Filomat 40:9 (2026), 3187–3195
https://doi.org/10.2298/FIL2609187B

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Affine nontrivial deformation of $\mathfrak{sl}(2)$ and $\mathfrak{osp}(1|2)$-modules of symbols

Khaled Basdouri[a]

[a] *University of Sfax, Faculty of Sciences of Sfax, BP 802, 3038 Sfax, Tunisia*

Abstract. In this paper, we introduce a new notion on deformation: *nonrelative* deformation. We consider the $\mathfrak{sl}(2)$-module structure on the spaces of symbols of differential operators acting on the spaces of weighted densities. If we restrict ourselves to the Lie subalgebra of $\mathfrak{sl}(2)$ generated by $\{X_1, X_x\}$, isomorphic to $\mathfrak{a}(1)$, we get a family of infinite dimensional $\mathfrak{a}(1)$ modules. We compute the necessary and sufficient integrability conditions of a given $\mathfrak{a}(1)$-*nonrelative* infinitesimal deformation of this structure and we prove that any $\mathfrak{a}(1)$-*nonrelative* formal deformation is equivalent to its infinitesimal part. We study also the super analogue of this problem getting the same results. This work is the simplest generalization *nonrelative* of a result by Imed Basdouri et al., [Deformations of $\mathfrak{sl}(2)$ and $\mathfrak{osp}(1|2)$-modules of symbols], **Acta Mathematica Hungarica, Volume 137, Issue 3, Year 2012**

1. Introduction

Let Vect($\mathbb{R}$) be the Lie algebra of vector fields on $\mathbb{R}$. Denote by $\mathcal{F}_\lambda = \left\{ f dx^\lambda \mid f \in C^\infty(\mathbb{R}) \right\}$ the space of weighted densities of weight $\lambda \in \mathbb{R}$. The space $\mathcal{F}_\lambda$ is a Vect($\mathbb{R}$)-module for the action defined by

$$L^\lambda_{g\frac{d}{dx}}(f dx^\lambda) = (g f' + \lambda g' f) dx^\lambda.$$

Any differential operator A on $\mathbb{R}$ can be viewed as the linear mapping $f(dx)^\lambda \mapsto (Af)(dx)^\mu$ from $\mathcal{F}_\lambda$ to $\mathcal{F}_\mu$ (λ, μ in $\mathbb{R}$). Thus the space of differential operators is a Vect($\mathbb{R}$)-module, denoted $\mathrm{D}_{\lambda,\mu} := \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)$. The Vect($\mathbb{R}$) action is:

$$L^{\lambda,\mu}_X(A) = L^\mu_X \circ A - A \circ L^\lambda_X. \tag{1}$$

Feigin and Fuchs computed $\mathrm{H}^1_{\mathrm{diff}}\left(\mathrm{Vect}(\mathbb{R}); \mathrm{D}_{\lambda,\mu}\right)$, see [16]. They showed that non-zero cohomology $\mathrm{H}^1_{\mathrm{diff}}\left(\mathrm{Vect}(\mathbb{R}); \mathrm{D}_{\lambda,\mu}\right)$ only appear if $\mu - \lambda \in \mathbb{N}$.

Each module $\mathrm{D}_{\lambda,\mu}$ has a natural filtration by the order of differential operators; the graded module $\mathcal{S}_{\lambda,\mu} := gr\mathrm{D}_{\lambda,\mu}$ is called the *space of symbols*. The quotient-module $\mathrm{D}^k_{\lambda,\mu}/\mathrm{D}^{k-1}_{\lambda,\mu}$ is isomorphic to module of tensor densities $\mathcal{F}_{\mu-\lambda-k}$, the isomorphism is provided by the principal symbol σ_{pr} defined by

$$A = \sum_{i=0}^{k} a_i(x)\partial^i_x \mapsto \sigma_{pr}(A) = a_k(x)(dx)^{\mu-\lambda-k}$$

2020 *Mathematics Subject Classification*. Primary 17B56 mandatory; Secondary 53D55, 58H15.
Keywords. *Nonrelative* Cohomology, The affine subalgebra, *nonrelative* Deformation, Weighted Densities, Symbols.
Received: 28 November 2025; Accepted: 29 December 2025
Communicated by Ljubica Velimirović
Email address: khaled.basdouri2016@gmail.com (Khaled Basdouri)
ORCID iD: https://orcid.org/0000-0003-2358-0223 (Khaled Basdouri)

As a Vect($\mathbb{R}$)-module, the space $\mathcal{S}_{\lambda,\mu}$ depends only on the difference $\gamma = \mu - \lambda$, so that $\mathcal{S}_{\lambda,\mu}$ can be written as $\mathcal{S}_\gamma$, and we have

$$\mathcal{S}_\gamma = \bigoplus_{k=0}^{\infty} \mathcal{F}_{\gamma-k}.$$

Denote by D_γ the Vect($\mathbb{R}$)-module of differential operators on $\mathcal{S}_\gamma$.

The space $\mathrm{D}_{\lambda,\mu}$ cannot be isomorphic as a Vect($\mathbb{R}$)-module to the corresponding space of symbols, but is a deformation of this space in the sense of Richardson-Neijenhuis [23].

If we restrict ourselves to the Lie subalgebra of Vect($\mathbb{R}$) generated by $\{X_1, X_x, X_{x^2}\}$, isomorphic to $\mathfrak{sl}(2)$, we get a family of infinite dimensional $\mathfrak{sl}(2)$ modules, add $\mathcal{F}_\lambda$, $\mathrm{D}_{\lambda,\mu}$ and D_γ. The affine subalgebra $\mathfrak{a}(1)$ of $\mathfrak{sl}(2)$ is

$$\mathfrak{a}(1) := \{X_1, X_x\}.$$

We are also interested in the study of the analogue super structures, namely, we consider the superspace $\mathbb{R}^{1|1}$ with coordinates (x, ξ) where ξ is the odd variable: $\xi^2 = 0$. This superspace is equipped with the standard contact structure given by the distribution $\langle \overline{D} \rangle$ generated by the vector field $\overline{D} = \partial_\xi - \xi\partial_x$. That is, the distribution $\langle \overline{D} \rangle$ is the kernel of the following 1-form:

$$\alpha = dx + \xi d\xi.$$

Consider the superspace of functions

$$C^\infty(\mathbb{R}^{1|1}) = \{F(x,\xi) = f_0(x) + \xi f_1(x) \mid f_0,\ f_1 \in C^\infty(\mathbb{R})\}$$

and consider the superspace $\mathcal{K}(1)$ of contact vector fields on $\mathbb{R}^{1|1}$. That is, $\mathcal{K}(1)$ is the superspace of vector fields on $\mathbb{R}^{1|1}$ preserving the distribution $\langle \overline{D} \rangle$:

$$\mathcal{K}(1) = \left\{X \in \mathrm{Vect}(\mathbb{R}^{(1|1)} \mid [X, \overline{D}] = F_X \overline{D} \quad \text{for some } F_X \in C^\infty(\mathbb{R}^{1|1})\right\}.$$

We introduce the superspace $\mathfrak{F}_\lambda = \{F\alpha^\lambda \mid F \in C^\infty(\mathbb{R}^{1|1})\}$ of λ-densities on $\mathbb{R}^{1|1}$. This space is a $\mathcal{K}(1)$-module for the action defined by

$$\mathfrak{L}^\lambda_{X_G}(F\alpha^\lambda) = (X_G + \lambda G')(F)\alpha^\lambda.$$

Similarly, we consider the $\mathcal{K}(1)$-module of linear differential operators, $\mathfrak{D}_{\nu,\mu} := \mathrm{Hom}_{\mathrm{diff}}(\mathfrak{F}_\nu, \mathfrak{F}_\mu)$, which is the super analogue of the space $\mathrm{D}_{\nu,\mu}$. The $\mathcal{K}(1)$-action on $\mathfrak{D}_{\nu,\mu}$ is given by

$$\mathfrak{L}^{\lambda,\mu}_{X_F}(A) = \mathfrak{L}^{\mu}_{X_F} \circ A - (-1)^{p(A)p(F)} A \circ \mathfrak{L}^{\lambda}_{X_F}. \tag{2}$$

The Lie superalgebra $\mathfrak{osp}(1|2)$, a super analogue of $\mathfrak{sl}(2)$, can be realized as a subsuperalgebra of $\mathcal{K}(1)$:

$$\mathfrak{osp}(1|2) = \mathrm{Span}\left(X_1,\ X_\xi,\ X_x,\ X_{x\xi},\ X_{x^2}\right).$$

The space of even elements of $\mathfrak{osp}(1|2)$ is isomorphic to $\mathfrak{sl}(2)$:

$$(\mathfrak{osp}(1|2))_0 = \mathrm{Span}(X_1,\ X_x,\ X_{x^2}) = \mathfrak{sl}(2).$$

The affine subalgebra $\mathfrak{a}(1|1)$ of $\mathfrak{osp}(1|2)$ is

$$\mathfrak{a}(1|1) := \{X_1, X_x, X_\xi\}.$$

The super analogue of the space $\mathcal{S}_\gamma$ is naturally the superspace (see [19]):

$$\mathfrak{S}_\delta = \bigoplus_{k\in\mathbb{N}} \mathfrak{F}_{\gamma-\frac{k}{2}}.$$

Denote by $\mathfrak{D}_\gamma$ the $\mathcal{K}(1)$-module of linear differential operators in $\mathfrak{S}_\gamma$. (For more details about supermanifold theory, see for example [1–3, 8–12, 14, 22]).

In this paper, we study the $\mathfrak{a}(1)$-*nonrelative* deformations of the structure of the $\mathfrak{sl}(2)$-modules $\mathcal{S}_\gamma$ and their analogues the $\mathfrak{osp}(1|2)$-modules $\mathfrak{S}_\gamma$. We exhibit the necessary and sufficient integrability conditions of a given $\mathfrak{a}(1)$-*nonrelative* infinitesimal deformation. We prove that any $\mathfrak{a}(1)$-*nonrelative* formal deformation is equivalent to its infinitesimal part and we give an example of $\mathfrak{a}(1)$-*nonrelative* deformation with one parameter. *Nonrelative* deformation is a very important problem in Physics and Science and Engineering [20, 28].

2. $\mathfrak{h}-$*nonrelative* Deformation of $\mathfrak{g}$-modules.

Deformation theory of Lie algebra homomorphisms was first considered with only one-parameter of deformation [2, 3, 11, 23, 27]. Recently, deformations of Lie (super)algebras with multi-parameters were intensively studied (see, e.g., [1, 5–10, 12, 13, 24–26]).

Let $\sigma_0 : \mathfrak{g} \to \mathrm{End}(\mathcal{M})$ be an action of a Lie (super)algebra $\mathfrak{g}$ on a vector (super)space $\mathcal{M}$. It is well known that the first cohomology space $\mathrm{H}^1(\mathfrak{g}; \mathrm{End}(\mathcal{M}))$ determines and classifies infinitesimal deformations up to equivalence. Thus, if $\dim \mathrm{H}^1(\mathfrak{g}; \mathrm{End}(\mathcal{M})) = m$, then choose 1-cocycles $\Lambda_1, \dots, \Lambda_m$ representing a basis of $\mathrm{H}^1(\mathfrak{g}; \mathrm{End}(\mathcal{M}))$ and consider the infinitesimal deformation

$$\sigma = \sigma_0 + \sum_{i=1}^{m} t_i \Lambda_i,$$

where $t_1, \dots, t_m$ are independent parameters with $p(t_i) = p(\Lambda_i)$. We try to extend this infinitesimal deformation to a formal one:

$$\sigma = \sigma_0 + \sum_{i=1}^{m} t_i \Lambda_i + \sum_{i,j} t_i t_j \sigma_{ij}^{(2)} + \cdots ,$$

where $\sigma_{ij}^{(2)}, \sigma_{ijk}^{(3)}, \dots$ are linear maps from $\mathfrak{g}$ to $\mathrm{End}(\mathcal{M})$ with $p(\sigma_{ij}^{(2)}) = p(t_i t_j), p(\sigma_{ijk}^{(3)}) = p(t_i t_j t_k), \dots$ such that

$$[\sigma(x), \sigma(y)] = \sigma([x, y]), \quad x,\ y \in \mathfrak{g}. \tag{3}$$

All the obstructions appear from the condition (3) and it is well known that they lie in $\mathrm{H}^2(\mathfrak{g}, \mathrm{End}(\mathcal{M}))$.

More generally, if $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, then the $\mathfrak{h}-$relative cohomology space $\mathrm{H}^1(\mathfrak{g}, \mathfrak{h}; End(\mathcal{M}))$ measures the infinitesimal deformations that become trivial once the action is restricted to $\mathfrak{h}$ ($\mathfrak{h}$-trivial deformations), while the obstructions to extension of any $\mathfrak{h}$-trivial infinitesimal deformation to a formal one are related to $\mathrm{H}^2(\mathfrak{g}, \mathfrak{h}; End(\mathcal{M}))$.

Definition 2.1. *Let $\varrho_0 : \mathfrak{g} \to \mathrm{End}(\mathcal{M})$ be an action of a Lie (super)algebra $\mathfrak{g}$ on a vector (super)space $\mathcal{M}$. It is well known that the first cohomology space $\mathrm{H}^1(\mathfrak{g}; \mathrm{End}(\mathcal{M}))$ and $H^1(\mathfrak{g}, \mathfrak{h}; End(\mathcal{M}))$. If $\dim (\mathrm{H}^1(\mathfrak{g}; \mathrm{End}(\mathcal{M}))/H^1(\mathfrak{g}, \mathfrak{h}; End(\mathcal{M}))) = \ell$, then choose 1-cocycles $\Lambda_1, \dots, \Lambda_\ell$ representing a basis of $\mathrm{H}^1_{\mathfrak{h}}(\mathfrak{g}; \mathrm{End}(\mathcal{M})) =: \mathrm{H}^1(\mathfrak{g}; \mathrm{End}(\mathcal{M}))/H^1(\mathfrak{g}, \mathfrak{h}; End(\mathcal{M}))$. Thus, $\mathrm{H}^1_{\mathfrak{h}}(\mathfrak{g}; \mathrm{End}(\mathcal{M}))$ determines and classifies $\mathfrak{h}-$nonrelative infinitesimal deformations up to equivalence. Consider the $\mathfrak{h}-$nonrelative infinitesimal deformation*

$$\varrho = \varrho_0 + \sum_{i=1}^{\ell} t_i \Lambda_i,$$

where $t_1, \dots, t_m$ are independent parameters with $p(t_i) = p(\Lambda_i)$. We try to extend this $\mathfrak{h}-$nonrelative infinitesimal deformation to a formal one:

$$\varrho = \varrho_0 + \sum_{i=1}^{\ell} t_i \Lambda_i + \sum_{i,j} t_i t_j \varrho_{ij}^{(2)} + \cdots ,$$

where $\varrho_{ij}^{(2)}, \varrho_{ijk}^{(3)}, \ldots$ *are linear maps from* $\mathfrak{g}$ *to* $\mathrm{End}(\mathcal{M})$ *with* $p(\varrho_{ij}^{(2)}) = p(t_it_j), p(\varrho_{ijk}^{(3)}) = p(t_it_jt_k), \ldots$ *such that*

$$[\varrho(x), \varrho(y)] = \varrho([x, y]), \quad x, y \in \mathfrak{g}. \tag{4}$$

All the obstructions appear from the condition (4) and it is well known that they lie in $\mathrm{H}^2_{\mathfrak{h}}(\mathfrak{g}; \mathrm{End}(\mathcal{M}))$.

2.1. $\mathfrak{a}(1)$-*nonrelative-Deformation of the* $\mathfrak{sl}(2)$-*Modules of Symbols*

In the following, the differential cohomology; that is, only cochains given by differential operators are considered. Now we study the $\mathfrak{a}(1)$-*nonrelative* formal deformations of the $\mathfrak{sl}(2)$-module structure on the space of symbols:

$$\mathcal{S}_\gamma = \bigoplus_{k\geq 0} \mathcal{F}_{\gamma-k}.$$

The $\mathfrak{a}(1)$-*nonrelative* infinitesimal deformations are described by the $\mathfrak{a}(1)$-*nonrelative* cohomology space

$$\mathrm{H}^1_{\mathfrak{a}(1)}(\mathfrak{sl}(2), \mathrm{D}_\gamma) = \bigoplus_{i,j\geq 0} \mathrm{H}^1_{\mathfrak{a}(1)}\left(\mathfrak{sl}(2), \mathrm{D}_{\gamma-j,\gamma-i}\right).$$

In fact, Lecomte computed $\mathrm{H}^1\left(\mathfrak{sl}(2), \mathrm{D}_{\lambda,\mu}\right)$, see [21]. He showed that non-zero $\mathfrak{a}(1)$-*nonrelative* cohomology $\mathrm{H}^1_{\mathfrak{a}(1)}\left(\mathfrak{sl}(2), \mathrm{D}_{\lambda,\mu}\right)$ only appear if $\lambda = \mu$ or $(\lambda, \mu) = (\frac{1-k}{2}, \frac{1+k}{2})$ where $k \in \mathbb{N}^*$. Thus, we distinguish two cases:

(i) If $\gamma \notin \frac{1}{2}(\mathbb{N}+2)$, then

$$\mathrm{H}^1_{\mathfrak{a}(1)}(\mathfrak{sl}(2), \mathrm{D}_\gamma) = \bigoplus_{k\geq 0} \mathrm{H}^1_{\mathfrak{a}(1)}\left(\mathfrak{sl}(2), \mathrm{D}_{\gamma-k,\gamma-k}\right).$$

The space $\mathrm{H}^1_{\mathfrak{a}(1)}(\mathfrak{sl}(2), \mathrm{D}_{\lambda,\lambda})$ is one dimensional and it is spanned by the $\mathfrak{a}(1)$-*nonrelative* cohomology classe of the cocycle Γ_λ given by

$$\Gamma_\lambda(F\tfrac{d}{dx})(f dx^\lambda) = F' f dx^\lambda.$$

(ii) If $2\gamma = \ell \in (\mathbb{N}+2)$, then

$$\mathrm{H}^1_{\mathfrak{a}(1)}(\mathfrak{sl}(2), \mathrm{D}_\gamma) = \bigoplus_{k=[\frac{\ell+1}{2}]}^{\ell-1} \mathrm{H}^1_{\mathfrak{a}(1)}\left(\mathfrak{sl}(2), \mathrm{D}_{\frac{\ell-2k}{2},\frac{2+2k-\ell}{2}}\right) \oplus \bigoplus_{k\geq 0} \mathrm{H}^1_{\mathfrak{a}(1)}\left(\mathfrak{sl}(2), \mathrm{D}_{\frac{\ell}{2}-k,\frac{\ell}{2}-k}\right).$$

The space $\mathrm{H}^1_{\mathfrak{a}(1)}\left(\mathfrak{sl}(2), \mathrm{D}_{\frac{\ell-2k}{2},\frac{2+2k-\ell}{2}}\right)$ is one dimensional and spanned by the $\mathfrak{a}(1)$-*nonrelative* cohomology classes of the 1-cocycles, Θ_k given by

$$\Theta_k(F\tfrac{d}{dx})(f dx^{\frac{\ell-2k}{2}}) = F' f^{(2k-\ell+1)} dx^{\frac{2+2k-\ell}{2}}.$$

In our study, a $\mathfrak{a}(1)$−*nonrelative* infinitesimal deformation of the $\mathfrak{sl}(2)$-module structure on the space $\mathcal{S}_\gamma$ is of the form

$$\mathcal{L}_X = L_X + \mathcal{L}_X^{(1)}, \tag{5}$$

where L_X is the Lie derivative of D_γ along the vector field X defined by (1), and

$$\mathcal{L}_X^{(1)} = \begin{cases} \sum_{k\geq 0} \beta_k \Gamma_{\frac{\ell}{2}-k}(X) & \text{if } \gamma \notin \frac{1}{2}(\mathbb{N}+2) \\ \sum_{k\geq 0} \beta_k \Gamma_{\frac{\ell}{2}-k}(X) + \sum_{k=[\frac{\ell+1}{2}]}^{\ell-1} \theta_k \Theta_k(X) & \text{if } 2\gamma = \ell \in (\mathbb{N}+2), \end{cases} \tag{6}$$

and where a_k and b_k are independent parameters.

Theorem 2.2. *The following conditions are necessary and sufficient for integrability of the* $\mathfrak{a}(1)-$*nonrelative infinitesimal deformation (5):*

$$(2k-\ell+1)\theta_k\beta_{\ell-k-1}=0,\quad [\tfrac{\ell+1}{2}]\leq k\leq \ell-1\ . \tag{7}$$

Moreover, any formal $\mathfrak{a}(1)-$*nonrelative deformation is equivalent to its infinitesimal part.*

Proof. Note that if $2\gamma=\ell\notin(\mathbb{N}+2)$ then the parameters a_k can be assumed to be zero, and then, there are no integrability conditions. Assume that the $\mathfrak{a}(1)-$*nonrelative* infinitesimal deformation (5) can be integrated to a $\mathfrak{a}(1)-$*nonrelative* formal deformation

$$\mathcal{L}_X=L_X+\mathcal{L}_X^{(1)}+\mathcal{L}_X^{(2)}+\mathcal{L}_X^{(3)}+\cdots$$

where $\mathcal{L}_X^{(1)}$ is given by (6) and $\mathcal{L}_X^{(2)}$ is a quadratic polynomial in a_k and b_k with coefficients in $\mathcal{D}_\gamma$. We compute the conditions for the second-order terms $\mathcal{L}^{(2)}$.

Consider the quadratic terms of the homomorphism condition

$$[\mathcal{L}_X,\mathcal{L}_Y]=\mathcal{L}_{[X,Y]}. \tag{8}$$

By a straightforward computation, the homomorphism condition (8) gives for the second-order terms the following equation

$$\delta(\mathcal{L}^{(2)})=-\mathcal{L}_X^{(1)}\vee\mathcal{L}_X^{(1)}, \tag{9}$$

where δ is the Chevalley-Eilenberg differential and $\vee$ stands for the cup-product of 1-cocycles, so that the right hand side of expression is automatically a 2-cocycle. In our case, we obtain explicitly:

$$\delta(\mathcal{L}^{(2)})=-(\sum_{k\geq0}\beta_k\,\Gamma_{\frac{\ell}{2}-k}(X)+\sum_{k=[\frac{m+1}{2}]}^{\ell-1}\theta_k\,\Theta_k(X))\vee(\sum_{k\geq0}\beta_k\,\Gamma_{\frac{\ell}{2}-k}(X)+\sum_{k=[\frac{\ell+1}{2}]}^{\ell-1}\theta_k\,\Theta_k(X)). \tag{10}$$

By a straightforward computation, the homomorphism condition (8) gives the following Maurer-Cartan equation for the second-order terms:

$$\delta(\mathcal{L}^{(2)})=\frac{1}{2}\sum_{k=[\frac{\ell+1}{2}]}^{\ell-1}(2k-m+1)\theta_k\beta_{\ell-k-1}\omega_{2k-\ell+1}, \tag{11}$$

where ω_k is the trivial 2 cocycle given by

$$\omega_k(F\tfrac{d}{dx},G\tfrac{d}{dx})(fdx^{\frac{1-k}{2}})=(F'G''-F''G')f^{(k-1)}dx^{\frac{1+k}{2}} \tag{12}$$

So there are integrability conditions, because $\omega_{2k-\ell+1}\in \mathrm{H}^2_{\mathfrak{a}(1)}\left(\mathfrak{sl}(2),\mathrm{D}_{\frac{\ell-2k}{2},\frac{2+2k-\ell}{2}}\right)$.

The solution $\mathcal{L}^{(2)}$ of (11) can be chosen identically zero. Now, we show that these conditions are sufficient. Choosing the highest-order terms $\mathcal{L}^{(m)}$ with $m\geq3$, also identically zero, one obviously obtains a $\mathfrak{a}(1)-$*nonrelative* deformation (which is of order 1 in β_k and θ_k). Any $\mathfrak{a}(1)-$*nonrelative* formal deformation is equivalent to its infinitesimal part since different choices of solutions of the Maurer-Cartan equation correspond to equivalent $\mathfrak{a}(1)-$*nonrelative* deformations. □

Example 2.3. *Let us consider* $\gamma=\frac{\ell}{2}\in\frac{1}{2}(\mathbb{N}+2)$ *and let* $(\alpha_k)_{k\geq0}$ *be a sequence of real numbers such that, for* $[\frac{\ell+1}{2}]\leq k\leq\ell-1$.*. Put* $\theta_k=t$ *and* $\beta_k=\alpha_k t$*. So, we obtain a* $\mathfrak{a}(1)-$*nonrelative deformation of* $\mathcal{S}_\gamma$ *with one parameter t:*

$$\mathcal{L}_X=L_X+t(\sum_{k\geq0}\alpha_k\,\Gamma_{\frac{m}{2}-k}(X)+\sum_{k=[\frac{m+1}{2}]}^{\ell-1}\Theta_k(X)).$$

Of course it is easy to give many other examples of true $\mathfrak{a}(1)-$*nonrelative deformations with one parameter or with several parameters.*

2.2. $\mathfrak{a}(1|1)$-*nonrelative Deformation of the* $\mathfrak{osp}(1|2)$-*Modules of Symbols*

We study the super analogous of the previous case. That is, we study $\mathfrak{a}(1|1)$-*nonrelative* deformations of the $\mathfrak{osp}(1|2)$-module of differential linear operators in the space of symbols on $\mathbb{R}^{1|1}$:

$$\mathfrak{S}_\gamma = \bigoplus_{k\geq 0} \mathfrak{F}_{\gamma-\frac{k}{2}}.$$

The infinitesimal $\mathfrak{a}(1|1)$-*nonrelative* deformations are described by the cohomology space

$$\mathrm{H}^1_{\mathfrak{a}(1|1)}(\mathfrak{osp}(1|2), \mathfrak{D}_\gamma) = \bigoplus_{i,j\geq 0} \mathrm{H}^1_{\mathfrak{a}(1|1)}\left(\mathfrak{osp}(1|2), \mathfrak{D}_{\gamma-\frac{j}{2},\gamma-\frac{i}{2}}\right).$$

In [4], it was proved that non-zero cohomology $\mathrm{H}^1\left(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu}\right)$ only appear if $\lambda = \mu$ or $(\lambda, \mu) = (\frac{1-k}{2}, \frac{k}{2})$ where $k \in \mathbb{N}$. Thus, as before, we have to distinguish two cases:

(i) If $\gamma \notin \frac{1}{2}(\mathbb{N} + 1)$, then

$$\mathrm{H}^1_{\mathfrak{a}(1|1)}(\mathfrak{osp}(1|2), \mathfrak{D}_\gamma) = \bigoplus_{k\geq 0} \mathrm{H}^1_{\mathfrak{a}(1|1)}\left(\mathfrak{osp}(1|2), \mathfrak{D}_{\gamma-\frac{k}{2},\gamma-\frac{k}{2}}\right).$$

The space $\mathrm{H}^1_{\mathfrak{a}(1|1)}\left(\mathfrak{osp}(1|2), \mathfrak{D}_{\frac{2\gamma-k}{2},\frac{2\gamma-k}{2}}\right)$ is one dimensional and it is spanned by the $\mathfrak{a}(1|1)-$*nonrelative* cohomology classe of the cocycle $\Lambda'_{2\gamma-k}$ given by

$$\Lambda'_{2\gamma-k}(F\tfrac{d}{dx}) = F'.$$

(ii) If $2\gamma = \ell \in (\mathbb{N} + 1)$, then

$$\mathrm{H}^1_{\mathfrak{a}(1|1)}(\mathfrak{osp}(1|2), \mathfrak{D}_\gamma) = \bigoplus_{k=1}^{\ell} \mathrm{H}^1_{\mathfrak{a}(1|1)}\left(\mathfrak{osp}(1|2), \mathfrak{D}_{\frac{1-k}{2},\frac{k}{2}}\right) \oplus \bigoplus_{k\geq 0} \mathrm{H}^1_{\mathfrak{a}(1|1)}\left(\mathfrak{osp}(1|2), \mathfrak{D}_{\frac{\ell-k}{2},\frac{\ell-k}{2}}\right).$$

The space $\mathrm{H}^1_{\mathfrak{a}(1|1)}\left(\mathfrak{osp}(1|2), \mathfrak{D}_{\frac{1-k}{2},\frac{k}{2}}\right)$ is one dimensional and spanned by the $\mathfrak{a}(1|1)-$*nonrelative* cohomology classes of the 1-cocycles, Λ_k given by

$$\Lambda_k(X_G) = (-1)^{|G|} D^2(G)\overline{D}^{2k-1}.$$

Any $\mathfrak{a}(1|1)$-*nonrelative* infinitesimal deformation of the $\mathfrak{osp}(1|2)$-module structure on $\mathfrak{S}_\gamma$ is of the form

$$\widetilde{\mathfrak{L}}_{X_F} = \mathfrak{L}_{X_F} + \mathfrak{L}^{(1)}_{X_F}, \tag{13}$$

where $\mathfrak{L}_{X_F}$ is the Lie derivative of $\mathfrak{D}_\gamma$ along the vector field X_F defined by (2), and

$$\mathfrak{L}^{(1)}_{X_F} = \begin{cases} \sum_{k\geq 0} \mathfrak{a}_{2\gamma-k}\, \Lambda'_{2\gamma-k}(X_F) & \text{if } \gamma \notin (\frac{1}{2}\mathbb{N} + 1) \\ \sum_{k\geq 0} \mathfrak{a}_{\ell-k}\, \Lambda'_{\ell-k}(X_F) + \sum_{k=1}^{\ell} \mathfrak{b}_k\, \Lambda_k(X_F) & \text{if } 2\gamma = \ell \in (\mathbb{N} + 1), \end{cases}$$

and where $\mathfrak{a}_k$ and $\mathfrak{b}_k$ are independent parameters.

Our main result in the super setting is the following

Theorem 2.4. *The following conditions are necessary and sufficient for integrability of the* $\mathfrak{a}(1|1)$-*nonrelative infinitesimal deformation (13):*

$$\mathfrak{b}_k \mathfrak{a}_{1-k} = 0, \quad 1 \leq k \leq m\ . \tag{14}$$

Moreover, any $\mathfrak{a}(1|1)$-*nonrelative formal deformation is equivalent to its infinitesimal part.*

Proof. Assume that the $\mathfrak{a}(1|1)$-*nonrelative* infinitesimal deformation (13) can be integrated to a $\mathfrak{a}(1|1)$-*nonrelative* formal deformation:

$$\widetilde{\mathfrak{L}}_{X_F} = \mathfrak{L}_{X_F} + \mathfrak{L}^{(1)}_{X_F} + \mathfrak{L}^{(2)}_{X_F} + \cdots$$

By a straightforward computation, the homomorphism condition

$$[\widetilde{\mathfrak{L}}_{X_F}, \widetilde{\mathfrak{L}}_{X_G}] = \widetilde{\mathfrak{L}}_{X_{\{F,G\}}}$$

gives for the second-order terms the following equation

$$\delta(\mathfrak{L}^{(2)}) = \sum_{k=1}^{m} \mathfrak{b}_k \mathfrak{a}_{1-k} \Phi_k$$

where $\Phi_k : \mathfrak{osp}(1|2) \times \mathfrak{osp}(1|2) \to \mathfrak{D}_{\frac{1-k}{2},\frac{k}{2}}$ is defined by

$$\Phi_k(X_F, X_G) = (-1)^{p(F)+p(G)}(k-1)(F'G'' - F''G')\overline{D}^{2k-3}.$$

□

We will prove the following lemma and then we conclude as for Theorem 2.2.

Lemma 2.5. *The map Φ_k is a nonrelative and nonrelative odd 2 cocycle:*

$$\Phi_k \in \mathrm{H}^2_{\mathfrak{a}(1|1)}\left(\mathfrak{osp}(1|2), \mathfrak{D}_{\frac{1-k}{2},\frac{k}{2}}\right).$$

Proof. The map Ω_k is a 2 cocycle since it is the cup-product of 1 cocycles. It is easy to see that Ω_k is an odd map, so, $\Phi_k(\mathfrak{sl}(2) \times \mathfrak{sl}(2)) \subset \left(\mathfrak{D}_{\frac{1-k}{2},\frac{k}{2}}\right)_1$. In [6] it was proved that, as $\mathfrak{sl}(2)$-module, we have

$$\left(\mathfrak{D}_{\frac{1-k}{2},\frac{k}{2}}\right)_1 \simeq \Pi\left(\mathrm{D}_{\frac{2-k}{2},\frac{k}{2}} \oplus \mathrm{D}_{\frac{1-k}{2},\frac{1+k}{2}}\right) \tag{15}$$

where Π is the change of parity operator. We check that the restriction of Φ_k to $\mathfrak{sl}(2) \times \mathfrak{sl}(2)$ is a *nonrelative* 2 cocycle. Indeed, let X_F, $X_G \in \mathfrak{sl}(2) \subset \mathfrak{osp}(1|2)$, it is easy to see that

$$(-1)^k \Phi_k(X_F, X_G) = (k-1)\omega_{k-1}(X_F, X_G) \circ \partial_\xi,$$

or equivalently, according to the decomposition (15), we have

$$(-1)^k \Phi_k|_{\mathfrak{sl}(2)\times\mathfrak{sl}(2)} = \Pi \circ ((k-1)\omega_{k-1})$$

where ω_k is the *nonrelative* 2 cocycle defined by (12). Thus, Φ_k is a *nonrelative* 2 cocycle. □

Obviously, as for the $\mathfrak{sl}(2)$-module $\mathcal{S}_\gamma$, it easy to construct many examples of true $\mathfrak{a}(1|1)$-*nonrelative* deformations of the $\mathfrak{osp}(1|2)$-module $\mathfrak{S}_\gamma$ with one parameter or with several parameters.

Example 2.6. *Let us consider $\gamma = \frac{\ell}{2} \in \frac{1}{2}(\mathbb{N}+1)$ and let $(\zeta_k)_{k\in\mathbb{Z}}$ be a sequence of real numbers such that, for $1 \le k \le \ell$, we have $\zeta_k \neq \zeta_{1-k}$. Put $\mathfrak{b}_k = t$ and $\mathfrak{a}_k = \zeta_k\, t$. So, we obtain a $\mathfrak{a}(1|1)$-nonrelative deformation of $\mathfrak{S}_\gamma$ with one parameter t:*

$$\mathfrak{L}^{(1)}_{X_F} = t\sum_{k\ge 0} \zeta_{\ell-k}\, \Lambda'_{\ell-k}(X_F) + t\sum_{k=1}^{\ell} \Lambda_k(X_F).$$

3. Open problems

3.1. Problem 1

The $\mathfrak{sl}(2)$-*nonrelative* cohomology $\mathrm{H}^1_{\mathfrak{sl}(2)}\left(\mathrm{Vect}(\mathbb{R}); \mathrm{D}_{\lambda,\mu}\right)$ determines and classifies $\mathfrak{sl}(2)$ *nonrelative* infinitesimal deformations up to equivalence.

Conjecture: Any $\mathfrak{sl}(2)$-*nonrelative* formal deformation is **no** equivalent to its infinitesimal part.

3.2. Problem 2

Determine the space of $\mathfrak{a}(1)$-*nonrelative* cohomology $\mathrm{H}^1_{\mathfrak{a}(1)}\left(\mathrm{Vect}(\mathbb{R}); \mathrm{D}_{\lambda,\mu}\right)$ and study the affine *nonrelative* deformation.

The authors have no conflict of interest to declare that are relevant to this article.

References

[1] M. Afi, O. Basdouri, On the isomorphism of non-abelian extensions of n-Lie algebras, J. Geom. Phys. Volume 173, March 2022, 104427.

[2] B. Agrebaoui I. Basdouri, S. Hammami, S. Saidi, *Deforming the orthosymplectic Lie superalgebra* $\mathfrak{osp}(3|2)$ *inside the Lie superalgebra of superpseudodifferential operators* $\mathcal{S}\psi\mathcal{DO}(3)$. Accept in Advanced Studies Euro-Tbilisi Mathematical Journal.

[3] I. Basdouri, S. Benabdelhafidh, A. Saghrouni, S. Silvestrov, *Cohomology and deformations of crossed homomorphisms on Lie conformal algebras*. October 2025 Advanced Studies Euro-Tbilisi Mathematical Journal 18(3).

[4] I. Basdouri, M. Ben Ammar, *Cohomology of* $\mathfrak{osp}(1|2)$ *Acting on Linear Differential Operators on the Supercercle* $S^{1|1}$. Letters in Mathematical Physics(2007)81:239–251.

[5] I. Basdouri, M. Ben Ammar, *Deformations of* $\mathfrak{sl}(2)$ *and* $\mathfrak{osp}(1|2)$*-modules of symbols.* Acta Mathematica Hungarica, Volume 137, Issue 3, Year 2012.

[6] I. Basdouri, M. Ben Ammar, N. Ben Fraj, M. Boujelbene and K. Kammoun, *Cohomology of the Lie Superalgebra of Contact Vector Fields on* $\mathbb{R}^{1|1}$ *and Deformations of the Superspace of Symbols* . Jour of Nonlinear Math Physics, Vol. 16, No. 4 (2009) 1–37.

[7] I. Basdouri, M. Ben Ammar, B. Dali and S. Omri, *Deformation of* $\mathfrak{vect}(1)$*-Modules of Symbols.* doi:10.1016/j.geomphys.2009.12.002.

[8] I. Basdouri, M. Ben Nasr, S. Chouaibi and H. Mechi, Construction of supermodular forms using differential operators from a given supermodular form. Journal of Geometry and Physics, August 2019. DOI: 10.1016/j.geomphys.2019.103488.

[9] I. Basdouri, A. Derbali, S. Saidi, *Second Cohomology of* $\mathfrak{aff}(1)$ *Acting on n-ary Differential Operators.* Bull. Korean Math. Soc. 56(1): 13-22 January 2019.

[10] K. Basdouri et al. *The Binary* $\mathfrak{aff}(n|1)$*-Invariant Differential Operators On Weighted Densities On The Superspace* $\mathbb{R}^{1|n}$ *And* $\mathfrak{aff}(n|1)$*-Relative Cohomology*. Journal of Geometry and Physics, Volume 123, p. 235-245.

[11] K. Basdouri et al. *Cohomology and Deformation of* $\mathfrak{aff}(1|1)$ *Acting on Differential Operators*. International Journal of Geometric Methods in Modern Physics, Dec 2017.

[12] K. Basdouri, G.Chaabane. *Cohomology of Bihom-Lie superbialgebras*. Journal of Geometry and Physics, Volume 221, March 2026, 105714.

[13] M. Ben Ammar, M. Boujelbene, $\mathfrak{sl}(2)$*-Trivial Deformation of* $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$*-Modules of Symbols.* SIGMA 4 (2008), 065, 19 pages.

[14] F. Berezin, Introduction to superanalysis. In: Mathematical Physics and Applied Mathematics, vol. 9. Reidel Dordrecht (1987)

[15] S. Chouaibi, A. Zbidi, Cohen brackets on supermodular forms, Journal of Geometry and Physics 146(1) (2019), DOI: 10.1016/j.geomphys.2019.07.004.

[16] B. L. Feigin, D. B. Fuchs, Homology of the Lie algebras of vector fields on the line, *Func. Anal. Appl.*, 14 (1980) 201–212.

[17] A. Fialowski, "Deformations of Lie algebras," Mat. Sbornyik USSR,, 127 (169), (1985), pp. 476–482; English translation: Math. USSR-Sb., 55, (1986), no. 2, 467–473

[18] A. Fialowski, "An example of formal deformations of Lie algebras" NATO Conference on Deformation Theory of Algebras and Applications, Il Ciocco, Italy, 1986, Proceedings. Kluwer, Dordrecht, 1988, pp. 375–401

[19] H. Gargoubi, N. Mellouli and V. Ovsienko *Differential Operators on Supercircle: Conformally Equivariant Quantization and Symbol Calculus*, Letters in Mathematical Physics (2007) **79**: 51–65.

[20] P. Gupta1, P. Dhar, D. Samanta1 *Rheology and electro-magnetism stimulated non-trivial deformation dynamics of viscoelastic compound droplets.* Proceedings of the Royal Society A, May 2025 Volume 481 Issue 2314.

[21] P. B. A. Lecomte, *On the cohomology of* $\mathfrak{sl}(n+1;\mathbb{R})$ *acting on differential operators and* $\mathfrak{sl}(n+1;\mathbb{R})$*-equivariant symbols,* Indag. Math. NS. 11 (1), (2000), 95 114.

[22] D. Leites, Supermanifold Theory. (in Russian) USSE Academy of Science Petrozavodsk (1983)

[23] A. Nijenuis, R. W. Richardson Jr., *Deformations of homomorphisms of Lie groups and Lie algebras*, Bull. Amer. Math. Soc. **73** (1967), 175–179.

[24] V. Ovsienko, C. Roger, *Deforming the Lie algebra of vector fields on* S^1 *inside the Lie algebra of pseudodifferential operators on* S^1, AMS Transl. Ser. 2, (Adv. Math. Sci.) vol. 194 (1999) 211–227.

[25] V. Ovsienko, C. Roger, *Deforming the Lie algebra of vector fields on* S^1 *inside the Poisson algebra on* $\dot{T}^*S^1$, Comm. Math. Phys., **198** (1998) 97–110.

[26] V. Ovsienko, C. Roger, *Deforming the Lie algebra of vector fields on* S^1 *inside the Poisson algebra on* $\dot{T}^*S^1$, Comm. Math. Phys., **198** (1998) 97–110.

[27] R.W. Richardson, *Deformations of subalgebras of Lie algebras*, J. Diff. Geom. , 3, (1969), 289-308.
[28] E.V. Vtorushin, V.N. Dorovsky *Application of non-stationary non-Euclidean model of inelastic deformations to rock cutting.* Journal of Petroleum Science and Engineering Volume 177, June 2019, Pages 508-517.