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Derivations and lower cohomology of Lie algebra crossed modules

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Abstract. In this paper, we define the derivation of a Lie algebra crossed module and prove that it can be represented by the augmentation ideal of a crossed module. Additionally, we introduce the first cohomology groups for $n = 0, 1$, of a crossed module with coefficients, and we establish their connection to the classical Chevalley-Eilenberg Lie algebra cohomology.

1. Introduction

J. H. C. Whitehead [14] introduced crossed modules of groups in the late 1940s as a tool for studying relative homotopy groups. Crossed modules of Lie algebras, referred to as Lie crossed modules, as well as crossed modules of associative algebras, are well-established as analogues of crossed modules of groups in different mathematical frameworks.

Lie crossed modules have been studied by various authors. For instance, Kassel and Loday [13] use Lie crossed modules as computational tools to interpret the third relative Chevalley-Eilenberg cohomology of Lie algebras. Additionally, Guin [10] developed a low-dimensional non-abelian cohomology of Lie algebras with coefficients in Lie crossed modules, a study that was later extended to higher dimensions in [12]. Furthermore, Lie crossed modules are significant in the "categorification" problem in the theory of Lie algebras, serving as an equivalent formulation of strict Lie 2-algebras, which are a two-dimensional generalisation of the concept of a Lie algebra.

In [1], the authors defined and studied the cohomology groups of a crossed module of groups (T, G, ∂) with coefficients in a trivial (T, G, ∂)-module. The internal (cotriple) homology and Chevalley-Eilenberg homology theories of Lie crossed modules with trivial coefficients were explored in [3, 7]. Recently, the cohomologies of a crossed module of groups (T, G, ∂) with coefficients in an arbitrary (T, G, ∂)-module have been defined and studied [8].

The motivation behind this paper is to develop the cohomology theory of a Lie crossed module with arbitrary coefficients. Recall that in [3], the n-th cotriple homology of the Lie algebra crossed module

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$\mathcal{G} = (\mathfrak{g}_1 \xrightarrow{\partial} \mathfrak{g}_0)$ with trivial coefficients is defined using the formula

$$H_n(\mathbf{G}) := H_{n-1}\big(\,\mathbf{Ab}(\mathbb{F}_*(G))\big),$$

where $\mathbb{F}_*(\mathcal{G})$ is the $\mathbb{F}$-cotriple resolution of $\mathcal{G}$ and $\mathbf{Ab}$ is the abelianisation functor.

Our goal is to similarly construct a cotriple cohomology theory with coefficients in a $\mathcal{G}$-module $\mathcal{A}$ (see Definition 2.5 below) within the category of crossed modules of Lie algebras. Following the analogy found in the category of Lie algebras, we aim to construct the functor $\operatorname{Der}(\mathcal{G},\mathcal{A})$, which represents the derivations of a Lie crossed module, and define the cotriple cohomology in the form

$$H^*(\mathcal{G},\mathcal{A}) := H^*\big(\operatorname{Der}(\mathbb{F}_*(\mathcal{G})),\mathcal{A}\big).$$

In this paper, we will address the first part of this project. The initial step involves constructing the derivations of a Lie crossed module conceptually. After demonstrating that the functor $\operatorname{Der}(\mathcal{G},-)$ is representable by the augmentation ideal, we will introduce the first cohomology groups and establish their relationship through an exact sequence.

The text is organised into five sections. After this introduction, Section 2 presents preliminaries by reviewing key concepts related to Lie crossed modules and examining actions between two such crossed modules. In Section 3, we revisit the notion of the universal enveloping crossed module of a Lie crossed module, as presented in [2]. We also introduce the augmentation ideal of a Lie crossed module and present several results regarding the universal enveloping crossed module, which will be crucial for studying the derivations of a Lie crossed module. Section 4 introduces the derivations $\operatorname{Der}(\mathcal{G},\mathcal{A})$, where $\mathcal{G} = (\mathfrak{g}_1 \xrightarrow{\partial} \mathfrak{g}_0)$ is a Lie algebra crossed module and $\mathcal{A}$ is a module over $\mathcal{G}$. We prove that the functor $\operatorname{Der}(\mathcal{G},-)$ is represented by the augmentation ideal of $\mathcal{G}$ (Proposition 4.5). Finally, in Section 5, we introduce the principal derivations $\operatorname{PDer}(\mathcal{G},\mathcal{A})$ and construct the zeroth and first cohomologies $H^0(\mathcal{G},\mathcal{A})$ and $H^1(\mathcal{G},\mathcal{A})$. We establish the connection between the cohomology of a Lie crossed module and the classical Chevalley–Eilenberg cohomology of Lie algebras (Proposition 5.4). In the end, we also obtain the following exact sequence

$$0 \to H^0(\mathcal{G},\mathcal{A}) \to H^0(\mathcal{G},\mathcal{B}) \to H^0(\mathcal{G},\mathcal{C}) \to H^1(\mathcal{G},\mathcal{A}) \to H^1(\mathcal{G},\mathcal{B}) \to H^1(\mathcal{G},\mathcal{C}),$$

for any extension $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ of $\mathcal{G}$-modules (Proposition 5.6).

Throughout the paper, we fix a field $\mathbf{K}$, and we consider Lie algebras over $\mathbf{K}$.

2. Preliminaries

Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras. An *action* of $\mathfrak{g}$ on $\mathfrak{h}$ is a $\mathbf{K}$-bilinear map $\mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}$, $(g,h) \mapsto {}^{g}h$ satisfying the following conditions:

$${}^{[g,g']}h = {}^{g}({}^{g'}h) - {}^{g'}({}^{g}h),$$

$${}^{g}[h,h'] = [{}^{g}h,h'] + [h,{}^{g}h'],$$

where $g,g' \in \mathfrak{g}$, $h,h' \in \mathfrak{h}$. If A is a vector space equipped with the trivial Lie bracket and $\mathfrak{g}$ acts on A, then we write ga instead of ${}^{g}a$, for $g \in \mathfrak{g}$, $a \in A$. In this case A is said to be a $\mathfrak{g}$-module.

A *Lie crossed module* $\mathcal{G}$ is a Lie homomorphism $\partial\colon \mathfrak{g}_1 \to \mathfrak{g}_0$ together with an action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ such that, for each $x,x' \in \mathfrak{g}_1$ and $g \in \mathfrak{g}_0$

$$\partial({}^{g}x) = [g,\partial(x)] \quad \text{and} \quad {}^{\partial(x)}x' = [x,x'].$$

Example 2.1.

1. *$\mathfrak{h} \xrightarrow{i} \mathfrak{g}$, where $\mathfrak{h}$ is an ideal of a Lie algebra $\mathfrak{g}$, i is the inclusion and $\mathfrak{g}$ acts on $\mathfrak{h}$ via the adjoint representation. Thus, a Lie algebra $\mathfrak{g}$ can be considered as a crossed module in two different ways:* $0 \xrightarrow{0} \mathfrak{g}$ *or* $\mathfrak{g} \xrightarrow{\mathrm{id}} \mathfrak{g}$.

2. *$A_1 \xrightarrow{\partial} A_0$ an abelian Lie crossed module, that is, both A_0 and A_1 are abelian Lie algebras, ∂ is a $\mathbf{K}$-linear map, and A_0 acts trivially on A_1.*

A *morphism of Lie crossed modules* $(\alpha_1,\alpha_0)\colon \mathcal{G} = (\mathfrak{g}_1 \xrightarrow{\partial_\mathfrak{g}} \mathfrak{g}_0) \to \mathcal{H} = (\mathfrak{h}_1 \xrightarrow{\partial_\mathfrak{h}} \mathfrak{h}_0)$ is a pair of Lie homomorphisms $\alpha_1\colon \mathfrak{g}_1 \to \mathfrak{h}_1$ and $\alpha_0\colon \mathfrak{g}_0 \to \mathfrak{h}_0$ such that $\partial_\mathfrak{h}\alpha_1 = \alpha_0\partial_\mathfrak{g}$ and $\alpha_1({}^{g}x) = {}^{\alpha_0(g)}\alpha_1(x)$ for each $g \in \mathfrak{g}_0$, $x \in \mathfrak{g}_1$.

An action of a Lie crossed module on another Lie crossed module is defined in [4] and an equivalent definition is given in [2].

Definition 2.2. *Let $\mathcal{G} = (\mathfrak{g}_1 \xrightarrow{\partial_\mathfrak{g}} \mathfrak{g}_0)$ and $\mathcal{H} = (\mathfrak{h}_1 \xrightarrow{\partial_\mathfrak{h}} \mathfrak{h}_0)$ be crossed modules. An action of $\mathcal{G}$ on $\mathcal{H}$ is defined by the following conditions:*

(i) There is an action of $\mathfrak{g}_0$ (and consequently $\mathfrak{g}_1$) on $\mathfrak{h}_1$ and $\mathfrak{h}_0$; $\partial_\mathfrak{h}\colon \mathfrak{h}_1 \to \mathfrak{h}_0$ is a $\mathfrak{g}_0$-equivariant homomorphism, that is, $\partial_\mathfrak{h}({}^{g}y) = {}^{g}\partial_\mathfrak{h}(y)$; and additionally, $\mathfrak{g}_0$, $\mathfrak{h}_0$ act compatibly on $\mathfrak{h}_1$, that is

$$ {}^{({}^{g}h)}y = {}^{g}({}^{h}y) - {}^{h}({}^{g}y), $$

for each $g \in \mathfrak{g}_0$, $h \in \mathfrak{h}_0$ and $y \in \mathfrak{h}_1$.

(ii) There is a $\mathbf{K}$-bilinear map $\xi\colon \mathfrak{g}_1 \times \mathfrak{h}_0 \to \mathfrak{h}_1$ such that

$$ \xi(x,\partial_\mathfrak{h}(y)) = {}^{x}y, \tag{2.1} $$

$$ \partial_\mathfrak{h}(\xi(x,h)) = {}^{x}h, \tag{2.2} $$

$$ {}^{g}\xi(x,h) = \xi({}^{g}x,h) + \xi(x,{}^{g}h), \tag{2.3} $$

$$ \xi([x,x'],h) = {}^{x}\xi(x',h) - {}^{x'}\xi(x,h), \tag{2.4} $$

$$ \xi(x,[h,h']) = {}^{h}\xi(x,h') - {}^{h'}\xi(x,h), \tag{2.5} $$

for $x,x' \in \mathfrak{g}_1$, $g \in \mathfrak{g}_0$, $y \in \mathfrak{h}_1$, $h,h' \in \mathfrak{h}_0$.

Example 2.3.

1. *An action of $0 \xrightarrow{0} \mathfrak{g}$ on an abelian Lie crossed module $A_1 \xrightarrow{\partial} A_0$ is an action of $\mathfrak{g}$ on A_1 and A_0, and $A_1 \xrightarrow{\partial} A_0$ is a morphism of $\mathfrak{g}$-modules.*

2. *An action of $\mathfrak{g}_1 \xrightarrow{\partial} \mathfrak{g}_0$ on the abelian Lie crossed module $0 \xrightarrow{0} A$ is an action of $\mathfrak{g}_0/\partial(\mathfrak{g}_1)$ on A since the map ξ is trivial, and identity* (2.2), *$\partial\xi(x,a) = \partial(x)a$, $x \in \mathfrak{g}_1$, $a \in A$, implies that $\partial(\mathfrak{g}_1)$ acts trivially on A.*

3. *An action of $\mathfrak{g} \xrightarrow{\mathrm{id}} \mathfrak{g}$ on an abelian Lie crossed module $A \xrightarrow{\mathrm{id}} A$ is an action of $\mathfrak{g}$ on A.*

4. *Let $\mathfrak{g}$ be an abelian Lie algebra. An action of $\mathfrak{g} \xrightarrow{0} 0$ on an abelian Lie crossed module $A_1 \xrightarrow{\partial} A_0$ is identified with a homomorphism of abelian groups $\mathfrak{g} \otimes \mathsf{Coker}(\partial) \xrightarrow{\xi} \mathsf{Ker}(\partial)$. This identification arises from identities* (2.1) *$\xi(\mathrm{id},\partial) = 0$, and* (2.2) *$\partial\xi = 0$.*

Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras equipped with an action of $\mathfrak{g}$ on $\mathfrak{h}$. The *semidirect product* $\mathfrak{h} \rtimes \mathfrak{g}$ is defined as the Lie algebra whose underlying vector space is $\mathfrak{h} \oplus \mathfrak{g}$ endowed with Lie bracket given by

$$ [(h,g),(h',g')] = ([h,h'] + {}^{g}h' - {}^{g'}h, [g,g']). $$

The following proposition is presented in [2].

Proposition 2.4. *Let $\mathcal{G}$ be a Lie crossed module acting on another Lie crossed module $\mathcal{H}$. Then, there is a Lie crossed module $\mathcal{H} \rtimes \mathcal{G}$, where $\partial\colon \mathfrak{h}_1 \rtimes \mathfrak{g}_1 \to \mathfrak{h}_0 \rtimes \mathfrak{g}_0$ is defined by $\partial(y,x) = (\partial_\mathfrak{h}(y),\partial_\mathfrak{g}(x))$, and the action $\mathfrak{h}_0 \rtimes \mathfrak{g}_0$ on $\mathfrak{h}_1 \rtimes \mathfrak{g}_1$ is given by the formula*

$$ {}^{(h,g)}(y,x) = ({}^{h}y + {}^{g}y - \xi(x,h), {}^{g}x), $$

for each $g \in \mathfrak{g}_0$, $h \in \mathfrak{h}_0$, $x \in \mathfrak{g}_1$, $y \in \mathfrak{h}_1$.

Definition 2.5. *Let $\partial\colon A_1 \to A_0$ be an abelian Lie crossed module. If a Lie crossed module $\mathcal{G}$ acts on $\mathcal{A}$, then $\mathcal{A}$ is said to be a $\mathcal{G}$-module.*

Given two $\mathcal{G}$-modules $\mathcal{A}$ and $\mathcal{B}$, a $\mathcal{G}$-module morphism from $\mathcal{A}$ into $\mathcal{B}$ is a pair of maps (f_1, f_0), $f_0\colon A_0 \to B_0$, $f_1\colon A_1 \to B_1$, such that both f_0 and f_1 are $\mathfrak{g}_0$-module homomorphisms, $f_0\partial_A = \partial_B f_1$ and the following diagram is commutative:

$$\begin{tikzcd}
\mathfrak{g}_1 \times A_0 \arrow[r, "\xi"] \arrow[d, "1\times f_0"'] & A_1 \arrow[d, "f_1"] \\
\mathfrak{g}_1 \times B_0 \arrow[r, "\xi"'] & B_1 .
\end{tikzcd}$$

We denote by $\mathrm{Hom}_{\mathcal{G}}(\mathcal{A},\mathcal{B})$ the set of all morphisms from $\mathcal{A}$ into $\mathcal{B}$. Obviously, $\mathrm{Hom}_{\mathcal{G}}(\mathcal{A},\mathcal{B})$ is a vector space.

Example 2.6.

1. *The category of $\mathcal{G}$-modules, with $\mathcal{G} = (0 \xrightarrow{0} \mathfrak{g})$, can be identified with the category of homomorphisms of $\mathfrak{g}$-modules.*
2. *The category of $\mathcal{G}$-modules, with $\mathcal{G} = (\mathfrak{g} \xrightarrow{0} 0)$, is equivalent to the category of pairs $(\mathcal{A}, \bar{\xi})$, where $\mathcal{A} = (A_1 \to A_0)$ and $\mathfrak{g} \otimes \mathsf{Coker}(\partial) \xrightarrow{\bar{\xi}} \mathsf{Ker}(\partial)$ are homomorphisms of abelian groups.*

3. Universal enveloping crossed module

We recall the fundamental concepts of crossed modules of associative algebras from [6] (also cf. [9]).

An associative algebra A *acts* on another associative algebra R if, as a $\mathbf{K}$-module, R possesses an A-bimodule structure $A \times R \to R$, $(a,r) \mapsto a \cdot r$, $R \times A \to R$, $(r,a) \mapsto r \cdot a$, and the following conditions are verified:

$$a \cdot (rr') = (a \cdot r)r', \qquad (r \cdot a)r' = r(a \cdot r'), \qquad (rr') \cdot a = r(r' \cdot a), \qquad a \in A,\ r, r' \in R.$$

A *crossed module of associative algebras* (R, A, ρ) is an algebra homomorphism $\rho\colon R \to A$, together with an action of A on R, such that the following conditions hold:

$$\begin{aligned} &\rho(a \cdot r) = a\rho(r), \qquad \rho(r \cdot a) = \rho(r)a, \\ &\rho(r) \cdot r' = rr' = r \cdot \rho(r'), \qquad a \in A,\ r, r' \in R. \end{aligned}$$

A *crossed submodule* (S, B, ρ') of a crossed module (R, A, ρ) is a crossed module such that S and B are respectively subalgebras of R and A, the action of B on S is induced by the action of A on R and $\rho|_S = \rho'$. (S, B, ρ') is an *ideal* if besides B is an ideal of A, S is closed under the action of A, i.e. $A \cdot S = S \cdot A \subset S$ and $B \cdot R = R \cdot B \subset B$. As a consequence of $A \cdot S = S \cdot A \subset S$, it follows that S is also an ideal of R, since $rs = \rho(r) \cdot s$.

Let $\mathfrak{g}$ be a Lie algebra. We have a natural epimorphism from the tensor algebra $\mathrm{T}(\mathfrak{g})$ into the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$. For an element $v_1 \otimes \cdots \otimes v_n$ of $\mathrm{T}(\mathfrak{g})$, we denote by $v_1 \cdots v_n$ the image of $v_1 \otimes \cdots \otimes v_n$ in $\mathrm{U}(\mathfrak{g})$.

Let $\mathcal{G}$ be a crossed module of Lie algebras. We identify an element $(x, 0)$ of $\mathfrak{g}_1 \rtimes \mathfrak{g}_0$ with x. We also identify an element $(0, g)$ of $\mathfrak{g}_1 \rtimes \mathfrak{g}_0$ with g. Using these notations, we have the following relation in $\mathrm{U}(\mathfrak{g}_1 \rtimes \mathfrak{g}_0)$:

$$g \cdot x - x \cdot g = {}^{g}x.$$

There are two homomorphisms $s, t\colon \mathfrak{g}_1 \rtimes \mathfrak{g}_0 \longrightarrow \mathfrak{g}_0$ defined by $s(x, g) = g$, $t(x, g) = \partial(x) + g$. Applying the universal enveloping algebra functor U, we get the homomorphisms $\mathrm{U}(s), \mathrm{U}(t)\colon \mathrm{U}(\mathfrak{g}_1 \rtimes \mathfrak{g}_0) \to \mathrm{U}(\mathfrak{g}_0)$ of associative algebras. We set

$$\mathrm{U}_1(\mathcal{G}) = \frac{\mathrm{Ker}\,\mathrm{U}(s)}{\mathrm{Ker}\,\mathrm{U}(s)\,\mathrm{Ker}\,\mathrm{U}(t) + \mathrm{Ker}\,\mathrm{U}(t)\,\mathrm{Ker}\,\mathrm{U}(s)}.$$

Then, the morphism $\partial\colon \mathrm{U}_1(\mathcal{G}) \to \mathrm{U}(\mathfrak{g}_0)$, where ∂ is induced by $\mathrm{U}(t)$, is a crossed module of associative algebras (see [2]). We denote it by $\mathrm{U}(\mathcal{G})$ because it is an analogue of the universal enveloping algebra in the context of crossed modules of associative algebras, and is called the *universal enveloping crossed module* of $\mathcal{G}$ (see [2]).

Next, note that $\partial\big(\mathrm{U}_1(\mathcal{G})\big) \subseteq I(\mathfrak{g}_0)$, where $I(\mathfrak{g}_0)$ represents the augmentation ideal of the Lie algebra $\mathfrak{g}_0$. Thus, we have a crossed module of associative algebras, denoted as $\partial\colon \mathrm{U}_1(\mathcal{G}) \to I(\mathfrak{g}_0)$. This crossed module is an ideal of the crossed module $\mathrm{U}(\mathcal{G}) = \big(\mathrm{U}_1(\mathcal{G}) \xrightarrow{\partial} \mathrm{U}(\mathfrak{g}_0)\big)$, which we will call the *augmentation ideal* of $\mathcal{G}$, and denote it as $I(\mathcal{G})$.

By definition, we have a short exact sequence of crossed modules of associative algebras:

$$0 \to I(\mathbf{G}) \to \mathrm{U}(\mathbf{G}) \to \big(0 \to \mathbf{K}\big) \to 0.$$

Lemma 3.1. *Let* Z *be a subspace of* $\mathrm{U}(\mathfrak{g}_1 \rtimes \mathfrak{g}_0) \otimes \mathfrak{g}_1$ *generated by all the elements of the form* $v \cdot x_1 \otimes x_2 - v \cdot x_2 \otimes x_1 - v \otimes [x_1, x_2]$, *where* $x_1, x_2 \in \mathfrak{g}_1$ *and* $v \in \mathrm{U}(\mathfrak{g}_1 \rtimes \mathfrak{g}_0)$. *Then, there is an epimorphism* $p\colon \mathrm{U}(\mathfrak{g}_1 \rtimes \mathfrak{g}_0) \otimes \mathfrak{g}_1 \to \{\mathsf{Ker}\,\mathrm{U}(s)\colon \mathrm{U}(\mathfrak{g}_1 \rtimes \mathfrak{g}_0) \to \mathrm{U}(\mathfrak{g}_0)\}$ *given by* $v \otimes x \mapsto v \cdot x$. *Moreover,* $\mathsf{Ker}\,p = \mathsf{Z}$.

Proof. It is easy to observe that $\mathsf{Ker}\,\mathrm{U}(s)$ is a subspace of $\mathrm{U}(\mathfrak{g}_1 \rtimes \mathfrak{g}_0)$ generated by the elements $v_1 \cdot v_2 \cdots v_n$, where $v_i \in \mathfrak{g}_1 \cup \mathfrak{g}_0$ and $v_j \in \mathfrak{g}_1$ for at least one j. Moreover, since $x \cdot g = g \cdot x - {}^{g}x$, we get that p is an epimorphism. Since $\mathsf{Z} \subseteq \mathsf{Ker}\,p$, it suffices to show that p induces a monomorphism from $\big(\mathrm{U}(\mathfrak{g}_1 \rtimes \mathfrak{g}_0) \otimes \mathfrak{g}_1\big)/\mathsf{Z}$ into $\mathrm{U}(\mathfrak{g}_1 \rtimes \mathfrak{g}_0)$. Let $\{x_i, i \in I_1\}$ be a basis of $\mathfrak{g}_1$ and $\{g_j, j \in I_2\}$ be a basis of $\mathfrak{g}_0$, where I_1 and I_2 are totally ordered sets. Then $\big(\mathrm{U}(\mathfrak{g}_1 \rtimes \mathfrak{g}_0) \otimes \mathfrak{g}_1\big)/\mathsf{Z}$ is a subspace generated by the following elements:

$$x_{i_1} \cdot x_{i_2} \cdots x_{i_{m-1}} \otimes x_{i_m} + \mathsf{Z},$$
$$g_{j_1} \cdot g_{j_2} \cdots g_{j_n} \cdot x_{i_1} \cdot x_{i_2} \cdots x_{i_{m-1}} \otimes x_{i_m} + \mathsf{Z},$$

where $j_1 \leq j_2 \leq \cdots \leq j_n$ and $i_1 \leq i_2 \leq \cdots \leq i_m$. By PBW theorem, their images (via p) are linearly independent in $\mathrm{U}(\mathfrak{g}_1 \rtimes \mathfrak{g}_0)$. This implies that the elements mentioned above form a basis of $\big(\mathrm{U}(\mathfrak{g}_1 \rtimes \mathfrak{g}_0) \otimes \mathfrak{g}_1\big)/\mathsf{Z}$ and that p induces a monomorphism from $\big(\mathrm{U}(\mathfrak{g}_1 \rtimes \mathfrak{g}_0) \otimes \mathfrak{g}_1\big)/\mathsf{Z}$ into $\mathrm{U}(\mathfrak{g}_1 \rtimes \mathfrak{g}_0)$. □

Lemma 3.2. *In* $\mathrm{U}(\mathfrak{g}_1 \rtimes \mathfrak{g}_0)$, *the following relation holds:*

$$x \cdot (y - \partial(y)) = (y - \partial(y)) \cdot x,$$

for each $x, y \in \mathfrak{g}_1$. *Moreover,* $\mathsf{Ker}(s)\,\mathsf{Ker}(t) \subseteq \mathsf{Ker}(t)\,\mathsf{Ker}(s)$.

Proof. In fact,

$$\begin{aligned} x \cdot \big(y - \partial(y)\big) - \big(y - \partial(y)\big) \cdot x &= \partial(y) \cdot x - x \cdot \partial(y) - (y \cdot x - x \cdot y) \\ &= {}^{\partial(y)}x - [y, x] = [y, x] - [y, x] = 0. \end{aligned}$$

Now, observe that $\mathsf{Ker}(t)$ is an ideal of $\mathrm{U}(\mathfrak{g}_1 \rtimes \mathfrak{g}_0)$ generated by $y - \partial(y)$, $y \in \mathfrak{g}_1$. Therefore, to prove the inclusion $\mathsf{Ker}(s)\,\mathsf{Ker}(t) \subseteq \mathsf{Ker}(t)\,\mathsf{Ker}(s)$, it suffices to show that

$$v_1 \cdots v_n \cdot x \cdot v_{n+1} \cdots v_{n+k} \cdot (y - \partial(y)) \cdot v_{n+k+1} \cdots v_{n+k+l} \in \mathsf{Ker}(t)\,\mathsf{Ker}(s),$$

for each $x, y \in \mathfrak{g}_1$, $v_i \in \mathfrak{g}_1 \cup \mathfrak{g}_0$. Next, $x \cdot v_{n+1} \cdots v_{n+k}$ is a linear combination of the elements of the form $w_1 \cdots w_{n_i} \cdot x_i$, where $x_i \in \mathfrak{g}_1$ and $w_j \in \mathfrak{g}_1 \cup \mathfrak{g}_0$. Since $x_i \cdot (y - \partial(y)) = (y - \partial(y)) \cdot x_i$, we have completed the proof. □

Lemma 3.3. *Let* $\mathcal{A}$ *be a* $\mathcal{G}$*-module and* $d_1\colon \mathfrak{g}_1 \to A_1$ *be a derivation.*

(i) Then there is a $\mathfrak{g}_0$*-module homomorphism* $f_1\colon \mathrm{U}(\mathfrak{g}_1 \rtimes \mathfrak{g}_0) \otimes \mathfrak{g}_1 \to A_1$ *given by*

$$f_1(v_1 \cdot v_2 \cdots v_n \otimes x) = t(v_1)\Big(t(v_2)\Big(\cdots\Big(t(v_n) d_1(x)\Big)\cdots\Big)\Big),$$

where $v_i \in \mathfrak{g}_0 \cup \mathfrak{g}_1$, $x \in \mathfrak{g}_1$.

(ii) *Let* Z *be as in Lemma 3.1. Then* $f_1(\mathsf{Z}) = 0$*, hence* f_1 *induces a* $\mathfrak{g}_0$*-module homomorphism* $f_1\colon \mathsf{Ker}\,\mathrm{U}(s) \to A_1$*. Moreover,* $f_1(\mathsf{Ker}(s)\,\mathsf{Ker}(t) + \mathsf{Ker}(t)\,\mathsf{Ker}(s)) = 0$*. Thus,* f_1 *induces a* $\mathfrak{g}_0$*-module homomorphism* $f_1\colon \mathrm{U}_1(\mathcal{G}) \to A_1$*.*

Proof. (i) Since A_1 is a $\mathfrak{g}_0$-module, we have

$$\begin{aligned} &g_1(g_2u) - g_2(g_1u) - [g_1,g_2]u = 0,\\ &\partial(x_1)(\partial(x_2)u) - \partial(x_2)(\partial(x_1)u) - \partial([x_1,x_2])u = 0,\\ &g(\partial(x)u) - \partial(x)(gu) = [g,\partial(x)]u = \partial({}^{g}x)u, \end{aligned}$$

where $u \in A_1$, $x, x_1, x_2 \in \mathfrak{g}_1$, $g, g_1, g_2 \in \mathfrak{g}_0$. This implies that $f_1\colon \mathrm{U}(\mathfrak{g}_1 \rtimes \mathfrak{g}_0) \otimes \mathfrak{g}_1 \to A_1$ is well defined.

(ii) Since d_1 is a derivation, $f_1(\mathsf{Z}) = 0$. In this way, we proved that f_1 induces a $\mathfrak{g}_0$-module homomorphism $f_1\colon \mathsf{Ker}\,\mathrm{U}(s) \to A_1$. We left to prove that $f_1(\mathsf{Ker}(s)\,\mathsf{Ker}(t) + \mathsf{Ker}(t)\,\mathsf{Ker}(s)) = 0$. By Lemma 3.2, it suffices to show that $f_1(\mathsf{Ker}(t)\,\mathsf{Ker}(s)) = 0$. Now, $\mathsf{Ker}(t)\,\mathsf{Ker}(s)$ is a subspace of $\mathrm{U}(\mathfrak{g}_1 \rtimes \mathfrak{g}_0)$ generated by the elements of the form $v_1 \cdots v_n \cdot \big(y - \partial(y)\big) \cdot v_{n+1} \cdots v_{n+k} \cdot x$, where $x, y \in \mathfrak{g}_1$, $v_i \in \mathfrak{g}_1 \cup \mathfrak{g}_0$. The images of these elements (via f_1) are trivial. □

4. Derivations

Let $\mathfrak{g}$ be a Lie algebra and A be a module over $\mathfrak{g}$. If we view A as an abelian Lie algebra, then $d\colon \mathfrak{g} \to A$ is a derivation if and only if $(d, 1_{\mathfrak{g}})\colon \mathfrak{g} \to A \rtimes \mathfrak{g}$ is a Lie algebra homomorphism. Therefore, if $\mathcal{G}$ is a Lie crossed module acting on an abelian Lie crossed module $\mathcal{A}$, then the intuition tells us to define a derivation from $\mathcal{G}$ into $\mathcal{A}$ as a pair of maps (d_1, d_0), $d_0\colon \mathfrak{g}_0 \to A_0$, $d_1\colon \mathfrak{g}_1 \to A_1$, so that $(d_1, 1_{\mathfrak{g}_1})\colon \mathfrak{g}_1 \to A_1 \rtimes \mathfrak{g}_1$ and $(d_0, 1_{\mathfrak{g}_0})\colon \mathfrak{g}_0 \to A_0 \rtimes \mathfrak{g}_0$ constitute a Lie crossed module morphism from $\mathcal{G}$ into $\mathcal{A} \rtimes \mathcal{G}$.

Definition 4.1. *A derivation* $\mathbf{d} = (d_1, d_0)$ *is a pair of linear maps* $d_0\colon \mathfrak{g}_0 \to A_0$ *and* $d_1\colon \mathfrak{g}_1 \to A_1$ *satisfying the following:*

$$d_0([g_1,g_2]) = g_1d_0(g_2) - g_2d_0(g_1), \tag{4.1}$$

$$d_1([x_1,x_2]) = \partial(x_1)d_1(x_2) - \partial(x_2)d_1(x_1), \tag{4.2}$$

$$d_1({}^{g}x) = gd_1(x) - \xi\big(x, d_0(g)\big), \tag{4.3}$$

$$\partial d_1(x) = d_0\partial(x), \tag{4.4}$$

where $g, g_1, g_2 \in \mathfrak{g}_0$*,* $x, x_1, x_2 \in \mathfrak{g}_1$*.*

This definition concurs with that introduced in [5] for two arbitrary crossed modules. Condition (4.3) matches condition (iii) in [5, p. 124]. In this context, because A is an abelian crossed module, the action of A_0 on A_1 is trivial.

Let us denote the set of all derivations by $\mathrm{Der}(\mathcal{G}, \mathcal{A})$. This set forms an abelian group under the operation of value-wise addition of functions.

Example 4.2.

1. *If* $\mathcal{G} = (0 \xrightarrow{0} \mathfrak{g})$*, then* $\mathrm{Der}\big(0 \to \mathfrak{g}, A_1 \to A_0\big) \cong \mathrm{Der}(\mathfrak{g}, A_0)$*. See Example 2.3 (1).*
2. $\mathrm{Der}\big(\mathfrak{g}_1 \xrightarrow{\partial} \mathfrak{g}_0, 0 \to A\big) \cong \mathrm{Der}(\mathfrak{g}_0/\partial(\mathfrak{g}_1), A)$*. See Example 2.3 (2).*
3. $\mathrm{Der}\big(\mathfrak{g} \xrightarrow{\mathrm{id}} \mathfrak{g}, A \xrightarrow{\mathrm{id}} A\big) \cong \mathrm{Der}(\mathfrak{g}, A)$*. See Example 2.3 (3).*
4. $\mathrm{Der}\big(\mathfrak{g} \xrightarrow{0} 0, A_1 \xrightarrow{\partial} A_0\big) \cong \mathrm{Hom}(\mathfrak{g}, \mathsf{Ker}\,\partial)$.

We will show that $\mathrm{Der}(\mathcal{G}, -)$ defines a functor, specifically $\mathrm{Der}(\mathcal{G}, -)\colon \mathcal{G}\text{-mod} \longrightarrow \mathrm{Ab}$, which maps between the category of $\mathcal{G}$-modules and the category of abelian groups. This fact follows from the lemma below.

Lemma 4.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{G}$-module and $(d_1, d_0) \in \mathrm{Der}(\mathcal{G}, \mathcal{A})$. Then, for each $(f_1, f_0) \in \mathrm{Hom}_{\mathcal{G}}(\mathcal{A}, \mathcal{B})$, $(f_1 d_1, f_0 d_0) \in \mathrm{Der}(\mathcal{G}, \mathcal{B})$.*

Proof. It is easy to see that $(f_1 d_1, f_0 d_0)$ satisfy identities (4.1), (4.2) and (4.4) of Definition 4.1. We will only check that identity (4.3) of Definition 4.1 holds.

$$\begin{aligned} f_1 d_1({}^{g}x) &= f\Big(gd_1(x) - \xi\big(x, d_0(g)\big)\Big) = f_1\big(gd_1(x)\big) - f_1\xi\big(x, d_0(g)\big) \\ &= gf_1 d_1(x) - \xi\big(x, f_0 d_0(x)\big). \end{aligned}$$

□

Now we define a $\mathcal{G}$-module structure on the augmentation ideal $I(\mathcal{G})$. If $v_1 \cdots v_n$ is an element of $\mathsf{Ker}\,\mathrm{U}(s)$, then we denote by $\langle\!\langle v_1 \cdots v_n \rangle\!\rangle$ the coset of $v_1 \cdots v_n$ in $\mathrm{U}_1(\mathcal{G})$. Define actions of $\mathfrak{g}_0$ on the underlying vector space of $\mathrm{U}_1(\mathcal{G})$ and that of $I(\mathfrak{g}_0)$:

$$\begin{aligned} &g\langle\!\langle v_1 \cdots v_n \rangle\!\rangle = \langle\!\langle g \cdot v_1 \cdots v_n \rangle\!\rangle, \\ &g(g_1 \cdots g_n) = g \cdot g_1 \cdots g_n, \end{aligned}$$

where $g, g_1, \dots, g_n \in \mathfrak{g}_0$, $v_1, \dots, v_n \in \mathfrak{g}_1 \cup \mathfrak{g}_0$. Moreover, we define a $\mathbf{K}$-bilinear map $\xi \colon \mathfrak{g}_1 \times I(\mathfrak{g}_0) \to \mathrm{U}_1(\mathcal{G})$ by the formula

$$\xi(x, g_1 \cdots g_n) = \langle\!\langle x \cdot g_1 \cdots g_n \rangle\!\rangle,$$

where $x \in \mathfrak{g}_1$ and $g_1, \dots, g_n \in \mathfrak{g}_0$.

Lemma 4.4. *Consider $\partial \colon \mathrm{U}_1(\mathcal{G}) \to I(\mathfrak{g}_0)$ as an abelian Lie crossed module. Then $\xi \colon \mathfrak{g}_1 \times I(\mathfrak{g}_0) \to \mathrm{U}_1(\mathcal{G})$ satisfies all the relations of Definition 2.2.*

Proof. We will only check identities (2.1) and (2.2). All other relations in Definition 2.2 are straightforward to verify.

(2.1): Let $x \in \mathfrak{g}_1$ and $v_1 \cdots v_n \in \mathsf{Ker}\,\mathrm{U}(s)$, where $v_1, \dots, v_n \in \mathfrak{g}_1 \cup \mathfrak{g}_0$. Then

$$\begin{aligned} \xi\Big(x, \partial\big(\langle\!\langle v_1 \cdots v_n \rangle\!\rangle\big)\Big) &= \xi\big(x, t(v_1) \cdots t(v_n)\big) = \langle\!\langle x \cdot t(v_1) \cdots t(v_n) \rangle\!\rangle \\ &= \langle\!\langle x \cdot v_1 \cdots v_n \rangle\!\rangle = \partial(x)\langle\!\langle v_1 \cdots v_n \rangle\!\rangle = x\langle\!\langle v_1 \cdots v_n \rangle\!\rangle. \end{aligned}$$

This implies that for each $u \in \mathrm{U}_1(\mathcal{G})$,

$$\xi(x, \partial(u)) = xu.$$

(2.2): For each $g_1, \dots, g_n \in \mathfrak{g}_0$, we have

$$\begin{aligned} \partial\xi(x, g_1 \cdots g_n) &= \partial(\langle\!\langle x \cdot g_1 \cdots g_n \rangle\!\rangle) = t(x) \cdot g_1 \cdots g_n \\ &= \partial(x) \cdot g_1 \cdots g_n = x(g_1 \cdots g_n). \end{aligned}$$

This implies that for each $a \in I(\mathfrak{g}_0)$,

$$\partial\xi(x, a) = xa.$$

□

As a consequence of Lemma 4.4, the augmentation ideal $I(\mathcal{G})$ has a $\mathcal{G}$-module structure.

The following proposition shows that the functor $\mathrm{Der}(\mathcal{G}, -)$ is represented by the $\mathcal{G}$-module $I(\mathcal{G})$.

Proposition 4.5. *There is a bijective linear map*

$$\mathrm{Hom}_{\mathcal{G}}(I(\mathcal{G}), \mathcal{A}) \approx \mathrm{Der}(\mathcal{G}, \mathcal{A})$$

Proof. We define a map $\phi\colon \mathrm{Hom}_{\mathcal{G}}(I(\mathcal{G}),\mathcal{A}) \to \mathrm{Der}(\mathcal{G},\mathcal{A})$. For a given pair $(f_1,f_0) \in \mathrm{Hom}_{\mathcal{G}}(I(\mathcal{G}),\mathcal{A})$, we set $(d_1,d_0) = \phi(f_1,f_0)$, where $d_0\colon \mathfrak{g}_0 \to A_0$ and $d_1\colon \mathfrak{g}_1 \to A_1$ are defined by $d_0(g) = f_0(g)$, $d_1(x) = f_1\big(\langle\!\langle x\rangle\!\rangle\big)$. We need to verify identities (4.1)–(4.4) of Definition 4.1.

(4.1): It is well-known in classical literature that d_0, defined as mentioned earlier, is a derivation (see, for example [11]).

(4.2): This is also fairly clear, but we will verify it.

$$\begin{aligned} d_1([x_1,x_2]) &= f_1\big(\langle\!\langle [x_1,x_2]\rangle\!\rangle\big) = f_1\big(\langle\!\langle x_1\cdot x_2\rangle\!\rangle - \langle\!\langle x_2\cdot x_1\rangle\!\rangle\big)\\ &= f_1\big(\langle\!\langle x_1\cdot x_2\rangle\!\rangle\big) - f_1\big(\langle\!\langle x_2\cdot x_1\rangle\!\rangle\big) = f_1\big(\langle\!\langle \partial(x_1)\cdot x_2\rangle\!\rangle\big) - f_1\big(\langle\!\langle \partial(x_2)\cdot x_1\rangle\!\rangle\big)\\ &= \partial(x_1)f_1\big(\langle\!\langle x_2\rangle\!\rangle\big) - \partial(x_2)f_1\big(\langle\!\langle x_1\rangle\!\rangle\big) = \partial(x_1)d_1(x_2) - \partial(x_2)d_1(x_1). \end{aligned}$$

(4.3): Using the following commutative diagram

$$\begin{tikzcd} \mathfrak{g}_1 \times I(\mathfrak{g}_0) \arrow[r, "\xi"] \arrow[d, "1\times f_0"'] & \mathrm{U}_1(\mathcal{G}) \arrow[d, "f_1"] \\ \mathfrak{g}_1 \times A_0 \arrow[r, "\xi"'] & A_1 \end{tikzcd}.$$

we have:

$$f_1\big(\langle\!\langle x\cdot g\rangle\!\rangle\big) = \xi(x, f_0(g)) = \xi(x, d_0(g)).$$

On the other hand,

$$\langle\!\langle g\cdot x\rangle\!\rangle - \langle\!\langle x\cdot g\rangle\!\rangle = \langle\!\langle {}^{g}x\rangle\!\rangle.$$

Thus,

$$\begin{aligned} d_1({}^{g}x) &= f_1\big(\langle\!\langle {}^{g}x\rangle\!\rangle\big) = f_1\big(\langle\!\langle g\cdot x\rangle\!\rangle\big) - f_1\big(\langle\!\langle x\cdot g\rangle\!\rangle\big)\\ &= gf_1\big(\langle\!\langle x\rangle\!\rangle\big) - \xi\big(x,d_0(g)\big) = gd_1(x) - \xi(x,d_0(g)). \end{aligned}$$

(4.4): This follows from the following commutative diagram:

$$\begin{tikzcd} \mathrm{U}_1(\mathcal{G}) \arrow[r, "\partial"] \arrow[d, "f_1"'] & I(\mathfrak{g}_0) \arrow[d, "f_0"] \\ A_1 \arrow[r, "\partial"'] & A_0 \end{tikzcd}.$$

Now we define a map $\psi\colon \mathrm{Der}(\mathcal{G},\mathcal{A}) \to \mathrm{Hom}_{\mathcal{G}}(I(\mathcal{G}),\mathcal{A})$. Given $(d_1,d_0) \in \mathrm{Der}(\mathcal{G},\mathcal{A})$, we set $(f_1,f_0) = \psi(d_1,d_0)$, where $f_0\colon I(\mathfrak{g}_0) \to A_0$ is defined by

$$f_0(g_1\cdots g_n) = g_1\Big(\cdots\big(g_{n-1}d_0(g_n)\big)\cdots\Big),$$

where $g_1,\dots,g_n \in \mathfrak{g}_0$, and $f_1\colon \mathrm{U}_1(\mathcal{G}) \to A_1$ is defined as in Lemma 3.3. Then f_0 and f_1 are $\mathfrak{g}_0$-module homomorphisms and $\partial f_1 = f_0\partial$. Thus, to show that $(f_1,f_0) \in \mathrm{Hom}_{\mathcal{G}}(I(\mathcal{G}),\mathcal{A})$ we have to prove that the following diagram is commutative:

$$\begin{tikzcd} \mathfrak{g}_1 \times I(\mathfrak{g}_0) \arrow[r, "\xi"] \arrow[d, "1\times f_0"'] & \mathrm{U}_1(\mathcal{G}) \arrow[d, "f_1"] \\ \mathfrak{g}_1 \times A_0 \arrow[r, "\xi"'] & A_1 \end{tikzcd}.$$

Thus, we need to check that

$$f_1\big(\langle\langle x\cdot g_1\cdots g_n\rangle\rangle\big)=\xi(x,f_0(g_1\cdots g_n)),$$

where $x\in\mathfrak{g}_1$ and $g_1,\dots,g_n\in\mathfrak{g}_0$. We will do this by induction on n. If $n=1$, then

$$\begin{aligned}f_1\big(\langle\langle x\cdot g\rangle\rangle\big)&=f_1\big(\langle\langle g\cdot x\rangle\rangle\big)-f_1\big(\langle\langle {}^{g}x\rangle\rangle\big)=gf_1\big(\langle\langle x\rangle\rangle\big)-d_1({}^{g}x)\\&=gd_1(x)-d_1({}^{g}x)=\xi\big(x,d_0(g)\big)=\xi\big(x,f_0(g)\big).\end{aligned}$$

Assume that $f_1\big(\langle\langle x\cdot g_1\cdots g_m\rangle\rangle\big)=\xi\big(x,f_0(g_1\cdots g_m)\big)$ for each $m\leq n$. Then

$$\begin{aligned}f_1\big(\langle\langle x\cdot g_1\cdot g_2\cdots g_{n+1}\rangle\rangle\big)&=f_1\big(\langle\langle (g_1\cdot x-{}^{g_1}x)\cdot g_2\cdots g_{n+1}\rangle\rangle\big)\\&=f_1\big(\langle\langle g_1\cdot x\cdot g_2\cdots g_{n+1}\rangle\rangle\big)-f_1\big(\langle\langle {}^{g_1}x\cdot g_2\cdots g_{n+1}\rangle\rangle\big)\\&=g_1f_1\big(\langle\langle x\cdot g_2\cdots g_{n+1}\rangle\rangle\big)-\xi\big({}^{g_1}x,f_0(g_2\cdots g_{n+1})\big)\\&=g_1\xi\big(x,f_0(g_2\cdots g_{n+1})\big)-\xi\big({}^{g_1}x,f_0(g_2\cdots g_{n+1})\big)\\&=\xi\big(x,g_1f_0(g_2\cdots g_{n+1})\big)=\xi\big(x,f_0(g_1\cdot g_2\cdots g_{n+1})\big).\end{aligned}$$

By the definition, we see that $\psi\phi=1_{\mathrm{Hom}_{\mathcal{G}}(I(\mathcal{G}),\mathcal{A})}$ and $\phi\psi=1_{\mathrm{Der}(\mathcal{G},\mathcal{A})}$. □

5. Principal derivations and cohomology

Let $\mathcal{A}$ be a $\mathcal{G}$-module. For each fixed element $a\in A_0$, we define a pair of linear maps (d_1,d_0) where $d_0(g)=ga$ for each $g\in\mathfrak{g}_0$, and $d_1(x)=\xi(x,a)$ for each $x\in\mathfrak{g}_1$.

We now show that (d_1,d_0) is a derivation by verifying that identities (4.1)–(4.4) in Definition 4.1 are satisfied.

$$\begin{aligned}&(4.1): && d_0([g,g'])=[g,g']a=g(g'a)-g'(ga)=gd_0(g')-g'd_0(g).\\&(4.2): && d_1([x,x'])=\xi([x,x'],a)=\partial(x)\xi(x',a)-\partial(x')\xi(x,a)\\& && \qquad=\partial(x)d_1(x')-\partial(x')d_1(x).\\&(4.3): && d_1({}^{g}x)=\xi({}^{g}x,a)=g\xi(x,a)-\xi(x,ga)=gd_1(x)-\xi\big(x,d_0(g)\big).\\&(4.4): && \partial d_1(x)=\partial\xi(x,a)=\partial(x)a=d_0\big(\partial(x)\big).\end{aligned}$$

Definition 5.1. *Let $\mathcal{A}$ be a $\mathcal{G}$-module. For each fixed $a\in A_0$, a principal derivation is a pair $\mathbf{d}_a=(d_0,d_1)\in\mathrm{Der}(G,A)$, where $d_0(g)=ga$, and $d_1(x)=\xi(x,a)$, for each $g\in\mathfrak{g}_0$ and $x\in\mathfrak{g}_1$.*

Denote the set of all principal derivations by $\mathrm{PDer}(\mathcal{G},\mathcal{A})$. Then $\mathrm{PDer}(\mathcal{G},\mathcal{A})$ is an abelian subgroup of $\mathrm{Der}(\mathcal{G},\mathcal{A})$. In particular, $\mathbf{d}_{(a+a')}=\mathbf{d}_a+\mathbf{d}_{a'}$, for each $a,a'\in A_0$. This implies that we have a K-linear map

$$\delta_0\colon A_0\to\mathrm{Der}(\mathcal{G},\mathcal{A}),\quad a\mapsto d_a.$$

Next, if $(f_1,f_0)\in\mathrm{Hom}_{\mathcal{G}}(\mathcal{A},\mathcal{B})$, then for each $a\in A_0$ and for the pair $(d_1,d_0)=\mathbf{d}_a$, we have $f_0d_0(g)=f_0(ga)=gf_0(a)$ and $f_1d_1(x)=f_1\xi(x,a)=\xi(x,f_0(a))$. Hence, $(f_1d_1,f_0d_0)=\mathbf{d}_{f_0(a)}$. This implies that $\mathrm{PDer}(\mathcal{G},-)$ is a functor. Moreover, the following diagram is commutative:

$$\begin{tikzcd}A_0 \arrow[r, "f_0"] \arrow[d, "\delta_0"'] & B_0 \arrow[d, "\delta_0"]\\ \mathrm{Der}(\mathcal{G},\mathcal{A}) \arrow[r] & \mathrm{Der}(\mathcal{G},\mathcal{B})\end{tikzcd}.$$

Example 5.2.

1. $\mathrm{PDer}\big(0 \to \mathfrak{g}, A_1 \to A_0\big) \cong \mathrm{PDer}(\mathfrak{g}, A_0)$. *See Example 2.3 (1).*
2. $\mathrm{PDer}\big(\mathfrak{g}_1 \xrightarrow{\partial} \mathfrak{g}_0, 0 \to A\big) \cong \mathrm{PDer}(\mathfrak{g}_0/\partial(\mathfrak{g}_1), A)$. *See Example 2.3 (2).*
3. $\mathrm{PDer}\big(\mathfrak{g} \xrightarrow{\mathrm{id}} \mathfrak{g}, A \xrightarrow{\mathrm{id}} A\big) \cong \mathrm{PDer}(\mathfrak{g}, A)$. *See Example 2.3 (3).*

Definition 5.3. *We define the zero and the first cohomologies of $\mathcal{G}$ with coefficients in $\mathcal{A}$ by the following formulas:*

$$H^0(\mathcal{G}, \mathcal{A}) := \mathrm{Ker}\{\delta_0 : A_0 \to \mathrm{Der}(G, A)\},$$
$$H^1(\mathcal{G}, \mathcal{A}) := \mathrm{Der}(\mathcal{G}, \mathcal{A}) / \mathrm{PDer}(\mathcal{G}, \mathcal{A}).$$

By the definition $H^0(\mathcal{G}, \mathcal{A}) = \{a \in A_0 \mid ga = 0,\ \xi(x, a) = 0, g \in \mathfrak{g}_0, x \in \mathfrak{g}_1\}$. Moreover, by the previous commutative diagram, we get that $H^0(\mathcal{G}, -)$ is a functor. Moreover, since $\mathrm{Der}(\mathcal{G}, -)$ and $\mathrm{PDer}(\mathcal{G}, -)$ are functors, $H^1(\mathcal{G}, -)$ is also a functor.

Let $\mathfrak{g}$ be a Lie algebra and A a $\mathfrak{g}$-module. Let us recall that the first cohomologies of Chevalley-Eilenberg are

$$H^0(\mathfrak{g}, A) \cong A^{\mathfrak{g}} = \{a \in A \mid ga = 0\}, \quad \text{and}$$
$$H^1(\mathfrak{g}, A) \cong \mathrm{Der}(\mathfrak{g}, A) / \mathrm{PDer}(\mathfrak{g}, A).$$

The following proposition establishes the connection with the classical Chevalley–Eilenberg cohomology of Lie algebras.

Proposition 5.4.

1. *Let $\mathfrak{g}$ be a Lie algebra and $A_1 \to A_0$ a $\mathcal{G}$-module, $\mathcal{G} = (0 \to \mathfrak{g})$. Then*

   $$H^i(0 \to \mathfrak{g}, A_1 \to A_0) \cong H^i(\mathfrak{g}, A_0), \quad i = 0, 1.$$

   In particular, if $\mathfrak{g}$ is a Lie algebra and A a $\mathfrak{g}$-module, then the abelian Lie crossed module $0 \to A$ will be a module over the Lie crossed module $0 \to \mathfrak{g}$, and it is satisfied that

   $$H^0(\mathfrak{g}, A) \cong H^0(0 \to \mathfrak{g}, 0 \to A) \quad \text{and} \quad H^1(\mathfrak{g}, A) \cong H^1(0 \to \mathfrak{g}, 0 \to A).$$

2. *Let $0 \to A$ be a $\mathcal{G}$-module. Then A is a module over the Lie algebra $\mathfrak{g}_0/\partial(\mathfrak{g}_1)$, and there is an isomorphism:*

   $$H^i(\mathcal{G}, 0 \to A) \cong H^i(\mathfrak{g}_0/\partial(\mathfrak{g}_1), A), \quad i = 0, 1.$$

3. *Let $\mathfrak{g}$ be a Lie algebra and A a $\mathfrak{g}$-module. Then*

   $$H^i(\mathfrak{g} \xrightarrow{\mathrm{id}} \mathfrak{g}, A \xrightarrow{\mathrm{id}} A) \cong H^i(\mathfrak{g}, A), \quad i = 0, 1.$$

Proof. The proof is deduced directly from the definition and Examples 4.2 and 5.2. □

Lemma 5.5. *Let $0 \longrightarrow \mathcal{A} \xrightarrow{(f_1, f_0)} \mathcal{B} \xrightarrow{(f_1', f_0')} \mathcal{C} \longrightarrow 0$ be an extension of $\mathcal{G}$-modules. Then we have the exact sequence of abelian groups:*

$$0 \to \mathrm{Der}(\mathcal{G}, \mathcal{A}) \to \mathrm{Der}(\mathcal{G}, \mathcal{B}) \to \mathrm{Der}(\mathcal{G}, \mathcal{C}).$$

Proof. Considering that both

$$0 \longrightarrow A_0 \xrightarrow{f_0} B_0 \xrightarrow{f_0'} C_0 \longrightarrow 0$$

and

$$0 \longrightarrow A_1 \xrightarrow{f_1} B_1 \xrightarrow{f_1'} C_1 \longrightarrow 0$$

are exact sequences, the proof follows easily by diagram chasing. □

Proposition 5.6. *Let* $0 \longrightarrow \mathcal{A} \xrightarrow{(f_1,f_0)} \mathcal{B} \xrightarrow{(f_1',f_0')} \mathcal{C} \longrightarrow 0$ *be an extension of* $\mathcal{G}$*-modules. Then we have the following exact sequence:*

$$0 \to H^0(\mathcal{G},\mathcal{A}) \to H^0(\mathcal{G},\mathcal{B}) \to H^0(\mathcal{G},\mathcal{C}) \to H^1(\mathcal{G},\mathcal{A}) \to H^1(\mathcal{G},\mathcal{B}) \to H^1(\mathcal{G},\mathcal{C}).$$

Proof. By Lemma 5.5, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A_0 \arrow[r] \arrow[d, "\delta_0"] & B_0 \arrow[r] \arrow[d, "\delta_0"] & C_0 \arrow[r] \arrow[d, "\delta_0"] & 0 \\
0 \arrow[r] & \mathrm{Der}(\mathcal{G},\mathcal{A}) \arrow[r] & \mathrm{Der}(\mathcal{G},\mathcal{B}) \arrow[r] & \mathrm{Der}(\mathcal{G},\mathcal{C}) &
\end{tikzcd}$$

Now, the proof follows from the snake lemma. □

Let $\mathcal{A}$ be a trivial $\mathcal{G}$-module, that is $\mathfrak{g}_0$ acts trivially on A_0 and on A_1, and $\xi\colon \mathfrak{g}_1 \times A_0 \to A_1$ is also trivial. Then it is easy to check that $H^0(\mathcal{G},\mathcal{A}) = A_0$ and

$$H^1(\mathcal{G},\mathcal{A}) = \mathrm{Hom}_{\mathcal{G}}(\mathcal{G}^{\mathrm{ab}},\mathcal{A}),$$

where $\mathcal{G}^{\mathrm{ab}}$ is the abelian crossed module $\mathfrak{g}_1/[\mathfrak{g}_0,\mathfrak{g}_1] \xrightarrow{\partial} \mathfrak{g}_0/[\mathfrak{g}_0,\mathfrak{g}_0]$. Here $[\mathfrak{g}_0,\mathfrak{g}_1]$ is the ideal of $\mathfrak{g}_1$ generated by the elements ${}^{g}x$, for each $g \in \mathfrak{g}_0$, $x \in \mathfrak{g}_1$.

In the next proposition $0 \to \mathbf{K}$ is assumed to be a trivial $\mathcal{G}$-module.

Proposition 5.7. *For each* $\mathcal{G}$*-module* $\mathcal{A}$*, we have*

$$H^0(\mathcal{G},\mathcal{A}) \cong \mathrm{Hom}_{\mathcal{G}}(0 \to K,\mathcal{A}).$$

Proof. By definition we have:

$$H^0(\mathcal{G},\mathcal{A}) = \{a \in A_0 \mid ga = 0 \ \text{ and } \ \xi(x,a) = 0 \ \text{ for each } \ g \in \mathfrak{g}_0, x \in \mathfrak{g}_1\}.$$

On the other hand, for each $(0,f_0) \in \mathrm{Hom}_{\mathcal{G}}(0 \to K,\mathcal{A})$, we have the following commutative diagrams:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "0"'] & \mathbf{K} \arrow[d, "f_0"] \\
A_1 \arrow[r, "\partial"'] & A_0 \ ,
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{g}_1 \times \mathbf{K} \arrow[r, "\xi"] \arrow[d, "1\times f_0"'] & 0 \arrow[d] \\
\mathfrak{g}_1 \times A_0 \arrow[r, "\xi"'] & A_1 \ .
\end{tikzcd}$$

The first diagram implies that $f_0\colon \mathbf{K} \to A_0$ is an arbitrary $\mathfrak{g}_0$-module homomorphism. Hence, $f_0(1)$ is such that $gf_0(1) = 0$ for each $g \in \mathfrak{g}_0$. The second diagram implies that $\xi\big(x, f_0(1)\big) = 0$ for each $x \in \mathfrak{g}_1$. Thus, the following map

$$\mathrm{Hom}_{\mathcal{G}}(0 \to K,\mathcal{A}) \to H^0(\mathcal{G},\mathcal{A}), \quad (0,f_0) \mapsto f_0(1),$$

is an isomorphism. □

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