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A panorama of generating functions for products of classical integer sequences

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Abstract. In this note, we present the generating function

$$\sum_{n=0}^{\infty} U_{n+\ell}(x)\, U_n(y)\, t^n = \frac{(1-t^2)\, U_\ell(x) + 2t(tx-y)\, U_{\ell-1}(x)}{1-4xy\,t + 2(2x^2+2y^2-1)\,t^2 - 4xy\,t^3 + t^4}\,,$$

where U_n denotes the Chebyshev polynomial of the second kind of order n and ℓ is an integer. Although this generating function can be deduced from classical results, the matrix-based derivation presented here provides a unified and alternative perspective, drawing on lesser-known references. We give a concise historical overview from algebraic, trigonometric, combinatorial, and matrix perspectives for this type of products involving Chebyshev polynomials. Particular cases and extensions are discussed.We also derive a unified family of generating functions for products of classical integer sequences, including Fibonacci, Lucas, Pell, Jacobsthal, Mersenne, and general Horadam families, culminating in a comprehensive list of explicit generating functions.

1. A brief historical context

Perhaps the best way to begin tracing the history of generating functions for products of polynomial families is with the influential work of the distinguished British mathematician G.N. Watson. In 1933, Watson published a remarkable set of three papers on generating functions of polynomials and their products. In these works, he introduced powerful new techniques, drawing on integral representations and

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elliptic integrals, that opened fresh avenues for studying generating functions of orthogonal polynomials. The third paper in the series [36], written in Watson's engaging and characteristically informal style, reveals the spark that set his investigation in motion: a question posed to him by another pioneering British mathematician, G.H. Hardy. Hardy had asked whether there might exist a function $f_n(t)$, with $f_n(1) = 1$, such that the series

$$\sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) f_n(t) P_n(\cos\theta) P_n(\cos\phi)$$

"had a simple sum", as Watson put it. Watson solved first a probably more simple question involving only the products $P_n(\cos\theta)P_n(\cos\phi)$ of Legendre polynomials evaluated at angular coordinates showing:

$$\begin{aligned}\sum_{n=0}^{\infty} P_n(\cos\theta)\, P_n(\cos\phi)\, t^n &= \frac{1}{\pi}\int_0^{\pi} \left(\sum_{n=0}^{\infty} t^n P_n(\cos\theta\cos\phi + \sin\theta\sin\phi\cos\omega)\right) d\omega \\ &= \frac{1}{\pi}\int_0^{\pi} \frac{d\omega}{\sqrt{1 - 2t(\cos\theta\cos\phi + \sin\theta\sin\phi\cos\omega) + t^2}}\,. \end{aligned} \tag{1}$$

He elegantly noted that this identity "is as old as Legendre", highlighting its origins in Adrien-Marie Legendre's 1789 memoir *Histoire de l'Académie Royale des Sciences* [19]. This seminal work laid the foundation for spherical harmonics and potential theory, and Watson's remark underscores the deep historical roots of the formulas he was investigating.

Notwithstanding, Watson goes further by extending the approach to Gegenbauer (or ultraspherical) polynomials $C_n^{\nu}(x)$ [8], defined by the generating function

$$(1 - 2xt + t^2)^{-\nu} = \sum_{n=0}^{\infty} t^n C_n^{\nu}(x)\,,$$

to the key algebraic simplified integral representation:

$$2^{2\nu-1}\,\Gamma^2(\nu) \sum_{n=0}^{\infty} \frac{n!(n+\nu)}{\Gamma(n+2\nu)}\, C_n^{\nu}(\cos\theta) C_n^{\nu}(\cos\phi)\, t^n = \int_0^{\pi} \frac{\nu(1-t^2)\sin^{2\nu-1}\omega\, d\omega}{[1 - 2t(\cos\theta\cos\phi + \sin\theta\sin\phi\cos\omega) + t^2]^{\nu+1}}\,, \tag{2}$$

where Γ refers to the Gamma function.

These identities are not only structurally refined but also analytically powerful, as they convert infinite sums into closed-form expressions, thereby facilitating deeper insights into both asymptotic behavior and functional relationships. In particular, they generalize the seminal work of the German mathematician Gustav Mehler [22], who provided the generating function for the product of two Hermite polynomials, each in different variables:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, H_n(x)\, H_n(y) = \frac{1}{\sqrt{1-t^2}}\, \exp\left(\frac{2txy - t^2(x^2+y^2)}{1-t^2}\right).$$

This remarkable formula has since found widespread applications in harmonic analysis, quantum mechanics, and probability theory.

From Watson's formula (2), we can derive the corresponding identity for Chebyshev polynomials of the second kind by setting $\nu = 1$. Indeed, recalling that

$$C_n^1(\cos\theta) = U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \tag{3}$$

defines the Chebyshev polynomial of the second kind of order n, and letting $x = \cos\theta$ and $y = \cos\phi$, we obtain

$$C_n^1(\cos\theta) = U_n(x), \quad C_n^1(\cos\phi) = U_n(y), \quad \sin\theta = \sqrt{1-x^2}, \quad \sin\phi = \sqrt{1-y^2}.$$

After straightforward simplifications and trigonometric manipulations, this yields

$$\begin{aligned}\sum_{n=0}^{\infty} U_n(x)\,U_n(y)\,t^n &= \int_0^{\pi} \frac{(1-t^2)\,\sin\omega\,d\omega}{\left[1-2t(\cos\theta\cos\phi+\sin\theta\sin\phi\cos\omega)+t^2\right]^2} \\ &= \frac{1-t^2}{\left[1-2t\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\right)+t^2\right]\left[1-2t\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)+t^2\right]} \\ &= \frac{1-t^2}{1-4xy\,t+2(2x^2+2y^2-1)\,t^2-4xy\,t^3+t^4}. \end{aligned} \tag{4}$$

About 25 years after Watson's seminal work, Leonard Carlitz [3] provided yet another perspective by employing hypergeometric functions rather than Watson's integral representation:

$$\sum_{n=0}^{\infty} \frac{n!}{(2\nu)_n} t^n C_n^{\nu}(\cos\alpha)\,C_n^{\nu}(\cos\beta) = \left(1-2t\cos(\alpha+\beta)+t^2\right)^{-\nu} \times {}_2F_1\left(\nu,\nu;2\nu;\frac{4t\sin\alpha\sin\beta}{1-2t\cos(\alpha+\beta)+t^2}\right), \tag{5}$$

where $(\cdot)_n$ denotes the Pochhammer symbol and ${}_2F_1$ is the Gauss hypergeometric function defined by the power series

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n.$$

Carlitz's note was largely motivated by Maximon, who two years earlier had independently published the special case for Legendre polynomials in [21].

Setting $\nu = 1$, Carlitz's formula (5) becomes:

$$\sum_{n=0}^{\infty} \frac{t^n}{n+1} U_n(\cos\alpha)\,U_n(\cos\beta) = \frac{1}{1-2t\cos(\alpha+\beta)+t^2} \cdot {}_2F_1\left(1,1;2;\frac{4t\sin\alpha\sin\beta}{1-2t\cos(\alpha+\beta)+t^2}\right).$$

Using the known identity for the hypergeometric function:

$${}_2F_1(1,1;2;z) = -\frac{\ln(1-z)}{z}, \quad |z|<1,$$

Carlitz's result can be transformed into a logarithmic form, providing a neat representation of the generating function for $\nu = 1$. In fact, (5) simplifies to a symmetric form:

$$\sum_{n=0}^{\infty} \frac{t^n}{n+1} U_n(\cos\alpha)\,U_n(\cos\beta) = \frac{1}{4t\sin\alpha\,\sin\beta} \ln\left(\frac{1-2t\cos(\alpha+\beta)+t^2}{1-2t\cos(\alpha-\beta)+t^2}\right). \tag{6}$$

Expressing this in algebraic form, with $x = \cos\alpha$ and $y = \cos\beta$ as before, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1} U_n(x)\,U_n(y) &= \frac{1}{4\sqrt{1-x^2}\,\sqrt{1-y^2}} \ln\left(\frac{1-2(xy-\sqrt{1-x^2}\sqrt{1-y^2})t+t^2}{1-2(xy+\sqrt{1-x^2}\sqrt{1-y^2})t+t^2}\right) \\ &= \frac{1}{2\sqrt{1-x^2}\,\sqrt{1-y^2}} \operatorname{arctanh}\left(\frac{2t\sqrt{1-x^2}\,\sqrt{1-y^2}}{1-2xyt+t^2}\right). \end{aligned}$$

Finally, differentiating the first equality with respect to t and simplifying the resulting expression through straightforward, though somewhat involved, algebraic manipulations produces the unweighted generating function (4).

Fast forward two decades, prompted by Marco Schützenberger to find a combinatorial proof of Mehler's formula, the French combinatorialist Dominique Foata developed an elegant and influential proof that combined combinatorial reasoning with graph-theoretic methods, bridging enumerative combinatorics and classical orthogonal polynomial theory [6]. His approach remains a cornerstone of modern enumerative techniques and has served as a model for numerous subsequent generalisations. Building on this direction shortly thereafter, Louis Shapiro [30] presented a concise and insightful combinatorial derivation of Watson's formula (4), motivated by the generating function problem concerning the square of two Fibonacci numbers, which Richard Stanley had brought to his attention. Inspired by Shapiro's approach, Kim [17] later provided a combinatorial proof of (4).

In 1971, Hautus and Klarner [11] introduced a systematic and powerful analytic approach using complex analysis, specifically contour integration, which can be used, in particular, to determine the generating function of $U_n(x)U_n(y)$. Their work, based on Hadamard's multiplication theorem [32, Sec. 4.6], avoids cumbersome algebraic expansions, encodes the combinatorial structure directly in the integral, and yields the explicit rational function (4). Indeed, we can write

$$\begin{aligned}\sum_{n=0}^{\infty} U_n(x)U_n(y)\,t^n &= \frac{1}{2\pi i}\oint_C \frac{1}{1-2xs+s^2}\cdot\frac{1}{1-2y(t/s)+(t/s)^2}\,\frac{ds}{s}\\ &= \frac{1}{2\pi i}\oint_C \frac{s}{(s^2-2xs+1)(s^2-2yts+t^2)}\,ds\,,\end{aligned}$$

where C is a positively oriented contour around the origin in the complex s-plane, chosen so that the integrand is analytic on and inside C except possibly at isolated poles. By computing the residues, after properly identifying the poles, their nature, and an appropriate choice of the contour C, we obtain

$$\sum_{n=0}^{\infty} U_n(x)U_n(y)\,t^n = \frac{1-t^2}{(2\gamma p+q)(2\delta p+q)},$$

where γ and δ are parameters that appear in the factorized form of the generating function after evaluating the residues, and are given by $\gamma = xy - \sqrt{1-x^2}\,\sqrt{1-y^2}$, $\delta = xy + \sqrt{1-x^2}\,\sqrt{1-y^2}$, together with $p = -t$, and $q = 1+t^2$. Substituting these expressions into the denominator of the above identity leads directly to the desired formula (4).

More recently, in 2022, Kar [16] algebraized the Hautus–Klarner method by replacing the contour-integral formulation with a symbolic algebraic framework. In this reformulation, the Hadamard product is no longer evaluated through complex integration or residue calculus, but expressed instead through explicit binomial-sum formulas derived via classical combinatorial identities (notably Vandermonde's identity). Recall that the Hadamard product of two formal power series

$$f(t) = \sum_{n=0}^{\infty} a_n\,t^n \quad \text{and} \quad g(t) = \sum_{n=0}^{\infty} b_n\,t^n$$

is defined by

$$(f * g)(t) = \sum_{n=0}^{\infty} a_n b_n\,t^n.$$

In particular, Kar's Theorem 5.2 constitutes a highly elaborated result obtained through a purely algebraic method for computing the Hadamard product of rational generating functions, especially those with quadratic denominators, without recourse to analytic methods. While conceptually clear, its practical implementation and computational use are far from straightforward. In the proof of that theorem, we find the following identity for the Hadamard product:

$$\frac{q+pt}{1-at+bt^2} * \frac{s+rt}{1-ct+dt^2} = \frac{qs+(pr+aqr+cps)\,t-(adps+bcqr+bdqs)\,t^2-bdpr\,t^3}{1-ac\,t+(a^2d+bc^2-2bd)\,t^2-abcd\,t^3+b^2d^2\,t^4}\,. \tag{7}$$

By applying the natural substitutions

$$p = r = 0, \quad b = d = q = s = 1, \quad a = 2x, \quad c = 2y,$$

in (7), we directly recover (4).

This selective retrospective, necessarily limited given the vast literature in this area, pauses, for now, with a gentle step back in time. Between the works of Hautus-Klarner and Kar just mentioned, István Mező's article [24] is especially noteworthy, providing a refined (and arguably the most natural) algebraic and combinatorial study of second-order recurrence sequences through the lens of generating functions.

Mező's paper builds on earlier work, notably Pantelimon Stănică's study of generating functions for powers and weighted sums of second-order sequences [31]. While Stănică focused on powers of a single sequence, largely influenced by Carlitz [4], Riordan [28] or Horadam [12], Mező extended the approach to include term-by-term products of second-order sequences based on their Binet representations. A central result, Proposition 7, applies to the Binet forms

$$U_n(x) = \frac{a^{n+1} - b^{n+1}}{a - b}, \qquad U_n(y) = \frac{c^{n+1} - d^{n+1}}{c - d},$$

where $a = x + \sqrt{x^2 - 1}$ and $b = x - \sqrt{x^2 - 1}$ are the characteristic roots of $U_n(x)$, and $c = y + \sqrt{y^2 - 1}$ and $d = y - \sqrt{y^2 - 1}$, of $U_n(y)$, and gives the generating function

$$\sum_{n=0}^{\infty} U_n(x)\, U_n(y)\, t^n = \frac{1 - t^2}{(1 - act)(1 - adt)(1 - bct)(1 - bdt)}.$$

Substituting these coefficients and simplifying leads to the compact rational generating function (4).

Although Mező does not reference other works specifically addressing products of sequences, his paper includes extensive tables listing generating functions for products of classical sequences, such as Fibonacci, Lucas, Pell, Jacobsthal, and others, highlighting the broad applicability of Proposition 7 and his method, providing a practical reference for both theoretical and computational applications.

2. A matricial perspective

In contrast to the first section, which did not utilize matrices, this section is devoted entirely to their use, thereby facilitating a more straightforward derivation of results. We then continue our historical exploration, with the general Horadam sequence (w_n) defined by the second-order recurrence relation

$$w_{n+1} = pw_n + qw_{n-1}, \quad \text{for } n \geqslant 1, \tag{8}$$

with initial conditions

$$w_0 = a \quad \text{and} \quad w_1 = b, \tag{9}$$

for arbitrary numbers a, b, p, q where, by definition, $q \neq 0$.

Suppose we have a sequence (a_n) defined by the general homogeneous recurrence relation

$$a_n = p_{n,n-1}\, a_{n-1} + \cdots + p_{n,n-r}\, a_{n-r}, \tag{10}$$

for $n > r$, with given initial conditions

$$a_1 = b_1, \ldots, a_r = b_r. \tag{11}$$

As shown in [15] (see also [23, 35]), the sequence (a_n) can be expressed explicitly as the determinant of a Hessenberg matrix:

$$a_n = \det\begin{pmatrix} b_1 & b_2 & \cdots & b_r & & & & \\ -1 & 0 & \cdots & 0 & p_{r+1,1} & & & \\ & -1 & \ddots & \vdots & \vdots & \ddots & & \\ & & \ddots & 0 & \vdots & & \ddots & \\ & & & -1 & p_{r+1,r} & & & p_{n,n-r} \\ & & & & -1 & \ddots & & \vdots \\ & & & & & \ddots & \ddots & \vdots \\ & & & & & & -1 & p_{n,n-1} \end{pmatrix}. \tag{12}$$

A general result along these lines can be found, for example, in [34, Theorem 4.20]. Using standard matrix-theoretic techniques, it is often possible to derive explicit formulas or generating functions for (a_n) based on (12). One such result, due to Getu [9], is the following:

Theorem 2.1. *Let (d_n) be the sequence defined by the determinants:*

$$d_n = \det\begin{pmatrix} b_0 & b_1 & \cdots & b_k & & & \\ -1 & c_1 & c_2 & \cdots & c_\ell & & \\ & -1 & c_1 & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & c_\ell \\ & & & \ddots & \ddots & \ddots & \vdots \\ & & & & -1 & c_1 & c_2 \\ & & & & & -1 & c_1 \end{pmatrix}_{(n+1)\times(n+1)}.$$

Then the generating function of (d_n) is

$$\sum_{n=0}^{\infty} d_n t^n = \frac{b_0 + b_1 t + \cdots + b_k t^k}{1 - c_1 t - \cdots - c_\ell t^\ell}.$$

For historical context and an earlier treatment of related results, the reader is referred to [14], while more recent extensions can be found in [20]. We note that, throughout, we assume $k < \ell$.

Returning to the Horadam sequence (w_n), we have, from (12) and a standard property of determinants,

$$\begin{aligned} w_n &= \det\begin{pmatrix} a & b & & & \\ -1 & 0 & q & & \\ & -1 & p & \ddots & \\ & & \ddots & \ddots & q \\ & & & -1 & p \end{pmatrix}_{(n+1)\times(n+1)} \\ &= \det\begin{pmatrix} a & b-ap & & & \\ -1 & p & q & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & q \\ & & & -1 & p \end{pmatrix}_{(n+1)\times(n+1)}. \end{aligned}$$

Applying Theorem 2.1, we immediately obtain the generating function

$$\sum_{n=0}^{\infty} w_n t^n = \frac{a+(b-ap)t}{1-pt-qt^2}\,. \tag{13}$$

This rational function was already presented by Horadam over 60 years ago [12, (22)]. For example, under the appropriate specialization, it reproduces the familiar generating function of the standard Fibonacci sequence.

We are now in position to incorporate a fully matrix-based approach into our discussion. This approach builds on the work of Elena A. Potekhina and Mikhail I. Tolovikov [27], a relatively little-known contribution published about a decade ago, with further refinements presented in [26]. In these papers, the authors develop explicit and computable formulae for the Hadamard product of rational power series, expressing these products as ratios of determinants of matrices constructed from the numerator and denominator coefficients.

A central element of their work is the use of specially constructed directed graphs, whose structure and weight matrices encode the dynamics of an oscillating random walk. By connecting weighted digraphs with their corresponding matrices, the authors derive determinantal formulas that extend the Hadamard product to matrix-valued series. Generating functions play a key role, and the main results are effectively reduced to problems of determinant evaluation.

By linking graphs, matrices, and generating functions, [27] presents a practical toolkit for explicit computations in combinatorics and probability. It is significant for bridging combinatorial algebra, graph theory, and probability theory, providing powerful computational tools for calculating distributions of statistics in oscillating random walks and related settings. One of the key results is stated next.

Theorem 2.2 ([27], Theorem 4). *Let $f(z)$ and*

$$g(z) = \frac{c_0 + c_1 z + \cdots + c_{n-1} z^{n-1}}{1 - d_1 z - \cdots - d_n z^n}$$

be formal power series. Define the row matrix

$$C = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-1} \end{pmatrix}$$

and the $n \times n$ companion matrix

$$D = \begin{pmatrix} d_1 & d_2 & \cdots & d_n \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

Then the Hadamard product $f(z) * g(z)$ is given by the first entry of the row matrix $Cf(zD)$.

This theorem illustrates that matrices are fundamental in expressing Hadamard products as ratios of determinants, effectively reducing the computation to standard linear algebra operations on these matrices.

In the case currently under consideration, Theorem 2.2 takes an even more elegant, compact, and efficient form.

Theorem 2.3. *Let*

$$f_1(t) = \frac{a_1 + (b_1 - a_1 p_1)t}{1 - p_1 t - q_1 t^2}, \quad f_2(t) = \frac{a_2 + (b_2 - a_2 p_2)t}{1 - p_2 t - q_2 t^2}$$

be two rational functions. Then their Hadamard product is given by

$$f_1(t) * f_2(t) = \frac{\begin{array}{l} a_1a_2 \\ + (b_1b_2 - a_1a_2p_1p_2)t \\ + (a_2q_2p_1(b_1 - a_1p_1) + a_1q_1(b_2p_2 - a_2(p_2^2 + q_2)))t^2 \\ - (b_2 - a_2p_2)(b_1 - a_1p_1)q_1q_2t^3 \end{array}}{\begin{array}{l} 1 - p_1p_2t \\ - (q_1(p_2^2 + 2q_2) + p_1^2q_2)t^2 \\ - p_1p_2q_1q_2t^3 + q_1^2q_2^2t^4 \end{array}}$$

Proof. From Theorem 2.2, we have

$$f_1(t) * f_2(t) = \begin{pmatrix} a_2 & b_2 - a_2p_2 \end{pmatrix} \cdot f_1(tD) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where

$$D = \begin{pmatrix} p_2 & q_2 \\ 1 & 0 \end{pmatrix},$$

and

$$f_1(tD) = \left[a_1I_2 + t(b_1 - a_1p_1)D\right] \cdot \left[I_2 - tp_1D - t^2q_1D^2\right]^{-1}.$$

Carrying out straightforward but somewhat lengthy calculations with this matrix representation immediately yields the explicit expression for $f_1(t) * f_2(t)$ □

The next corollary provides the generating function for the square of the Horadam sequence (8)–(9).

Corollary 2.4. *The generating function of* (w_n^2) *is given by*

$$\sum_{n=0}^{\infty} w_n^2\, t^n = \frac{a^2 - \left(a^2(p^2 + q) - b^2\right)t - q(ap - b)^2t^2}{(1 + qt)\left(1 - (p^2 + 2q)t + q^2t^2\right)}. \tag{14}$$

In other words, (14) shows that the generating function of (w_n^2) is invariably a rational function, represented as the ratio of a quadratic polynomial to a cubic polynomial. This result has been known for more than six decades; it also appears in [12, (65)] as a special case of the generating function for powers of (w_n).

Since the Chebyshev polynomials of the second kind, $U_n(x)$, satisfy the recurrence relation

$$U_n(x) = 2x\, U_{n-1}(x) - U_{n-2}(x), \quad n \geqslant 2.$$

with initial conditions

$$U_0(x) = 1 \quad \text{and} \quad U_1(x) = 2x,$$

we derive (4) from Theorem 2.3. In particular from Corollary 2.4, we get the well-known identity

$$\sum_{n=0}^{\infty} U_n^2(x)\, t^n = \frac{1 + t}{(1 - t)\,(1 + 2(1 - 2x^2)t + t^2)}.$$

From (13) and using the recurrence relation for $U_n(x)$, we have

$$\sum_{n=0}^{\infty} U_{n+\ell}(x)\, t^n = \frac{tU_{\ell+1}(x) + (1 - 2xt)U_\ell(x)}{1 - 2xt + t^2} = \frac{U_\ell(x) - tU_{\ell-1}(x)}{1 - 2xt + t^2}. \tag{15}$$

The reader may refer to [5, Proposition 3] for an alternative approach. By applying (15) together with Theorem 2.2, or its specialization given in Theorem 2.3, we obtain the following general result.

Theorem 2.5.

$$\sum_{n=0}^{\infty} U_{n+\ell}(x)\,U_n(y)\,t^n = \frac{(1-t^2)\,U_\ell(x) + 2t(tx-y)\,U_{\ell-1}(x)}{1-4xy\,t+2(2x^2+2y^2-1)t^2-4xy\,t^3+t^4}$$

In particular, for any integer ℓ,

$$\begin{aligned}\sum_{n=0}^{\infty} U_{n+\ell}(x)\,U_n(x)\,t^n &= \frac{(1+t)\,U_\ell(x) - 2xt\,U_{\ell-1}(x)}{(1-t)(1+2(1-2x^2)t+t^2)} \\ &= \frac{U_\ell(x) - t\,U_{\ell-2}(x)}{(1-t)(1+2(1-2x^2)t+t^2)}\,,\end{aligned} \tag{16}$$

as disclosed in [7].

3. The trigonometric way

While in the previous section we relied on linear algebra to describe the generating functions following recent literature, in this section we will use exclusively trigonometric methods, which, in a certain sense, can be considered the more natural framework. In doing so, we will reverse the usual order of development for finding the generating functions: we will start from the Chebyshev polynomials and then indicate how to derive the general results.

We begin by employing the trigonometric representation of the Chebyshev polynomials of the second kind to derive product-to-sum identities and explicit generating functions in terms of sines and cosines.

Taking into account (3), with $x = \cos\alpha$, $y = \cos\beta$, and using a standard product-to-sum identity, we have

$$\begin{aligned}U_{n+\ell}(x)\,U_n(y) &= \frac{\sin((n+\ell+1)\alpha)\,\sin((n+1)\beta)}{\sin\alpha\,\sin\beta} \\ &= \frac{1}{2\sin\alpha\,\sin\beta}\Big(\cos((n+1)(\alpha-\beta)+\ell\alpha) - \cos((n+1)(\alpha+\beta)+\ell\alpha)\Big).\end{aligned} \tag{17}$$

This decomposition transforms the product of Chebyshev polynomials into a sum of cosines, paving the way for generating functions via geometric series, as we will see in the next theorem. First, recall the well-known closed geometric–trigonometric series formula

$$\sum_{n=0}^{\infty} \cos(n\theta+\phi)t^n = \frac{\cos\phi - t\cos(\theta-\phi)}{1-2t\cos\theta+t^2}\,, \tag{18}$$

often considered part of the mathematical folklore. A concise proof uses the sum of a trigonometric series:

$$\sum_{n=0}^{\infty} t^n\cos(n\theta+\phi) = \operatorname{Re}\sum_{n=0}^{\infty}\left(te^{i\theta}\right)^n e^{i\phi} = \operatorname{Re}\frac{e^{i\phi}}{1-te^{i\theta}} = \frac{\cos\phi - t\cos(\theta-\phi)}{1-2t\cos\theta+t^2}.$$

Alternatively, (18) can be obtained from the well-known sums

$$\sum_{n=0}^{\infty} t^n\cos n\theta = \frac{1-t\cos\theta}{1-2t\cos\theta+t^2}, \quad \sum_{n=0}^{\infty} t^n\sin n\theta = \frac{t\sin\theta}{1-2t\cos\theta+t^2},$$

together with the cosine addition formula $\cos(n\theta+\phi) = \cos(n\theta)\cos\phi - \sin(n\theta)\sin\phi$.

Theorem 3.1. *Let $x = \cos\alpha$ and $y = \cos\beta$, and let ℓ be an integer. Then, as a formal power series in t, one has*

$$\sum_{n=0}^{\infty} U_{n+\ell}(x)\,U_n(y)\,t^n = \frac{1}{2\sin\alpha\,\sin\beta}\left(\frac{\cos((\ell+1)\alpha-\beta) - t\cos(\ell\alpha)}{1-2t\cos(\alpha-\beta)+t^2} - \frac{\cos((\ell+1)\alpha+\beta) - t\cos(\ell\alpha)}{1-2t\cos(\alpha+\beta)+t^2}\right). \tag{19}$$

Proof. Summing the product-to-sum identity (17) over $n \geqslant 0$ and applying the geometric–trigonometric series formula (18) to each cosine term, noting that $(\alpha \pm \beta) - ((\ell + 1)\alpha \pm \beta) = -\ell\alpha$, a straightforward simplification yields (19). □

Substituting $x = \cos\alpha$ and $y = \cos\beta$ into (19) immediately recovers Theorem 2.5 and, in particular, with $\ell = 0$, formula (4). A similar reasoning applies to the subsequent two corollaries when setting $y = x$ and then $\ell = 0$.

Corollary 3.2.

$$\sum_{n=0}^{\infty} U_{n+\ell}(x)\, U_n(x)\, t^n = \frac{\sin\big((\ell+1)\alpha\big) - t\,\sin\big((\ell-1)\alpha\big)}{\sin\alpha\,(1-t)\big(1 - 2t\cos(2\alpha) + t^2\big)}, \qquad x = \cos\alpha.$$

Corollary 3.3.

$$\sum_{n=0}^{\infty} U_n^2(x)\, t^n = \frac{1+t}{(1-t)\,(1-2t\cos(2\alpha)+t^2)}, \qquad x = \cos\alpha.$$

An explicit formula for the Horadam sequence (w_n), defined by (8)–(9), which includes the degenerate cases, is

$$w_n = \big(\sqrt{-q}\big)^n \left(\frac{b}{\sqrt{-q}}\, U_{n-1}\Big(\frac{p}{2\sqrt{-q}}\Big) - a\, U_{n-2}\Big(\frac{p}{2\sqrt{-q}}\Big)\right), \tag{20}$$

as shown, for example, in [33] or even earlier in [13] and discussed for a bi-periodic extension in [1]. Together with (4), and performing the substitutions

$$x = \frac{p_1}{2i\sqrt{q_1}}, \qquad y = \frac{p_2}{2i\sqrt{q_2}}, \qquad t \mapsto -\sqrt{q_1 q_2}\, t,$$

along with the linear combination dictated by (20), one recovers, after straightforward but somewhat involved calculations, the rational generating function presented in Theorem 2.3.

In conclusion, this section has shown that the generating function of the product of any two Horadam-type sequences can be reduced to the corresponding case of Chebyshev polynomials of the second kind, underscoring the central role of these polynomials in the general analysis.

4. Generating functions for products of classical number sequences

Over the past few centuries, researchers have found numerous motivations for studying generating functions of classical number sequences, particularly those involving their products. As often occurs in the history of mathematics, important results have at times arisen independently and remained unrecognized. Within this context, the work embodied in Henry W. Gould's Fibonacci Bibliographical Project stands out. In 1963, encouraged by Vern Hoggatt to collect formulas, compile a bibliography, and coordinate research on Fibonacci numbers, Gould published a seminal paper [10]. In it, he sought to construct a comprehensive framework capable of generating, unifying, and extending the many known results on Fibonacci and Lucas numbers.

Gould's paper was also strongly influenced by Riordan's earlier work [28], which investigated the arithmetic properties of certain classes of coefficients arising from the analysis of the generating functions defined by the powers of Fibonacci numbers.

Of special relevance in [10] is identity (4.4), from which the following specific formula can be derived:

$$\sum_{n=0}^{\infty} F_{n+\ell} F_n\, t^n = \frac{t((1-t)F_{\ell+1} + tF_\ell)}{(1+t)(1-3t+t^2)}. \tag{21}$$

Here, the Fibonacci numbers are defined by the initial conditions $F_0 = 0$ and $F_1 = 1$. Notice that we can deduce (21) from Theorem 2.3 with the appropriate specialization, or more directly from (16), taking into account that

$$F_n = (-i)^{n-1}\, U_{n-1}\Big(\frac{i}{2}\Big),$$

as we can find in [2], for example.

The case $\ell = 1$ in (21) corresponds to the generating function

$$\sum_{n=0}^{\infty} F_{n+1}F_n\, t^n = \frac{t}{(1+t)(1-3t+t^2)} \tag{22}$$

which enumerates the Golden (convergents to) rectangle numbers, listed in the On-Line Encyclopedia of Integer Sequences (OEIS) as sequence A001654.

The generating function for the product of successive Fibonacci numbers (22) illustrates the broader potential of generating functions in characterizing numerical sequences. In 1992, Simon Plouffe published his Ph.D. thesis [25], whose most impactful and enduring contribution was its monumental 500-page appendix documenting 1,031 generating functions. With the generating function (22) appearing as entry A.184, these were systematically derived from the initial terms of integer sequences later incorporated into Neil J.A. Sloane's pioneering OEIS. As a technical collaborator, Plouffe played a crucial role in transforming the OEIS into a comprehensive, accessible, and modern online resource.

While Plouffe's approach, later exemplified by the BBP (Bailey-Borwein-Plouffe) formula for π, was groundbreaking, it was not without limitations. His work occasionally contained minor errors, including inaccuracies in initial sequence terms, mismatches between proposed generating functions and full sequences, computational mistakes, and cases where rational approximations held only over a finite range of terms.

Before the internet era, generating functions were often scattered across dispersed publications or remained in unpublished notes, making organized discovery and comparison nearly impossible. Plouffe's thesis transformed this landscape, assembling a vast and comprehensive, say "Rosetta Stone", for experimental mathematics, a centralized repository that preserved dispersed knowledge and revealed unexpected relationships among sequences. His methodology, combining high-precision computation with conjectural identification of closed-form expressions, extended the influence of his work far beyond pure mathematics, inspiring new approaches in computational and applied fields.

In this spirit, we will present several identities for the generating functions of products of classical integer sequences of distinct orders, where one of them is shifted. This list is by no means exhaustive, as many other identities could also be derived. Beyond the Fibonacci numbers, we will consider the Lucas numbers (L_n), which satisfy the same recurrence as the Fibonacci numbers but with initial conditions $L_0 = 2$ and $L_1 = 1$. In addition, with the same initial conditions as the Fibonacci numbers, we analyse the Pell numbers (P_n), the Jacobsthal numbers (J_n), and the Mersenne numbers (M_n) satisfying the recurrence relations

$$P_n = 2P_{n-1} + P_{n-2}\,, \quad J_n = J_{n-1} + 2J_{n-2}\,, \quad M_n = 3M_{n-1} - 2M_{n-2}\,,$$

respectively [18].

We begin with the product with shifted Fibonacci numbers.

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{n+\ell}F_n\, t^n &= \frac{t\Big((1-t)F_{\ell+1}+tF_\ell\Big)}{(1+t)(1-3t+t^2)} \\
\sum_{n=0}^{\infty} F_{n+\ell}L_n\, t^n &= \frac{t(1+t)F_{\ell+1}+(2-4t-t^2)F_\ell}{(1+t)(1-3t+t^2)} \\
\sum_{n=0}^{\infty} F_{n+\ell}P_n\, t^n &= \frac{t\Big((1-t^2)F_{\ell+1}+t(2+t)F_\ell\Big)}{1-2t-7t^2-2t^3+t^4} \\
\sum_{n=0}^{\infty} F_{n+\ell}J_n\, t^n &= \frac{t\Big((1-2t^2)F_{\ell+1}+t(1+2t)F_\ell\Big)}{(1+t-t^2)\,(1-2t-4t^2)} \\
\sum_{n=0}^{\infty} F_{n+\ell}M_n\, t^n &= \frac{t\Big((1+2t^2)F_{\ell+1}+t(3-2t)F_\ell\Big)}{(1-t-t^2)\,(1-2t-4t^2)}\,.
\end{aligned}$$

For the shifted Lucas sequence, the corresponding identities are obtained by replacing F with L, reflecting the fact that both sequences satisfy the same recurrence relation. We proceed then with the shifted Pell sequence:

$$\begin{aligned}
\sum_{n=0}^{\infty} P_{n+\ell}F_n\, t^n &= \frac{t\Big((1-t^2)P_{\ell+1}+t(1+2t)P_\ell\Big)}{1-2t-7t^2-2t^3+t^4} \\
\sum_{n=0}^{\infty} P_{n+\ell}L_n\, t^n &= \frac{t(1+4t+t^2)P_{\ell+1}+(2-4t-11t^2-2t^3)P_\ell}{1-2t-7t^2-2t^3+t^4} \\
\sum_{n=0}^{\infty} P_{n+\ell}P_n\, t^n &= \frac{t\,((1-t)P_{\ell+1}+2tP_\ell)}{(1+t)\,(1-6t+t^2)} \\
\sum_{n=0}^{\infty} P_{n+\ell}J_n\, t^n &= \frac{t\Big((1-2t^2)P_{\ell+1}+t(1+4t)P_\ell\Big)}{(1+2t-t^2)\,(1-4t-4t^2)} \\
\sum_{n=0}^{\infty} P_{n+\ell}M_n\, t^n &= \frac{t\Big((1+2t^2)P_{\ell+1}+t(3-4t)P_\ell\Big)}{(1-2t-t^2)\,(1-4t-4t^2)}\,.
\end{aligned}$$

Proceeding with the shifted Jacobsthal numbers, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} J_{n+\ell}F_n\, t^n &= \frac{t\Big((1-2t^2)J_{\ell+1}+2t(1+t)J_\ell\Big)}{(1+t-t^2)\,(1-2t-4t^2)} \\
\sum_{n=0}^{\infty} J_{n+\ell}L_n\, t^n &= \frac{t(1+2t+2t^2)J_{\ell+1}+2(1-t-4t^2-t^3)J_\ell}{(1+t-t^2)\,(1-2t-4t^2)} \\
\sum_{n=0}^{\infty} J_{n+\ell}P_n\, t^n &= \frac{t\Big((1-2t^2)J_{\ell+1}+2t(2+t)J_\ell\Big)}{(1+2t-t^2)\,(1-4t-4t^2)} \\
\sum_{n=0}^{\infty} J_{n+\ell}J_n\, t^n &= \frac{t\,((1-2t)J_{\ell+1}+2tJ_\ell)}{(1-t)(1+2t)(1-4t)} \\
\sum_{n=0}^{\infty} J_{n+\ell}M_n\, t^n &= \frac{t\Big((1+4t^2)J_{\ell+1}+2t(3-2t)J_\ell\Big)}{(1+t)(1-2t)(1+2t)(1-4t)}\,.
\end{aligned}$$

We conclude with the shifted Mersenne sequence:

$$\begin{aligned}
\sum_{n=0}^{\infty} M_{n+\ell} F_n\, t^n &= \frac{(2^{\ell+1}-1)t-(2^{\ell+1}-2)t^2-(2^{\ell+1}-4)t^3}{(1-t-t^2)(1-2t-4t^2)} \\
\sum_{n=0}^{\infty} M_{n+\ell} L_n\, t^n &= \frac{(2^{\ell+1}-4)t^3+6t^2+(5-2^{\ell+2})t+(2^{\ell+1}-2)}{(1-t-t^2)(1-2t-4t^2)} \\
\sum_{n=0}^{\infty} M_{n+\ell} P_n\, t^n &= \frac{(4-2^{\ell+1})t^3+(4-2^{\ell+2})t^2+(2^{\ell+1}-1)t}{(1-2t-t^2)(1-4t-4t^2)} \\
\sum_{n=0}^{\infty} M_{n+\ell} J_n\, t^n &= \frac{(8-2^{\ell+2})t^3+(2-2^{\ell+1})t^2+(2^{\ell+1}-1)t}{(t+1)(2t-1)(2t+1)(4t-1)} \\
\sum_{n=0}^{\infty} M_{n+\ell} M_n\, t^n &= \frac{(2^{\ell+1}-1)t+\left(4-2^{\ell+1}\right)t^2}{(1-t)(1-2t)(1-4t)}\,.
\end{aligned}$$

More generally, we have for the shifted Horadam sequence (w_n):

$$\begin{aligned}
\sum_{n=0}^{\infty} w_{n+\ell} F_n\, t^n &= \frac{t((1-qt^2)w_{\ell+1}+qt(1+pt)w_\ell)}{1-pt-(p^2+3q)t^2-pqt^3+q^2t^4} \\
\sum_{n=0}^{\infty} w_{n+\ell} L_n\, t^n &= \frac{t(1+2pt+qt^2)w_{\ell+1}+(2-2pt-(2p^2+3q)t^2-pqt^3)w_\ell}{1-pt-(p^2+3q)t^2-pqt^3+q^2t^4} \\
\sum_{n=0}^{\infty} w_{n+\ell} P_n\, t^n &= \frac{t((1-qt^2)w_{\ell+1}+qt(2+pt)w_\ell)}{1-2pt-(p^2+6q)t^2-2pqt^3+q^2t^4} \\
\sum_{n=0}^{\infty} w_{n+\ell} J_n\, t^n &= \frac{t((1-2qt^2)w_{\ell+1}+qt(1+2pt)w_\ell)}{(1+pt-qt^2)(1-2pt-4qt^2)} \\
\sum_{n=0}^{\infty} w_{n+\ell} M_n\, t^n &= \frac{t((1+2qt^2)w_{\ell+1}+qt(3-2pt)w_\ell)}{(1-pt-qt^2)(1-2pt-4qt^2)}\,.
\end{aligned}$$

Explicit formulas for generating functions associated with the shifted Horadam sequence have received significant attention in recent literature (see, for instance, [29]). All such formulas can be systematically derived via appropriate specializations.

The last identity is a natural extension of Corollary 2.4:

$$\sum_{n=0}^{\infty} w_{n+\ell}\, w_n\, t^n = \frac{t(b+q(ap-b)t)w_{\ell+1}+(a-a(p^2+q)t+pq(b-ap)t^2)w_\ell}{(1+qt)\,(1-(p^2+2q)t+q^2t^2)}\,.$$

This identity, like the others discussed, is probably not new; variations of these identities may be found scattered throughout the mathematical literature.

5. Conclusion: The intertwined paths of discovery

As in many areas of science, mathematics exhibits a recurring pattern: new concepts and their developments often arise independently, across different times, but sometimes almost simultaneously, and in different places. This reflects the structural nature of discovery, which is rarely purely individual. Progress frequently follows a collective intellectual momentum, in which the necessary tools, frameworks, or questions become available to multiple thinkers at once.

Parallel paths in mathematics can occur for many reasons. Seminal results may remain hidden in less accessible publications for generations, expressed in the technical language of distinct disciplines, or framed in approaches not immediately recognized as equivalent. Even today, despite rapid communication and broad access to research, independent rediscoveries persist. The sheer volume of work and increasing specialization mean that the unity of mathematical knowledge often emerges only in retrospect.

The study of generating functions for products of sequences provides a clear example of this phenomenon. In this work, we have collected, unified, and discussed the most significant contributions in this area. Our account traces the thread of this idea from its discernible origins in the 19th century, though its roots likely extend even deeper into the past, and follows its development through to contemporary research.

By revisiting these works together, we aim to highlight conceptual links between seemingly distinct results, demonstrate how different notations or motivations often conceal a common core, and place recent advances within the historical narrative that shaped them. Inspired by the exploratory spirit behind the On-Line Encyclopedia of Integer Sequences, we have also derived several new identities, continuing a tradition of playful and insightful investigation that has always propelled the field forward.

Ultimately, we emphasize that the story of generating functions for products is not a linear chronology, but a rich tapestry of interconnected ideas. We structured the paper so that its comprehensive survey component not only serves as a reference tool for experts but also acts as an easily navigable guide for students and those new to the subject. The journey of these ideas, from independent beginnings to their synthesis in works such as this, stands as a testament to the collaborative and cumulative nature of the mathematical endeavor.

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