.5. *Let* $1 \le u < \infty, 0 < \beta < 1$, *for all* $\eta(\cdot), q_1(\cdot), q_2(\cdot) \in C^{log}(\mathbb{R})$ *with* $q_1(\cdot) \in \mathcal{P}(\mathbb{R}), (q_1)_+ < 1/\beta$, $1/q_2(\cdot) = 1/q_1(\cdot) - \beta$, $1/q_1(\cdot) + 1/q_1^{'}(\cdot) = 1$ *and* $1/q_1(\infty) + 1/q_1^{'}(\infty) = 1$. *If* $b \in \Lambda_\beta$ *and* $\eta(\cdot)$ *satisfies the following conditions*

(1) $\beta - \frac{n}{q_1(0)} < \eta(0) < \frac{n}{q_1'(0)}$;

(2) $\beta - \frac{n}{q_1(\infty)} < \eta(\infty) < \frac{n}{q_1'(\infty)}$.

Then the commutator $[b, T]$ *is bounded from* $K_{q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R})$ *to* $K_{q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R})$.

2. Preliminaries

Let $1 \le q < \infty$, for any measurable function f on $\mathbb{R}^n$, the Lebesgue space can be defined by

$$L^q(\mathbb{R}^n) = \left\{ f : \|f\|_{L^q(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f(x)|^q \, dx \right)^{\frac{1}{q}} < \infty \right\}, 1 \le q < \infty;$$

$$L^\infty(\mathbb{R}^n) = \left\{ f : \|f\|_{L^\infty(\mathbb{R}^n)} := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \left| f(x) \right| = \inf\left\{ \beta \ge 0 : \left| \left\{ x \in \mathbb{R}^n : \left| f(x) \right| > \beta \right\} \right| = 0 \right\} < \infty \right\}.$$

Definition 2.1. *[2] Given a measurable function* $q(\cdot)$ *defined on* $\mathbb{R}^n$, *we denote by*

$$q_- := \operatorname*{ess\,inf}_{x \in \mathbb{R}^n} q(x),$$

$$q_+ := \underset{x\in\mathbb{R}^n}{ess\,sup}\, q(x).$$

(1) $q'_- := \underset{x\in\mathbb{R}^n}{ess\ inf} q'(x) = \frac{q_+}{q_+-1}$, $q'_+ := \underset{x\in\mathbb{R}^n}{ess\ inf} q'(x) = \frac{q_-}{q_--1}$.

(2) Denote by $\mathcal{P}$ *the set of all measurable function* $q(\cdot) : \mathbb{R}^n \to (1,\infty)$ *such that*

$$1 < q_- \le q(x) \le q_- < \infty.$$

Definition 2.2. *(variable exponent Lebesgue spaces) [6, 7] Let* $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$*, then the variable exponent Lebesgue space can be defined by*

$$L^{q(\cdot)}(\mathbb{R}^n) = \left\{ f : f \text{ is measurable function} : \mathcal{F}_q\left(f/\eta\right) < \infty \text{ for some constant } \eta > 0 \right\},$$

where

$$\mathcal{F}_q(f) := \int_{\mathbb{R}^n} \left|f(x)\right|^{q(x)} dx.$$

It is well-known that the Lebesgue sapce $L^{q(\cdot)}(\mathbb{R}^n)$ *is a Banach function space with respect to the Luxemburg norm*

$$\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} = \inf\left\{ \eta > 0 : \mathcal{F}_q\left(f/\eta\right) = \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\eta}\right)^{q(x)} dx \le 1 \right\}.$$

Definition 2.3. *(log-Hölder continuity) [7] Let a real-valued measurable function* $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$.

(1) The function $q(\cdot)$ *is locally log-Hölder continuous if there exsists a universal constant C such that*

$$\left|q(x) - q(y)\right| \le \frac{C}{\log\left(e + \frac{1}{|x-y|}\right)}, x, y \in \mathbb{R}^n, \left|x - y\right| < \frac{1}{2}. \tag{2.1}$$

Denote by $C^{\log}_{loc}(\mathbb{R}^n)$ *the set of all locally log-Hölder continuous function.*

(2) The function $q(\cdot)$ *is log-Hölder continuous at the origin if there exsists a universal constant C such that*

$$\left|q(x) - q(0)\right| \le \frac{C}{\log\left(e + \frac{1}{|x|}\right)}, \forall x \in \mathbb{R}^n. \tag{2.2}$$

Denote by $C^{\log}_0(\mathbb{R}^n)$ *the set of all log-Hölder continuous function at the origin.*

(3)The function $q(\cdot)$ *is log-Hölder continuous at infinity if there exsists* $q_\infty := q(\infty) = \lim_{|x|\to\infty} q(x)$*, and there is a universal constant C such that*

$$\left|q(x) - q(\infty)\right| \le \frac{C}{\log\left(e + |x|\right)}, \forall x \in \mathbb{R}^n.$$

Denote by $C^{\log}_\infty(\mathbb{R}^n)$ *the set of all log-Hölder continuous function at infinity.*

(4)The function $q(\cdot)$ *is global continuous if* $q(\cdot)$ *are both locally log-Hölder continuous and log-Hölder continuous at infinity. Denote by* $C^{\log}(\mathbb{R}^n)$ *the set of all global log-Hölder continuous functions.*

Definition 2.4. *[13] Suppose* $\eta(\cdot) : \mathbb{R}^n \to \mathbb{R}, 1 \le u < \infty, s(\cdot) \in \mathcal{P}(\mathbb{R}^n), \theta > 0$*. A grand variable exponent Herz space* $K^{\eta(\cdot),u),\theta}_{s(\cdot)}(\mathbb{R}^n)$ *can be defined by*

$$K^{\eta(\cdot),u),\theta}_{s(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{s(\cdot)}_{loc}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{K^{\eta(\cdot),u),\theta}_{s(\cdot)}}(\mathbb{R}^n) < \infty \right\},$$

where the norm

$$\|g\|_{K^{\eta(\cdot),u),\theta}_{s(\cdot)}(\mathbb{R}^n)} = \sup_{\varepsilon>0}\left(\varepsilon^\theta \sum_{k\in\mathbb{Z}} \left\|2^{\eta(\cdot)k} g\chi_k\right\|^{u(1+\varepsilon)}_{L^{s(\cdot)}(\mathbb{R}^n)} \right)^{\frac{1}{u(1+\varepsilon)}}.$$

Proposition 2.5. *[4] Let $\eta(\cdot), s(\cdot), u$ are as defined in Definition 2.4, then*

$$\begin{aligned}\|g\|_{K_{s(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R}^n)} &= \sup_{\varepsilon>0}\left(\varepsilon^\theta \sum_{k\in\mathbb{Z}} \left\|2^{\eta(\cdot)k} g\chi_k\right\|_{L^{s(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}\right)^{\frac{1}{u(1+\varepsilon)}} \\ &\approx \sup_{\varepsilon>0}\left(\varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{\eta(0)ku(1+\varepsilon)} \left\|g\chi_k\right\|_{L^{s(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}\right)^{\frac{1}{u(1+\varepsilon)}} \\ &+ \sup_{\varepsilon>0}\left(\varepsilon^\theta \sum_{k=0}^{\infty} 2^{\eta(\infty)ku(1+\varepsilon)} \left\|g\chi_k\right\|_{L^{s(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}\right)^{\frac{1}{u(1+\varepsilon)}} .\end{aligned}$$

Definition 2.6. *[17] Let $0 < \beta < 1$, the Lipschitz space $\Lambda_\beta(\mathbb{R}^n)$ can be defined by*

$$\Lambda_\beta(\mathbb{R}^n) := \left\{ f \in L^1_{loc}(\mathbb{R}^n) : \|f\|_{\Lambda_\beta(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{\Lambda_\beta(\mathbb{R}^n)} = \sup_{x,y\in\mathbb{R}^n, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\beta}.$$

Next, we state some auxiliary propositions and lemmas which will be used in the proofs of our main theorems. And we only describe partial results we need.

Lemma 2.7. *(Generalized Hölder's inequality in $\mathbb{R}^n$) [6] (1) Assume that $1 \le q \le \infty$ with $\frac{1}{q} + \frac{1}{q'} = 1$, and measurable functions $f \in L^q(\mathbb{R}^n)$ and $g \in L^{q'}(\mathbb{R}^n)$. Then there exists a positive constant C such that*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, \mathrm{d}x \le C\|f\|_{L^q(\mathbb{R}^n)}\|g\|_{L^{q'}(\mathbb{R}^n)}.$$

(2) Assume that G is a measurable subset of $\mathbb{R}^n$, and $1 \le p_-(G) \le p_+(G) \le \infty$. Then

$$\|fg\|_{L^{r(\cdot)}(G)} \le C\|f\|_{L^{p(\cdot)}(G)}\|g\|_{L^{q(\cdot)}(G)}.$$

holds, where $f \in L^{p(\cdot)}(G), g \in L^{q(\cdot)}(G)$, and $\frac{1}{r(z)} = \frac{1}{p(z)} + \frac{1}{q(z)}$ for every $z \in G$.

(3) When $r(\cdot) = 1$ in (2) as mentioned above, we have $p(\cdot), q(\cdot) \in C^{log}(\mathbb{R}^n)$ and $\frac{1}{p(z)} + \frac{1}{q(z)} = 1$ almost everywhere. Then there exists a positive constant C such that the inequality

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, \mathrm{d}x \le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{q(\cdot)}(\mathbb{R}^n)}$$

holds for all $g \in L^{p(\cdot)}(\mathbb{R}^n)$ and $f \in L^{q(\cdot)}(\mathbb{R}^n)$.

Lemma 2.8. *(Norms of characteristic functions) [7] Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy conditions (2.1) and (2.2), then*

$$\|\chi_{B_r}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \approx \begin{cases} |B_r|^{\frac{1}{q(0)}}, & |B_r| \le 2^n \text{ and } x \in B_r, \\ |B_r|^{\frac{1}{q(\infty)}}, & |B_r| > 1. \end{cases}$$

According to [2], we can directly obtain the following results.

Lemma 2.9. *Let $0 < \beta < 1, 0 < \alpha < \alpha + \beta < n$, for all $q_1(\cdot), q_2(\cdot) \in C^{log}(\mathbb{R}^n)$ with $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n), (q_1)_+ < n/(\alpha+\beta)$, and $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha+\beta)/n$, $1/q_1(\cdot) + 1/q_1'(\cdot) = 1$ and $1/q_1(\infty) + 1/q_1'(\infty) = 1$, taking $k, l \in Z$, then*

$$2^{k(\alpha+\beta-n)}\, \|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\, \|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \le \begin{cases} C2^{\frac{(l-k)n}{q_1'(0)}}, & k<0, l<0, \\ C2^{\frac{-kn}{q_1'(\infty)} + \frac{ln}{q_1'(0)}}, & k \ge 0, l < 0, \end{cases}$$

and

$$2^{l(\alpha+\beta-n)}\,\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\,\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \le \begin{cases} C2^{k\left(\frac{n}{q_1(0)}-\alpha-\beta\right)-l\left(\frac{n}{q_1(\infty)}-\alpha-\beta\right)}, & k<0, l\ge 0,\\ C2^{(k-l)\left(\frac{n}{q_1(\infty)}-\alpha-\beta\right)}, & k\ge 0, l\ge 0.\end{cases}$$

Proof. Let $0<\beta<1, 0<\alpha<\alpha+\beta<n$, for all $q_1(\cdot), q_2(\cdot)\in C^{log}(\mathbb{R}^n)$ with $q_1(\cdot)\in\mathcal{P}(\mathbb{R}^n), (q_1)_+<n/(\alpha+\beta)$, by Definition 2.3. When $k<0, l<0$, we have

$$\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \approx 2^{\frac{kn}{q_2(0)}}, \quad \|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \approx 2^{\frac{kn}{q_1'(0)}}.$$

Since $1/q_2(\cdot)=1/q_1(\cdot)-(\alpha+\beta)/n$ and $1/q_1(\cdot)+1/q_1'(\cdot)=1$, we have

$$\begin{aligned} 2^{k(\alpha+\beta-n)}\,\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\,\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} &\le C2^{k(\alpha+\beta-n)}\cdot 2^{\frac{kn}{q_2(0)}}\cdot 2^{\frac{ln}{q_1'(0)}}\\ &= C2^{k(\alpha+\beta-n)}\cdot 2^{kn(\frac{1}{q_1(0)}-\frac{\alpha+\beta}{n})}\cdot 2^{\frac{ln}{q_1'(0)}}\\ &= C2^{k(\alpha+\beta)-kn+\frac{kn}{q_1(0)}-k(\alpha+\beta)}\cdot 2^{\frac{ln}{q_1'(0)}}\\ &= C2^{\frac{kn}{q_1(0)}-kn}\cdot 2^{\frac{ln}{q_1'(0)}}\\ &= C2^{-kn(1-\frac{1}{q_1(0)})}\cdot 2^{\frac{ln}{q_1'(0)}}\\ &= C2^{\frac{-kn}{q_1'(0)}}\cdot 2^{\frac{ln}{q_1'(0)}}\\ &= C2^{\frac{(l-k)n}{q_1'(0)}}. \end{aligned}$$

When $k\ge 0, l<0$, we have

$$\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \approx 2^{\frac{kn}{q_2(\infty)}}, \quad \|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \approx 2^{\frac{kn}{q_1'(0)}}.$$

Since $1/q_2(\cdot)=1/q_1(\cdot)-(\alpha+\beta)/n$ and $1/q_1(\infty)+1/q_1'(\infty)=1$, we have

$$\begin{aligned} 2^{k(\alpha+\beta-n)}\,\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\,\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} &\le C2^{k(\alpha+\beta-n)}\cdot 2^{\frac{kn}{q_2(\infty)}}\cdot 2^{\frac{ln}{q_1'(0)}}\\ &= C2^{k(\alpha+\beta-n)}\cdot 2^{kn(\frac{1}{q_1(\infty)}-\frac{\alpha+\beta}{n})}\cdot 2^{\frac{ln}{q_1'(0)}}\\ &= C2^{k(\alpha+\beta)-kn+\frac{kn}{q_1(\infty)}-k(\alpha+\beta)}\cdot 2^{\frac{ln}{q_1'(0)}}\\ &= C2^{\frac{kn}{q_1(\infty)}-kn}\cdot 2^{\frac{ln}{q_1'(0)}}\\ &= C2^{-kn(1-\frac{1}{q_1(\infty)})}\cdot 2^{\frac{ln}{q_1'(0)}}\\ &= C2^{\frac{-kn}{q_1'(\infty)}}\cdot 2^{\frac{ln}{q_1'(0)}}\\ &= C2^{\frac{-kn}{q_1'(\infty)}+\frac{ln}{q_1'(0)}}. \end{aligned}$$

When $k<0, l\ge 0$, we have

$$\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \approx 2^{\frac{kn}{q_2(0)}}, \quad \|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \approx 2^{\frac{kn}{q_1'(\infty)}}.$$

Since $1/q_2(\cdot)=1/q_1(\cdot)-(\alpha+\beta)/n$ and $1/q_1(\infty)+1/q_1'(\infty)=1$, we have

$$2^{k(\alpha+\beta-n)}\,\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\,\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \le C2^{l(\alpha+\beta-n)}\cdot 2^{\frac{kn}{q_2(0)}}\cdot 2^{\frac{ln}{q_1'(\infty)}}$$

$$\begin{aligned}
&= C2^{l(\alpha+\beta-n)} \cdot 2^{kn(\frac{1}{q_1(0)}-\frac{\alpha+\beta}{n})} \cdot 2^{ln(1-\frac{1}{q_1(\infty)})}\\
&= C2^{l(\alpha+\beta)-ln+\frac{kn}{q_1(0)}-k(\alpha+\beta)+ln-\frac{ln}{q_1(\infty)}}\\
&= C2^{k(\frac{n}{q_1(0)}-\alpha-\beta)-l(\frac{n}{q_1(\infty)}-\alpha-\beta)}.
\end{aligned}$$

When $k \geq 0, l \geq 0$, we have

$$\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \approx 2^{\frac{kn}{q_2(\infty)}}, \quad \|\chi_l\|_{L^{q_1'(\infty)}(\mathbb{R}^n)} \approx 2^{\frac{kn}{q_1'(\infty)}}$$

Since $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha+\beta)/n$ and $1/q_1(\infty) + 1/q_1'(\infty) = 1$, we have

$$\begin{aligned}
2^{k(\alpha+\beta-n)} \|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} &\leq C2^{l(\alpha+\beta-n)} \cdot 2^{\frac{kn}{q_2(\infty)}} \cdot 2^{\frac{ln}{q_1'(\infty)}}\\
&= C2^{l(\alpha+\beta-n)} \cdot 2^{kn(\frac{1}{q_1(\infty)}-\frac{\alpha+\beta}{n})} \cdot 2^{ln(1-\frac{1}{q_1(\infty)})}\\
&= C2^{l(\alpha+\beta)-ln+\frac{kn}{q_1(\infty)}-k(\alpha+\beta)+ln-\frac{ln}{q_1(\infty)}}\\
&= C2^{(k-l)(\frac{n}{q_1(\infty)}-\alpha-\beta)}.
\end{aligned}$$

□

3. Proofs of the principal results

Proof. [Proof of Theorem 1.1] If $f \in K_{q_2(\cdot)}^{\eta(\cdot),u),\theta}\big(\mathbb{R}^n\big)$, we write $f(x) = \sum_{l=-\infty}^{\infty} f(x)\chi_l(x)$. Suppose $k_0 > 0$, since it is similar to non-positive case, for any $k \in \mathbb{Z}$ and $a.e.x \in S_k, t \in B_k$, we have $|x| \approx 2^k$, $|t| \leq 2^k$, then $|x-t| \approx 2^k$. Since $b \in \Lambda_\beta\big(\mathbb{R}^n\big)$, by Lemma 2.7 (3) with exponent $q_1(\cdot)$ and $q_2(\cdot)$, we obtain

$$\begin{aligned}
\Big|[b,\mathcal{H}_\alpha]f(x)\Big| &\leq \frac{1}{|x|^{n-\alpha}} \int_{|t|\leq|x|} \Big|b(x)-b(t)\Big|\Big|f(t)\Big|dt\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \frac{1}{|x|^{n-\alpha}} \int_{B_k} |x-t|^\beta \Big|f(t)\Big|dt\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} 2^{k(\alpha+\beta-n)} \sum_{l=-\infty}^{k} \Big\|f\chi_l\Big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \Big\|\chi_l\Big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}. \qquad (3.1)
\end{aligned}$$

By virtue of Minkowski's inequality, Lemma 2.9 and (3.1), and by using the fact $2^{k\eta(x)} \approx 2^{k\eta(0)}$, $k < 0$, $x \in S_k$ implies that $\Big\|2^{k\eta(\cdot)} f\chi_l\Big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \approx 2^{k\eta(0)}\Big\|f\chi_l\Big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}$ and the fact $2^{k\eta(x)} \approx 2^{k\eta(\infty)}$, $k \geq 0$, $x \in S_k$ implies that $\Big\|2^{k\eta(\cdot)} f\chi_l\Big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \approx 2^{k\eta(\infty)}\Big\|f\chi_l\Big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}$, we obtain

$$\begin{aligned}
\Big\|[b,\mathcal{H}_\alpha]f\Big\|_{K_{q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R}^n)} &= \sup_{\varepsilon>0}\Big(\varepsilon^\theta \sum_{k\in\mathbb{Z}} \Big\|2^{\eta(\cdot)k}[b,\mathcal{H}_\alpha]f\chi_k\Big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\sup_{\varepsilon>0}\Big(\varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{\eta(0)ku(1+\varepsilon)} \Big\|[b,\mathcal{H}_\alpha]f\chi_k\Big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&+ C\sup_{\varepsilon>0}\Big(\varepsilon^\theta \sum_{k=0}^{\infty} 2^{\eta(\infty)ku(1+\varepsilon)} \Big\|[b,\mathcal{H}_\alpha]f\chi_k\Big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\Big(\varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{\eta(0)ku(1+\varepsilon)} \Big(\sum_{l=-\infty}^{k} \Big\|\chi_k\Big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} 2^{k(\alpha+\beta-n)}
\end{aligned}$$

$$
\begin{aligned}
&\times \big\|f\chi_l\big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\big\|\chi_l\big\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\
&+ C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\Bigg(\varepsilon^\theta \sum_{k=0}^{\infty} 2^{\eta(\infty)ku(1+\varepsilon)}\Big(\sum_{l=-\infty}^{k} \big\|\chi_k\big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}2^{k(\alpha+\beta-n)} \\
&\times \big\|f\chi_l\big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\big\|\chi_l\big\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\
&:= E_1 + E_2.
\end{aligned}
$$

For E_1, taking $\gamma = \frac{n}{q_1'(0)} - \eta(0)$, by Lemma 2.9 and the fact $\eta(0) < n/q_1^{'}(0)$, we obtain

$$
\begin{aligned}
E_1 &\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\Bigg(\varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{\eta(0)ku(1+\varepsilon)}\Big(\sum_{l=-\infty}^{k} 2^{\frac{(l-k)n}{q_1'(0)}}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\
&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\Bigg(\varepsilon^\theta \sum_{k=-\infty}^{-1} \Big(\sum_{l=-\infty}^{k} 2^{l\eta(0)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}2^{\gamma(l-k)}\Big)^{u(1+\varepsilon)}\Bigg)^{\frac{1}{u(1+\varepsilon)}}.
\end{aligned}
$$

According to Fubini's theorem, by Hölder's inequality with $\frac{1}{u(1+\varepsilon)} + \frac{1}{(u(1+\varepsilon))'} = 1$, we have

$$
\begin{aligned}
E_1 &\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\Bigg(\varepsilon^\theta \sum_{k=-\infty}^{-1} \Big(\sum_{l=-\infty}^{k} 2^{l\eta(0)u(1+\varepsilon)}\|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}2^{\frac{\gamma(l-k)u(1+\varepsilon)}{2}}\Big) \\
&\quad\times \Big(\sum_{l=-\infty}^{k} 2^{\frac{\gamma(l-k)(u(1+\varepsilon))'}{2}}\Big)^{\frac{u(1+\varepsilon)}{(u(1+\varepsilon))'}}\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\
&= C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\Bigg(\varepsilon^\theta \sum_{k=-\infty}^{-1} \Big(\sum_{l=-\infty}^{k} 2^{l\eta(0)u(1+\varepsilon)}\|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}2^{\frac{\gamma(l-k)u(1+\varepsilon)}{2}}\Big)\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\
&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\Bigg(\varepsilon^\theta \sum_{l=-\infty}^{-1} \Big(2^{l\eta(0)u(1+\varepsilon)}\|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\sum_{k=l}^{-1} 2^{\frac{\gamma(l-k)u(1+\varepsilon)}{2}}\Big)\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\
&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\Bigg(\varepsilon^\theta \sum_{l=-\infty}^{-1} 2^{l\eta(0)u(1+\varepsilon)}\|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\
&\approx C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\Bigg(\varepsilon^\theta \sum_{l\in\mathbb{Z}} 2^{l\eta(\cdot)u(1+\varepsilon)}\|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\
&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\Bigg(\varepsilon^\theta \sum_{l\in\mathbb{Z}} \|2^{l\eta(\cdot)}f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\
&= C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\|f\|_{K^{\eta(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{R}^n)},
\end{aligned}
$$

where $\gamma > 0$, $l \le k$ and $l - k \le 0$, then $\sum_{l=-\infty}^{k} 2^{\frac{\gamma(l-k)(u(1+\varepsilon))'}{2}}$ and $\sum_{k=l}^{-1} 2^{\frac{\gamma(l-k)u(1+\varepsilon)}{2}}$ converge.

Next, for the sake of estimating E_2, by using Minkowski's inequality, we get

$$
\begin{aligned}
E_2 &= C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\Bigg(\varepsilon^\theta \sum_{k=0}^{\infty} 2^{\eta(\infty)ku(1+\varepsilon)}\Big(\sum_{l=-\infty}^{k} \|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}2^{k(\alpha+\beta-n)} \\
&\quad\times \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Bigg)^{\frac{1}{u(1+\varepsilon)}}
\end{aligned}
$$

$$
\begin{aligned}
&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=0}^{\infty}2^{\eta(\infty)ku(1+\varepsilon)}\Big(\sum_{l=-\infty}^{-1}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}2^{k(\alpha+\beta-n)}\\
&\times\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&+C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=0}^{\infty}2^{\eta(\infty)ku(1+\varepsilon)}\Big(\sum_{l=0}^{k}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}2^{k(\alpha+\beta-n)}\\
&\times\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&=: E_{21}+E_{22}.
\end{aligned}
$$

We need to estimate E_{21} and E_{22}. Indeed, the methods of E_{22} and E_1 are similar, and it is only necessary to replace $q_1'(0)$ with $q_1'(\infty)$.

For the estimation of E_{21}, According to Fubini's theorem, the fact $\eta(0) < n/q_1'(0)$ and $\eta(\infty) < n/q_1'(\infty)$, by Lemma 2.9 and Hölder's inequality with $\frac{1}{u(1+\varepsilon)}+\frac{1}{(u(1+\varepsilon))'}=1$, we have

$$
\begin{aligned}
E_{21}&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=0}^{\infty}2^{k\eta(\infty)u(1+\varepsilon)}\Big(\sum_{l=-\infty}^{-1}2^{\frac{-kn}{q_1(\infty)}+\frac{ln}{q_1(0)}}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=0}^{\infty}\Big(\sum_{l=-\infty}^{-1}2^{l\eta(0)}2^{\frac{-kn}{q_1'(\infty)}+k\eta(\infty)}2^{\frac{ln}{q_1'(0)}-l\eta(0)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=0}^{\infty}\Big(\sum_{l=-\infty}^{-1}2^{l\eta(0)u(1+\varepsilon)}\Big(2^{\frac{-kn}{q_1'(\infty)}+k\eta(\infty)}2^{\frac{ln}{q_1'(0)}-l\eta(0)}\Big)^{\frac{u(1+\varepsilon)}{2}}\Big)\\
&\times\Big\|f\chi_l\Big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big(\sum_{l=-\infty}^{-1}\Big(2^{\frac{-kn}{q_1'(\infty)}+k\eta(\infty)}2^{\frac{ln}{q_1'(0)}-l\eta(0)}\Big)^{\frac{(u(1+\varepsilon))'}{2}}\Big)^{\frac{u(1+\varepsilon)}{(u(1+\varepsilon))'}}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=0}^{\infty}\sum_{l=-\infty}^{-1}2^{l\eta(0)u(1+\varepsilon)}\Big(2^{\frac{-kn}{q_1'(\infty)}+k\eta(\infty)}2^{\frac{ln}{q_1'(0)}-l\eta(0)}\Big)^{\frac{u(1+\varepsilon)}{2}}\Big\|f\chi_l\Big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{l=-\infty}^{-1}2^{l\eta(0)u(1+\varepsilon)}\Big\|f\chi_l\Big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\sum_{k=0}^{\infty}\Big(2^{\frac{-kn}{q_1'(\infty)}+k\eta(\infty)}2^{\frac{ln}{q_1'(0)}-l\eta(0)}\Big)^{\frac{u(1+\varepsilon)}{2}}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{l=-\infty}^{-1}2^{l\eta(0)u(1+\varepsilon)}\Big\|f\chi_l\Big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\approx C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{l\in\mathbb{Z}}2^{l\eta(\cdot)u(1+\varepsilon)}\Big\|f\chi_l\Big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{l\in\mathbb{Z}}\Big\|2^{l\eta(\cdot)}f\chi_l\Big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&= C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\|f\|_{K^{\eta(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{R}^n)},
\end{aligned}
$$

where $k>0, l<0, \frac{-n}{q_1'(\infty)}+\eta(\infty)<0$ and $\frac{n}{q_1'(0)}-\eta(0)>0$, then $\sum_{k=0}^{\infty}\Big(2^{\frac{-kn}{q_1'(\infty)}+k\eta(\infty)}2^{\frac{ln}{q_1'(0)}-l\eta(0)}\Big)^{\frac{u(1+\varepsilon)}{2}}$ converges and $\sum_{l=-\infty}^{-1}\Big(2^{\frac{-kn}{q_1'(\infty)}+k\eta(\infty)}2^{\frac{ln}{q_1'(0)}-l\eta(0)}\Big)^{\frac{(u(1+\varepsilon))'}{2}}$ converges.

For the estimation of E_{22}, applying Lemmas 2.9, the fact $\eta(\infty) < n/q_1'(\infty)$ and Hölder's inequality with

$\frac{1}{u(1+\varepsilon)}+\frac{1}{(u(1+\varepsilon))'}=1$, we have

$$\begin{aligned}
E_{22} \leq{}& C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Bigg(\varepsilon^\theta\sum_{k=0}^{\infty}2^{k\eta(\infty)u(1+\varepsilon)}\Bigg(\sum_{l=0}^{k}2^{k(\alpha+\beta-n)}2^{-l(\alpha+\beta-n)}2^{(k-l)(\frac{n}{q_1(\infty)}-\alpha-\beta)}\\
&\times\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Bigg)^{u(1+\varepsilon)}\Bigg)^{\frac{1}{u(1+\varepsilon)}}\\
={}& C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Bigg(\varepsilon^\theta\sum_{k=0}^{\infty}2^{k\eta(\infty)u(1+\varepsilon)}\Bigg(\sum_{l=0}^{k}2^{(k-l)(\frac{n}{q_1(\infty)}-n)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Bigg)^{u(1+\varepsilon)}\Bigg)^{\frac{1}{u(1+\varepsilon)}}\\
\leq{}& C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Bigg(\varepsilon^\theta\sum_{k=0}^{\infty}2^{k\eta(\infty)u(1+\varepsilon)}\Bigg(\sum_{l=0}^{k}2^{(k-l)(-\frac{n}{q_1'(\infty)})}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Bigg)^{u(1+\varepsilon)}\Bigg)^{\frac{1}{u(1+\varepsilon)}}.
\end{aligned}$$

Taking $\omega=\frac{n}{q_1'(\infty)}-\eta(\infty)$, we obtain

$$\begin{aligned}
E_{22} \leq{}& C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Bigg(\varepsilon^\theta\sum_{k=0}^{\infty}\Bigg(\sum_{l=0}^{k}2^{k\eta(\infty)}2^{(l-k)\frac{n}{q_1'(\infty)}}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Bigg)^{u(1+\varepsilon)}\Bigg)^{\frac{1}{u(1+\varepsilon)}}\\
\leq{}& C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Bigg(\varepsilon^\theta\sum_{k=0}^{\infty}\Bigg(\sum_{l=0}^{k}2^{l\eta(\infty)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}2^{(l-k)\omega}\Bigg)^{u(1+\varepsilon)}\Bigg)^{\frac{1}{u(1+\varepsilon)}}\\
\leq{}& C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Bigg(\varepsilon^\theta\sum_{k=0}^{\infty}\Bigg(\sum_{l=0}^{k}2^{l\eta(\infty)u(1+\varepsilon)}\Big\|f\chi_l\Big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}2^{\frac{\omega(l-k)u(1+\varepsilon)}{2}}\Bigg)\\
&\times\Bigg(\sum_{l=0}^{k}2^{\frac{\omega(l-k)(u(1+\varepsilon))'}{2}}\Bigg)^{\frac{u(1+\varepsilon)}{(u(1+\varepsilon))'}}\Bigg)^{\frac{1}{u(1+\varepsilon)}}\\
\leq{}& C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Bigg(\varepsilon^\theta\sum_{k=0}^{\infty}\Bigg(\sum_{l=0}^{k}2^{l\eta(\infty)u(1+\varepsilon)}\Big\|f\chi_l\Big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}2^{\frac{\omega(l-k)u(1+\varepsilon)}{2}}\Bigg)\Bigg)^{\frac{1}{u(1+\varepsilon)}}\\
\leq{}& C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Bigg(\varepsilon^\theta\sum_{l=0}^{\infty}2^{l\eta(\infty)u(1+\varepsilon)}\Big\|f\chi_l\Big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}\sum_{l=k}^{\infty}2^{\frac{\omega(l-k)u(1+\varepsilon)}{2}}\Bigg)^{\frac{1}{u(1+\varepsilon)}}\\
\leq{}& C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Bigg(\varepsilon^\theta\sum_{l=0}^{\infty}2^{l\eta(\infty)u(1+\varepsilon)}\Big\|f\chi_l\Big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}\Bigg)^{\frac{1}{u(1+\varepsilon)}}\\
\approx{}& C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Bigg(\varepsilon^\theta\sum_{l\in\mathbb{Z}}2^{l\eta(\cdot)u(1+\varepsilon)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}\Bigg)^{\frac{1}{u(1+\varepsilon)}}\\
\leq{}& C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Bigg(\varepsilon^\theta\sum_{l\in\mathbb{Z}}\|2^{l\eta(\cdot)}f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}\Bigg)^{\frac{1}{u(1+\varepsilon)}}\\
={}& C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\|f\|_{K_{q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R}^n)}
\end{aligned}$$

where $\omega>0, l-k\leq 0$, then $\sum_{l=0}^{k}2^{\frac{\omega(l-k)(u(1+\varepsilon))'}{2}}$ and $\sum_{l=k}^{\infty}2^{\frac{\omega(l-k)u(1+\varepsilon)}{2}}$ converge.

Combining E_{21} and E_{22}, we get the estimation of E_2, namely

$$E_2\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\|f\|_{K_{q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R}^n)}.$$

In summary, combining the estimation of E_1, E_2, we can obtain

$$\Big\|[b,\mathcal{H}_\alpha]f\Big\|_{K_{q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R}^n)}\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\|f\|_{K_{q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R}^n)},$$

which implies the proof of Theorem 1.1. □

Proof. [Proof of Theorem 1.3] If $f \in K_{q_2(\cdot)}^{\eta(\cdot),u),\theta}\big(\mathbb{R}^n\big)$, similarly to Theorem 1.1, we write $f(x) = \sum_{l=-\infty}^{\infty} f(x)\chi_l(x)$. Suppose $k_0 > 0$, since it is similar to non-positive case. For any $k \in \mathbb{Z}$ and $a.e. x \in S_k$, $t \in \mathbb{R}^n \setminus B_k$, $l = k+1$, we have $|x| \approx 2^k \approx 2^{l-1}$, $|t| \approx 2^l$, then $|x-t| \leq |x| + |t| \leq 2^l$. Since $b \in \Lambda_\beta\big(\mathbb{R}^n\big)$, by Lemma 2.7 (3) with exponent $q_1(\cdot)$ and $q_2(\cdot)$, we obtain

$$\begin{aligned}
\Big|[b, \mathcal{H}_\alpha^*]f(x)\Big| &\leq \int_{|t|>|x|} \frac{\big|f(t)\big|}{\big|t\big|^{n-\alpha}} \big|b(x) - b(t)\big| dt \\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus B_k} \big|x - t\big|^\beta \frac{\big|f(t)\big|}{\big|t\big|^{n-\alpha}} dt \\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sum_{l=k+1}^{\infty} 2^{l(\alpha+\beta-n)} \big\|f\chi_l\big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \big\|\chi_l\big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}. \qquad (3.2)
\end{aligned}$$

Where we make the use of the fact

$$|t - x| \approx |t|,$$

otherwise, $|t| = 2^k$ and the condition $t \in \mathbb{R}^n \setminus B_k$ are contradictory.

By virtue of Minkowski's inequality, Lemma 2.9 and (3.2), and by using the fact $2^{k\eta(x)} \approx 2^{k\eta(0)}$, $k < 0$, $x \in S_k$ implies that $\big\|2^{k\eta(\cdot)} f\chi_l\big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \approx 2^{k\eta(0)} \big\|f\chi_l\big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}$ and the fact $2^{k\eta(x)} \approx 2^{k\eta(\infty)}$, $k \geq 0$, $x \in S_k$ implies that $\big\|2^{k\eta(\cdot)} f\chi_l\big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \approx 2^{k\eta(\infty)} \big\|f\chi_l\big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}$, we obtain

$$\begin{aligned}
\Big\|[b, \mathcal{H}_\alpha^*]f\Big\|_{K_{q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R}^n)} &= \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k=-\infty}^{\infty} \Big\|2^{\eta(\cdot)k} [b, \mathcal{H}_\alpha^*] f\chi_k\Big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}} \\
&\leq C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{\eta(0)ku(1+\varepsilon)} \Big\|[b, \mathcal{H}_\alpha^*] f\chi_k\Big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}} \\
&+ C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k=0}^{\infty} 2^{\eta(\infty)ku(1+\varepsilon)} \Big\|[b, \mathcal{H}_\alpha^*] f\chi_k\Big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}} \\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{\eta(0)ku(1+\varepsilon)} \Big(\sum_{l=k+1}^{\infty} \big\|\chi_k\big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} 2^{l(\alpha+\beta-n)} \\
&\times \big\|f\chi_l\big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \big\|\chi_l\big\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}} \\
&+ C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k=0}^{\infty} 2^{\eta(\infty)ku(1+\varepsilon)} \Big(\sum_{l=k+1}^{\infty} \big\|\chi_k\big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} 2^{l(\alpha+\beta-n)} \\
&\times \big\|f\chi_l\big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \big\|\chi_l\big\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}} \\
&:= E_1 + E_2.
\end{aligned}$$

For the estimation of E_2, assuming $\mu = \frac{n}{q_1(\infty)} + \eta(\infty) - \alpha - \beta$, by Lemma 2.9 and the fact $\alpha + \beta - n/q_1(\infty) < \eta(\infty)$, we obtain

$$E_2 \leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k=0}^{\infty} 2^{\eta(\infty)ku(1+\varepsilon)} \Big(\sum_{l=k+1}^{\infty} 2^{(k-l)(\frac{n}{q_1(\infty)} - \alpha - \beta)} \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}$$

$$\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=0}^{\infty}\Big(\sum_{l=k+1}^{\infty}2^{l\eta(\infty)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}2^{\mu(k-l)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}.$$

According to Fubini's theorem, by Hölder's inequality with $\frac{1}{u(1+\varepsilon)}+\frac{1}{(u(1+\varepsilon))'}=1$, we obtain

$$\begin{aligned}
E_2 &\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=0}^{\infty}\Big(\sum_{l=k+1}^{\infty}2^{l\eta(\infty)u(1+\varepsilon)}\|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}2^{\frac{\mu(k-l)u(1+\varepsilon)}{2}}\Big)\\
&\quad\times\Big(\sum_{l=k+1}^{\infty}2^{\frac{\mu(k-l)(u(1+\varepsilon))'}{2}}\Big)^{\frac{u(1+\varepsilon)}{(u(1+\varepsilon))'}}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&= C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=0}^{\infty}\Big(\sum_{l=k+1}^{\infty}2^{l\eta(\infty)u(1+\varepsilon)}\|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}2^{\frac{\mu(k-l)u(1+\varepsilon)}{2}}\Big)\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{l=0}^{\infty}\Big(2^{l\eta(\infty)u(1+\varepsilon)}\|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\sum_{k=0}^{l-1}2^{\frac{\mu(k-l)u(1+\varepsilon)}{2}}\Big)\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{l=0}^{\infty}2^{l\eta(\infty)u(1+\varepsilon)}\|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\approx C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{l\in\mathbb{Z}}2^{l\eta(\cdot)u(1+\varepsilon)}\|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{l\in\mathbb{Z}}\|2^{l\eta(\cdot)}f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&= C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\|f\|_{K^{\eta(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{R}^n)},
\end{aligned}$$

where $\mu>0$ and $k-l<0$, then $\sum_{l=k+1}^{\infty}2^{\frac{\mu(k-l)(u(1+\varepsilon))'}{2}}$ and $\sum_{k=0}^{l-1}2^{\frac{\mu(k-l)u(1+\varepsilon)}{2}}$ converge.

Next, we estimate E_1, by using Lemma 2.9 and Minkowski's inequality, we get

$$\begin{aligned}
E_1 &= C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=-\infty}^{-1}2^{\eta(0)ku(1+\varepsilon)}\Big(\sum_{l=k+1}^{\infty}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}2^{l(\alpha+\beta-n)}\\
&\quad\times\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=-\infty}^{-2}2^{\eta(0)ku(1+\varepsilon)}\Big(\sum_{l=k+1}^{-1}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}2^{l(\alpha+\beta-n)}\\
&\quad\times\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\quad+C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=-\infty}^{-2}2^{\eta(0)ku(1+\varepsilon)}\Big(\sum_{l=0}^{\infty}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}2^{l(\alpha+\beta-n)}\\
&\quad\times\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\quad+C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=-1}^{-1}2^{\eta(0)ku(1+\varepsilon)}\Big(\sum_{l=k+1}^{\infty}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}2^{l(\alpha+\beta-n)}\\
&\quad\times\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&=: E_{11}+E_{12}+E_{13}.
\end{aligned}$$

Indeed, the estimation methods of E_{11} and E_2 are similar, and it is only necessary to replace $q_1(\infty)$ with $q_1(0)$. According to Fubini's theorem, the fact $\alpha+\beta-n/q_1(0)<\eta(0)<n/q_1'(0)$, by Lemma 2.9 and Hölder's inequality with $\frac{1}{u(1+\varepsilon)}+\frac{1}{(u(1+\varepsilon))'}=1$, we obtain

$$
\begin{aligned}
E_{11} &\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\bigg(\varepsilon^\theta \sum_{k=-\infty}^{-2} 2^{k\eta(0)u(1+\varepsilon)}\bigg(\sum_{l=k+1}^{-1} 2^{l(\alpha+\beta-n)}2^{-k(\alpha+\beta-n)}2^{\frac{(l-k)n}{q_1'(0)}} \\
&\quad \times \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\bigg)^{u(1+\varepsilon)}\bigg)^{\frac{1}{u(1+\varepsilon)}} \\
&= C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\bigg(\varepsilon^\theta \sum_{k=-\infty}^{-2} 2^{k\eta(0)u(1+\varepsilon)}\bigg(\sum_{l=k+1}^{-1} 2^{(l-k)(\frac{n}{q_1'(0)}+\alpha+\beta-n)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\bigg)^{u(1+\varepsilon)}\bigg)^{\frac{1}{u(1+\varepsilon)}} \\
&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\bigg(\varepsilon^\theta \sum_{k=-\infty}^{-2} 2^{k\eta(0)u(1+\varepsilon)}\bigg(\sum_{l=k+1}^{-1} 2^{(k-l)(\frac{n}{q_1(0)}-\alpha-\beta)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\bigg)^{u(1+\varepsilon)}\bigg)^{\frac{1}{u(1+\varepsilon)}} \\
&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\bigg(\varepsilon^\theta \sum_{k=-\infty}^{-2} \bigg(\sum_{l=k+1}^{-1} 2^{k\eta(0)}2^{(k-l)(\frac{n}{q_1(0)}-\alpha-\beta)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\bigg)^{u(1+\varepsilon)}\bigg)^{\frac{1}{u(1+\varepsilon)}}.
\end{aligned}
$$

Taking $\rho=\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta$, we have

$$
\begin{aligned}
E_{11} &\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\bigg(\varepsilon^\theta \sum_{k=-\infty}^{-2} \bigg(\sum_{l=k+1}^{-1} 2^{l\eta(0)}2^{(k-l)(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\bigg)^{u(1+\varepsilon)}\bigg)^{\frac{1}{u(1+\varepsilon)}} \\
&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\bigg(\varepsilon^\theta \sum_{k=-\infty}^{-2} \bigg(\sum_{l=k+1}^{-1} 2^{l\eta(0)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}2^{(k-l)\rho}\bigg)^{u(1+\varepsilon)}\bigg)^{\frac{1}{u(1+\varepsilon)}} \\
&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\bigg(\varepsilon^\theta \sum_{k=-\infty}^{-2} \bigg(\sum_{l=k+1}^{-1} 2^{l\eta(0)u(1+\varepsilon)}\|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}2^{\frac{\rho(k-l)u(1+\varepsilon)}{2}}\bigg) \\
&\quad \times \bigg(\sum_{l=k+1}^{-1} 2^{\frac{\rho(k-l)(u(1+\varepsilon))'}{2}}\bigg)^{\frac{u(1+\varepsilon)}{(u(1+\varepsilon))'}}\bigg)^{\frac{1}{u(1+\varepsilon)}} \\
&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\bigg(\varepsilon^\theta \sum_{k=-\infty}^{-2} \bigg(\sum_{l=k+1}^{-1} 2^{l\eta(0)u(1+\varepsilon)}\|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}2^{\frac{\rho(k-l)u(1+\varepsilon)}{2}}\bigg)\bigg)^{\frac{1}{u(1+\varepsilon)}} \\
&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\bigg(\varepsilon^\theta \sum_{l=-\infty}^{-1} \bigg(2^{l\eta(0)u(1+\varepsilon)}\|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\sum_{k=-\infty}^{l-1}2^{\frac{\rho(k-l)u(1+\varepsilon)}{2}}\bigg)\bigg)^{\frac{1}{u(1+\varepsilon)}} \\
&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\bigg(\varepsilon^\theta \sum_{l=-\infty}^{-1} 2^{l\eta(0)u(1+\varepsilon)}\|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\bigg)^{\frac{1}{u(1+\varepsilon)}} \\
&\approx C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\bigg(\varepsilon^\theta \sum_{l\in\mathbb{Z}} 2^{l\eta(\cdot)u(1+\varepsilon)}\|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\bigg)^{\frac{1}{u(1+\varepsilon)}} \\
&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\bigg(\varepsilon^\theta \sum_{l\in\mathbb{Z}} \|2^{l\eta(\cdot)}f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\bigg)^{\frac{1}{u(1+\varepsilon)}} \\
&= C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\|f\|_{K^{\eta(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{R}^n)},
\end{aligned}
$$

where $\rho>0$ and $k-l<0$, then $\sum_{l=k+1}^{-1}2^{\frac{\rho(k-l)(u(1+\varepsilon))'}{2}}$ and $\sum_{k=\infty}^{l-1}2^{\frac{\rho(k-l)u(1+\varepsilon)}{2}}$ converge.

Moreover, the estimation of E_{12} and E_{13} is totally comparable. For the estimation of E_{12}, according to Fubini's theorem, the fact $\alpha+\beta-n/q_1(0)<\eta(0)$ and $\alpha+\beta-n/q_1(\infty)<\eta(\infty)$, by Lemma 2.9 and Hölder's inequality with $\frac{1}{u(1+\varepsilon)}+\frac{1}{(u(1+\varepsilon))'}=1$, we have

$$
\begin{aligned}
E_{12} &\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\bigg(\varepsilon^\theta\sum_{k=-\infty}^{-2}2^{k\eta(0)u(1+\varepsilon)}\bigg(\sum_{l=0}^{\infty}2^{k(\frac{n}{q_1(0)}-\alpha-\beta)-l(\frac{n}{q_1(\infty)}-\alpha-\beta)}\\
&\quad\times\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\bigg)^{u(1+\varepsilon)}\bigg)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\bigg(\varepsilon^\theta\sum_{k=-\infty}^{-2}\bigg(\sum_{l=0}^{\infty}2^{l\eta(\infty)}2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\\
&\quad\times\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\bigg)^{u(1+\varepsilon)}\bigg)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\bigg(\varepsilon^\theta\sum_{k=-\infty}^{-2}\bigg(\sum_{l=0}^{\infty}2^{l\eta(\infty)u(1+\varepsilon)}\bigg(2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\bigg)^{\frac{u(1+\varepsilon)}{2}}\bigg)\\
&\quad\times\Big\|f\chi_l\Big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\bigg(\sum_{l=0}^{\infty}\bigg(2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\bigg)^{\frac{(u(1+\varepsilon))'}{2}}\bigg)^{\frac{u(1+\varepsilon)}{(u(1+\varepsilon))'}}\bigg)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\bigg(\varepsilon^\theta\sum_{k=-\infty}^{-2}\bigg(\sum_{l=0}^{\infty}2^{l\eta(\infty)u(1+\varepsilon)}\Big\|f\chi_l\Big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\bigg(2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)}\\
&\quad\times 2^{-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\bigg)^{\frac{u(1+\varepsilon)}{2}}\bigg)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\bigg(\varepsilon^\theta\sum_{l=0}^{\infty}2^{l\eta(\infty)u(1+\varepsilon)}\Big\|f\chi_l\Big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\sum_{k=-\infty}^{-2}\bigg(2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)}\\
&\quad\times 2^{-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\bigg)^{\frac{u(1+\varepsilon)}{2}}\bigg)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\bigg(\varepsilon^\theta\sum_{l=0}^{\infty}2^{l\eta(\infty)u(1+\varepsilon)}\Big\|f\chi_l\Big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\bigg)^{\frac{1}{u(1+\varepsilon)}}\\
&\approx C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\bigg(\varepsilon^\theta\sum_{l\in\mathbb{Z}}2^{l\eta(\cdot)u(1+\varepsilon)}\|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\bigg)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\bigg(\varepsilon^\theta\sum_{l\in\mathbb{Z}}\|2^{l\eta(\cdot)}f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\bigg)^{\frac{1}{u(1+\varepsilon)}}\\
&= C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\|f\|_{K^{\eta(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{R}^n)},
\end{aligned}
$$

where $k<0, l>0$, $\frac{n}{q_1'(0)}+\eta(0)-\alpha-\beta>0$, then $\sum_{l=0}^{\infty}\Big(2^{k(\frac{n}{q_1'(0)}+\eta(0)-\alpha-\beta)-l(\frac{n}{q_1'(\infty)}+\eta(\infty)-\alpha-\beta)}\Big)^{\frac{(u(1+\varepsilon))'}{2}}$ converge. Since $\frac{n}{q_1'(\infty)}+\eta(\infty)-\alpha-\beta>0$, then $\sum_{k=-\infty}^{-2}\Big(2^{k(\frac{n}{q_1'(0)}+\eta(0)-\alpha-\beta)-l(\frac{n}{q_1'(\infty)}+\eta(\infty)-\alpha-\beta)}\Big)^{\frac{u(1+\varepsilon)}{2}}$ converge.

For the eatimation of E_{13}, according to Fubini's theorem, the fact $\alpha+\beta-n/q_1(0)<\eta(0)$ and $\alpha+\beta-n/q_1(\infty)<\eta(\infty)$, by Lemma 2.9 and Hölder's inequality with $\frac{1}{u(1+\varepsilon)}+\frac{1}{(u(1+\varepsilon))'}=1$, we have

$$
E_{13}\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\bigg(\varepsilon^\theta\sum_{k=-1}^{-1}2^{k\eta(0)u(1+\varepsilon)}\bigg(\sum_{l=k+1}^{\infty}2^{k(\frac{n}{q_1(0)}-\alpha-\beta)-l(\frac{n}{q_1(\infty)}-\alpha-\beta)}
$$

$$
\begin{aligned}
&\times \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^{\theta}\sum_{k=-1}^{-1}\Big(\sum_{l=k+1}^{\infty}2^{l\eta(\infty)}2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\\
&\times \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^{\theta}\sum_{k=-1}^{-1}\Big(\sum_{l=k+1}^{\infty}2^{l\eta(\infty)u(1+\varepsilon)}\Big(2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\Big)^{\frac{u(1+\varepsilon)}{2}}\Big)\\
&\times \Big\|f\chi_l\Big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big(\sum_{l=k+1}^{\infty}\Big(2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\Big)^{\frac{(u(1+\varepsilon))'}{2}}\Big)^{\frac{u(1+\varepsilon)}{(u(1+\varepsilon))'}}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^{\theta}\sum_{k=-1}^{-1}\Big(\sum_{l=k+1}^{\infty}2^{l\eta(\infty)u(1+\varepsilon)}\Big\|f\chi_l\Big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big(2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)}\\
&\times 2^{-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\Big)^{\frac{u(1+\varepsilon)}{2}}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^{\theta}\sum_{l=0}^{\infty}2^{l\eta(\infty)u(1+\varepsilon)}\Big\|f\chi_l\Big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\sum_{k=-1}^{-1}\Big(2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)}\\
&\times 2^{-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\Big)^{\frac{u(1+\varepsilon)}{2}}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^{\theta}\sum_{l=0}^{\infty}2^{l\eta(\infty)u(1+\varepsilon)}\Big\|f\chi_l\Big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\approx C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^{\theta}\sum_{l\in\mathbb{Z}}2^{l\eta(\cdot)u(1+\varepsilon)}\|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\Big(\varepsilon^{\theta}\sum_{l\in\mathbb{Z}}\|2^{l\eta(\cdot)}f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&= C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\|f\|_{K^{\eta(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{R}^n)},
\end{aligned}
$$

where $k<0, l>0, \frac{n}{q_1(0)}+\eta(0)-\alpha-\beta>0$, then $\sum_{l=k+1}^{\infty}\Big(2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\Big)^{\frac{(u(1+\varepsilon))'}{2}}$ converge. Since $\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta>0$, then $\sum_{k=-1}^{-1}\Big(2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\Big)^{\frac{u(1+\varepsilon)}{2}}$ converge.

Combining E_{11}, E_{12} and E_{13}, we get the estimation of E_1,

$$E_1 \leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\|f\|_{K^{\eta(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.$$

In summary, combining the estimation of E_1, E_2, we can obtain

$$\Big\|[b,\mathcal{H}^*_\alpha]f\Big\|_{K^{\eta(\cdot),u),\theta}_{q_2(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\|f\|_{K^{\eta(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{R}^n)},$$

which implies the proof of Theorem 1.3. □

Proof. [Proof of Theorem 1.5] For $x \in R$, by triangle inequality, we have

$$\max\{|x|,|t|\} = \frac{1}{2}\Big(|x|+|t|+\Big||x|-|t|\Big|\Big).$$

When $|t| \leq |x|$, one hand, we have

$$\begin{aligned}\frac{1}{2}\Big(|x|+|t|+\big||x|-|t|\big|\Big) &\leq \frac{1}{2}\Big(|x|+|t|+|x|+|t|\Big)\\ &= |x|+|t|\\ &\leq 2|x|,\end{aligned}$$

the other hand, we have $|x| \leq \max\{|x|,|t|\}$, then $\max\{|x|,|t|\} \approx |x|$.

When $|x| < |t|$, one hand, we have

$$\begin{aligned}\frac{1}{2}\Big(|x|+|t|+\big||x|-|t|\big|\Big) &\leq \frac{1}{2}\Big(|x|+|t|+|t|+|x|\Big)\\ &= |x|+|t|\\ &< 2|t|,\end{aligned}$$

the other hand, we have $|t| \leq \max\{|x|,|t|\}$, then $\max\{|x|,|t|\} \approx |t|$.

In summary, we obtain

$$\frac{1}{\max\{|x|,|t|\}} \approx \begin{cases}\frac{1}{|x|}, & |t| \leq |x|,\\ \frac{1}{|t|}, & |t| > |x|.\end{cases}$$

By Definition 2.4, if $b \in \Lambda_\beta(\mathbb{R})$, we can obtian

$$\begin{aligned}|[b,T]f(x)| &= \Big|b(x)Tf(x) - T(bf)(x)\Big|\\ &\leq \left|b(x)\int_{\mathbb{R}\setminus\{0\}} \frac{f(t)}{\max\{|x|,|t|\}}dt - \int_{\mathbb{R}\setminus\{0\}} \frac{b(t)f(t)}{\max\{|x|,|t|\}}dt\right|\\ &\leq \left|\int_{\mathbb{R}\setminus\{0\}} \frac{f(t)}{\max\{|x|,|t|\}}\Big(b(x)-b(t)\Big)dt\right|\\ &\leq \left|\frac{1}{|x|}\int_{|t|\leq|x|}\Big(b(x)-b(t)\Big)f(t)dt\right| + \left|\int_{|t|>|x|}\frac{f(t)}{|t|}\Big(b(x)-b(t)\Big)dt\right|\\ &= \Big|\mathcal{H}_b f(x)\Big| + \Big|\mathcal{H}_b^* f(x)\Big|. \end{aligned} \tag{3.3}$$

By virtue of Minkowski's inequality and (3.3), we obtain

$$\Big\|[b,T]f\Big\|_{K_{q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R})} \leq C\Big\|\mathcal{H}_b f\Big\|_{K_{q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R})} + C\Big\|\mathcal{H}_b^* f\Big\|_{K_{q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R})}.$$

It follows from Corollaries 1.2 and 1.4 ($n = 1$), then the proof of Theorem 1.5 is finished. □

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