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Zalcman-type rescaling and normality via total derivatives in $\mathbb{C}^n$

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Abstract. We study normal families of holomorphic functions in $\mathbb{C}^n$ through a generalization of Zalcman's Lemma involving total derivatives. A new rescaling theorem is established, leading to a criterion for normality based on value separation and derivative bounds.

1. Introduction and Main Results

The concept of normal families plays a fundamental role in complex analysis, particularly in understanding the compactness properties of families of meromorphic or holomorphic functions. Let $\mathcal{F}$ be a family of meromorphic functions defined on a domain $\Omega \subset \mathbb{C}$. The family $\mathcal{F}$ is said to be normal on Ω if every sequence in $\mathcal{F}$ contains a subsequence that converges spherically uniformly on each compact subset of Ω to a meromorphic function f, which may be identically equal to ∞. For a comprehensive treatment of the theory of normal families of meromorphic functions, we refer the reader to [6] and [8]. One of the classical criteria for normality is Marty's Theorem, due to Marty [5], which provides a necessary and sufficient condition in terms of the spherical derivative.

Marty's Theorem: A family $\mathcal{F}$ of meromorphic function on Ω is normal on Ω if and only if the family $\{f^\sharp : f \in \mathcal{F}\}$ is uniformly bounded on compact subsets of Ω, where $f^\sharp$ denotes the spherical derivative of f.

Building upon Marty's Theorem, Zalcman [9] introduced a powerful and elegant result now widely known as Zalcman's Lemma which has significantly influenced the development of the theory of normal families.

Zalcman's Lemma: A family $\mathcal{F}$ of meromorphic functions on Ω is not normal if, and only if there exist a number $r : 0 < r < 1$, $z_n : |z_n| < r$, $f_n \in \mathcal{F}$ and a positive numbers ρ_n tending to 0 such that

$$f_n(z_n + \rho_n \zeta) \to g(\zeta)$$

normally on $\mathbb{C}$ with respect to the spherical metric, where g is nonconstant meromorphic function on $\mathbb{C}$.

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The classical theory of normal families has been further developed in the context of several complex variables. Let D be a domain in $\mathbb{C}^n$ and let $\mathcal{F}$ be a family of holomorphic functions on D. Then $\mathcal{F}$ is said to be normal in D if every sequence in $\mathcal{F}$ admits a subsequence that converges locally uniformly on D. Aladro and Krantz [4] extended the concept of normality to several complex variables by employing the Kobayashi metric. We denote by $\mathcal{H}(D)$ the class of holomorphic functions $f : D \to \mathbb{C}$, where $D \subset \mathbb{C}^n$ is a domain. The open unit ball in $\mathbb{C}^n$ will be denoted by $\mathbb{B} := \{z \in \mathbb{C}^n : ||z|| < 1\}$.

In 2003, Jin [3] introduced the total derivative of entire functions in several complex variables.

Definition 1.1. *Let $f \in \mathcal{H}(\mathbb{C}^n)$ and $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$. The total derivative of f, denoted by Df, is defined as $Df(z) = \sum_{j=1}^{n} z_j f_{z_j}$, where f_{z_j} is the partial derivative of f with respect to z_j. The l-th order total derivative $D^l f$ of f is defined by $D^l f = D(D^{l-1} f)$ respectively.*

Liu and Cao [1] obtained an extension of Zalcman's lemma concerning the total derivative in several complex variables as

Theorem A Let $\mathcal{F}$ be a family of holomorphic functions in the unit ball $\mathbb{B}$ and $f(z) \neq 0$ for every $f \in \mathcal{F}$ and for all $z \in D(0, s) = \{z \in \mathbb{C}^n : ||z|| < s\}$ $(0 < s < 1)$. If $\mathcal{F}$ is not normal in $\mathbb{B}$, then for all $-1 < k < 1$, there exist real number $0 < r < 1$, a sequence $\{z_j\} \subseteq \mathbb{B}$ satisfying $0 < ||z_j|| < r$, a sequence $\{f_j\} \subset \mathcal{F}$, a sequence $\rho_j \to 0$ such that

$$g_j(\zeta) := \frac{f_j(z_j e^{\rho_j \zeta})}{\rho_j^k} \ (\zeta \in \mathbb{C})$$

converges locally uniformly to a nonconstant entire function $g(\zeta)$ in $\mathbb{C}$, where $z_j e^{\rho_j \zeta} \in \mathbb{B}$. Further, if $f(z) \neq 0$ for all $f \in \mathcal{F}$ and $z \in \mathbb{B}$, then k can be chosen in $(-1, \infty)$.

We now present a generalization of Theorem A:

Theorem 1.2. *Let $\mathcal{F}$ be a family of holomorphic functions in the unit ball $\mathbb{B}$ and $f(z) \neq 0$ for every $f \in \mathcal{F}$ and for all $z \in D(0, \delta) = \{z \in \mathbb{C}^n : ||z|| < \delta\}$ $(0 < \delta < 1)$. If $\mathcal{F}$ is not normal in $\mathbb{B}$, then for all $k \geq 0$ and $m \geq k + 1$, there exist real number $0 < r < 1$, a sequence $\{z_j\} \subseteq \mathbb{B}$ satisfying $0 < ||z_j|| < r$, a sequence $\{f_j\} \subset \mathcal{F}$, a sequence $\rho_j \to 0$ such that*

$$g_j(\zeta) := \frac{f_j(z_j e^{\rho_j^m \zeta})}{\rho_j^k} \ (\zeta \in \mathbb{C})$$

converges locally uniformly to a nonconstant entire function g in $\mathbb{C}$, where $z_j e^{\rho_j^m \zeta} \in \mathbb{B}$.

As an application of Theorem 1.2, and inspired by the ideas in [7], we establish a normality criterion for families of holomorphic functions of several complex variables.

Theorem 1.3. *Let $\mathcal{F}$ be a family of holomorphic functions in the unit ball $\mathbb{B}$ and $f(z) \neq 0$ for every $f \in \mathcal{F}$ and for all $z \in D(0, \delta)$ $(0 < \delta < 1)$. Assume that for each compact subset $K \subset \mathbb{B}$, there are*

(i) positive integers (or ∞) $l_1, l_2, \ldots, l_q$ satisfying $\sum_{m=1}^{q} \frac{1}{l_m} < q - 2$,

(ii) holomorphic functions $a_{1f}, a_{2f}, \ldots, a_{qf}$ $(f \in \mathcal{F})$ in $\mathbb{B}$, positive constant ϵ and M such that $\sigma(a_{if}(z), a_{tf}(z)) \geq \epsilon$ for all $z \in \mathbb{B}$, $1 \leq i, t \leq q$, $i \neq t$ and $\sup_{z \in K : f(z) = a_{mf}(z)} \frac{|D^{k+1} f(z)|}{1 + |D^k f(z)|^2} \leq M$, for all $f \in \mathcal{F}$, $m \in \{1, 2, \ldots, q\}$, and $k = 0, \ldots, l_m - 2$.

Then $\mathcal{F}$ is a normal family.

2. Auxiliary Results

Let f be a nonconstant meromorphic function on the complex plane $\mathbb{C}$. It is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory (see [2]), such as $T(r, f)$, $m(r, f)$, $N(r, f)$, $\bar{N}(r, f)$ and $S(r, f)$. In particular, $T(r, f)$ denotes the characteristic function, and $\bar{N}(r, f)$ is the counting function with respect to the poles of $f(z)$, ignoring multiplicities. We shall use the notation $S(r, f)$ to denote any quantity satisfying

$$S(r, f) = o(T(r, f)) \quad \text{as} \quad r \to \infty,$$

possibly outside a set E of finite linear measure.

In order to prove our theorems, we need the following lemmas:

First Main Theorem: Let f be a nonconstant meromorphic function on $\mathbb{C}$ and let a be a complex number. Then

$$T\left(r, \frac{1}{f - a}\right) = T(r, f) + O(1).$$

Second Main Theorem: Let f be a nonconstant meromorphic function on $\mathbb{C}$. Let $a_1, a_2, \ldots, a_q$ be q distinct values in $\mathbb{C}$. Then

$$(q - 1)T(r, f) \leq \bar{N}(r, f) + \sum_{i=1}^{q} \bar{N}\left(r, \frac{1}{f - a_i}\right) + o(T(r, f)),$$

for all $r \in [1, \infty)$, excluding a set of finite Lebesgue measure.

Lemma 2.1. *[1] Let $\mathcal{F}$ be a family of holomorphic function in a domain D containing the origin in $\mathbb{C}^n$.*

(i) If there exist $r(> 0)$ such that $f(z) \neq 0$ for all $f \in \mathcal{F}$ and all $z \in D(0, r) \subset D$, and if $\alpha > 0$ and

$$\mathcal{F}_{\alpha,1} := \{\frac{|Df(z)|}{1 + |f(z)|^{\alpha}} : f \in \mathcal{F}\}$$

is locally uniformly bounded in D, then $\mathcal{F}$ is normal in D.

(ii) If $f(z) \neq 0$ for any $f \in \mathcal{F}$ and all $z \in D$, and if $\alpha > 0$, $k \in \mathbb{N}$ and

$$\mathcal{F}_{\alpha,k} := \{\frac{|D^k f(z)|}{1 + |f(z)|^{\alpha}} : f \in \mathcal{F}\}$$

is locally uniformly bounded in D, then $\mathcal{F}$ is normal in D.

Lemma 2.2. *Let f be a holomorphic function on $\mathbb{B}$ and let $0 \leq k < m$. If there exist a point $w : \|w\| < r < 1$ such that*

$$\frac{(ln\frac{r}{\|w\|})^{m+k}|Df(w)|}{(ln\frac{r}{\|w\|})^{2k} + |f(w)|^2} > 1$$

Then there exist a point $z_0 : \|z_0\| < r$ and a number $s_0 : 0 < s_0 < 1$ such that

$$\sup_{\|z\|<r} \frac{(ln\frac{r}{\|z\|})^{m+k}s_0^{m+k}|Df(z)|}{(ln\frac{r}{\|z\|})^{2k}s_0^{2k} + |f(z)|^2}$$

$$= \frac{(ln\frac{r}{\|z_0\|})^{m+k}s_0^{m+k}|Df(z_0)|}{(ln\frac{r}{\|z_0\|})^{2k}s_0^{2k} + |f(z_0)|^2} = 1$$

Proof. Let $A = \{(z,s) : z \in \mathbb{B},\ \|z\| < r < 1,\ 0 < s \leq 1\}$ and define a real-valued function on A as

$$\Phi(z,s) := \frac{(ln\frac{r}{\|z\|})^{m+k}s^{m+k}|Df(z)|}{(ln\frac{r}{\|z\|})^{2k}s^{2k} + |f(z)|^2}.$$

Then Φ is continuous on A. It is sufficient to prove that there exists $(z_0, s_0) \in A$ with $s_0 \in (0,1)$ such that $\sup_{\|z\|\leq r} \Phi(z, s_0) = \Phi(z_0, s_0) = 1$.

First, we show that $\lim_{(ln\frac{r}{\|z\|})s \to 0} \Phi(z,s) = 0$.

For, let $(ln\frac{r}{\|z_j\|})s_j \to 0$ as $j \to \infty$ where $\|z_j\| < r,\ 0 < s_j < 1,\ z_j \to w_0\ (j \to \infty)$ then $\|w_0\| \leq r$. Now,

$$\begin{aligned} 0 &\leq \lim_{j\to\infty} \Phi(z_j, s_j) \\ &\leq \lim_{j\to\infty} (ln\frac{r}{\|z_j\|})^{m-k} s_j^{m-k} |Df(z_j)| \\ &= 0 \end{aligned}$$

Hence

$$\lim_{(ln\frac{r}{\|z\|})s \to 0} \Phi(z,s) = 0 \tag{1}$$

Let $S = \{(z,s) \in A : \Phi(z,s) > 1\}$. Then $(w,1) \in S$ and thus the set $S \neq \emptyset$.

Taking $s_0 = \inf\{s : (z,s) \in S\}$, then $s_0 \neq 0$.

If $s_0 = 1$, then there exist a sequence $\{s_j\}$ (< 1) such that $s_j \to s_0$ $(j \to \infty)$ and $\Phi(w, s_j) \leq 1$. Let $j \to \infty$, $\Phi(w, s_j) \to \Phi(w, 1) \leq 1$, this contradicts that $(w,1) \in S$. Hence, we have $0 < s_0 < 1$.

Take $z_0 \in \{z : \|z\| \leq r\}$ satisfying

$$\sup_{\|z\|\leq r} \Phi(z, s_0) = \Phi(z_0, s_0).$$

If $\Phi(z_0, s_0) < 1$, then by definition of s_0, there exist $(z_j, s_j) \in S$ such that $s_j \to s_0$. Suppose $z_j \to w_1$. Then $\|w_1\| \leq r$.

Now $\Phi(w_1, s_0) \leq \Phi(z_0, s_0) < 1$ and $\lim_{j\to\infty} \Phi(z_j, s_j) = \Phi(w_1, s_0)$. Thus for sufficiently large j, $\Phi(z_j, s_j) < 1$ a contradiction.

If $\Phi(z_0, s_0) > 1$. We know $\Phi(z_0, 0) = 0$. Then for the continuity of $\Phi(z,s)$ with respect to s in set A, there exist $s_1 : 0 < s_1 < s_0$ such that

$$\Phi(z_0, s_1) = 1 + \frac{\Phi(z_0, s_0) - 1}{2}.$$

This is not compatible with the definition of s_0 and, therefore, there exist (z_0, s_0) such that

$$\sup_{\|z\|<r} \Phi(z, s_0) = \Phi(z_0, s_0) = 1. \tag{2}$$

□

3. Proof of Theorems

Proof of Theorem 1.2 Suppose $\mathcal{F}$ is not normal at the point $z_0 = 0$. Then by Lemma 2.1, there exist $\lambda : 0 < \lambda < 1$, $\|\zeta_j\| < \lambda$, $\{f_j\} \subseteq \mathcal{F}$ such that

$$\lim_{j\to\infty} \frac{|Df_j(\zeta_j)|}{1 + |f_j(\zeta_j)|^2} = \infty.$$

Choose r such that $0 < \lambda < r < 1$ and define

$$G_j(z,s) := \frac{(ln\frac{r}{\|z\|})^{m+k}s^{m+k}|Df_j(z)|}{(ln\frac{r}{\|z\|})^{2k}s^{2k} + |f_j(z)|^2}$$

where $\|z\| < r,\ 0 < s \leq 1$.

$$\begin{aligned} G_j(\zeta_j,1) &= \frac{(ln\frac{r}{\|\zeta_j\|})^{m+k}|Df_j(\zeta_j)|}{(ln\frac{r}{\|\zeta_j\|})^{2k} + |f_j(\zeta_j)|^2} \\ &= \frac{(ln\frac{r}{\|\zeta_j\|})^{m-k}|Df_j(\zeta_j)|}{1 + \frac{|f_j(\zeta_j)|^2}{(ln\frac{r}{\|\zeta_j\|})^{2k}}} \\ &> \frac{(ln\frac{r}{\|\lambda\|})^{m-k}|Df_j(\zeta_j)|}{1 + \frac{|f_j(\zeta_j)|^2}{(ln\frac{r}{\|\lambda\|})^{2k}}} \to \infty \text{ as } j \to \infty \end{aligned}$$

It follows that $\lim_{j\to\infty} G_j(\zeta_j, 1) = \infty$. Thus for sufficiently large values of j, we have

$$G_j(\zeta_j, 1) > 1.$$

By Lemma 2.2, there exist z_j and a_j satisfying $\|z_j\| < r,\ 0 < a_j < 1$ related to every $f_j \in \mathcal{F}$ such that

$$\sup_{\|z\|<r} G_j(z, a_j) = G_j(z_j, a_j) = 1.$$

Hence

$$\begin{aligned} 1 &= G_j(z_j, a_j) \\ &\geq G_j(\zeta_j, a_j) \\ &= \frac{(ln\frac{r}{\|\zeta_j\|})^{m+k}a_j^{m+k}|Df_j(\zeta_j)|}{(ln\frac{r}{\|\zeta_j\|})^{2k}a_j^{2k} + |f_j(\zeta_j)|^2} \\ &\geq \frac{a_j^{m+k}(ln\frac{r}{\|\zeta_j\|})^{m+k}|Df_j(\zeta_j)|}{(ln\frac{r}{\|\zeta_j\|})^{2k} + |f_j(\zeta_j)|^2} \\ &= a_j^{m+k}G_j(\zeta_j, 1) \end{aligned}$$

It follows that $\lim_{j\to\infty} a_j = 0$.
Let $\rho_j = ln\frac{r}{\|z_j\|}a_j \to 0$ and it follows that $\lim_{j\to\infty} \rho_j/ln\frac{r}{\|z_j\|} = 0$.
Thus the function $g_j(\zeta) := f_j(z_je^{\rho_j^m\zeta})/\rho_j^k$ is defined in $|\zeta| < R_j = ln\frac{r}{\|z_j\|}/\rho_j^m \to \infty$.
Now

$$\begin{aligned} g_j^\sharp(\zeta) &= \frac{|g_j'(\zeta)|}{1 + |g_j(\zeta)|^2} \\ &= \frac{\rho_j^{m+k}|Df_j(z_je^{\rho_j^m\zeta})|}{\rho_j^{2k} + |f_j(z_je^{\rho_j^m\zeta})|^2} \end{aligned}$$

Now

$$\frac{ln\frac{r}{\|z_j\|}}{ln\frac{r}{\|z_je^{\rho_j^m\zeta}\|}} \to 1.$$

Hence there exist $\epsilon_j \to 0$ such that

$$\rho_j^{m+k} \leq (1+\epsilon_j)^{m+k}(ln\frac{r}{||z_je^{\rho_j^m\zeta}||})^{m+k}a_j^{m+k}$$

$$\rho_j^{2k} \geq (1-\epsilon_j)^{2k}(ln\frac{r}{||z_je^{\rho_j^m\zeta}||})^{2k}a_j^{2k}.$$

Thus

$$g_j^\sharp(\zeta) \leq \frac{(1+\epsilon_j)^{m+k}(ln\frac{r}{||z_je^{\rho_j^m\zeta}||})^{m+k}a_j^{m+k}|Df_j(z_je^{\rho_j^m\zeta})|}{(1-\epsilon_j)^{2k}(ln\frac{r}{||z_je^{\rho_j^m\zeta}||})^{2k}a_j^{2k}+|f_j(z_je^{\rho_j^m\zeta})|^2} \leq \frac{(1+\epsilon_j)^{m+k}}{(1-\epsilon_j)^{2k}}.$$

By Marty's Theorem, $\{g_j\}$ is normal on $\mathbb{C}$. Without loss of generality we may assume that $\{g_j\}$ converges normally to a holomorphic function g on $\mathbb{C}$ or ∞.
Now,

$$\begin{aligned} g_j^\sharp(0) &= \frac{|g_j'(0)|}{1+|g_j(0)|^2} \\ &= \frac{\rho_j^{m+k}|Df_j(z_j)|}{\rho_j^{2k}+|f_j(z_j)|^2} \\ &= \frac{(ln\frac{r}{||z_j||})^{m+k}a_j^{m+k}|Df_j(z_j)|}{(ln\frac{r}{||z_j||})^{2k}a_j^{2k}+|f_j(z_j)|^2} \\ &= G_j(z_j,a_j)=1 \end{aligned}$$

That is, $g^\sharp(0)=1$, showing that g is a nonconstant entire function on $\mathbb{C}$. □

proof of Theorem 1.3 Suppose $\mathcal{F}$ is not normal at $z_0 \in \mathbb{B}$. By Theorem 1.2, there exist sequences $\{f_j\} \subset \mathcal{F}$, $\{\rho_j\} \subset (0,1) : \rho_j \to 0$ and $\{z_j\} \subset \mathbb{B} : z_j \to z_0$ as $j \to \infty$ such that $g_j(\zeta) = f_j(z_je^{\rho_j\zeta})$ converges locally uniformly to a nonconstant entire function g in $\mathbb{C}$. Let $K \subset \mathbb{B}$ be any compact set containing z_0. Then by the assumptions, there are

(i) positive integers (or ∞) $l_1, l_2, \dots, l_q$ satisfying $\sum_{m=1}^{q}\frac{1}{l_m} < q-2$,

(ii) holomorphic functions $a_{1f}, a_{2f}, \dots, a_{qf}$ $(f \in \mathcal{F})$ in $\mathbb{B}$, positive constant ϵ and M such that $\sigma(a_{if}(z), a_{tf}(z)) \geq \epsilon$ for all $z \in \mathbb{B}$, $1 \leq i,t \leq q$, $i \neq t$ and $\sup\limits_{z\in K: f(z)=a_{mf}(z)} \frac{|D^{k+1}f(z)|}{1+|D^kf(z)|^2} \leq M$, for all $f \in \mathcal{F}$, $m \in \{1,2,\dots,q\}$, and $k = 0,\dots,l_m-2$.

Without loss of generality, we may assume that $q \geq 3$ and $l_m \geq 2$ for all $m = 1,\dots,q$. We may assume that $\{a_{mf_j}\}$ converges locally uniformly on $\mathbb{C}^n$ to a holomorphic function a_m (or ∞) for all $m = 1,\dots,q$. Then $a_{mf_j}(z_je^{\rho_j\zeta})$ converges locally uniformly on $\mathbb{C}$ to the constant $a_m(z_0)$. By the assumption on the spherical metric, $a_1(z_0),\dots,a_q(z_0)$ are distinct. If $a_m(z_0)=\infty$, then g omit $a_m(z_0)$.
Claim: For any $m \in \{1,\dots,q\}$, if $a_m(z_0) \neq \infty$ then all zeros of $g - a_m(z_0)$ has multiplicity at least l_m.

Let ζ_0 be any zero of $g(\zeta) - a_m(z_0)$. Hurwitz's theorem ensure the existence of sequence ζ_j converging to ζ_0 such that for sufficiently large j, $f_j(z_je^{\rho_j\zeta_j}) - a_{mf_j}(z_je^{\rho_j\zeta_j}) = 0$. Also $z_je^{\rho_j\zeta_j} \in K$ for sufficiently large j. Since $a_{mf_j}(z_je^{\rho_j\zeta_j})$ converges to $a_m(z_0)$, we may assume that

$$|f_j(z_je^{\rho_j\zeta_j})| = |a_{mf_j}(z_je^{\rho_j\zeta_j})| \leq 1+|a_m(z_0)|. \quad (3)$$

We have

$$\frac{|D^{k+1}f_j(z_je^{\rho_j\zeta_j})|}{1+|D^kf_j(z_je^{\rho_j\zeta_j})|^2} \leq M \tag{4}$$

for all $k = 0, \ldots, l_m - 2$ and for all sufficiently large j.
Set $M_1 := M(1 + (1 + |a_m(z_0)|)^2)$, and $M_{n+1} := M(1 + M_n^2)$, for all positive integer n.
Using the induction method we prove the following inequality.

$$|D^kf_j(z_je^{\rho_j\zeta_j})| \leq M_k, \text{ for all } k = 1, \ldots, l_m - 1. \tag{5}$$

Indeed, for $k = 0$, by 4 we have

$$\frac{|Df_j(z_je^{\rho_j\zeta_j})|}{1+|f_j(z_je^{\rho_j\zeta_j})|^2} \leq M.$$

Combining with 3, we have

$$|Df_j(z_je^{\rho_j\zeta_j})| \leq M.(1 + |f_j(z_je^{\rho_j\zeta_j})|^2) \leq M.(1 + (1 + |a_m(z_0)|)^2) = M_1.$$

We get 5 for $k = 1$.
Assume that 5 holds for some $k(k \leq l_m - 2)$. Then, by 4 and by induction hypothesis, we have

$$|D^{k+1}f_j(z_je^{\rho_j\zeta_j})| \leq M(1 + |D^kf_j(z_je^{\rho_j\zeta_j})|^2) \leq M(1 + M_k^2) = M_{k+1}.$$

Hence by induction, we get 5.
By 5 we have

$$\begin{aligned}
\frac{|g_j^{(k)}(\zeta_j)|}{1+|g_j^{(k-1)}(\zeta_j)|^2} &= \rho_j^k\frac{|D^kf_j(z_je^{\rho_j\zeta_j})|}{1+\rho_j^{2(k-1)}|D^{(k-1)}f_j(z_je^{\rho_j\zeta_j})|^2} \\
&\leq \rho_j|D^kf_j(z_je^{\rho_j\zeta_j})| \\
&\leq \rho_j^kM_k, \text{ for all } k = 1, \ldots, l_m - 1.
\end{aligned}$$

Therefore,

$$|g^{(k)}(\zeta_0)| = \lim_{j\to\infty}|g_j^{(k)}(\zeta_j)| \leq \lim_{j\to\infty}\rho_j^k.M_k(1 + |g_j^{(k-1)}(\zeta_j)|^2) = 0.$$

Then, $g^{(k)}(\zeta_0) = 0$ for all $k = 1, \ldots, l_m - 1$. Hence, the zero ζ_0 of $g - a_m(z_0)$ has multiplicity atleast l_m. This establishes the claim. By second fundamental theorem of Nevanlinna, we have

$$\begin{aligned}
(q-2)T(r,g) &\leq \sum_{m=1}^{q}\bar{N}(r, \frac{1}{g-a_m(z_0)}) + o(T(r,g)) \\
&\leq \sum_{m=1}^{q}\frac{1}{l_m}N(r, \frac{1}{g-a_m(z_0)}) + o(T(r,g)) \\
&\leq \sum_{m=1}^{q}\frac{1}{l_m}T(r,g) + o(T(r,g)).
\end{aligned}$$

This is impossible by the fact that $\sum_{m=1}^{q}\frac{1}{l_m} < q - 2$. Thus, $\mathcal{F}$ is a normal family. □

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