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Well-posedness and energy decay of solutions to a coupled system of heat and Schrödinger equations

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Abstract. Our goal is to examine a system of a Schrödinger equation coupled with a heat equation in a bounded region. The coupling involves an operator that is parameterized by a real number in [0,1]. For $0 \leq \theta < 1$, we show that the associated semigroup of the system is not exponentially stable. We then propose a specific non-uniform decay rate. When $\theta = 1$, the linked semigroup is exponentially stable rather than analytic.

1. Introduction

Our objective is to investigate a system composed by a heat equation coupled by a Schrödinger equation, they are coupled through a fractional power of the Laplacian. The system reads:

$$\begin{cases} \phi_t + i\kappa\Delta\phi - \xi\Theta = 0, & \text{on} \quad \Omega \times (0, +\infty), \\ \Theta_t - \omega\Delta\Theta + \varsigma\,(-\Delta)^{\theta}\,\phi = 0, & \text{on} \quad \Omega \times (0, +\infty), \\ \phi = \Theta = 0, & \text{in} \quad \partial\Omega \times (0, +\infty), \\ \phi(x, 0) = \phi_0(x), \Theta(x, 0) = \Theta_0(x), & \text{on} \quad x \in \Omega, \end{cases} \tag{1}$$

where $\xi, \varsigma \in \mathbb{R}^*$ such that $\xi.\varsigma > 0$, $\kappa, \omega \in \mathbb{R}^+$ are physical constants, $(-\Delta)^{\theta}$ is the spectral fractional Laplace operator for which we can consult **[29]** for (definition, properties.....), $t > 0$ denotes the time variable and $x \in \Omega$ is the space variable, Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$ and $0 \leq \theta \leq 1$. The energy associated to problem (1) is:

$$\mathcal{E}_{\theta}(t) = \frac{1}{2}\left[\frac{\varsigma}{\xi}\left\|(-\Delta)^{\frac{\theta}{2}}\phi\right\|^2_{L^2(\Omega)} + \|\Theta\|^2_{L^2(\Omega)}\right]. \tag{2}$$

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1.1. Literature

Lebeau et al. [21] considered the two- and three-dimensional system of linear thermoelasticity in a bounded smooth domain with Dirichlet boundary conditions

$$\begin{cases} y_{tt} - \mu\Delta y - (\lambda+\mu)\nabla \mathrm{div} y + \alpha\nabla\theta = 0, & \text{in} \quad \Omega\times(0,+\infty),\\ \theta_t - \Delta\theta + \theta \mathrm{div} y_t = 0, & \text{in} \quad \Omega\times(0,+\infty),\\ y = \theta = 0, & \text{on} \quad \partial\Omega\times(0,+\infty),\\ y(\cdot,0) = y_0,\ y_t(\cdot,0) = y_1,\ \theta(\cdot,0) = \theta_0, & \text{in} \quad \Omega, \end{cases}$$

where λ and μ are the Lame coefficients, which are assumed to satisfy $\mu > 0,\ \ \lambda + 2\mu > 0$. The constants $\alpha,\ \theta > 0$ are the coupling parameters. The authors analyzed whether the energy of solutions decays exponentially or uniformly to zero as $t \to \infty$, they showed that when the domain is convex, the decay rate is never uniform. In fact, the lack of uniform decay may also be due to a critical polarization of the energy on the transversal component of the displacement.

Mansouri et al. [26] considered a coupled system consisting of a Kirchhoff thermoelastic plate and an undamped wave equation

$$\begin{cases} y_{tt} - \gamma\Delta y_{tt} + a\Delta^2 y + \alpha\Delta\theta + \mu z = 0, & \text{in} \quad \Omega\times(0,+\infty),\\ \theta_t - \sigma\Delta\theta - \theta\Delta y_t = 0, & \text{in} \quad \Omega\times(0,+\infty),\\ z_{tt} - \mu\Delta z + \mu y = 0, & \text{in} \quad \Omega\times(0,+\infty),\\ y = \partial_\nu y = z = \theta = 0, & \text{on} \quad \partial\Omega\times(0,+\infty),\\ y(\cdot,0) = y_0,\ y_t(\cdot,0) = y_1,\ \theta(\cdot,0) = \theta_0, & \text{in} \quad \Omega,\\ z(\cdot,0) = z_0,\ z_t(\cdot,0) = z_1, & \text{in} \quad \Omega, \end{cases}$$

they showed that the coupled system is not exponentially stable. Afterwards, they proved that the coupled system is polynomial stable and provides an explicit polynomial decay rate of the associated semigroup.

Tebou [30] studied a coupled system of the wave and heat equations given by

$$\begin{cases} y_{tt} - c^2\Delta y + \alpha(-\Delta)^\mu\theta = 0, & \text{in} \quad \Omega\times(0,+\infty),\\ \theta_t - \nu\Delta\theta - \theta y_t = 0, & \text{in} \quad \Omega\times(0,+\infty),\\ y = \theta = 0, & \text{on} \quad \partial\Omega\times(0,+\infty),\\ y(\cdot,0) = y_0,\ y_t(\cdot,0) = y_1,\ \theta(\cdot,0) = \theta_0, & \text{in} \quad \Omega, \end{cases}$$

where c and ν are positive physical constants. For $0 \le \mu < 1$, he showed that the semigroup associated with the system is not uniformly stable, as shown in [1], and he proposed an explicit non-uniform decay rate, completing the work in ([1], Remark 5) and improving the one in ([22], Example 2). When $\mu = 1$, the author showed that in this case, the semigroup is exponentially stable. In addition, he examined a partially clamped Kirchhoff thermoelastic plate without mechanical feedback controls, and he proved that the semigroup is also exponentially stable in this case, using a constructive frequency domain method to prove the stabilization result along with an explicit decay rate.

Tebou et al. [31] considered a thermoelastic plate with rotational forces in a bounded domain Ω. This rotational forces involves, the spectral fractional Laplacian, with a power parameter $0 \le \theta \le 1$

$$\begin{cases} y_{tt} + (-\Delta)^\theta y_{tt} + \Delta^2 y + \alpha\Delta z = 0, & \text{in} \quad \Omega\times(0,+\infty),\\ z_t - \kappa\Delta z - \theta\Delta y_t = 0, & \text{in} \quad \Omega\times(0,+\infty),\\ y(\cdot,0) = y_0,\ y_t(\cdot,0) = y_1,\ z(\cdot,0) = z_0, & \text{in} \quad \Omega, \end{cases}$$

the authors distinguished two particular cases of this problem that models the thermoelastic plate: either Euler-Bernoulli when $\theta = 0$ or Kirchhoff if $\theta = 1$. They showed that the semigroup studied in this case is of Gevrey class δ for every $\delta > (2-\theta)/(2-4\theta)$ and proved that it is exponentially stable.We can also consult **[6–12]** for other systems coupled with the heat equation.

The present work is organized as follows: In Section 2, we deal with the existence of of solutions to the the above system. In section 3, we treat the strong stability and appealing either Arendt-batty**[2]**, Lyubich-Vu**[25]** or the note of C. D. Benchimol**[4]**, whether or not the system is exponential, analytic depending on the value of θ. We conclude this section by establishing a polynomial decay of the energy in the case when $0 \le \theta < 1$ via the theorem of A. Borichev and Y. Tomilov **[5]**. In Section 4, we give some remarks and present some open questions.

2. Global existence

This section focuses on establishing the global existence of solutions to system (1)using a combination of key analytical tools. The Lax-Milgram lemma ensures the existence and uniqueness of weak solutions through a variational framework. Elliptic regularity theory improves the smoothness of these solutions, bridging the gap to stronger solution concepts. Finally, semigroup theory handles the time evolution, enabling the extension from local to global solutions by treating the problem as an abstract Cauchy problem. Together, these methods provide a solid foundation for proving global existence rigorously. We define energy space of system (1) by

$$\mathcal{H}_\theta = L^2(\Omega) \times H^\theta(\Omega).$$

The space $\mathcal{H}_\theta$ is equipped by an inner-product for any $\mathcal{X}_1, \mathcal{X}_2 \in \mathcal{H}_\theta$ with $\mathcal{X} = (\phi_1, \Theta_1)^T$ and $\mathcal{X}_2 = (\phi_2, \Theta_2)^T$ as follows

$$< \mathcal{X}_1, \mathcal{X}_2 >_{\mathcal{H}_\theta} = \frac{\varsigma}{\xi} \int_\Omega (-\Delta)^{\frac{\theta}{2}} \phi_1 \overline{(-\Delta)^{\frac{\theta}{2}} \phi_2} dx + \int_\Omega \Theta_1 \overline{\Theta_2} dx. \tag{3}$$

Let us write problem (1) in an abstract form in the Hilbert space $\mathcal{H}_\theta$:

$$\begin{cases} \mathcal{X}'(t) = \mathcal{G}_\theta \mathcal{X}(t), \\ \mathcal{X}(0) = \mathcal{X}_0, \end{cases} \tag{4}$$

where $\mathcal{X}_0 = (\phi_0, \Theta_0)$ and $\mathcal{G}_\theta$ is the operator

$$\mathcal{G}_\theta : \mathcal{D}(\mathcal{G}_\theta) \subset \mathcal{H}_\theta \to \mathcal{H}_\theta, \tag{5}$$

given by

$$\mathcal{G}_\theta \begin{bmatrix} \phi \\ \Theta \end{bmatrix} = \begin{bmatrix} -i\kappa\Delta\phi + \xi\Theta \\ \omega\Delta\Theta - \varsigma(-\Delta)^\theta \phi \end{bmatrix}, \tag{6}$$

with domain

$$\mathcal{D}(\mathcal{G}_\theta) = \left\{ (\phi, \Theta)^T \in \mathcal{H}_\theta \;\middle|\; \begin{array}{l} \phi \in H_0^1(\Omega), \Theta \in H_0^1(\Omega) \\ \omega\Delta\Theta - \varsigma(-\Delta)^\theta\phi \in H^\theta(\Omega); \\ -i\kappa\Delta\phi + \xi\Theta \in L^2(\Omega); \end{array} \right\}. \tag{7}$$

The formula for the energy associated with new notations now takes the form

$$\mathcal{E}_\theta(t) = \frac{1}{2} \|\mathcal{X}\|^2_{\mathcal{H}_\theta}. \tag{8}$$

We begin this study by the following lemma.

Lemma 2.1. *Let* (ϕ, Θ) *be a regular solutions of (1), then*

$$\mathcal{E}'_\theta(t) = -\omega \left\|\nabla\phi\right\|^2_{L^2(\Omega)}. \tag{9}$$

Proof. Multiplying equation $(1)_1$ by $\frac{\varsigma}{\xi}\overline{(-\Delta)^\theta\phi}$, integrating over Ω and using integration by parts, we obtain

$$\frac{\varsigma}{2\xi} \frac{d}{dt} \|(-\Delta)^{\frac{\theta}{2}} \phi\|^2_{L^2(\Omega)} + i\frac{\kappa\varsigma}{\xi} \|(-\Delta)^{\frac{\theta+1}{2}} \phi\|^2_{L^2(\Omega)} + \varsigma \int_\Omega \Theta(-\Delta)^\theta \overline{\phi} dx = 0. \tag{10}$$

Multiplying equation $(1)_2$ by $\overline{\Theta}$, integrating over Ω and using integration by parts, we get

$$\frac{1}{2} \frac{d}{dt} \|\Theta\|^2_{L^2(\Omega)} + \omega \|\nabla\Theta\|^2_{L^2(\Omega)} - \varsigma \int_\Omega \overline{\Theta}(-\Delta)^\theta \phi dx = 0. \tag{11}$$

Now we take the real part of the sum of (10), (11) and then integrate over (0,t), we get

$$\begin{aligned} \mathcal{E}_\theta(t) &= \frac{1}{2}\|\mathcal{X}\|^2_{\mathcal{H}_\theta}. \\ \mathcal{E}'_\theta(t) &= -\omega\,\|\nabla\Theta\|^2_{L^2(\Omega)}. \end{aligned} \tag{12}$$

□

Theorem 2.2. *(Existence and uniqueness)*

- *For $\mathcal{X}_0 \in \mathcal{H}_\theta$, the system (1) has a unique weak solution $\mathcal{X} \in C^0(\mathbb{R}^+;\mathcal{H}_\theta)$.*
- *For more regular $\mathcal{X}_0 \in \mathcal{D}(\mathcal{G}_\theta)$, the system (1) has a unique strong solution*

$$\mathcal{X} \in C^0(\mathbb{R}^+;\mathcal{D}(\mathcal{G}_\theta)) \cap C^1(\mathbb{R}^+;\mathcal{H}_\theta).$$

Proof. We can easily conclude this theorem from the well-known (Lumer-Philips theorem) if we show that $\mathcal{G}_\theta$ is maximal and monotone, we are going to prove this in three steps.
Step 1 Let $\mathcal{X} \in \mathcal{D}(\mathcal{G}_\theta)$, then

$$\Re e < \mathcal{G}_\theta\mathcal{X}, \mathcal{X} >_{\mathcal{H}_\theta} = \mathcal{E}'_\theta(t) = -\omega\,\|\nabla\Theta\|^2_{L^2(\Omega)} \leq 0. \tag{13}$$

Step 2: It remains to prove that $\mathcal{R}(\gamma\mathcal{I} - \mathcal{G}_\theta) = \mathcal{S}_\theta$, for any $\gamma > 0$. Let $\mathcal{Y} = (y_1, y_2)^T \in \mathcal{H}_\theta$, $\gamma > 0$ and $0 \leq \theta \leq 1$,

$$\mathcal{R}(\gamma\mathcal{I} - \mathcal{G}_\theta) = \mathcal{H}_\theta \Longleftrightarrow \gamma\mathcal{X} - \mathcal{G}_\theta\mathcal{X} = \mathcal{Y}. \tag{14}$$

Equivalently, we can look for a solution of the following system

$$\begin{cases} \gamma\phi + i\kappa\Delta\phi - \xi\Theta = y_1, \\ \gamma\Theta - \omega\Delta\Theta + \varsigma(-\Delta)^\theta\phi = y_2. \end{cases} \tag{15}$$

Let $(\widetilde{\phi}, \widetilde{\Theta}) \in H^\theta(\Omega) \times H^1_0(\Omega)$, multiply equations $(15)_1$ and $(15)_2$ by $\frac{\varsigma}{\xi}\overline{(-\Delta)^\theta\,\widetilde{\phi}}, \overline{\widetilde{\Theta}}$, respectively, and integrate over Ω, we get

$$\begin{aligned} \frac{\varsigma}{\xi}\int_\Omega (-\Delta)^{\frac{\theta}{2}}\, y_1\overline{(-\Delta)^{\frac{\theta}{2}}\,\widetilde{\phi}}dx = \;& \frac{\varsigma\gamma}{\xi}\int_\Omega (-\Delta)^{\frac{\theta}{2}}\,\phi\overline{(-\Delta)^{\frac{\theta}{2}}\,\widetilde{\phi}}dx + i\frac{\varsigma\kappa}{\xi}\int_\Omega (-\Delta)^{\frac{\theta+1}{2}}\,\phi\overline{(-\Delta)^{\frac{\theta+1}{2}}\,\widetilde{\phi}}dx \\ & -\varsigma\int_\Omega (-\Delta)^{\frac{\theta}{2}}\,\Theta\overline{(-\Delta)^{\frac{\theta}{2}}\,\widetilde{\phi}}dx, \end{aligned} \tag{16}$$

and

$$\int_\Omega y_2\overline{\widetilde{\Theta}}dx = \gamma\int_\Omega \Theta\overline{\widetilde{\Theta}}dx + \omega\int_\Omega \nabla\Theta\overline{\nabla\widetilde{\Theta}}dx + \varsigma\int_\Omega (-\Delta)^{\frac{\theta}{2}}\phi\overline{(-\Delta)^{\frac{\theta}{2}}\widetilde{\Theta}}dx. \tag{17}$$

Let $\mathcal{H}_* = H^\theta(\Omega) \times H^1_0(\Omega)$. We define a sesquilinear form $\mathcal{B} : \mathcal{H}_* \times \mathcal{H}_* \to \mathbb{C}$, by

$$\begin{aligned} \mathcal{B}((\phi,\Theta),(\widetilde{\phi},\widetilde{\Theta})) \;&= \gamma\int_\Omega \Theta\overline{\widetilde{\Theta}}dx + \omega\int_\Omega \nabla\Theta\overline{\nabla\widetilde{\Theta}}dx + \varsigma\int_\Omega (-\Delta)^{\frac{\theta}{2}}\phi\overline{(-\Delta)^{\frac{\theta}{2}}\widetilde{\Theta}}dx \\ & + \frac{\varsigma\gamma}{\xi}\int_\Omega (-\Delta)^{\frac{\theta}{2}}\,\phi\overline{(-\Delta)^{\frac{\theta}{2}}\,\widetilde{\phi}}dx + i\frac{\varsigma\kappa}{\xi}\int_\Omega (-\Delta)^{\frac{\theta+1}{2}}\,\phi\overline{(-\Delta)^{\frac{\theta+1}{2}}\,\widetilde{\phi}}dx \\ & -\varsigma\int_\Omega (-\Delta)^{\frac{\theta}{2}}\,\Theta\overline{(-\Delta)^{\frac{\theta}{2}}\,\widetilde{\phi}}dx, \end{aligned} \tag{18}$$

and an anti-linear form $\mathcal{L} : \mathcal{H}_* \to \mathbb{C}$, by

$$\mathcal{L}(\widetilde{\phi},\widetilde{\Theta}) = \frac{\varsigma}{\xi}\int_\Omega (-\Delta)^{\frac{\theta}{2}}\, y_1\overline{(-\Delta)^{\frac{\theta}{2}}\,\widetilde{\phi}}dx + \int_\Omega y_2\overline{\widetilde{\Theta}}dx. \tag{19}$$

Consequently we can write equations (16) and (17) as

$$\mathcal{B}((\phi,\Theta),(\widetilde{\phi},\widetilde{\Theta})) = \mathcal{L}(\widetilde{\phi},\widetilde{\Theta}). \tag{20}$$

Now, we have to prove that $\mathcal{L}$ is bounded and $\mathcal{B}$ is bounded and coercive, for this end we use the Cauchy-Schwarz inequality, to get

$$\begin{aligned} \left|\mathcal{L}(\widetilde{\phi},\widetilde{\Theta})\right| &= \left|\frac{\varsigma}{\xi}\int_\Omega (-\Delta)^{\frac{\theta}{2}} y_1\overline{(-\Delta)^{\frac{\theta}{2}}\widetilde{\phi}}dx + \int_\Omega y_2\overline{\widetilde{\Theta}}dx.\right| \\ &\leq \frac{\varsigma}{\xi}\left\|(-\Delta)^{\frac{\theta}{2}} y_1\right\|_{L^2(\Omega)}\left\|(-\Delta)^{\frac{\theta}{2}}\widetilde{\phi}\right\|_{L^2(\Omega)} + \left\|y_2\right\|_{L^2(\Omega)}\left\|\widetilde{\Theta}\right\|_{L^2(\Omega)} \\ &\leq c\left(\left\|\widetilde{\phi}\right\|_{H^\theta(\Omega)} + \left\|\widetilde{\Theta}\right\|_{H_0^1(\Omega)}\right) = c\left\|(\widetilde{\phi},\widetilde{\Theta})\right\|_{\mathcal{H}_*}. \end{aligned} \tag{21}$$

From the fact that $\mathcal{D}((-\Delta)^{\frac{\theta}{2}}) \hookrightarrow \mathcal{D}((-\Delta)^{\frac{1}{2}})$, and a Similar argument as above, we get

$$\begin{aligned} \left|\mathcal{B}((\phi,\Theta),(\widetilde{\phi},\widetilde{\Theta}))\right| &\leq (\gamma + \frac{\varsigma\kappa}{\xi})\left\|(-\Delta)^{\frac{\theta}{2}}\phi\right\|_{L^2(\Omega)}\left\|(-\Delta)^{\frac{\theta}{2}}\widetilde{\phi}\right\|_{L^2(\Omega)} + \max\{\gamma,\omega\}\left\|\Theta\right\|_{H_0^1(\Omega)}\left\|\widetilde{\Theta}\right\|_{H_0^1(\Omega)} \\ &+|\varsigma|\left\|(-\Delta)^{\frac{\theta}{2}}\phi\right\|_{L^2(\Omega)}\left\|(-\Delta)^{\frac{\theta}{2}}\widetilde{\Theta}\right\|_{L^2(\Omega)} + |\varsigma|\left\|(-\Delta)^{\frac{\theta}{2}}\Theta\right\|_{L^2(\Omega)}\left\|(-\Delta)^{\frac{\theta}{2}}\widetilde{\phi}\right\|_{L^2(\Omega)} \\ &\leq c\left(\left\|\phi\right\|_{H^\theta(\Omega)} + \left\|\Theta\right\|_{H_0^1(\Omega)}\right)\left(\left\|(-\Delta)^{\frac{\theta}{2}}\widetilde{\phi}\right\|_{L^2(\Omega)} + \left\|\widetilde{\Theta}\right\|_{H_0^1(\Omega)}\right) \\ &\leq c\left\|(\phi,\Theta)\right\|_{\mathcal{H}_*}\left\|(\widetilde{\phi},\widetilde{\Theta})\right\|_{\mathcal{H}_*}. \end{aligned} \tag{22}$$

Finally we have

$$\begin{aligned} \Re e\mathcal{B}((\phi,\Theta),(\phi,\Theta)) &= \gamma\left\|(-\Delta)^{\frac{\theta}{2}}\phi\right\|^2_{L^2(\Omega)} + \omega\left\|\nabla\Theta\right\|^2_{L^2(\Omega)} + \gamma\left\|\widetilde{\Theta}\right\|^2_{L^2(\Omega)} \\ &\geq \min\{\gamma,\omega\}\left(\left\|\phi\right\|^2_{H^\theta(\Omega)} + \left\|\widetilde{\Theta}\right\|^2_{H_0^1(\Omega)}\right) = \min\{\gamma,\omega\}\left\|(\phi,\Theta)\right\|^2_{\mathcal{H}_*}. \end{aligned} \tag{23}$$

By the Lax-Milgram Lemma, equation (20) has a unique solution in $\mathcal{H}_*$, and the regularity of elliptic equations leads us to the conclusion of step 2.

Step 3 From of Step 1 and Step 2 we conclude that $\mathcal{G}_\theta$ is m-dissipative. Following the general theory of abstract semigroup in [14, 28] Theorem 2.2 follows. □

3. Stability

This section is dedicated to studying various types of stability for the system. It begins with the characterization of the exponential and polynomial stability also the analytic regularity of the associated semigroup, which provides insight into the system's smoothness and decay properties . The analysis then addresses exponential stability, using criteria such as those by **[15, 27]**, which guarantee rapid decay of solutions. Finally, the section explores polynomial stability based on the Borichev-Tomilov **[5, 22]** framework, describing slower decay rates under less restrictive conditions. Together, these results offer a comprehensive understanding of the system's stability behavior. First, we begin by characterizing exponential stability as stated in the following theorem

Theorem 3.1. *[15, 27] Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on a Hilbert space $\mathcal{S}$. Then $S(t)$ is exponentially stable if and only if*

$$i\mathbb{R} \in \rho(\mathcal{A}), \tag{24}$$

and

$$\overline{\lim_{|s|\longrightarrow\infty}} \|(is\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{S})} < \infty. \tag{25}$$

Another important type of stability to consider is polynomial stability. Unlike exponential stability, which implies rapid decay of solutions, polynomial stability characterizes a slower, algebraic rate of decay. This type of stability is crucial for understanding the long-term behavior of the system when exponential decay cannot be guaranteed, and it is described as follows

Theorem 3.2. *[5, 22] Let $S(t)$ be a bounded C_0-semigroup on a Hilbert space $\mathcal{Z}$ with generator $\mathcal{G}$. If*

$$\begin{cases} i\mathbb{R} \in \rho(\mathcal{G}), \\ \overline{\lim\limits_{|s|\longrightarrow\infty}} \dfrac{1}{|s|^{\rho}} \|(is\mathcal{I} - \mathcal{G})^{-1}\|_{\mathcal{L}(\mathcal{Z})} < \infty, \end{cases} \tag{26}$$

for some ρ>0, then there exists a positive constant C such that

$$\|e^{\mathcal{G}t}U_0\|^2_{\mathcal{Z}} \leq \frac{C}{t^{\frac{2}{\rho}}} \|U_0\|^2_{\mathcal{D}(\mathcal{G})}. \tag{27}$$

Finally, we present the characterization of analytic semigroups established by Liu and Zheng. This result provides essential criteria to identify when a semigroup possesses analytic regularity, which is fundamental for understanding the enhanced smoothness and stability properties of the system. The precise statement is given in the following theorem

Theorem 3.3. *[23] Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on a Hilbert space $\mathcal{S}$. Then $S(t)$ is analytic if and only if*

$$i\mathbb{R} \in \rho(\mathcal{A}), \tag{28}$$

and

$$\overline{\lim\limits_{|s|\longrightarrow\infty}} |s| \|(is\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{S})} < \infty. \tag{29}$$

Now, we present and prove the following lemma, which will play a crucial role in many contexts

Lemma 3.4. *For each $0 \leq \theta < 1$, let $\mathcal{G}_\theta$ be the operator defined in equation (6), then there exist $(\gamma_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\mathcal{X}_n \subset \mathcal{D}(\mathcal{G}_\theta)$ such that*

1. *For all $n \in \mathbb{N}$ and $\theta \in [0, 1]$, we have*
$$\lim_{n\longrightarrow\infty} \gamma_n = +\infty, \quad \|\mathcal{X}_n\|^2_{\mathcal{D}(\mathcal{G}_\theta)} = 1. \tag{30}$$
2. *For all $n \in \mathbb{N}$ and $\theta \in [0, 1)$, we have*
$$\lim_{n\longrightarrow\infty} \|i\gamma_n \mathcal{X}_n - \mathcal{G}_\theta \mathcal{X}_n\|^2_{\mathcal{D}(\mathcal{G}_\theta)} = 0. \tag{31}$$
3. *For all $n \in \mathbb{N}$ and $\theta = 1$, we have*
$$\lim_{n\longrightarrow\infty} \|\gamma_n^{-1}(i\gamma_n \mathcal{X}_n - \mathcal{G}_1 \mathcal{X}_n)\|^2_{\mathcal{D}(\mathcal{G}_1)} = 0. \tag{32}$$
4. *For all $n \in \mathbb{N}$, $\theta \in [0, 1)$ and $1 - \theta > r$, we have*
$$\lim_{n\longrightarrow\infty} \|\gamma_n^{r}(i\gamma_n \mathcal{X}_n - \mathcal{G}_\theta \mathcal{X}_n)\|^2_{\mathcal{D}(\mathcal{G}_\theta)} = 0. \tag{33}$$

Proof. The idea of the proof is to use the well-known properties of the eigenvalues and the eigenfunctions associated with the Laplacian and modify them to construct what we claimed above. To do so, we introduce first the eigenfunctions $(\phi_n)_{n\geq 1}$ solutions of the spectral problem

$$\begin{cases} -\kappa\Delta\phi_n = \gamma_n\phi_n, & \Omega, \\ \phi_n = 0, & \partial\Omega, \\ \left\|\phi_n\right\|^2_{L^2(\Omega)} = 1, & n \geq 1, \\ \lim\limits_{n\longrightarrow\infty} \gamma_n = +\infty. \end{cases} \tag{34}$$

We are going to borrow some ideas from [30]; we have to seek $\mathcal{X}_n = \left(\varrho_n\phi_n, \lambda_n\phi_n\right)$ for $\lambda_n, \varrho_n \in \mathbb{C}$ which are going to be precised later in a way to fulfills (30). For $\sigma_1, \sigma_2 \in \mathbb{R}$ satisfying

$$\omega^2\left(\sigma_1^2 + \sigma_2^2\right) = \xi^2\kappa^{4-\theta}, \tag{35}$$

we set

$$\lambda_n = \sqrt{\frac{\varsigma}{\xi}}\frac{(\sigma_1 + i\sigma_2)}{\gamma_n^{\frac{2-\theta}{2}}}, \quad \varrho_n = \sqrt{\frac{\xi\kappa^\theta}{\varsigma}}\frac{(\mu_{1,n} + i\mu_{2,n})}{\gamma_n^{\frac{\theta}{2}}}, \tag{36}$$

where

$$\begin{cases} \mu_{1,n}^2 = \dfrac{\omega^2}{\kappa^2\xi^2}\left(\dfrac{\sigma_1}{\sqrt{\kappa^{2-\theta}}} - \dfrac{\kappa^\theta\sigma_2}{\gamma_n\omega}\right)^2 - \dfrac{\kappa^{\theta+2}}{2\gamma_n^2} - \dfrac{\varsigma\xi\kappa^{4-\theta}\gamma_n^{\theta-2}}{2\omega^2}, \\ \mu_{2,n}^2 = \dfrac{\omega^2}{\kappa^2\xi^2}\left(\dfrac{\kappa^\theta\sigma_1}{\gamma_n\omega} + \dfrac{\sigma_2}{\sqrt{\kappa^{2-\theta}}}\right)^2 - \dfrac{\kappa^{\theta+2}}{2\gamma_n^2} - \dfrac{\varsigma\xi\kappa^{4-\theta}\gamma_n^{\theta-2}}{2\omega^2}, \\ \lim\limits_{n\longrightarrow\infty}\sqrt{\gamma_n^\theta}\left(\mu_{1,n} - \dfrac{\omega}{\kappa\xi}\left(\dfrac{\sigma_1}{\sqrt{\kappa^{2-\theta}}} - \dfrac{\kappa^\theta\sigma_2}{\gamma_n\omega}\right)\right) = 0, \\ \lim\limits_{n\longrightarrow\infty}\sqrt{\gamma_n^\theta}\left(\mu_{2,n} - \dfrac{\omega}{\kappa\xi}\left(\dfrac{\kappa^\theta\sigma_1}{\gamma_n\omega} + \dfrac{\sigma_2}{\sqrt{\kappa^{2-\theta}}}\right)\right) = 0. \end{cases} \tag{37}$$

Now N_0 is chosen to get the positivity of equations $(37)_1$ and $(37)_2$. For the case $n < N_0$, we can take $\lambda_n = 0$ and $\varrho_n = 1$. Let us check now that up to the given equations (35)-(37), all claims of (30) hold

$$\begin{aligned} \|\mathcal{X}_n\|_{\mathcal{D}(\mathcal{G}_\theta)}^2 &= \frac{\varsigma}{\xi}|\lambda_n|^2\left\|(-\Delta)^{\frac{\theta}{2}}\phi_n\right\|_{L^2(\Omega)}^2 + |\varrho_n|^2\left\|\phi_n\right\|_{L^2(\Omega)}^2 \\ &= \frac{\varsigma}{\xi\kappa^\theta}\gamma_n^\theta|\varrho_n|^2 + |\lambda_n|^2 \\ &= \frac{\varsigma\gamma_n^{\theta-2}}{\xi}\left(\sigma_1^2 + \sigma_2^2\right) + \left(\mu_{1,n}^2 + \mu_{2,n}^2\right) \\ &= \frac{\varsigma\xi\kappa^{4-\theta}\gamma_n^{\theta-2}}{\omega^2} + \frac{\omega^2}{\kappa^2\xi^2}\left(\frac{\sigma_1}{\sqrt{\kappa^{2-\theta}}} - \frac{\kappa^\theta\sigma_2}{\gamma_n\omega}\right)^2 - \frac{\kappa^{\theta+2}}{2\gamma_n^2} - \frac{\varsigma\xi\kappa^{4-\theta}\gamma_n^{\theta-2}}{2\omega^2} + \frac{\omega^2}{\kappa^2\xi^2}\left(\frac{\kappa^\theta\sigma_1}{\gamma_n\omega} + \frac{\sigma_2}{\sqrt{\kappa^{2-\theta}}}\right)^2 \\ &\quad -\frac{\kappa^{\theta+2}}{2\gamma_n^2} - \frac{\varsigma\xi\kappa^{4-\theta}\gamma_n^{\theta-2}}{2\omega^2} \\ &= 1. \end{aligned} \tag{38}$$

For the second condition we have

$$i\gamma_n\mathcal{X}_n - \mathcal{G}_\theta\mathcal{X}_n = \begin{bmatrix} \xi\lambda_n\phi_n \\ \left(i\gamma_n\lambda_n + \dfrac{\omega}{\kappa}\lambda_n\gamma_n - \varsigma\varrho_n\gamma_n^\theta\right)\phi_n \end{bmatrix}. \tag{39}$$

From the above calculation and any $0 \leq \theta < 1$, we can deduce that

$$\begin{aligned} \|i\gamma_n\mathcal{X}_n - \mathcal{G}_\theta\mathcal{X}_n\|_{\mathcal{D}(\mathcal{G}_\theta)}^2 &= \xi\varsigma\kappa^\theta\gamma_n^\theta\left|\mu_{1,n} - \frac{\omega}{\kappa\xi}\left(\frac{\sigma_1}{\sqrt{\kappa^{2-\theta}}} - \frac{\kappa^\theta\sigma_2}{\omega}\right)\right|^2 + \xi\varsigma\kappa^\theta\gamma_n^\theta\left|\mu_{2,n} - \frac{\omega}{\kappa\xi}\left(\frac{\kappa^\theta\sigma_1}{\omega} + \frac{\sigma_2}{\sqrt{\kappa^{2-\theta}}}\right)\right|^2 \\ &\quad + \frac{\xi^4}{2\omega^2\kappa^{\theta-2}}\gamma_n^{2(\theta-1)}. \end{aligned} \tag{40}$$

In this same case and for $1 - \theta > r$, we have also

$$\begin{aligned} \|\gamma_n^r(i\gamma_n\mathcal{X}_n - \mathcal{G}_\theta\mathcal{X}_n)\|_{\mathcal{D}(\mathcal{G}_\theta)}^2 &= \frac{\xi^4}{2\omega^2\kappa^{\theta-2}}\gamma_n^{2(\theta-1+r)} + \xi\varsigma\kappa^{\theta+2r}\gamma_n^\theta\left|\mu_{1,n} - \frac{\omega}{\kappa\xi}\left(\frac{\sigma_1}{\sqrt{\kappa^{2-\theta}}} - \frac{\kappa^\theta\sigma_2}{\omega}\right)\right|^2 \\ &\quad +\xi\varsigma\kappa^\theta\gamma_n^{\theta+2r}\left|\mu_{2,n} - \frac{\omega}{\kappa\xi}\left(\frac{\kappa^\theta\sigma_1}{\omega} + \frac{\sigma_2}{\sqrt{\kappa^{2-\theta}}}\right)\right|^2. \end{aligned} \tag{41}$$

In the case when $\theta = 1$, we have

$$\begin{aligned}\|\gamma_n^{-1}(i\gamma_n\mathcal{X}_n - \mathcal{G}_1\mathcal{X}_n)\|^2_{\mathcal{D}(\mathcal{G}_1)} &= \frac{\xi\varsigma\kappa}{\gamma_n}\left|\mu_{1,n} - \frac{\omega}{\kappa\xi}\left(\frac{\sigma_1}{\sqrt{\kappa}} - \frac{\kappa\sigma_2}{\omega}\right)\right|^2 + \frac{\xi\varsigma\kappa}{\gamma_n}\left|\mu_{2,n} - \frac{\omega}{\kappa\xi}\left(\frac{\kappa\sigma_1}{\omega} + \frac{\sigma_2}{\sqrt{\kappa}}\right)\right|^2 \\ &+ \frac{\kappa\xi^4}{2\omega^2}\gamma_n^{-2}.\end{aligned} \tag{42}$$

Now we pass to the limit in equations (40) and (42), taking into account equations $(37)_3$, $(37)_4$, we get

$$\lim_{n\longrightarrow\infty}\|i\gamma_n\mathcal{X}_n - \mathcal{G}_\theta\mathcal{X}_n\|^2_{\mathcal{D}(\mathcal{G}_\theta)} = \lim_{n\longrightarrow\infty}\|\gamma_n^{-1}(i\gamma_n\mathcal{X}_n - \mathcal{G}_1\mathcal{X}_n)\|^2_{\mathcal{D}(\mathcal{G}_1)} = \lim_{n\longrightarrow\infty}\|\gamma_n^{r}(i\gamma_n\mathcal{X}_n - \mathcal{G}_\theta\mathcal{X}_n)\|^2_{\mathcal{D}(\mathcal{G}_\theta)} = 0. \tag{43}$$

This concludes the proof. □

3.1. Strong stability of the system

Theorem 3.5. *The C_0-semigroup $e^{\mathcal{G}_\theta t}$ is strongly stable in $\mathcal{H}_\theta$.*

The proof of this theorem is based on the following lemma.

Lemma 3.6. *Let $0 \le \theta \le 1$ and $\mathcal{G}_\theta$ be the operator defined by equation (6). Then*

- *$\mathcal{G}_\theta$ does not have eigenvalues on $i\mathbb{R}$;*
- *The resolvent is compact;*
- *$i\mathbb{R} \subset \rho(\mathcal{G}_\theta)$.*

Proof. To prove the first statement we assume the opposite and use a contradiction argument. Let $\gamma \in \mathbb{R}$ and $\mathcal{X} \in \mathcal{D}(\mathcal{G}_\theta)$ be such that

$$\mathcal{G}_\theta\mathcal{X} = i\gamma\mathcal{X}. \tag{44}$$

Alternatively, we may write

$$\begin{cases} i\gamma\phi + i\kappa\Delta\phi - \xi\Theta = 0, \\ i\gamma\Theta - \omega\Delta\Theta + \varsigma(-\Delta)^\theta\phi = 0. \end{cases} \tag{45}$$

Taking the inner product by $\mathcal{X} \in \mathcal{H}_\theta$ in equation (44) and the consideration in equation (9), we obtain

$$0 = \Re e < \mathcal{G}_\theta\mathcal{X}, \mathcal{X} >_{\mathcal{H}_\theta} = \mathcal{E}'_\theta(t) = -\omega\,\|\nabla\Theta\|^2_{L^2(\Omega)}, \quad \text{for any} \quad 0 \le \theta \le 1, \tag{46}$$

From Poincare's inequality we can easily get

$$\phi = 0 \;\; in \;\; \Omega. \tag{47}$$

From equation $(45)_2$, we get

$$\Theta = 0 \;\; in \;\; \Omega, \tag{48}$$

which means that $\mathcal{X} = 0$ and contradicts the hypotheses. The second claim is an easy application of the Sobolev compact embeddings theorem. The third claim is derived from the fact that the resolvent is compact. The proof is complete. □

Proof. Theorem 3.5
Since $\mathcal{G}_\theta$ is of compact resolvent and by use of Lemma 3.6, theorems due to Arent-batty[2], Lyubich-Vu[25] or the note of C. D. Benchimol[4] guarantee that the C_0-semigroup $e^{\mathcal{G}_\theta t}$ is strongly stable for each $0 \le \theta \le 1$. □

3.2. Exponential Stability $\theta = 1$

Theorem 3.7. *The C_0-semigroup $e^{\mathcal{G}_1 t}$ is exponentially stable in $\mathcal{H}_1$, but is not analytic.*

Remark 3.8. *We remark that in this case equation (12) boil to*

$$\begin{aligned} &\mathcal{E}_1(t) = \frac{1}{2}\left[\left\|\nabla\phi\right\|^2_{L^2(\Omega)} + \|\Theta\|^2_{L^2(\Omega)}\right], \\ &\mathcal{E}_1'(t) = -\omega\,\|\nabla\Theta\|^2_{L^2(\Omega)}. \end{aligned} \tag{49}$$

Proof. **Step 1:** Since Lemma 3.6 asserts that condition (24) is satisfied, then to prove the exponential stability, we need to get the resolvent estimates. Let $\gamma \in \mathbb{R}$, $\mathcal{Y} \in \mathcal{H}_1$, and there exists $\mathcal{X} \in \mathcal{D}(\mathcal{G}_1)$ such that $(i\gamma\mathcal{I} - \mathcal{G}_1)\mathcal{X} = \mathcal{Y}$, or equivalently

$$\begin{cases} i\gamma\phi + i\kappa\Delta\phi - \xi\Theta = y_1, \\ i\gamma\Theta - \omega\Delta\Theta + \varsigma\Delta\phi = y_2. \end{cases} \tag{50}$$

From equation (13), we can deduce that

$$\|\nabla\Theta\|^2_{L^2(\Omega)} \leq C\|\mathcal{X}\|_{\mathcal{D}(\mathcal{G}_1)}\|\mathcal{Y}\|_{\mathcal{H}_1}. \tag{51}$$

Take the product of equation $(50)_2$ by $\overline{\phi}$ and integrating over Ω, we get

$$\left\|\nabla\phi\right\|^2_{L^2(\Omega)} = \frac{1}{\varsigma}\Re e\left(-\int_\Omega y_2\overline{\phi}dx + \omega\int_\Omega \nabla\Theta\nabla\overline{\phi}dx - \int_\Omega \Theta\overline{(i\gamma\phi)}dx\right). \tag{52}$$

Gathering equations $(50)_1$, (3.9) and using the Cauchy-Schwarz and Young's inequalities, we get

$$\begin{aligned} \left\|\nabla\phi\right\|^2_{L^2(\Omega)} &= \frac{1}{\varsigma}\Re e\left(-\int_\Omega y_2\overline{\phi}dx + \omega\int_\Omega \nabla\Theta\nabla\overline{\phi}dx - \int_\Omega \Theta\left(-i\kappa\Delta\overline{\phi} + \xi\overline{\Theta} + \overline{y_1}\right)dx\right) \\ &\leq C\|\mathcal{X}\|_{\mathcal{D}(\mathcal{G}_1)}\|\mathcal{Y}\|_{\mathcal{H}_1} + C\|\mathcal{X}\|^{\frac{3}{2}}_{\mathcal{D}(\mathcal{G}_1)}\|\mathcal{Y}\|^{\frac{1}{2}}_{\mathcal{H}_1}. \end{aligned} \tag{53}$$

Now, from the consideration in equations (51) and (53), we get

$$\|\mathcal{X}\|^2_{\mathcal{D}(\mathcal{G}_1)} \leq C\|\mathcal{X}\|_{\mathcal{D}(\mathcal{G}_1)}\|\mathcal{Y}\|_{\mathcal{H}_1} + C\|\mathcal{X}\|^{\frac{3}{2}}_{\mathcal{D}(\mathcal{G}_1)}\|\mathcal{Y}\|^{\frac{1}{2}}_{\mathcal{H}_1}. \tag{54}$$

Applying Young's inequality in equation (54), we reach the desired inequality, which is

$$\|\mathcal{X}\|^2_{\mathcal{D}(\mathcal{G}_1)} \leq C\|\mathcal{Y}\|^2_{\mathcal{H}_1}. \tag{55}$$

Step 2: From Lemma 3.6 condition 24 is satisfied. Then to prove that $e^{\mathcal{G}_1 t}$ is not analytic, we prove that

$$\overline{\lim_{|s|\longrightarrow\infty}}\|s(is\mathcal{I} - \mathcal{G}_\theta)^{-1}\|_{\mathcal{L}(\mathcal{H}_1)} = +\infty. \tag{56}$$

From Lemma 3.4, we set $\mathcal{Y}_n = \dfrac{\gamma_n^{-1}(i\gamma_n\mathcal{X}_n - \mathcal{G}_\theta\mathcal{X}_n)}{\|\gamma_n^{-1}(i\gamma_n\mathcal{X}_n - \mathcal{G}_\theta\mathcal{X}_n)\|_{\mathcal{D}(\mathcal{G}_1)}}$, then $\|\mathcal{Y}_n\|_{\mathcal{D}(\mathcal{G}_1)} = 1$ and

$$\lim_{n\longrightarrow\infty}\|\gamma_n(i\gamma_n\mathcal{I} - \mathcal{G}_\theta)^{-1}\mathcal{Y}_n\|_{\mathcal{L}(\mathcal{H}_1)} = \lim_{n\longrightarrow\infty}\frac{1}{\|\gamma_n^{-1}(i\gamma_n\mathcal{X}_n - \mathcal{G}_\theta\mathcal{X}_n)\|_{\mathcal{D}(\mathcal{G}_1)}} = +\infty. \tag{57}$$

□

3.3. Lack of Exponential Stability $0 \leq \theta < 1$

Theorem 3.9. *The C_0-semigroup $e^{\mathcal{G}_\theta t}$ is not exponentially stable in $\mathcal{G}_\theta$ whenever $0 \leq \theta < 1$.*

Proof. From Lemma 3.6, condition 24 is satisfied. Then to prove that $e^{\mathcal{G}_\theta t}$ is not exponentially stable when $0 \leq \theta < 1$, is equivalent to prove that

$$\overline{\lim_{|s|\longrightarrow\infty}} \|(is\mathcal{I} - \mathcal{G}_\theta)^{-1}\|_{\mathcal{L}(\mathcal{G}_\theta)} = +\infty. \tag{58}$$

From Lemma 3.4, we set $\mathcal{Y}_n = \dfrac{(i\gamma_n\mathcal{X}_n - \mathcal{G}_\theta\mathcal{X}_n)}{\|i\gamma_n\mathcal{X}_n - \mathcal{G}_\theta\mathcal{X}_n\|_{\mathcal{D}(\mathcal{G}_\theta)}}$; then $\|\mathcal{Y}_n\|_{\mathcal{D}(\mathcal{G}_1)} = 1$ and

$$\lim_{n\longrightarrow\infty} \|(i\gamma_n\mathcal{I} - \mathcal{G}_\theta)^{-1}\mathcal{Y}_n\|_{\mathcal{H}_\theta} = \lim_{n\longrightarrow\infty} \frac{1}{\|i\gamma_n\mathcal{X}_n - \mathcal{G}_\theta\mathcal{X}_n\|_{\mathcal{D}(\mathcal{G}_\theta)}} = +\infty, \tag{59}$$

which completes the proof. □

3.4. Polynomial Stability for $0 \leq \theta < 1$

The study of polynomial stability is motivated by the absence of exponential stability established in the previous section. When exponential decay cannot be guaranteed, polynomial stability provides a valuable framework to understand the slower, algebraic decay behavior of the system over time. This analysis is essential for capturing the system's long-term dynamics under less restrictive conditions. Based on theorem 3.2 we have the following result

Theorem 3.10. *The C_0-semigroup $\mathcal{S}_\theta(t)$ generated by $\mathcal{G}_\theta$ is polynomially stable in $\mathcal{H}_\theta$ and we have*

$$\|\mathcal{S}_\theta(t)U_0\|^2_{\mathcal{H}_\theta} \leq \frac{1}{t}\|U_0\|^2_{\mathcal{D}(\mathcal{G}_\theta)}. \tag{60}$$

Proof. Since Lemma 3.6 asserts that condition $(26)_1$ is satisfied, then to prove the polynomial stability, we need to get the resolvent estimate. Let $\gamma \in \mathbb{R}$, $\mathcal{Y} \in \mathcal{G}_\theta$, and there exists $\mathcal{X} \in \mathcal{D}(\mathcal{G}_\theta)$ such that $(i\gamma\mathcal{I} - \mathcal{G}_\theta)\mathcal{X} = \mathcal{Y}$, or equivalently

$$\begin{cases} i\gamma\phi + i\kappa\Delta\phi - \xi\Theta = y_1, \\ i\gamma\Theta - \omega\Delta\Theta + \varsigma(-\Delta)^\theta\phi = y_2. \end{cases} \tag{61}$$

Equation (13) implies that

$$\|\nabla\Theta\|^2_{L^2(\Omega)} \leq C\|\mathcal{X}\|_{\mathcal{D}(\mathcal{G}_\theta)}\|\mathcal{Y}\|_{\mathcal{H}_\theta}. \tag{62}$$

Employing Poincare's inequality in equation (13), we get

$$\|\Theta\|^2_{L^2(\Omega)} \leq C\|\mathcal{X}\|_{\mathcal{D}(\mathcal{G}_\theta)}\|\mathcal{Y}\|_{\mathcal{H}_\theta}. \tag{63}$$

Multiply Eq.$(61)_2$ by $\frac{1}{\xi}\overline{\phi}$ and integrate over Ω to get

$$\frac{\varsigma}{\xi}\left\|(-\Delta)^{\frac{\theta}{2}}\phi\right\|^2_{L^2(\Omega)} = \frac{1}{\xi}\Re e\left(\int_\Omega y_2\overline{\phi}dx + \omega\int_\Omega \Theta\Delta\overline{\phi}dx - i\gamma\int_\Omega \Theta\overline{\phi}dx\right). \tag{64}$$

From Eq.$(61)_1$, Eq.(62), Eq.(63) and by means of Cauchy-Schwarz and Young's inequalities we deduce

$$\begin{aligned} \frac{\varsigma}{\xi}\left\|(-\Delta)^{\frac{\theta}{2}}\phi\right\|^2_{L^2(\Omega)} &= \frac{1}{\xi}\Re e\left(\int_\Omega y_1\overline{\phi}dx + \int_\Omega \Theta\left(-\gamma\overline{\phi} - i\xi\overline{\Theta} - i\overline{y_2}\right)dx - i\gamma\int_\Omega \Theta\overline{\phi}dx\right) \\ &\leq C\|\mathcal{X}\|_{D(\mathcal{H}_\theta)}\|\mathcal{Y}\|_{\mathcal{G}_\theta} + C|\gamma|\|\mathcal{X}\|^{\frac{3}{2}}_{D(\mathcal{G}_\theta)}\|\mathcal{Y}\|^{\frac{1}{2}}_{\mathcal{H}_\theta} + C\|\mathcal{X}\|^{\frac{1}{2}}_{D(\mathcal{G}_\theta)}\|\mathcal{Y}\|^{\frac{3}{2}}_{\mathcal{H}_\theta} \\ &\leq \varepsilon\|\mathcal{X}\|^2_{D(\mathcal{G}_\theta)} + C(\varepsilon)|\gamma|^4\|\mathcal{Y}\|^2_{\mathcal{H}_\theta}. \end{aligned} \tag{65}$$

Now from the consideration in Eq.(63) and Eq.(65) we get

$$\|\mathcal{X}\|^2_{D(\mathcal{G}_\theta)} \leq \varepsilon\|\mathcal{X}\|^2_{D(\mathcal{G}_\theta)} + C(\varepsilon)|\gamma|^4\|\mathcal{Y}\|^2_{\mathcal{H}_\theta}. \tag{66}$$

Choose ε small enough in Eq.(66) we reach the desired inequality, which is

$$\|\mathcal{X}\|_{D(\mathcal{G}_\theta)} \leq C|\gamma|^2\|\mathcal{Y}\|_{\mathcal{H}_\theta}. \tag{67}$$

From this equation we can directly conclude, and the proof now is complete. □

4. Conclusion and Outlook

We analyzed a coupled system of Schrödinger and heat equations using the frequency domain technique and multiplier method to determine the uniformity of stability, which depends on the parameter θ. Strong asymptotic stability was established through the application of the Benchimol and Arendt-Batty theorems. For the range $0 \leq \theta < 1$, we obtained a polynomial energy decay rate of order t^{-1}. It is important to note that this decay rate is not optimal, as indicated by the eigenvalue expansion presented in Lemma 3.4. In a future work we will address a system which is quietly related to system (1)

$$\begin{cases} \phi_t + i\Delta\phi - \xi\,(-\Delta)^\theta\,\Theta = 0, & \text{on} \quad \Omega\times(0,+\infty), \\ \Theta_t - \Delta\Theta + \xi\phi = 0, & \text{on} \quad \Omega\times(0,+\infty), \\ \phi = \Theta = 0, & \text{in} \quad \partial\Omega\times(0,+\infty), \\ \phi(x,0) = \phi_0(x), \Theta(x,0) = \Theta_0(x), & \text{on} \quad x\in\Omega, \end{cases} \tag{68}$$

In this system we will come across the optimality of the decay and demonstrated using an approach used in [24]. We raise also two interesting questions given as follows, the first one is the non-linear version of system (1) which reads

$$\begin{cases} \phi_t(x,t) + i\kappa\Delta\phi(x,t) - \xi\Theta(x,t) = 0, & \text{on} \quad \Omega\times(0,+\infty), \\ \Theta_t(x,t) - \omega\Delta_p\Theta(x,t) + \varsigma\,(-\Delta)^\theta\,\phi(x,t) = 0, & \text{on} \quad \Omega\times(0,+\infty), \\ \phi = \Theta = 0, & \text{in} \quad \partial\Omega\times(0,+\infty), \\ \phi(x,0) = \phi_0(x), \Theta(x,0) = \Theta_0(x), & \text{on} \quad x\in\Omega. \end{cases} \tag{69}$$

Where $\Delta_p = \nabla\left(|\nabla\Theta|^{p-2}\,\nabla\Theta\right)$, for $p \geq 2$, is the well-known quasi linear p-Laplace operator. This system generalizes system (1) and presents numerous intriguing aspects, among which the issue of optimality stands out as particularly challenging, especially in higher-dimensional settings. As highlighted in **[31]**, establishing optimality in such contexts remains one of the most difficult problems to resolve. Another compelling question that arises in the analysis is

$$\begin{cases} \phi_t(x,t) + i\kappa\Delta\phi(x,t) - \xi\Theta(x,t) = 0, & \text{on} \quad \Omega\times(0,+\infty), \\ \Theta_t(x,t) - \omega\Delta\Theta(x,t-\tau) + \varsigma\,(-\Delta)^\theta\,\phi(x,t) = 0, & \text{on} \quad \Omega\times(0,+\infty), \\ \phi = \Theta = 0, & \text{in} \quad \partial\Omega\times(0,+\infty), \\ \phi(x,0) = \phi_0(x), \Theta(x,0) = \Theta_0(x), & \text{on} \quad x\in\Omega. \end{cases} \tag{70}$$

This system is also of great interest as it provides insight into the impact that delays in the diffusion operator may have on the stability behavior. Understanding whether such delays influence stability is a critical aspect of the analysis, as it can significantly affect the system's long-term dynamics and robustness.

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Authors' contributions
Mohamed Hocine Braiki: Conceptualization, methodology, formal analysis, investigation, and writing—original draft preparation.
Zakaria Chedjara : Review and editing of the manuscript, and validation of results.
Svetlin Georgiev Georgiev: Supervision, project administration and resources.
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