



Second order determinants for three-term recurrence polynomial sequences

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Abstract. In this paper, we present some sufficient conditions for the sign of second order determinants for Hankel matrices of three-term recurrence polynomial sequences. As applications, we confirm two conjectures of Z.-H. Sun on the reverse Turán expression for two polynomial sequences $G_n(x)$ and $V_n(x)$. In addition, we show that all the zeros of $G_n(x)$ and $V_n(x)$ lie on the line $\Re(x) = -1/2$.

1. Introduction

A classical result of Turán states that the Legendre polynomials $P = (P_n(x))_{n \geq 0}$ satisfy

$$P_{n-1}(x)P_{n+1}(x) - P_n(x)^2 \leq 0, \quad -1 \leq x \leq 1, \quad (1.1)$$

with equality only for $x = \pm 1$ [29]. Szegő [26] provided four different proofs of (1.1), and established similar inequalities for the ultraspherical polynomials, the generalized Laguerre polynomials, and the Hermite polynomials. There have been extensive and remarkable refinements of the Turán inequalities (1.1) for classical orthogonal polynomials and special functions, see [1, 6, 13, 14, 22] and the references therein. Gasper [11] showed that

$$R_{n-1}(x)R_{n+1}(x) - R_n(x)^2 \leq 0, \quad -1 \leq x \leq 1$$

if and only if $\beta \geq \alpha > -1$, where $R_n(x) = P_n^{(\alpha, \beta)}(x)/P_n^{(\alpha, \beta)}(1)$ and $P_n^{(\alpha, \beta)}(x)$ is the Jacobi polynomial of order (α, β) . This result contains the special case of the Turán type inequalities for the ultraspherical polynomials ($\alpha = \beta$), and also contains the case of (1.1) ($\alpha = \beta = 0$).

Given a polynomial sequence $(f_n(x))_{n \geq 0}$, its m th Hankel matrix is given by

$$[f_{i+j}(x)]_{0 \leq i, j \leq m} = \begin{bmatrix} f_0(x) & f_1(x) & \cdots & f_m(x) \\ f_1(x) & f_2(x) & \cdots & f_{m+1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_m(x) & f_{m+1}(x) & \cdots & f_{2m}(x) \end{bmatrix}.$$

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Let $f = (f_n(x))_{n \geq 0}$ be a polynomial sequence and $k, h \in \mathbb{N}^+$. Denote the second order determinant for the Hankel matrix of f by

$$\mathcal{T}(n, h, k; f) := \begin{vmatrix} f_n(x) & f_{n+h}(x) \\ f_{n+k}(x) & f_{n+k+h}(x) \end{vmatrix} = f_n(x)f_{n+k+h}(x) - f_{n+h}(x)f_{n+k}(x)$$

and denote the n th reverse Turán expression of f by

$$\mathcal{T}_n(f) := \mathcal{T}(n - 1, 1, 1; f) = f_{n-1}(x)f_{n+1}(x) - f_n^2(x), \quad n = 1, 2, \dots$$

Several results were obtained as the generalization of the Turán type inequalities of $\mathcal{T}_n(f)$ to the inequalities of the second order determinant $\mathcal{T}(n, h, k; f)$. Forsythe [9] showed that the Legendre polynomials satisfy

$$\mathcal{T}(n, 1, 2; P) < 0 \quad \text{and} \quad \mathcal{T}(2n + 1, 2, 2; P) < 0$$

in the interval $[0, 1]$. For the ultraspherical polynomials $U_n^{(\lambda)}(x)$, let $u_n^{(\lambda)}(x) = U_n^{(\lambda)}(x)/U_n^{(\lambda)}(1)$. Gasper [10] showed that

$$\mathcal{T}(2n - 1, 2, 2; u^\lambda) \geq 0, \quad -1 \leq x \leq 1, -\frac{1}{2} < \lambda \leq 1$$

and

$$\mathcal{T}(2n - 2, 2, 2; u^\lambda) \geq 0, \quad -1 \leq x \leq 1, -\frac{1}{2} < \lambda \leq 0.$$

Let $f = (f_n(x))_{n \geq 0}$ be a polynomial sequence. The polynomial sequence is called *x-log-concave* (*x-log-convex*, resp.) if all coefficients of $\mathcal{T}_n(f)$ ($-\mathcal{T}_n(f)$, resp.) are nonpositive. The polynomial sequence is called *strongly x-log-concave* (*strongly x-log-convex*, resp.) if all coefficients of $\mathcal{T}(n, 1, k; f)$ ($-\mathcal{T}(n, 1, k; f)$, resp.) are nonpositive. The concept of *x-log-concavity* and *strongly x-log-concavity* were suggested by Stanley and Sagan, respectively, see [2, 21] for more detail. It is known that the q -analogues of many combinatorial numbers are *x-log-concave* (we consider the q -analogues as polynomials in x), including the q -binomial coefficients [2], the q -Stirling numbers of the first kind [15] and the second kind [20], the q -Catalan numbers [3] and the q -Kaplansky numbers [12]. The concept of *x-log-convexity* and *strongly x-log-convexity* were introduced by Liu-Wang [18] and Chen-Wang-Yang [5], respectively. Liu and Wang [18] proved the *x-log-convexity* of certain famous polynomial sequences, including the Bell polynomials, the Eulerian polynomials, the q -Schröder numbers and the q -central Delannoy numbers. Chen, Wang and Yang [4, 5] proved the *x-log-convexity* of the Bessel polynomials, the Ramanujan polynomials, the Dowling polynomials and the Narayana polynomials.

The motivation of this paper is to prove two conjectures of Z.-H. Sun [23, 24] on the n th reverse Turán expression of the two polynomial sequences $G = (G_n(x))_{n \geq 0}$ and $V = (V_n(x))_{n \geq 0}$.

In their study of special values of spectral zeta functions, Kimoto and Wakayama [19] introduced the number

$$\sum_{k=0}^n \binom{n}{k} (-1)^k \binom{-1/2}{k}^2.$$

As one-parameter generalization of the number, Z.-W. Sun [25] defined the polynomial sequence $G = (G_n(x))_{n \geq 0}$ with

$$G_n(x) = \sum_{k=0}^n \binom{n}{k} (-1)^k \binom{x}{k} \binom{-1-x}{k}, \tag{1.2}$$

and investigated the supercongruences involving the polynomial $G_n(x)$. Z.-H. Sun [23] introduced the polynomial sequence $V = (V_n(x))_{n \geq 0}$ with

$$V_n(x) = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (-1)^k \binom{x}{k} \binom{-1-x}{k}. \tag{1.3}$$

Using Zeilberger’s algorithm, Z.-H. Sun [23, 24] established the following recurrence relations

$$(n + 1)^2 G_{n+1}(x) = (2n^2 + 2n + x^2 + x + 1)G_n(x) - n^2 G_{n-1}(x), \quad G_{-1}(x) = 0, \quad G_0(x) = 1, \tag{1.4}$$

$$(n + 1)^3 V_{n+1}(x) = (2n + 1)(n^2 + n + 2x^2 + 2x + 1)V_n(x) - n^3 V_{n-1}(x), \quad V_{-1}(x) = 0, \quad V_0(x) = 1. \tag{1.5}$$

These recurrence relations imply that $G_n(x)$ and $V_n(x)$ are both Apéry-like numbers for a fixed x , see [7, 30] for more details on Apéry-like numbers. Z.-H. Sun obtained congruences involving $G_n(x)$ and $V_n(x)$. Furthermore, he proposed the following two conjectures.

Conjecture 1.1. [24, Conjecture 7.22] Let $G = (G_n(x))_{n \geq 0}$ be the polynomial sequence defined in (1.2). Suppose $n \in \mathbb{N}^+$. If $x \in (-1, 0)$, then $\mathcal{T}_n(G) > 0$. If $x \notin [-1, 0]$, then $\mathcal{T}_n(G) < 0$.

Conjecture 1.2. [23, Conjecture 4.12] Let $V = (V_n(x))_{n \geq 0}$ be the polynomial sequence defined in (1.3). Suppose $n \in \mathbb{N}^+$. If $x \in (-1, 0)$, then $\mathcal{T}_n(V) > 0$. If $x \notin [-1, 0]$, then $\mathcal{T}_n(V) < 0$.

This paper is organized as follows. In Section 2, we give some criteria for the sign of $\mathcal{T}(n, h, k; f)$, where $f = (f_n(x))_{n \geq 0}$ is a polynomial sequence satisfying certain three-term recurrence relation. As applications, we obtain the sign of $\mathcal{T}(n, h, k; G)$ and $\mathcal{T}(n, h, k; V)$ and confirm the Conjecture 1.1 and 1.2. In the last section, we show that all the zeros of $G_n(x)$ and $V_n(x)$ lie on the line $\Re(x) = -1/2$, and propose a problem on the zero distribution of the reverse Turán expression of polynomial sequences related to $(G_n(x))_{n \geq 0}$ and $(V_n(x))_{n \geq 0}$.

2. Main results

Consider a polynomial sequence $f = (f_n(x))_{n \geq 0}$ defined by

$$\theta(x)f_n(x) = \gamma_n f_{n+1}(x) - (\gamma_n + \alpha_n)f_n(x) + \alpha_n f_{n-1}(x), \quad f_{-1}(x) = 0, \quad f_0(x) = 1, \tag{2.1}$$

where α_n, γ_n are independent on x and $\theta(x)$ is independent on n . As is shown in (1.4), the polynomial sequence $G = (G_n(x))_{n \geq 0}$ satisfy the recurrence relation (2.1) with

$$\gamma_n = (n + 1)^2, \quad \alpha_n = n^2, \quad \theta(x) = x^2 + x.$$

Similarly by (1.5), the polynomial sequence $V = (V_n(x))_{n \geq 0}$ satisfy the recurrence relation (2.1) with

$$\gamma_n = \frac{(n + 1)^3}{2n + 1}, \quad \alpha_n = \frac{n^3}{2n + 1}, \quad \theta(x) = 2x^2 + 2x.$$

The following recurrence relation of $\mathcal{T}_n(f)$ is inspired by Szwarc [27, 28]. In this paper, we denote $f_n(x)$ by f_n for short without ambiguity.

Proposition 2.1. Let $f = (f_n(x))_{n \geq 0}$ be the polynomial sequence satisfying (2.1). Then we have

$$\gamma_n \mathcal{T}_n(f) = -(f_n - f_{n-1}) [(\gamma_n - \gamma_{n-1})f_n - (\alpha_n - \alpha_{n-1})f_{n-1}] + \alpha_{n-1} \mathcal{T}_{n-1}(f). \tag{2.2}$$

Proof. By the recurrence relation (2.1), we get

$$\begin{aligned} \gamma_n \mathcal{T}_n(f) &= \gamma_n f_{n+1} f_{n-1} - \gamma_n f_n^2 \\ &= f_{n-1} [(\theta(x) + \gamma_n + \alpha_n)f_n - \alpha_n f_{n-1}] - \gamma_n f_n^2 \\ &= (\theta(x) + \gamma_n + \alpha_n)f_n f_{n-1} - \gamma_n f_n^2 - \alpha_n f_{n-1}^2. \end{aligned}$$

Note that

$$(\theta(x) + \gamma_n + \alpha_n)f_n f_{n-1} = (\theta(x) + \gamma_{n-1} + \alpha_{n-1})f_n f_{n-1} + (\gamma_n + \alpha_n - \gamma_{n-1} - \alpha_{n-1})f_n f_{n-1}.$$

Again applying (2.1), we derive

$$(\theta(x) + \gamma_{n-1} + \alpha_{n-1})f_n f_{n-1} = \gamma_{n-1}f_n^2 + \alpha_{n-1}f_n f_{n-2} = \gamma_{n-1}f_n^2 + \alpha_{n-1}\mathcal{T}_{n-1}(f) + \alpha_{n-1}f_{n-1}^2.$$

Thus we obtain that

$$\begin{aligned} \gamma_n \mathcal{T}_n(f) &= (\gamma_{n-1} - \gamma_n)f_n^2 + (\alpha_{n-1} - \alpha_n)f_{n-1}^2 + (\gamma_n + \alpha_n - \gamma_{n-1} - \alpha_{n-1})f_n f_{n-1} + \alpha_{n-1}\mathcal{T}_{n-1}(f) \\ &= -(f_n - f_{n-1})[(\gamma_n - \gamma_{n-1})f_n - (\alpha_n - \alpha_{n-1})f_{n-1}] + \alpha_{n-1}\mathcal{T}_{n-1}(f). \end{aligned}$$

The proof is complete. \square

Now we are ready to give the sufficient conditions for the sign of second order determinants for Hankel matrices $\mathcal{T}(n, h, k; f)$ of three-term recurrence polynomial sequences $f = (f_n(x))_{n \geq 0}$.

Theorem 2.2. Let $f = (f_n(x))_{n \geq 0}$ be the polynomial sequence satisfying the recurrence relation (2.1). Assume that

$$\gamma_n > 0 \text{ for } n \geq 0; \quad \alpha_n > 0 \text{ for } n \geq 1, \text{ and } \alpha_0 = 0.$$

Case1. Suppose that

- $\gamma_n - \gamma_{n-1} \geq \alpha_n - \alpha_{n-1} \geq 0$;
- $\theta(x)$ has nonnegative coefficients.

If $\mathcal{T}_1(f)$ has nonpositive coefficients, then $\mathcal{T}(n, h, k; f)$ has nonpositive coefficients for $n \in \mathbb{N}$ and $k, h \in \mathbb{N}^+$.

Case2. Suppose that there exists $x_0 < 0$ such that for $x \in [x_0, 0)$,

- $(\gamma_n - \gamma_{n-1})f_n > (\alpha_n - \alpha_{n-1})f_{n-1}$;
- $\theta(x) < 0$;
- $f_n > 0$.

If $\mathcal{T}_1(f) > 0$ for $x \in [x_0, 0)$, then we have $\mathcal{T}(n, h, k; f) > 0$ for $x \in [x_0, 0)$ and for $n \in \mathbb{N}$ and $k, h \in \mathbb{N}^+$.

Proof. For Case 1, firstly we show that the polynomial sequence f is x -log-concave, and secondly we prove that $\mathcal{T}(n, h, k; f)$ has nonpositive coefficients.

Rewriting the recurrence relation (2.1) yields

$$\gamma_n(f_{n+1} - f_n) = \alpha_n(f_n - f_{n-1}) + \theta(x)f_n. \tag{2.3}$$

To show that the polynomial sequence f is x -log-concave is equality to show that all coefficients of $\mathcal{T}_n(f)$ are nonpositive. Firstly, we show that the polynomials $f_{n+1} - f_n$ and f_n have nonnegative coefficients by induction on n . Clearly, the initial terms

$$f_0 = 1, \quad f_1 = 1 + \frac{\theta(x)}{\gamma_0}, \quad f_1 - f_0 = \frac{\theta(x)}{\gamma_0}$$

have nonnegative coefficients by the fact that $\gamma_0 > 0$ and $\theta(x)$ has nonnegative coefficients. Assume that $f_n - f_{n-1}$ and f_n have nonnegative coefficients. Since $\gamma_n > 0$ for $n \geq 0$ and $\alpha_n > 0$ for $n \geq 1$, we obtain that $f_{n+1} - f_n$ have nonnegative coefficients by the recurrence relation (2.3) and so as f_{n+1} . Next we show that all coefficients of $\mathcal{T}_n(f)$ are nonpositive by induction on n . By Proposition 2.1,

$$\begin{aligned} \gamma_n \mathcal{T}_n(f) &= -(f_n - f_{n-1})((\gamma_n - \gamma_{n-1})f_n - (\alpha_n - \alpha_{n-1})f_{n-1}) + \alpha_{n-1}\mathcal{T}_{n-1}(f) \\ &= -(\alpha_n - \alpha_{n-1})(f_n - f_{n-1})^2 - (\gamma_n - \gamma_{n-1} - \alpha_n + \alpha_{n-1})f_n(f_n - f_{n-1}) + \alpha_{n-1}\mathcal{T}_{n-1}(f). \end{aligned} \tag{2.4}$$

Combining the condition $\gamma_n - \gamma_{n-1} \geq \alpha_n - \alpha_{n-1} \geq 0$ with the facts the $f_n - f_{n-1}$ and f_n have nonnegative coefficients, we obtain that

$$(\alpha_n - \alpha_{n-1})(f_n - f_{n-1})^2 + (\gamma_n - \gamma_{n-1} - \alpha_n + \alpha_{n-1})f_n(f_n - f_{n-1})$$

has nonnegative coefficients. Since $\mathcal{T}_1(f)$ has nonpositive coefficients and $\gamma_n > 0, \alpha_n > 0$ for $n \geq 1$, we derive that $\mathcal{T}_n(f)$ has nonpositive coefficients for $n \in \mathbb{N}^+$.

Next we show that $\mathcal{T}(n, h, k; f)$ has nonpositive coefficients for $n, k, h \in \mathbb{N}^+$ by induction on both h and k . For $h = k = 1$, we have proved that

$$\mathcal{T}(n, 1, 1; f) = \mathcal{T}_{n+1}(f)$$

has nonpositive coefficients. Fix $h = 1$. For $k \geq 2$, assume that

$$\mathcal{T}(n, 1, k - 1; f) = \begin{vmatrix} f_n & f_{n+1} \\ f_{n+k-1} & f_{n+k} \end{vmatrix} = f_n f_{n+k} - f_{n+1} f_{n+k-1}$$

has nonpositive coefficients. Then we have

$$\begin{aligned} f_{n+k} \mathcal{T}(n, 1, k; f) &= f_{n+k} f_n f_{n+k+1} - f_{n+k}^2 f_{n+1} \\ &= f_{n+1} f_{n+k-1} f_{n+k+1} - f_{n+k}^2 f_{n+1} + f_{n+k+1} (f_n f_{n+k} - f_{n+1} f_{n+k-1}) \\ &= f_{n+1} (f_{n+k-1} f_{n+k+1} - f_{n+k}^2) + f_{n+k+1} (f_n f_{n+k} - f_{n+1} f_{n+k-1}) \\ &= f_{n+1} \mathcal{T}_{n+k}(f) + f_{n+k+1} \mathcal{T}(n, 1, k - 1; f). \end{aligned} \tag{2.5}$$

By the fact that f_n has nonnegative coefficients, $\mathcal{T}_{n+k}(f)$ has nonpositive coefficients and the assumption that $\mathcal{T}(n, 1, k - 1; f)$ has nonpositive coefficients, we obtain that $\mathcal{T}(n, 1, k; f)$ has nonpositive coefficients for $n, k \in \mathbb{N}^+$. Next we prove that $\mathcal{T}(n, h, k; f)$ has nonpositive coefficients for $n, k, h \in \mathbb{N}^+$ by induction on h . For $h \geq 2$, suppose

$$\mathcal{T}(n, h - 1, k; f) = f_n f_{n+h-1+k} - f_{n+h-1} f_{n+k}$$

has nonpositive coefficients. Then we have

$$\begin{aligned} f_{n+h-1+k} \mathcal{T}(n, h, k; f) &= f_{n+h-1+k} f_n f_{n+k+h} - f_{n+h-1+k} f_{n+k} f_{n+h} \\ &= f_{n+h-1} f_{n+k} f_{n+h+k} - f_{n+h-1+k} f_{n+k} f_{n+h} \\ &\quad + f_{n+h+k} (f_n f_{n+h-1+k} - f_{n+h-1} f_{n+k}) \\ &= f_{n+k} (f_{n+k+h} f_{n+h-1} - f_{n+h+k-1} f_{n+h}) \\ &\quad + f_{n+h+k} (f_n f_{n+h-1+k} - f_{n+h-1} f_{n+k}) \\ &= f_{n+k} \mathcal{T}(n + h - 1, 1, k; f) + f_{n+h+k} \mathcal{T}(n, h - 1, k; f). \end{aligned}$$

By the fact that f_n has nonnegative coefficients, $\mathcal{T}(n + h - 1, 1, k; f)$ has nonpositive coefficients and the assumption that $\mathcal{T}(n, h - 1, k; f)$ has nonpositive coefficients, we obtain that $\mathcal{T}(n, h, k; f)$ has nonpositive coefficients for $n, k, h \in \mathbb{N}^+$.

The proof of Case 2 is in a similar way to that of Case 1. Firstly we show that the reverse Turán expressions $\mathcal{T}_n(f) > 0$, and secondly we prove that $\mathcal{T}(n, h, k; f) > 0$. First we show that the polynomials $f_{n+1} - f_n < 0$ for $x \in [x_0, 0)$ by induction on n . It is easy to see that the initial term $f_1 - f_0 = \theta(x)/\gamma_0 < 0$ by the condition that $\gamma_0 > 0$ and $\theta(x) < 0$. Assume that $f_n - f_{n-1} < 0$ for $x \in [x_0, 0)$. Since $f_n > 0$ for $x \in [x_0, 0)$, we have $\theta(x)f_n < 0$ for $x \in [x_0, 0)$. From the recurrence relation (2.3), we can see that $f_{n+1} - f_n < 0$ for $x \in [x_0, 0)$. Recall the condition $(\gamma_n - \gamma_{n-1})f_n > (\alpha_n - \alpha_{n-1})f_{n-1}$ for $x \in [x_0, 0)$. From equation (2.4), we obtain the reverse Turán expression $\mathcal{T}_n(f) > 0$ for $x \in [x_0, 0)$ by induction on n . The proof of $\mathcal{T}(n, h, k; f) > 0$ is similar to the proof above. This completes the proof. \square

Remark 2.3. It is obvious that $\mathcal{T}(n, h, k; f)$ has nonpositive coefficients for $n \in \mathbb{N}$ and $k, h \in \mathbb{N}^+$ implies that the polynomial sequence f is strongly x -log-concave. Conversely, by the proof of Theorem 2.2 in Case 1, $\mathcal{T}(n, h, k; f)$ has nonpositive coefficients for $n \in \mathbb{N}$ and $k, h \in \mathbb{N}^+$ can be obtained from the polynomial sequence f is strongly x -log-concave by induction on h . Hence, $\mathcal{T}(n, h, k; f)$ has nonpositive coefficients for $n \in \mathbb{N}$ and $k, h \in \mathbb{N}^+$ is equality to the polynomial sequence f is strongly x -log-concave under the Case 1 in Theorem 2.2.

3. Applications

Now we focus on the polynomials $G_n(x)$ and $V_n(x)$. Clearly, by equations (1.2) and (1.3),

$$G_n(x) = G_n(-1 - x), \quad V_n(x) = V_n(-1 - x) \tag{3.1}$$

for $n \in \mathbb{N}$. In [24, Theorem 3.2] and [23, Theorem 4.1], Z.-H. Sun presented the other expressions of $G_n(x)$ and $V_n(x)$ respectively,

$$G_n(x) = \sum_{k=0}^n \binom{x}{k}^2 (-1)^{n-k} \binom{-1-x}{n-k} \quad \text{and} \quad V_n(x) = \sum_{k=0}^n \binom{x}{k}^2 \binom{-1-x}{n-k}^2. \tag{3.2}$$

Lemma 3.1. For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we have $G_n(x) > 0$ and $V_n(x) > 0$.

Proof. For $x \geq -1/2$,

$$(-1)^{n-k} \binom{-1-x}{n-k} = \frac{(x+1)(x+2)\dots(x+n-k)}{(n-k)!} > 0, \quad \binom{x}{k}^2 \geq 0, \quad \binom{x}{0}^2 = 1.$$

Then $G_n(x) > 0$ for $x \geq -1/2$ by (3.2). Since $G_n(x) = G_n(-1 - x)$, we obtain that $G_n(x) > 0$ for $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Similarly we can prove that $V_n(x) > 0$ for $x \in \mathbb{R}$ and $n \in \mathbb{N}$. \square

We show that the sign of the second order determinants for Hankel matrices $\mathcal{T}(n, h, k; f)$ in the following, where $f \in \{G, V\}$.

Theorem 3.2. Let $G = (G_n(x))_{n \geq 0}$ be the polynomial sequence defined in (1.2) and $k, h \in \mathbb{N}^+$. If $x \in (-1, 0)$, then $\mathcal{T}(n, h, k; G) > 0$. If $x \notin [-1, 0]$, then $\mathcal{T}(n, h, k; G) < 0$.

Proof. Since $G_n(x) = G_n(-1 - x)$, it suffices to prove that $\mathcal{T}(n, h, k; G) < 0$ for $x \in (0, \infty)$ and $\mathcal{T}(n, h, k; G) > 0$ for $x \in [-1/2, 0)$. Rewriting the recurrence relation (1.4) yields

$$(x^2 + x)G_n(x) = (n + 1)^2 G_{n+1}(x) - ((n + 1)^2 + n^2) G_n(x) + n^2 G_{n-1}(x).$$

Note that $\gamma_n = (n + 1)^2 > 0$ for $n \geq 0$, $\alpha_n = n^2 > 0$ for $n \geq 1$, and $\alpha_0 = 0$.

First we prove that $\mathcal{T}(n, h, k; G)$ has nonpositive coefficients for $n \in \mathbb{N}$. Consider the conditions that

- $\gamma_n - \gamma_{n-1} = 2n + 1 > 2n - 1 = \alpha_n - \alpha_{n-1} > 0$ for $n \geq 1$;
- $\theta(x) = x^2 + x$ has nonnegative coefficients.

Note that

$$\mathcal{T}_1(G) = -\frac{(x^2 + x)(3x^2 + 3x + 2)}{4}$$

has nonpositive coefficients. Thus $\mathcal{T}(n, h, k; G)$ has nonpositive coefficients for $n \in \mathbb{N}$ and $k, h \in \mathbb{N}^+$ by Theorem 2.2. Now we show that $\mathcal{T}(n, h, k; G) \neq 0$. Suppose $\mathcal{T}(n, h, k; G) = 0$, then the equation (2.5) implies that $\mathcal{T}_n(G) = 0$ for $n \geq 1$, which is contradictory since $\mathcal{T}_1(G) \neq 0$. So if $x \in (0, \infty)$, then $\mathcal{T}(n, h, k; G) < 0$ for $n \in \mathbb{N}$ and $k, h \in \mathbb{N}^+$.

Next we show that $\mathcal{T}(n, h, k; G) > 0$ for $x \in [-1/2, 0)$. For $x \in [-1/2, 0)$, we have

- $\theta(x) = x^2 + x \in [-1/4, 0)$;
- $G_n(x) > 0$ for $x \in [-1/2, 0)$ and $n \in \mathbb{N}^+$ by Lemma 3.1;
- $\mathcal{T}_1(G) > 0$.

Thus it suffices to prove $(2n + 1)G_n(x) - (2n - 1)G_{n-1}(x) > 0$ by Theorem 2.2, i.e.,

$$\frac{G_{n-1}(x)}{G_n(x)} < \frac{2n + 1}{2n - 1}, \quad \text{for } -\frac{1}{2} \leq x < 0$$

(since $G_n(x) > 0$ by Lemma 3.1). Claim that

$$\frac{G_{n-1}(x)}{G_n(x)} < \frac{2n + 1}{2n} < \frac{2n + 1}{2n - 1}. \tag{3.3}$$

We proceed by induction on n . Clearly,

$$\frac{G_0(x)}{G_1(x)} = \frac{1}{1 + x + x^2} \leq \frac{1}{1 - 1/4} = \frac{4}{3} < \frac{3}{2}.$$

Assume that (3.3) is true. Then by the recurrence relation (1.4), we derive that

$$\begin{aligned} \frac{G_{n+1}(x)}{G_n(x)} - \frac{2n + 2}{2n + 3} &= 1 + \frac{x^2 + x}{(n + 1)^2} + \frac{n^2}{(n + 1)^2} \left(1 - \frac{G_{n-1}(x)}{G_n(x)}\right) - \frac{2n + 2}{2n + 3} \\ &> 1 + \frac{x^2 + x}{(n + 1)^2} - \frac{n}{2(n + 1)^2} - \frac{2n + 2}{2n + 3} \\ &> 1 - \frac{1}{4(n + 1)^2} - \frac{n}{2(n + 1)^2} - \frac{2n + 2}{2n + 3} = \frac{1}{4(n + 1)^2(2n + 3)} > 0. \end{aligned}$$

This completes the proof. \square

Note that $\mathcal{T}_n(f) := \mathcal{T}(n - 1, 1, 1; f)$, so the Conjecture 1.1 is confirmed by Theorem 3.2.

Corollary 3.3. *Let $G = (G_n(x))_{n \geq 0}$ be the polynomial sequence defined in (1.2). Suppose $n \in \mathbb{N}^+$. If $x \in (-1, 0)$, then $\mathcal{T}_n(G) > 0$. If $x \notin [-1, 0]$, then $\mathcal{T}_n(G) < 0$.*

Actually in the proof of Theorem 3.2, we prove that $\mathcal{T}(n, h, k; G)$ has nonpositive coefficients for $n \in \mathbb{N}$ and $k, h \in \mathbb{N}^+$. Thus we obtain the following result immediately.

Corollary 3.4. *The polynomial sequence $G = (G_n(x))_{n \geq 0}$ is strongly x -log-concave.*

Remark 3.5. *Note that*

$$G_n(-1/2) = \sum_{k=0}^n \binom{n}{k} (-1)^k \binom{-1/2}{k}^2$$

arise in the calculation of spectral zeta functions, see [19, 25]. By Theorem 3.2, we have

$$G_n^2(-1/2) < G_{n-1}(-1/2)G_{n+1}(-1/2).$$

Similarly, we present the sign of $\mathcal{T}(n, h, k; V)$ as follows.

Theorem 3.6. *Let $V = (V_n(x))_{n \geq 0}$ be the polynomial sequence defined in (1.3) and $k, h \in \mathbb{N}^+$. If $x \in (-1, 0)$, then $\mathcal{T}(n, h, k; V) > 0$. If $x \notin [-1, 0]$, then $\mathcal{T}(n, h, k; V) < 0$.*

Proof. By equation (1.3), it suffices to prove that $\mathcal{T}(n, h, k; V) < 0$ for $x \in (0, \infty)$ and $\mathcal{T}(n, h, k; V) > 0$ for $x \in [-1/2, 0)$. Rewriting the recurrence relation (1.5) yields

$$(2x^2 + 2x)V_n(x) = \frac{(n + 1)^3}{2n + 1}V_{n+1}(x) - \left(\frac{(n + 1)^3}{2n + 1} + \frac{n^3}{2n + 1}\right)V_n(x) + \frac{n^3}{2n + 1}V_{n-1}(x).$$

Note that $\gamma_n = (n + 1)^3/(2n + 1) > 0$ for $n \geq 0$, $\alpha_n = n^3/(2n + 1) > 0$ for $n \geq 1$, $\alpha_0 = 0$.

Consider the conditions

- $\gamma_n - \gamma_{n-1} = \frac{4n^3+3n^2-n-1}{4n^2-1} > \frac{4n^3-3n^2-n+1}{4n^2-1} = \alpha_n - \alpha_{n-1} > 0$ for $n \geq 1$;
- $\theta(x) = 2x^2 + 2x$ has nonnegative coefficients.

Note that

$$\mathcal{T}_1(V) = -\frac{(2x^2 + 2x)(10x^2 + 10x + 4)}{8}$$

has nonpositive coefficients. Thus $\mathcal{T}(n, h, k; V)$ has nonpositive coefficients for $n \in \mathbb{N}$ and $k, h \in \mathbb{N}^+$ by Theorem 2.2. And we can derive that if $x \in (0, \infty)$, then $\mathcal{T}(n, h, k; V) < 0$ in the same way as $G = (G_n(x))_{n \geq 0}$.

Similarly, we show that $\mathcal{T}(n, h, k; V) > 0$ for $x \in [-1/2, 0)$. We only show that $(\gamma_n - \gamma_{n-1})V_n(x) > (\alpha_n - \alpha_{n-1})V_{n-1}(x)$, i.e.,

$$\frac{V_{n-1}(x)}{V_n(x)} < \frac{4n^3 + 3n^2 - n - 1}{4n^3 - 3n^2 - n + 1} \quad \text{for } -\frac{1}{2} \leq x < 0.$$

Other conditions can be checked easily. Note that

$$\frac{4n^3 + 3n^2 - n - 1}{4n^3 - 3n^2 - n + 1} - \frac{n + 1}{n} = \frac{2n^3 + 3n^2 - n - 1}{n(4n^3 - 3n^2 - n + 1)} > 0 \quad \text{for } n \geq 1.$$

Claim that

$$\frac{V_{n-1}(x)}{V_n(x)} < \frac{n + 1}{n} < \frac{4n^3 + 3n^2 - n - 1}{4n^3 - 3n^2 - n + 1}. \tag{3.4}$$

We proceed by induction on n . Clearly,

$$\frac{V_0(x)}{V_1(x)} = \frac{1}{1 + 2x + 2x^2} \leq \frac{1}{1 - 1/2} = \frac{2}{1}.$$

Assume that (3.4) is true. By the recurrence relation (1.5), we have

$$\begin{aligned} \frac{V_{n+1}(x)}{V_n(x)} - \frac{n + 1}{n + 2} &= 1 + \frac{(2n + 1)(2x^2 + 2x)}{(n + 1)^3} + \frac{n^3}{(n + 1)^3} \left(1 - \frac{V_{n-1}(x)}{V_n(x)} \right) - \frac{n + 1}{n + 2} \\ &> 1 - \frac{(2n + 1)}{2(n + 1)^3} - \frac{n^2}{(n + 1)^3} - \frac{n + 1}{n + 2} = \frac{n}{2(n + 1)^3(n + 2)} > 0. \end{aligned}$$

This completes the proof. \square

Hence, we confirm the Conjecture 1.2 by Theorem 3.6 and obtain the strong x -log concavity for the polynomial sequence $V = (V_n(x))_{n \geq 0}$.

Corollary 3.7. *Let $V = (V_n(x))_{n \geq 0}$ be the polynomial sequence defined in (1.2). Suppose $n \in \mathbb{N}^+$. If $x \in (-1, 0)$, then $\mathcal{T}_n(V) > 0$. If $x \notin [-1, 0]$, then $\mathcal{T}_n(V) < 0$.*

Corollary 3.8. *The polynomial sequence $V = (V_n(x))_{n \geq 0}$ is strongly x -log-concave.*

4. Further works

Define the polynomial sequences $\tilde{G} = (\tilde{G}_n(x))_{n \geq 0}$ and $\tilde{V} = (\tilde{V}_n(x))_{n \geq 0}$ by

$$(n + 1)^2 \tilde{G}_{n+1}(x) = (2n^2 + 2n + x + 1) \tilde{G}_n(x) - n^2 \tilde{G}_{n-1}(x), \quad \tilde{G}_{-1}(x) = 0, \quad \tilde{G}_0(x) = 1, \tag{4.1}$$

$$(n + 1)^3 \tilde{V}_{n+1}(x) = (2n + 1)(n^2 + n + x + 1) \tilde{V}_n(x) - n^3 \tilde{V}_{n-1}(x), \quad \tilde{V}_{-1}(x) = 0, \quad \tilde{V}_0(x) = 1. \tag{4.2}$$

It is clear that $G_n(x)$ ($V_n(x)$, resp.) and $\widetilde{G}_n(x^2 + x)$ ($\widetilde{V}_n(2x^2 + 2x)$, resp.) share the same recurrence relation and initial conditions. Thus we obtain that

$$G_n(x) = \widetilde{G}_n(x^2 + x), \quad V_n(x) = \widetilde{V}_n(2x^2 + 2x). \tag{4.3}$$

According to [22], any orthogonal system of real valued polynomials $p_n(x)$ satisfies

$$p_{-1}(x) = 0, \quad p_0(x) = 1, \quad \text{and} \quad xp_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x) \quad (n \geq 0), \tag{4.4}$$

where A_n, B_n, C_n are real numbers and $A_n C_n > 0$. Conversely, if the condition (4.4) is satisfied by a sequence of polynomials $p_n(x)$, and A_n, B_n, C_n are real numbers with $A_n C_{n+1} > 0$, then there is a weight function $\omega(x)$ such that

$$\int_{-\infty}^{\infty} \omega(x) p_m(x) p_n(x) dx = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{1}{v_n} \int_{-\infty}^{\infty} \omega(x) dx & \text{if } m = n, \end{cases}$$

where $v_0 = 1$ and $v_n = \frac{A_0 A_1 \cdots A_{n-1}}{C_1 \cdots C_n}$ ($n \geq 1$). Hence we obtain that $(\widetilde{G}_n(x))_{n \geq 0}$ and $(\widetilde{V}_n(x))_{n \geq 0}$ are both orthogonal polynomials. It is well known [6, 17] that $\widetilde{G}_n(x)$ and $\widetilde{V}_n(x)$ have only real zeros. As in the proof of Theorems 3.2 and 3.6, we can show that $\widetilde{G}_n(x)$ and $\widetilde{V}_n(x)$ have nonnegative coefficients. So all the zeros of $\widetilde{G}_n(x)$ and $\widetilde{V}_n(x)$ are real, negative and simple.

Theorem 4.1. *Let $G = (G_n(x))_{n \geq 0}$ and $V = (V_n(x))_{n \geq 0}$ be the polynomial sequences defined in (1.2) and (1.3) respectively. Then for $n \in \mathbb{N}^+$, all the zeros of $G_n(x)$ and $V_n(x)$ lie on the line $\Re(x) = -1/2$, where $\Re(x)$ is the real part of x .*

Proof. By Lemma 3.1, $G_n(x) > 0$ for $x \in \mathbb{R}$ and all $n \in \mathbb{N}^+$. Since $x^2 + x \geq -1/4$, we obtain that $\widetilde{G}_n(x) > 0$ for $x \geq -1/4$ by (4.3). Suppose that $\widetilde{G}_n(r) = 0$, then $r < -1/4$. Again by (4.3),

$$-\frac{1}{2} \pm i \sqrt{-\frac{1}{4} - r}$$

are the two conjugate zeros of $G_n(x) > 0$, where $i^2 = -1$. It is easy to see that the degree of $\widetilde{G}_n(x)$ is n and that of $G_n(x)$ is $2n$. Since $\widetilde{G}_n(x)$ has n negative zeros that are smaller than $-1/4$, all the zeros of $G_n(x)$ lie on the line $\Re(x) = -1/2$. The proof of $V_n(x)$ is similar and we omit it here. \square

We say a real polynomial is (Hurwitz) stable if all of its zeros lie in the open left half complex plane and a real polynomial is weakly (Hurwitz) stable if all of its zeros lie in the closed left half complex plane. It is clear that coefficients of any (Hurwitz) stable polynomial have the same sign. Recently, Liu and Yan [16], Ding and Zhu [8] studied the weakly (Hurwitz) stability of the (reverse strong) Turán expressions of recursive polynomials. Recall that $G = (G_n(x))_{n \geq 0}$ and $V = (V_n(x))_{n \geq 0}$ are both x -log-concave by Theorems 3.2 and 3.6, it is natural to consider (Hurwitz) stability of $\mathcal{T}_n(G)$ and $\mathcal{T}_n(V)$. We can direct calculate that $\mathcal{T}_3(G)$ and $\mathcal{T}_6(V)$ exist zeros lie in the right half complex plane, so $\mathcal{T}_n(G)$ and $\mathcal{T}_n(V)$ are not weakly (Hurwitz) stable. The x -log-concavity of $\widetilde{G} = (\widetilde{G}_n(x))_{n \geq 0}$ and $\widetilde{V} = (\widetilde{V}_n(x))_{n \geq 0}$ can be verify in the same way as proof of $G = (G_n(x))_{n \geq 0}$ and $V = (V_n(x))_{n \geq 0}$. So, an interesting problem is to consider the weak (Hurwitz) stability of $\mathcal{T}_n(\widetilde{G})$ and $\mathcal{T}_n(\widetilde{V})$.

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