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Solutions of a double phase singular Kirchhoff type equation with nonstandard growth

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Abstract. A specific category of singular class of double phase Kirchhoff type equations characterized by variable singularities is studied. By employing variational method, we prove both the uniqueness and existence of positive solutions for cases of weak variable singularities, where singularity η lies within the interval $(0, 1)$, as well as for strong variable singularities, where η exceeds 1. Additionally, we present an application from composite materials.

1. Introduction

Here, a degenerate Kirchhoff-type equation

$$\begin{cases} -\Re\left(\jmath_{\mathcal{H}}(u)\right) div\left(\mathcal{B}\left(u,p,q,\nu\right)\right) = \hbar(x,u,\eta) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain with Lipschitz boundary $\partial\Omega$, $N \geq 2$, $\Re \in C^1(0,+\infty)$, $0 \leq \nu(\cdot) \in L^\infty(\Omega)$ and

$$\jmath_{\mathcal{H}}(u) := \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \nu(x)\frac{|\nabla u|^{q(x)}}{q(x)} \right) dx \text{ and}$$
$$\mathcal{B}\left(u,p,q,\nu\right) := |\nabla u|^{p(x)-2}\nabla u + \nu(x)|\nabla u|^{q(x)-2}\nabla u.$$

Suppose

(H_1) $p, q \in C_+(\overline{\Omega})$ with $1 < p^- \leq p(x) \leq q(x) \leq q^+ < \min\{N, p^*(x)\}$, where $p^*(x) := \frac{Np(x)}{N-p(x)}$.

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(H_2) $\mathfrak{R} : (0, \infty) \to (0, \infty)$ is a nondecreasing, C^1 function and

$$\kappa_1 s^{\alpha_1 - 1} \leq \mathfrak{R}(s) \leq \kappa_2 s^{\alpha_2 - 1}, \tag{2}$$

where $\kappa_1, \kappa_2, \alpha_1, \alpha_2$ are real numbers such that $\kappa_2 \geq \kappa_1 > 0$ and $\alpha_2 \geq \alpha_1 > 1$.

We suppose two types of singularities of the nonlinearity $\hbar(x, u, \eta)$ as defined below:

$$\hbar(x, u, \eta) = \begin{cases} \mathfrak{I}(x) u^{-\eta(x)} & \text{if } 0 < \eta(x) < 1, \\ \wp(x) u^{-\eta(x)} & \text{if } 1 < \eta(x) < \infty. \end{cases} \tag{3}$$

where η is a continuous function over $\overline{\Omega}$, $\wp \in C^1(\overline{\Omega})$ is a nontrivial nonnegative function, $\mathfrak{I} \in L^1(\overline{\Omega})$ is a positive function (see [8, 19, 23–25, 28]).

The double phase operator, expressed as

$$div\left(|\nabla u|^{p(x)-2}\nabla u + \nu(x)|\nabla u|^{q(x)-2}\nabla u\right),\ u \in W_0^{1,\mathcal{H}}(\Omega),$$

is associated with

$$u \to \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \nu(x) \frac{|\nabla u|^{q(x)}}{q(x)} \right) dx,\ \ u \in W_0^{1,\mathcal{H}}(\Omega). \tag{4}$$

This functional exhibits variations in ellipticity based on the regions where the weight function $\nu(\cdot)$ is null, transitioning through two distinct phases of elliptic behavior (see [1, 9–11, 16–18, 21, 29, 30]).

The problem (1) encompasses various models that illustrate intriguing phenomena examined in mathematical physics (refer to the works of [15, 20, 22]). Singular Kirchhoff-type issues concerning the double phase operator have been examined extensively see [3]. When $\nu(x) = 0$, Avci [4–7] by using the Ekelands variational principle and a constrained minimization approach (introduced by Yijing and Duanzhi in [27]), studied the positive solutions for the weak and strong variable singularities.

The primary findings of this article demonstrate the existence of a unique solution within the function space $W_0^{1,\mathcal{H}}(\Omega)$. This space is significant as it provides a coherent framework for understanding the problem (3).

The following theorems are presented in this paper.

Theorem 1.1. *Suppose* (H_1), (H_2), *and*

(A_1) $\eta \in C(\overline{\Omega})$ *and* $1 < \eta^- \leq \eta(x) \leq \eta^+ < \infty$,

(A_2) $\wp(x) \in L^1(\overline{\Omega})$ *and* $\wp(x) > 0$ *a.e. in* Ω,

hold. The problem

$$\begin{cases} -\mathfrak{R}\left(\jmath_{\mathcal{H}}(u)\right) div\left(\mathcal{B}(u, p, q, \nu)\right) = \wp(x) u^{-\eta(x)} & in\ \Omega, \\ u > 0 & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{5}$$

admit a unique positive $W_0^{1,\mathcal{H}}(\Omega)$*-solution iff there exists* $\overline{u} \in W_0^{1,\mathcal{H}}(\Omega)$ *satisfying* $\displaystyle\int_\Omega \wp(x)|\overline{u}|^{1-\eta(x)} dx < \infty$.

Theorem 1.2. *Suppose* (H_1), (H_2), *and*

(A_3) $\eta \in C(\overline{\Omega})$ *and* $0 < \eta^- \leq \eta(x) \leq \eta^+ < 1$,

(A_4) $0 < \mathfrak{I}(x) \in C^1(\overline{\Omega})$, *hold. The problem*

$$\begin{cases} -\Re\left(\jmath_{\mathcal{H}}(u)\right) div\left(\mathcal{B}(u,p,q,\nu)\right) = \Im(x)u^{-\eta(x)} & in\ \Omega, \\ u > 0 & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{6}$$

admit a unique positive $W_0^{1,\mathcal{H}}(\Omega)$-solution with a negative energy level.

Section 2 delves into the analysis of the space $W_0^{1,\mathcal{H}}(\Omega)$. The problem (5) is studied in 3 and the existence of a unique solution in $W_0^{1,\mathcal{H}}(\Omega)$ is proved, contingent upon the existence of a function $\overline{u} \in W_0^{1,\mathcal{H}}(\Omega)$ such that $\int_\Omega \wp(x)|\overline{u}|^{1-\eta(x)}dx < \infty$, thereby proving Theorem 1.1. In Subsection 3.1, we provide an example to illustrate Theorem 1.1. Finally, Section 4 addresses problem (6), demonstrating that it possesses a unique positive solution in the space $W_0^{1,\mathcal{H}}(\Omega)$ with a negative energy level, which corresponds to the proof of Theorem 1.2.

2. The space $W_0^{1,\mathcal{H}}(\Omega)$

Let $\mathcal{H} : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ be given by

$$\mathcal{H}(x,t) := t^{p(x)} + \nu(x)t^{q(x)} \text{ for all } (x,t) \in \Omega \times \mathbb{R}^+,$$

and $\rho_{\mathcal{H}}(\cdot)$ is defined by

$$\rho_{\mathcal{H}}(u) := \int_\Omega \mathcal{H}(x,|u|)dx = \int_\Omega \left(|u|^{p(x)} + \nu(x)|u|^{q(x)}\right)dx.$$

One can define the Musielak-Orlicz space

$$L^{\mathcal{H}}(\Omega) := \{u : \Omega \to \mathbb{R} \text{ measurable};\ \rho_{\mathcal{H}}(u) < +\infty\},$$

endowed with the Luxemburg norm

$$\|u\|_{\mathcal{H}} = \inf\left\{\tau > 0 : \rho_{\mathcal{H}}\left(\frac{u}{\tau}\right) \leq 1\right\}.$$

We recall [2, Proposition 2.2] or [14, Proposition 2.13]:

Proposition 2.1. *Assume (H_1) hold, $u \in L^{\mathcal{H}}(\Omega)$ and $\lambda \in \mathbb{R}$. Then*

- *If $u \neq 0$, then $\|u\|_{\mathcal{H}} = \lambda \Leftrightarrow \rho_{\mathcal{H}}(\frac{u}{\lambda}) = 1$ and $\|u\|_{\mathcal{H}} \to 1 \Leftrightarrow \rho_{\mathcal{H}}(u) \to 1$,*
- $\|u\|_{\mathcal{H}} < 1$ (*resp.* $> 1, = 1$) $\Leftrightarrow \rho_{\mathcal{H}}(\frac{u}{\lambda}) < 1$ (*resp.* $> 1, = 1$),
- *If $\|u\|_{\mathcal{H}} > 1 \Rightarrow \|u\|_{\mathcal{H}}^{p^-} \leq \rho_{\mathcal{H}}(u) \leq \|u\|_{\mathcal{H}}^{q^+}$,*
- *If $\|u\|_{\mathcal{H}} < 1 \Rightarrow \|u\|_{\mathcal{H}}^{q^+} \leq \rho_{\mathcal{H}}(u) \leq \|u\|_{\mathcal{H}}^{p^-}$,*
- *$\|u\|_{\mathcal{H}} \to +\infty \Leftrightarrow \rho_{\mathcal{H}}(u) \to +\infty$ and $\|u\|_{\mathcal{H}} \to 0 \Leftrightarrow \rho_{\mathcal{H}}(u) \to 0$,*
- *If $u_n \to u$ in $L^{\mathcal{H}}(\Omega)$, then $\rho_{\mathcal{H}}(u_n) \to \rho_{\mathcal{H}}(u)$.*

One can define the Musielak-Orlicz Sobolev space

$$W^{1,\mathcal{H}}(\Omega) := \left\{u \in L^{\mathcal{H}}(\Omega) : |\nabla u| \in L^{\mathcal{H}}(\Omega)\right\},$$

equipped with the norm

$$\|u\|_{1,\mathcal{H}} = \|\nabla u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}},$$

with $\|\nabla u\|_{\mathcal{H}} = \|\,|\nabla u|\,\|_{\mathcal{H}}$ and by $W_0^{1,\mathcal{H}}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{1,\mathcal{H}}(\Omega)}}$. The spaces $L^{\mathcal{H}}(\Omega), W^{1,\mathcal{H}}(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)$ are reflexive, uniformly convex and Banach spaces.

Proposition 2.2. *(see [14, Proposition 2.16]) Assume (H_1) hold. Then*

(i) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega), W^{1,\mathcal{H}}(\Omega) \hookrightarrow W^{1,r(\cdot)}(\Omega), W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow W_0^{1,r(\cdot)}(\Omega)$ *are continuous when* $1 \leq r(x) \leq p(x)$ *for all* $x \in \overline{\Omega}$ *and* $r \in C(\overline{\Omega})$.

(ii) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *is compact when* $1 \leq r(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$ *and for all* $r \in C(\overline{\Omega})$.

By [14, Proposition 2.18], we can equip the space $W_0^{1,\mathcal{H}}(\Omega)$ with

$$\|u\|_{1,\mathcal{H},0} = \|\nabla u\|_{\mathcal{H}}.$$

Now, for any $r \in C(\overline{\Omega})$ for which the continuous embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ hold (see [2, Proposition 2.3]), and $C_{\mathcal{H}}$ denotes the best constant for which one has

$$\|u\|_{r(\cdot)} \leq C_{\mathcal{H}}\|u\|_{1,\mathcal{H},0}.$$

Proposition 2.3. *(see [14]) For the convex functional* $J_{\mathcal{H}}(\cdot)$, *we have* $J_{\mathcal{H}} \in C^1(W_0^{1,\mathcal{H}}(\Omega), \mathbb{R})$ *with the derivative*

$$\langle J'_{\mathcal{H}}(u), v\rangle = \int_{\Omega} \mathcal{B}(u, p, q, v) \cdot \nabla v dx,$$

for all $u, v \in W_0^{1,\mathcal{H}}(\Omega)$.

Lemma 2.4. *[4] Let* V *be a vector space, and let* $I : V \to \mathbb{R}$. *Then* I *is convex if and only if*

$$I((1-\lambda)u + \lambda v) \leq (1-\lambda)\epsilon_1 + \lambda\epsilon_2, \quad 0 < \lambda < 1, \tag{7}$$

whenever $I(u) < \epsilon_1$ *and* $I(v) < \epsilon_2$, *for all* $u, v \in V$ *and* $\epsilon_1, \epsilon_2 \in \mathbb{R}$.

In the rest of the paper, we let $X := W_0^{1,\mathcal{H}}(\Omega)$.

3. The Strong Singularity: $1 < \eta(x) < \infty$

Corresponding to equation (5), we define $\mathcal{E} : X \to \mathbb{R}$ as the singular energy functional

$$\mathcal{E}(u) = \widehat{\mathfrak{R}}\left(J_{\mathcal{H}}(u)\right) + \int_{\Omega} \frac{\wp(x)|u|^{1-\eta(x)}}{\eta(x)-1} dx,$$

where $\widehat{\mathfrak{R}}(t) = \int_0^t \mathfrak{R}(s)ds$.
For any $u \in X$ provided $\int_{\Omega} \wp(x)|u|^{1-\eta(x)}dx < \infty$, the functional $\mathcal{E}$ is well-defined and coercive on X. Indeed, using (H_2), Proposition 2.1, and (A_3) we have

$$|\mathcal{E}(u)| \leq \frac{\kappa_2}{\alpha_2(p^-)^{\alpha_2}}\|u\|_{1,\mathcal{H},0}^{\alpha_2 q^+} + \frac{1}{\eta^- - 1}\int_{\Omega} \wp(x)|u|^{1-\eta(x)}dx < \infty.$$

Now assume that $\|u\|_{1,\mathcal{H},0} > 1$. Then using (H_2), Proposition 2.1, and (A_3), (A_4) together provides

$$\mathcal{E}(u) \geq \frac{\kappa_1}{\alpha_1(q^+)^{\alpha_1}}\|u\|_{1,\mathcal{H},0}^{\alpha_1 p^-} + \frac{1}{\eta^+ - 1}\int_{\Omega} \wp(x)|u|^{1-\eta(x)}dx,$$

which implies $\mathcal{E}(u) \to \infty$ as $\|u\|_{1,\mathcal{H},0} \to \infty$; that is, $\mathcal{E}$ is coercive.

Definition 3.1. *A weak solution to problem (5) is a function $u \in X$ such that $\operatorname{essinf}_Q u > 0$ for any domain $Q \Subset \Omega$ it holds*

$$\mathfrak{R}\left(\jmath_{\mathcal{H}}(u)\right) \int_{\Omega} \mathcal{B}(u,p,q,\nu) \cdot \nabla\varphi dx = \int_{\Omega} \wp(x) u^{-\eta(x)} \varphi dx, \tag{8}$$

for all $\varphi \in X$.

Definition 3.2. *According to the constrained minimization for problem (5), we define*

$$X_1 = \left\{ u \in X : \int_{\Omega} \left(\mathfrak{R}(\jmath_{\mathcal{H}}(u)) \left[|\nabla u|^{p(x)} + \nu(x)|\nabla u|^{q(x)} \right] - \wp(x)|u|^{1-\eta(x)} \right) dx \geq 0 \right\}$$

and

$$X_2 = \left\{ u \in X : \int_{\Omega} \left(\mathfrak{R}(\jmath_{\mathcal{H}}(u)) \left[|\nabla u|^{p(x)} + \nu(x)|\nabla u|^{q(x)} \right] - \wp(x)|u|^{1-\eta(x)} \right) dx = 0 \right\}.$$

Remark 3.3. *Notice that X_1 is closed in X.*

Lemma 3.4. *Assume that (H_2) holds. Then*

(i) $\widehat{\mathfrak{R}}$ *is increasing on $(0, \infty)$ and convex.*

(ii) *The operator $\mathcal{G} : X \to \{x \in \mathbb{R} | x > 0\}$ defined by $\mathcal{G}(u) := \widehat{\mathfrak{R}}(\jmath_{\mathcal{H}}(u))$ is convex. Moreover, $\mathcal{G}$ is of class $C^1(X, \mathbb{R})$ and $\mathcal{G}' : X \to X^*$ is given by*

$$\langle \mathcal{G}'(u), \varphi \rangle = \mathfrak{R}\left(\jmath_{\mathcal{H}}(u)\right) \langle \jmath'_{\mathcal{H}}(u), \varphi \rangle, \tag{9}$$

is monotone for $u, \varphi \in X$.

Proof. Here we are working on the function space, X, and provide a concise outline of the proof whenever a slightly different approach is required (see [4, Lemma 3.4]). In this regard, we skip part (*i*), and continue with part (*ii*). Now, by the continuity of $\widehat{\mathfrak{R}}$ and the Mean Value Theorem

$$\begin{aligned} \langle \mathcal{G}'(u), \varphi \rangle = \lim_{t \to 0} \Big[\mathfrak{R}\left(\jmath_{\mathcal{H}}(u + \varepsilon t \varphi)\right) \times \int_{\Omega} \Big(& |\nabla(u + \varepsilon t\varphi)|^{p(x)-2} \nabla(u + \varepsilon t\varphi) \\ & + \nu(x)|\nabla(u + \varepsilon t\varphi)|^{q(x)-2}\nabla(u + \varepsilon t\varphi)\Big) \cdot \nabla\varphi dx \Big], \end{aligned} \tag{10}$$

for all $u, \varphi \in X$, and $0 \leq \varepsilon \leq 1$.
Put

$$\mathfrak{I}(u + \varepsilon t\varphi) := \Big[|\nabla(u + \varepsilon t\varphi)|^{p(x)-2}\nabla(u + \varepsilon t\varphi) + \nu(x)|\nabla(u + \varepsilon t\varphi))|^{q(x)-2}\nabla(u + \varepsilon t\varphi) \Big] \cdot \nabla\varphi. \tag{11}$$

By the Young's inequality, it reads

$$|\mathfrak{I}(u + \varepsilon t\varphi)| \leq C_q[|\nabla u|^{p(x)} + |\nabla\varphi|^{p(x)} + |\nu|_{\infty}^{\hat{q}} |\nabla u|^{q(x)} + |\nabla\varphi|^{q(x)}], \tag{12}$$

where $C_q := \frac{2^{q^+}(q^+-1)+1}{2q^-}$. By $L^{q(x)}(\Omega) \subset L^{p(x)}(\Omega) \subset L^1(\Omega)$, the right-hand side of (12) belongs to $L^1(\Omega)$. In conclusion, the Lebesgue's Dominated Convergence Theorem and the continuity of $\widehat{\mathfrak{R}}$ yield

$$\begin{aligned} \langle \mathcal{G}'(u), \varphi \rangle &= \Big[\lim_{t \to 0} \mathfrak{R}\left(\jmath_{\mathcal{H}}(u + \varepsilon t\varphi)\right) \int_{\Omega} \lim_{t \to 0} \{ |\nabla(u + \varepsilon t\varphi)|^{p(x)-2}\nabla(u + \varepsilon t\varphi) \\ &\qquad + \nu(x)|\nabla(u + \varepsilon t\varphi))|^{q(x)-2}\nabla(u + \varepsilon t\varphi)\} \cdot \nabla\varphi dx \Big] \\ &= \mathfrak{R}\left(\jmath_{\mathcal{H}}(u)\right) \langle \jmath'_{\mathcal{H}}(u), \varphi \rangle. \end{aligned} \tag{13}$$

Since $0 \leq \nu \in L^{\infty}(\Omega)$, $\mathcal{G}$ is of class $C^1(X, \mathbb{R})$ and $\mathcal{G}'$ is monotone. □

Lemma 3.5. *If $(u_n \geq 0) \subset X_1$ is a minimizing sequence for the problem $\lim_{n\to\infty} \mathcal{E}(u_n) = \inf_{X_1} \mathcal{E}$, then there exist $\delta_1, \delta_2 \geq 0$ for which*

$$\delta_1 \leq \|u_n\|_{1,\mathcal{H},0} \leq \delta_2.$$

Proof. Considering that $(u_n) \subset X_1$ is a nonnegative minimizing sequence for $\mathcal{E}$ which is coercive, (u_n) would be forced to have a bounded norm, otherwise, $\mathcal{E}(u_n)$ would not approach the minimum value, but would diverge to infinity as $\|u_n\|_{1,\mathcal{H},0} \to \infty$. Thus, for a minimizing sequence $(u_n) \subset X_1$ of $\mathcal{E}$, there exists $\delta_2 > 0$ such that $\|u_n\|_{1,\mathcal{H},0} \leq \delta_2$. However, we still must rule out the possibility of having a minimizing sequence which tends to zero. To this end, by contradiction, suppose there exists a subsequence $(u_n) \subset X_1$ such that $u_n \to 0$ (*strongly*) in X.
Then, by Fatou's lemma,

$$\begin{aligned}
&\int_\Omega \mathfrak{R}(J_{\mathcal{H}}(u_n))[|\nabla u_n|^{p(x)} + \nu(x)|\nabla u_n|^{q(x)}]dx \geq \int_\Omega \wp(x)|u_n|^{1-\eta(x)}dx \\
&\liminf_{n\to\infty} J_{\mathcal{H}}(u_n)^{\alpha_2} \geq \frac{p^-}{\kappa_2 q^+} \int_\Omega \liminf_{n\to\infty} \wp(x)|u_n|^{1-\eta(x)}dx.
\end{aligned} \tag{14}$$

However, if $f \in L^1(\Omega)$ and $\wp(x) > 0$ a.e. in Ω, the assumption $u_n \to 0$ in X, and Proposition 2.1, (14) gives a contradiction. Thus, there exists $\delta_1 > 0$ such that $\|u_n\|_{1,\mathcal{H},0} \geq \delta_1$. □

Lemma 3.6. *Suppose $u \in X$ and $\int_\Omega \wp(x)|u|^{1-\eta(x)}dx < \infty$. Thus there exists a unique continuous scaling function $X \to \{x \in \mathbb{R} | x > 0\} : u \longmapsto t(u)$ such that $t(u)u \in X_2$, and $t(u)u$ is the minimizer of the functional $\mathcal{E}$ along the ray $\{tu : t > 0\}$, i.e., $\inf_{t>0} \mathcal{E}(tu) = \mathcal{E}(t(u)u)$.*

Proof. The proof is much the same as Lemma 3.8's of [4], so we omit it. □

Lemma 3.7. *Suppose that $(u_n) \subset X_1$ is the nonnegative minimizing sequence for the minimization problem $\lim_{n\to\infty} \mathcal{E}(u_n) = \inf_{X_1} \mathcal{E}$. Then, there exists a subsequence (u_n) such that $u_n \to \hat{u}$ in X.*

Proof. Notice that (u_n) is bounded in X, since it is a minimizing sequence. Thus, due to the reflexivity of X, $\exists (u_n)$, not relabelled, and $\hat{u} \in X$ such that

- $u_n \rightharpoonup \hat{u}$ (weakly) in X,
- $u_n \to \hat{u}$ in $L^{\mathcal{H}}(\Omega)$,
- $u_n(x) \to \hat{u}(x)$ a.e. in Ω.

Using Fatou's Lemma and Lemma 3.5, it follows

$$\begin{aligned}
\inf_{X_1} \mathcal{E} &= \lim_{n\to\infty} \mathcal{E}(u_n) \\
&\geq \liminf_{n\to\infty} \left[\widehat{\mathfrak{R}}\left(J_{\mathcal{H}}(u_n)\right) + \int_\Omega \frac{\wp(x)|u_n|^{1-\eta(x)}}{\eta(x)-1}dx \right] \\
&\geq \widehat{\mathfrak{R}}\left(J_{\mathcal{H}}(u)\right) + \int_\Omega \frac{\wp(x)|\hat{u}|^{1-\eta(x)}}{\eta(x)-1}dx \\
&= \mathcal{E}(\hat{u}) \geq \mathcal{E}(t(\hat{u})\hat{u}) \geq \inf_{X_1} \mathcal{E},
\end{aligned} \tag{15}$$

since the functional $\|\cdot\|_{1,\mathcal{H},0}$ is weakly lower semicontinuous. This result means that

$$\lim_{n\to\infty} \|u_n\|_{1,\mathcal{H},0} = \|\hat{u}\|_{1,\mathcal{H},0}. \tag{16}$$

In conclusion, (16) and the fact that $u_n \rightharpoonup \hat{u}$ in X ensure that $u_n \to \hat{u}$ in X. □

Now we present the proof of Theorem 1.1.

Proof. **Necessity**. If $u \in X$ is a weak solution to problem (5), then replacing u with φ in Definition 3.1 immediately leads to $\int_\Omega \wp(x)|u|^{1-\eta(x)}dx < \infty$.
Sufficiency. Assume there exists $\overline{u} \in X$ with $\int_\Omega \wp(x)|\overline{u}|^{1-\eta(x)}dx < \infty$. Then $\exists$ a unique number $t(\overline{u}) > 0$ s.t. $t(\overline{u})\overline{u} \in X_2$ (by Lemma 3.6). Since $\mathcal{E}$ is and bounded below on X_1 and coercive, one can apply Ekeland's variational principle to $\inf_{X_1} \mathcal{E}$. Therefore, there exists a minimizing sequence $(u_n) \subset X_1$ for which:

(E_1) $\mathcal{E}(u_n) - \inf_{X_1} \mathcal{E} \leq \frac{1}{n}$,

(E_2) $\mathcal{E}(u_n) - \mathcal{E}(w) \leq \frac{1}{n}\|u_n - w\|_{1,\mathcal{H},0}, \ \ \forall w \in X_1$.

Since $\mathcal{E}(|u_n|) = \mathcal{E}(u_n)$, one can assume that $u_n(x) \geq 0$ a.e. in Ω. Moreover, considering that $(u_n) \subset X_1$, it reads

$$\jmath_{\mathcal{H}}(u_n)^{\alpha_2} \geq \frac{p^-}{\kappa_2 q^+}\int_\Omega \wp(x)u_n^{1-\eta(x)}dx. \tag{17}$$

However, if we consider Lemma 3.5, (A_2), and Proposition 2.1, then $u_n(x) > 0$ a.e. in Ω.
Next, building up on the result of Lemma 3.7, we claim that when n is large enough $\hat{u}$ is contained in X_2 regardless $(u_n) \subset X_1 \setminus X_2$ or $(u_n) \subset X_2$. As a result, it is a weak solution to (5).
Case I: Assume that $(u_n) \subset X_1 \setminus X_2$ when n is large enough. Then $\hat{u} \in X_2$.
For a function $\varphi \in X$ with $\varphi \geq 0$, and $t \geq 0$, using (A_1), (A_2) give

$$\begin{aligned}\int_\Omega \wp(x)(u_n + t\varphi)^{1-\eta(x)}dx &\leq \int_\Omega \wp(x)u_n^{1-\eta(x)}dx \\ &< \Re(\jmath_{\mathcal{H}}(u_n))\int_\Omega [|\nabla u_n|^{p(x)} + \nu(x)|\nabla u_n|^{q(x)}]dx.\end{aligned} \tag{18}$$

Thus, for small enough $t > 0$

$$\begin{aligned}&\int_\Omega \wp(x)(u_n + t\varphi)^{1-\eta(x)}dx \\ &< \Re(\jmath_{\mathcal{H}}(u_n + t\varphi))\int_\Omega \left[|\nabla(u_n + t\varphi)|^{p(x)} + \nu(x)|\nabla(u_n + t\varphi)|^{q(x)}\right]dx,\end{aligned} \tag{19}$$

that is, $u_n + t\varphi \in X_1 \setminus X_2$.
Next, we apply (E_2). Thus

$$\frac{1}{n}\|t\varphi\| \geq \widehat{\Re}\left(\jmath_{\mathcal{H}}(u_n)\right) - \widehat{\Re}\left(\jmath_{\mathcal{H}}(u_n + t\varphi)\right) + \int_\Omega \frac{\wp(x)u_n^{1-\eta(x)}}{\eta(x) - 1}dx - \int_\Omega \frac{\wp(x)(u_n + t\varphi)^{1-\eta(x)}}{\eta(x) - 1}dx.$$

By dividing both sides of the inequality by t and passing to the limit infimum as $t \to 0$, and applying Fatou's lemma, we obtain

$$\frac{\|\varphi\|}{n} + \Re\left(\jmath_{\mathcal{H}}(u_n)\right)\int_\Omega \mathcal{B}(u_n, p, q, \nu)\cdot\nabla\varphi dx \geq \int_\Omega \wp(x)u_n^{-\eta(x)}\varphi dx.$$

Using Lemma 3.7, and considering (H_2) and Fatou's Lemma, we have

$$\Re\left(\jmath_{\mathcal{H}}(\hat{u})\right)\int_\Omega \mathcal{B}\left(\hat{u}, p, q, \nu\right)\cdot\nabla\varphi dx - \int_\Omega \wp(x)(\hat{u})^{-\eta(x)}\varphi dx \geq 0. \tag{20}$$

If we let $\varphi = \hat{u}$ in (20), we obtain that $\hat{u} \in X_1$.
Hence, by applying Lemma 3.7, it reads

$$\hat{u} \in X_2 \ \ \text{(where } t(\hat{u}) = 1\text{)}, \tag{21}$$

which concludes **Case I**.
Case II: There is a subsequence of (u_n) (without relabeling) contained in X_2. Then $\hat{u} \in X_2$.
For a function $\varphi \in X$ with $\varphi \geq 0$, and $t \geq 0$, using (A_1), (A_2) gives

$$\begin{aligned}\int_\Omega \wp(x)(u_n + t\varphi)^{1-\eta(x)}dx \leq& \int_\Omega \wp(x)u_n^{1-\eta(x)}dx \\ =&\mathfrak{R}(J_{\mathcal{H}}(u_n)) \int_\Omega [|\nabla u_n|^{p(x)} + \nu(x)|\nabla u_n|^{q(x)}]dx \\ <&\infty.\end{aligned} \tag{22}$$

According to Lemma 3.6 there exists a unique continuous scaling function, denoted by $\sigma_n(t) := t(u_n + t\varphi) > 0$, corresponding to $(u_n + t\varphi)$ so that $\sigma_n(t)(u_n + t\varphi) \in X_2$ for $n = 1, 2, \cdots$. Then, by the means of the definition of X_2, we can write

$$\begin{aligned}0 \geq \mathfrak{R}(J_{\mathcal{H}}(\sigma_n(u_n + t\varphi))) \int_\Omega \Big[|\nabla \sigma_n(t)(u_n + t\varphi)|^{p(x)} + \nu(x)|\nabla \sigma_n(t)(u_n + t\varphi)|^{q(x)}\Big]dx \\ - \sigma_n^{1-\check{\eta}}(t) \int_\Omega \wp(x)(u_n + t\varphi)^{1-\eta(x)}dx,\end{aligned} \tag{23}$$

and

$$0 = \mathfrak{R}(J_{\mathcal{H}}(u_n)) \int_\Omega [|\nabla u_n|^{p(x)} + \nu(x)|\nabla u_n|^{q(x)}]dx - \int_\Omega \wp(x)u_n^{1-\eta(x)}dx. \tag{24}$$

Then by (23) and (24) we have

$$\begin{aligned}0 \geq& \Big[-(1 - \check{\eta})\, [\sigma_n(0) + \tau(\sigma_n(t) - \sigma_n(0))]^{-\check{\eta}} \int_\Omega \wp(x)(u_n + t\varphi)^{1-\eta(x)}dx\Big](\sigma_n(t) - \sigma_n(0)) \\ &+ \mathfrak{R}(J_{\mathcal{H}}(\sigma_n(u_n + t\varphi))) \int_\Omega \Big[|\nabla \sigma_n(t)(u_n + t\varphi)|^{p(x)} + \nu(x)|\nabla \sigma_n(t)(u_n + t\varphi)|^{q(x)}\Big]dx \\ &- \mathfrak{R}(J_{\mathcal{H}}(u_n + t\varphi)) \int_\Omega \Big[|\nabla (u_n + t\varphi)|^{p(x)} + \nu(x)|\nabla (u_n + t\varphi)|^{q(x)}\Big]dx \\ &+ \mathfrak{R}(J_{\mathcal{H}}(u_n + t\varphi)) \int_\Omega \Big[|\nabla (u_n + t\varphi)|^{p(x)} + \nu(x)|\nabla (u_n + t\varphi)|^{q(x)}\Big]dx \\ &- \mathfrak{R}(J_{\mathcal{H}}(u_n)) \int_\Omega \Big[|\nabla u_n|^{p(x)} + \nu(x)|\nabla u_n|^{q(x)}\Big]dx \\ &- \int_\Omega \wp(x)(u_n + t\varphi)^{1-\eta(x)}dx + \int_\Omega \wp(x)u_n^{1-\eta(x)}dx,\end{aligned} \tag{25}$$

for some constant $0 < \tau < 1$. The function σ_n is a continuous real-valued function (Lemma 3.6). However, we don't have any information about $\sigma_n'(0)$. Therefore, suppose $\sigma_n'(0) = \lim_{t\to 0} \frac{\sigma_n(t)-\sigma_n(0)}{t} \in [-\infty, \infty]$. If this limit doesn't exist, we can instead use a subsequence $t_k > 0$ of t such that $t_k \to 0$ as $k \to \infty$.
(*i*) $\sigma_n'(0) \neq \infty$.
By (H_2), Proposition 2.1, and dividing the both sides of (25) by t and passing to the limit as $t \to 0$ leads to

$$\begin{aligned}0 \geq& \Bigg[p^- \mathfrak{R}(J_{\mathcal{H}}(u_n)) \int_\Omega [|\nabla u_n|^{p(x)} + \nu(x)|\nabla u_n|^{q(x)}]dx + (\check{\eta} - 1) \int_\Omega \wp(x)u_n^{1-\eta(x)}dx \\ &+\mathfrak{R}'\left(J_{\mathcal{H}}(u_n)\right)\left(\int_\Omega [|\nabla u_n|^{p(x)} + \nu(x)|\nabla u_n|^{q(x)}]dx\right)^2\Bigg]\sigma_n'(0) \\ &+ \Bigg[p^- \mathfrak{R}\left(J_{\mathcal{H}}(u_n)\right) + \mathfrak{R}'\left(J_{\mathcal{H}}(u_n)\right) \int_\Omega [|\nabla u_n|^{p(x)} + \nu(x)|\nabla u_n|^{q(x)}]dx\Bigg]\end{aligned}$$

$$\times \int_\Omega \mathcal{B}(u_n, p, q, \nu) \cdot \nabla\varphi dx + (\check{\eta} - 1) \int_\Omega \wp(x) u_n^{-\eta(x)} \varphi dx. \tag{26}$$

Considering Lemma 3.5, (26) means that there is a constant $\bar{c} > 0$ such that $\sigma_n'(0) \leq \bar{c}$ uniformly in all large n.
(ii) $\sigma_n'(0) \neq -\infty$.
Using Ekeland's variational principle, the definition of X_2, it reads

$$\begin{aligned}
&\frac{1}{n}\left[|\sigma_n(t) - 1|\|u_n\| + t\sigma_n(t)\|\varphi\|\right] \\
\geq &\widehat{\mathfrak{R}}\left(\jmath_{\mathcal{H}}(u_n)\right) + \int_\Omega \frac{\wp(x) u_n^{1-\eta(x)}}{\eta(x) - 1} dx - \widehat{\mathfrak{R}}\left(\jmath_{\mathcal{H}}(\sigma_n(u_n + t\varphi))\right) - \int_\Omega \frac{\wp(x)[\sigma_n(t)(u_n + t\varphi)]^{1-\eta(x)}}{\eta(x) - 1} dx \\
\geq &\widehat{\mathfrak{R}}\left(\jmath_{\mathcal{H}}(u_n)\right) - \widehat{\mathfrak{R}}\left(\jmath_{\mathcal{H}}(\sigma_n(u_n + t\varphi))\right) - \frac{1}{\eta^- - 1} \mathfrak{R}\left(\jmath_{\mathcal{H}}(\sigma_n(u_n + t\varphi))\right) \\
&\times \int_\Omega \left[|\nabla\sigma_n(t)(u_n + t\varphi)|^{p(x)} + \nu(x)|\nabla\sigma_n(t)(u_n + t\varphi)|^{q(x)}\right] dx.
\end{aligned} \tag{27}$$

Or, using (H_2) and Lemma 3.5 gives

$$\begin{aligned}
&\frac{1}{n}\left[|\sigma_n(t) - 1|\|u_n\| + t\sigma_n(t)\|\varphi\|\right] + \widehat{\mathfrak{R}}\left(\jmath_{\mathcal{H}}(u_n + t\varphi)\right) - \widehat{\mathfrak{R}}\left(\jmath_{\mathcal{H}}(u_n)\right) + \frac{\kappa_2 \sigma_n^{\alpha_2 \hat{p}}(t) C(\delta_2)^{\alpha_2 \hat{p}}}{(\eta^- - 1)(p^-)^{(\alpha_2 - 1)}} t\|\varphi\|_{1,\mathcal{H},0}^{\alpha_2 \hat{q}} \\
&\geq \widehat{\mathfrak{R}}\left(\jmath_{\mathcal{H}}(u_n + t\varphi)\right) - \widehat{\mathfrak{R}}\left(\jmath_{\mathcal{H}}(\sigma_n(u_n + t\varphi))\right).
\end{aligned} \tag{28}$$

Thus when $t \to 0$, it follows

$$\begin{aligned}
&\frac{1}{n}\|\varphi\| + \mathfrak{R}\left(\jmath_{\mathcal{H}}(u_n)\right) \int_\Omega \mathcal{B}(u_n, p, q, \nu) \cdot \nabla\varphi dx + \frac{\kappa_2 C(\delta_2)^{\alpha_2 \hat{p}}}{(\eta^- - 1)(p^-)^{(\alpha_2 - 1)}} \|\varphi\|_{1,\mathcal{H},0}^{\alpha_2 \hat{q}} \\
\geq &\left(-\mathfrak{R}\left(\jmath_{\mathcal{H}}(u_n)\right) \int_\Omega \left[|\nabla u_n|^{p(x)} + \nu(x)|\nabla u_n|^{q(x)}\right] dx - \frac{\|u_n\|}{n} \operatorname{sgn}[\sigma_n(t) - 1]\right) \sigma_n'(0),
\end{aligned} \tag{29}$$

which shows that $\sigma_n'(0) \neq -\infty$. Thus, $\sigma_n'(0) \geq \underline{c}$, $\underline{c}$ is a constant, uniformly in large n. Putting all these together implies that there exists a constant $C_0 > 0$ such that $|\sigma_n'(0)| \leq C_0$ when n is large enough.
(iii) $\hat{u} \in X_2$.
Using Ekeland's variational principle once more,

$$\begin{aligned}
&\frac{1}{n}\left[|\sigma_n(t) - 1|\|u_n\| + t\sigma_n(t)\|\varphi\|\right] \\
\geq &\widehat{\mathfrak{R}}\left(\jmath_{\mathcal{H}}(u_n)\right) - \widehat{\mathfrak{R}}\left(\jmath_{\mathcal{H}}(u_n + t\varphi)\right) + \int_\Omega \frac{\wp(x) u_n^{1-\eta(x)}}{\eta(x) - 1} dx - \int_\Omega \frac{\wp(x)(u_n + t\varphi)^{1-\eta(x)}}{\eta(x) - 1} dx \\
&+ \widehat{\mathfrak{R}}\left(\jmath_{\mathcal{H}}(u_n + t\varphi)\right) - \widehat{\mathfrak{R}}\left(\jmath_{\mathcal{H}}(\sigma_n(u_n + t\varphi))\right) \\
&+ \int_\Omega \frac{\wp(x)(u_n + t\varphi)^{1-\eta(x)}}{\eta(x) - 1} dx - \int_\Omega \frac{\wp(x)[\sigma_n(t)(u_n + t\varphi)]^{1-\eta(x)}}{\eta(x) - 1} dx.
\end{aligned} \tag{30}$$

Then when $t \to 0$, we get

$$\begin{aligned}
&\frac{1}{n}\left[|\sigma_n'(0)|\|u_n\| + \|\varphi\|\right] + \mathfrak{R}\left(\jmath_{\mathcal{H}}(u_n)\right) \int_\Omega \mathcal{B}(u_n, p, q, \nu) \cdot \nabla\varphi dx \\
\geq &\left[-\mathfrak{R}\left(\jmath_{\mathcal{H}}(u_n)\right) \int_\Omega [|\nabla u_n|^{p(x)} + \nu(x)|\nabla u_n|^{q(x)}] dx + \int_\Omega \wp(x) u_n^{1-\eta(x)} dx\right] \sigma_n'(0) + \int_\Omega \wp(x) u_n^{-\eta(x)} \varphi dx.
\end{aligned} \tag{31}$$

However, since $|\sigma_n'(0)| \le C_0$ uniformly in n, by letting $n \to \infty$

$$\mathfrak{R}\left(\jmath_{\mathcal{H}}(\hat{u})\right)\int_{\Omega}\mathcal{B}(\hat{u},p,q,\nu)\cdot\nabla\varphi dx-\int_{\Omega}\wp(x)(\hat{u})^{-\eta(x)}\varphi dx\ge 0, \tag{32}$$

for all $\varphi \in X$, $\varphi \ge 0$. Substituting $\varphi = \hat{u}$ in (32) implies that $\hat{u} \in X_1$. Then by Lemma 3.7

$$\hat{u} \in X_2 \quad (\text{where } t(\hat{u}) = 1). \tag{33}$$

Putting it all together, we conclude that $\hat{u} \in X_2$, and (32) holds for both cases. As well, since $\hat{u} \ge 0$ and $\hat{u} \ne 0$, by the strong maximum principle it follows that $\hat{u}(x) > 0$ almost everywhere in Ω.
Next, we shall show that $\hat{u} \in X$ is a weak solution to problem (5).
Let $\phi \in X$ be a random function, and let $\varphi = (\hat{u}+\varepsilon\phi)^+ = \max\{0, \hat{u}+\varepsilon\phi\}$, $\varepsilon > 0$. Let $\Omega = \Omega_{\ge} \cup \Omega_{<}$, where $\Omega_{\ge} := \{x \in \Omega : \hat{u}(x)+\varepsilon\phi(x) \ge 0\}$ and $\Omega_{<} := \{x \in \Omega : \hat{u}(x)+\varepsilon\phi(x) < 0\}$.
To this end, switching φ with $(\hat{u}+\varepsilon\phi)$ in (32), we have

$$\begin{aligned}0 \le &\mathfrak{R}(\jmath_{\mathcal{H}}(\hat{u}))\int_{\Omega_{\ge}}\mathcal{B}(\hat{u},p,q,\nu)\cdot\nabla(\hat{u}+\varepsilon\phi)dx-\int_{\Omega_{\ge}}\wp(x)(\hat{u})^{-\eta(x)}(\hat{u}+\varepsilon\phi)dx\\ =&\mathfrak{R}(\jmath_{\mathcal{H}}(\hat{u}))\Big[\int_{\Omega}[|\nabla\hat{u}|^{p(x)}+\nu(x)|\nabla\hat{u}|^{q(x)}]dx+\varepsilon\int_{\Omega}\mathcal{B}(\hat{u},p,q,\nu)\cdot\nabla\phi dx\Big]\\ &-\int_{\Omega}\wp(x)(\hat{u})^{1-\eta(x)}dx-\varepsilon\int_{\Omega}\wp(x)(\hat{u})^{-\eta(x)}\phi dx-\mathfrak{R}(\jmath_{\mathcal{H}}(\hat{u}))\int_{\Omega_{<}}\mathcal{B}(\hat{u},p,q,\nu)\cdot\nabla(\hat{u}+\varepsilon\phi)dx\\ &-\int_{\Omega_{<}}\wp(x)(\hat{u})^{-\eta(x)}(\hat{u}+\varepsilon\phi)dx.\end{aligned} \tag{34}$$

However, since $\hat{u} \in X_2$, it reads

$$\begin{aligned}0 \le &\varepsilon\left[\mathfrak{R}(\jmath_{\mathcal{H}}(\hat{u}))\int_{\Omega}\mathcal{B}(\hat{u},p,q,\nu)\cdot\nabla\phi dx-\int_{\Omega}\wp(x)(\hat{u})^{-\eta(x)}\phi dx\right]\\ &-\varepsilon\left[\mathfrak{R}(\jmath_{\mathcal{H}}(\hat{u}))\int_{\Omega_{<}}\mathcal{B}(\hat{u},p,q,\nu)\cdot\nabla\phi dx+\int_{\Omega_{<}}\wp(x)(\hat{u})^{-\eta(x)}\phi dx\right].\end{aligned} \tag{35}$$

Dividing (35) by ε and considering that $|\Omega_{<}| \to 0$ as $\varepsilon \to 0$, we obtain

$$\mathfrak{R}\left(\jmath_{\mathcal{H}}(\hat{u})\right)\int_{\Omega}\mathcal{B}(\hat{u},p,q,\nu)\cdot\nabla\phi dx\ge\int_{\Omega}\wp(x)(\hat{u})^{-\eta(x)}\phi dx, \tag{36}$$

for all $\phi \in X$.
However, due to the randomness of the function $\phi \in X$,

$$\mathfrak{R}\left(\jmath_{\mathcal{H}}(\hat{u})\right)\int_{\Omega}\mathcal{B}(\hat{u},p,q,\nu)\cdot\nabla\phi dx=\int_{\Omega}\wp(x)(\hat{u})^{-\eta(x)}\phi dx. \tag{37}$$

Thus, $\hat{u} \in X$ is a weak solution to problem (5).
Lastly, we show that $\hat{u} \in X$ is the unique positive solution to problem (5).
Argue by contradiction. Assume that $\hat{v}$ is an another positive weak solution to (5). Thus

$$\begin{aligned}&\mathfrak{R}\left(\jmath_{\mathcal{H}}(\hat{u})\right)\int_{\Omega}\mathcal{B}(\hat{u},p,q,\nu)\cdot\nabla\phi dx-\mathfrak{R}\left(\jmath_{\mathcal{H}}(\hat{v})\right)\int_{\Omega}\mathcal{B}(\hat{v},p,q,\nu)\cdot\nabla\phi dx\\ =&\int_{\Omega}\wp(x)\left[(\hat{u})^{-\eta(x)}-(\hat{v})^{-\eta(x)}\right]\phi dx.\end{aligned} \tag{38}$$

Then, by replacing ϕ with $\hat{u}-\hat{v}$ in (38), and applying Lemma 3.4, we obtain

$$\begin{aligned}0 \ge &\int_{\Omega}\wp(x)\left[(\hat{u})^{-\eta(x)}-(\hat{v})^{-\eta(x)}\right](\hat{u}-\hat{v})dx\\ =&\langle\mathcal{G}'(\hat{u})-\mathcal{G}'(\hat{v}),\hat{u}-\hat{v}\rangle\ge 0.\end{aligned} \tag{39}$$

Thus $\hat{u} = \hat{v}$ in X. □

3.1. Application to Composite Materials

We provide an application from composite materials to illustrate the problem (5) and the result of Theorem 1.1 in a concrete way.

Description of the Problem and Modeling

In composite materials, the mechanical properties can vary significantly from one region to another due to the presence of different constituents (e.g., fibers embedded in a matrix). The double phase operator is particularly useful for modeling such materials because it can capture the variation in material behavior across different phases (see, e.g. [13, 26]).

We can consider a composite material occupying a domain Ω in $\mathbb{R}^N$, consisting of two distinct phases:

Soft Phase ($\nu(x) = 0$). Represented by regions where the material is more compliant or softer.

Hard Phase ($\nu(x) > 0$). Represented by regions where the material is stiffer or harder.

As mentioned before, we can model the mechanical response of this type composite materials using

$$div\left(|\nabla u|^{p(x)-2}\nabla u + \nu(x)|\nabla u|^{q(x)-2}\nabla u\right).$$

A weighted version of the double phase operator might lead to the following equation, i.e. problem (5), that governs the equilibrium state of the material under external forces

$$\begin{cases} -\mathfrak{R}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} + \nu(x)\frac{|\nabla u|^{q(x)}}{q(x)}\right)div\left[|\nabla u|^{p(x)-2}\nabla u + \nu(x)|\nabla u|^{q(x)-2}\nabla u\right] = \wp(x)u^{-\eta(x)} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{40}$$

where:

- u represents the displacement field in the material.
- ∇u is related to the strain in the material.
- $p(x)$ and $q(x)$ satisfy $1 < p(x) < q(x)$.
- $\nu(x)$ is a measurable weight function defined as:

$$\nu(x) = \begin{cases} 0 & \text{if } x \text{ is in the soft phase (matrix)}, \\ \nu_0 > 0 & \text{if } x \text{ is in the hard phase (inclusion)}. \end{cases}$$

The functional $\mathfrak{R}$ depends on the integral of the energy functional over the entire domain Ω, that is, it introduces a nonlocal effect into the equation. In the context of composite materials, this might suggests that the stress-strain relationship at a point is influenced by the overall deformation energy of the material. On the other hand, the singular external force, $u^{-\eta(x)}$, could represent stress concentrations or defects in the material where the strain (related to ∇u) becomes large.

Now, we shall present a specified example for problem (40).

Let $\Omega := B_{1/2} = \{x \in \mathbb{R}^N : |x| < 1/2, \ N \geq 3\}$. We split Ω into two disjoint regions as follows:

$$\Omega_{\nu=0} := \text{The soft phase (matrix)} \quad \text{and}$$
$$\Omega_{\nu>0} := \text{The hard phase (inclusion)}.$$

Assume the following:

- $\nu(x) = \begin{cases} 0, & \text{if } x \text{ is in } \Omega_{\nu=0}, \\ \nu_0 > 0, & \text{if } x \text{ is in } \Omega_{\nu>0}. \end{cases}$
- $p(x) = 2$ in Ω.

- $q(x) = \begin{cases} 2, & \text{if } x \text{ is in } \Omega_{\nu=0}, \\ 4, & \text{if } x \text{ is in } \Omega_{\nu>0}. \end{cases}$
- $\wp(x) = (e^{x^2} \log|x|^{-1})^k$, $k \geq 1$; and $\bar{u} = (e^{x^2} \log|x|^{-1})^{\alpha}$, $x \neq 0$, $\alpha \geq 1$.
- $\Re(t) = a + bt$, $a \geq 0$, $t, b > 0$ are constants.

Theorem 3.8. *Suppose (A_1) holds. Additionally, if $1 < \eta^+ \leq 1 + \frac{k}{\alpha}$ and $\alpha \geq \frac{\hat{q}+1}{\hat{q}}$, then problem (40) has a unique positive X-solution.*

Proof. Since $\wp(x) = (e^{x^2} \log|x|^{-1})^k \leq 2^{k+1}$ for all $x \in \Omega$, $\wp \in L^1(\Omega)$. Therefore, conditions (H_1), (H_2) and (A_2) hold. It is straightforward to show that $\bar{u} \in X$, so it's omitted.
To show that $\int_\Omega \wp(x)|\bar{u}|^{1-\eta(x)}dx < \infty$, we switch to spherical coordinates, that is,

$$\begin{aligned} &\int_\Omega (e^{x^2} \log|x|^{-1})^k (e^{x^2} \log|x|^{-1})^{\alpha(1-\eta(x))} dx \\ &\leq \Omega_{N-1} \int_0^{1/2} (e^{r^2} \log r^{-1})^{k+\alpha(1-\eta^+)} r^{N-1} dr \\ &< \infty, \end{aligned} \tag{41}$$

where Ω_{N-1} is the surface area of the unit sphere S^{N-1} in $\mathbb{R}^N$ given by the formula $\Omega_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)}$ and Γ is the gamma function. Therefore, by Theorem 1.1, problem (40) has a unique positive X-solution. □

4. The Weak Singularity: $0 < \eta(x) < 1$

We study problem (6) and prove it has a unique positive X-solution with a negative energy level. We define the singular energy functional $\mathcal{E} : X \to \mathbb{R}$ corresponding to equation (6) by

$$\mathcal{E}(u) = \widehat{\Re}\left(\jmath_{\mathcal{H}}(u)\right) + \int_\Omega \frac{\Im(x)|u|^{1-\eta(x)}}{\eta(x) - 1} dx.$$

Definition 4.1. *A weak solution to problem (6) is a function $u \in X$ such that $\operatorname{essinf}_Q u > 0$ for any domain $Q \Subset \Omega$ it holds*

$$\Re\left(\jmath_{\mathcal{H}}(u)\right) \int_\Omega B(u, p, q, \nu) \cdot \nabla\varphi dx = \int_\Omega \Im(x) u^{-\eta(x)} \varphi dx, \tag{42}$$

for all $\varphi \in X$.

Lemma 4.2. *For the functional $\mathcal{E}$, we have the following:*

(*i*) *$\mathcal{E}$ is well-defined on X.*

(*ii*) *$\mathcal{E}$ is continuous on X.*

(*iii*) *$\mathcal{E}$ is coercive on X.*

Proof. Hölder inequality and the continuous embedding $X \hookrightarrow L^{\frac{q(x)}{q(x)+\eta(x)-1}}(\Omega)$ together give

$$\begin{aligned} |\mathcal{E}(u)| &\leq \widehat{\Re}\left(\jmath_{\mathcal{H}}(u)\right) + \left|\int_\Omega \frac{\Im(x)|u|^{1-\eta(x)}}{\eta(x) - 1} dx\right| \\ &\leq \frac{\kappa_2}{\alpha_2}\left(\jmath_{\mathcal{H}}(u)\right)^{\alpha_2} + \left|\int_\Omega \frac{\Im(x)|u|^{1-\eta(x)}}{\eta(x) - 1} dx\right| \\ &\leq \frac{\kappa_2}{\alpha_2 (p^-)^{\alpha_2}} \|u\|_{1,\mathcal{H},0}^{\alpha_2 q^+} + \frac{|g|_\infty}{1 - \eta^+} \|u\|_{1,\mathcal{H},0}^{1-\eta^-} < +\infty, \end{aligned} \tag{43}$$

that is, $\mathcal{E}$ is well-defined on X.
Considering that $\mathcal{G}$ is of class $C^1(X,\mathbb{R})$, and using the inequality $|a^k - b^k| \le |a-b|^k$ with $0 < k < 1$, for any $a, b \ge 0$, and the continuous embedding $X \hookrightarrow L^{\frac{q(x)}{q(x)+\eta(x)-1}}(\Omega)$, we get

$$|\mathcal{E}(u) - \mathcal{E}(v)| \le |\mathcal{G}(u) - \mathcal{G}(v)| + \frac{|g|_\infty}{1-\eta^+}\|u-v\|_{1,\mathcal{H},0}^{1-\eta^-} < \epsilon,$$

for any $u, v \in X$, provided $\|u - v\| < C(\epsilon)$. Therefore, $\mathcal{E}$ is continuous on X.
Let $u \in X$. In the similar fashion, one can get

$$\mathcal{E}(u) \ge \frac{\kappa_1}{\alpha_1(q^+)^{\alpha_1}}\|u\|_{1,\mathcal{H},0}^{\alpha_1 p^-} - \frac{|g|_\infty}{1-\eta^+}\|u\|_{1,\mathcal{H},0}^{1-\eta^-}. \tag{44}$$

Considering that $\alpha_1 p^- > 1 - \eta^+$, we obtain that $\mathcal{E}$ is bounded below on X and coercive. □

Lemma 4.3. *If (H_2) holds then $\mathcal{E}$ is convex.*

Proof. We apply Lemma 2.4, thus, it is enough to show that $\mathcal{E}$ satisfies

$$\mathcal{E}((1-\lambda)u + \lambda v)) \le (1-\lambda)\sigma + \lambda\delta,$$

whenever $\mathcal{E}(u) < \sigma$ and $\mathcal{E}(v) < \delta$, for all real numbers $0 < \lambda < 1$ and $\sigma, \delta > 0$.
Using (H_2) and Proposition 2.1 we obtain

$$\mathcal{E}(u) < \frac{c_1\kappa_2}{\alpha_2(p^-)^{\alpha_2}}\left(\|u\|_{1,\mathcal{H},0}^{\alpha_2 q^+} + \|u\|_{1,\mathcal{H},0}^{\alpha_2 p^-}\right) := \hat{\sigma},$$

and

$$\mathcal{E}(v) < \frac{c_2\kappa_2}{\alpha_2(p^-)^{\alpha_2}}\left(\|v\|_{1,\mathcal{H},0}^{\alpha_2 q^+} + \|v\|_{1,\mathcal{H},0}^{\alpha_2 p^-}\right) := \hat{\delta},$$

for all $u, v \in X$. Therefore,

$$\mathcal{E}((1-\lambda)u + \lambda v)) \le (1-\lambda)\frac{c_3\kappa_2}{\alpha_2(p^-)^{\alpha_2}}\left(\|u\|_{1,\mathcal{H},0}^{\alpha_2 q^+} + \|u\|_{1,\mathcal{H},0}^{\alpha_2 p^-}\right) + \lambda\frac{c_4\kappa_2}{\alpha_2(p^-)^{\alpha_2}}\left(\|v\|_{1,\mathcal{H},0}^{\alpha_2 q^+} + \|v\|_{1,\mathcal{H},0}^{\alpha_2 p^-}\right)$$

If we set $\sigma := \max\{\frac{\hat{\sigma}}{c_1}, \frac{\hat{\sigma}}{c_3}\}$ and $\delta := \max\{\frac{\hat{\delta}}{c_2}, \frac{\hat{\delta}}{c_4}\}$, it reads

$$\mathcal{E}((1-\lambda)u + \lambda v)) \le (1-\lambda)\sigma + \lambda\delta.$$

Thus, by Lemma 2.4, $\mathcal{E}$ is convex on X. □

Considering the Lemmas 4.2 and 4.3, the next lemma implies that $\mathcal{E}$ attains a global minimum in X.

Lemma 4.4. *$\mathcal{E}$ attains a global minimum in X, that is, there exists a function $u_0 \in X$ such that*

$$m = \mathcal{E}(u_0) = \inf_{u\in X}\mathcal{E}(u) < 0. \tag{45}$$

Proof. Notice that since $\mathcal{E}$ is convex, coercive and continuous, it has a global minimum in X.
Set

$$m := \inf_{u\in X}\mathcal{E}(u), \tag{46}$$

where m is a real number by (43).
For a nontrivial function $\varphi \in X$, we have

$$\mathcal{E}(t\varphi) \le \frac{\kappa_2\|\varphi\|_{1,\mathcal{H},0}^{\alpha_2\hat{q}}}{\alpha_2(p^-)^{\alpha_2}}t^{\alpha_2 p^-} + \frac{t^{1-\eta^+}}{\eta^- - 1}\int_\Omega \Im(x)|\varphi|^{1-\eta(x)}dx,$$

which shows that for $t > 0$ small enough, $\mathcal{E}(tu) < 0$. Let $t\varphi = u$ with $\|u\|_{1,\mathcal{H},0} < 1$. Then, $m = \inf_{u\in X} \mathcal{E}(u) < 0$. As well, by (46), there is a minimizing sequence (u_n) of X such that

$$m = \lim_{n\to\infty} \mathcal{E}(u_n) < 0. \tag{47}$$

Considering the fact that $\mathcal{E}(u_n) = \mathcal{E}(|u_n|)$, suppose $u_n \geq 0$.
Due to (44) and (47), (u_n) is bounded in X. Thus, there is a subsequence (u_n), with the same label, and $u_0 \in X$ for which

$$\begin{aligned}
&u_n \rightharpoonup u_0 \text{ in } X,\\
&u_n \to u_0 \text{ in } L^{r(x)}(\Omega),\ 1 \leq r(x) < p^*(x),\\
&u_n(x) \to u_0(x) \text{ a.e. in } \Omega.
\end{aligned}$$

The function $\mathcal{E}$ is weakly lower semi-continuous on X, Since it is convex, continuous on X. This implies

$$\begin{aligned}
m \leq& \mathcal{E}(u_0)\\
=& \widehat{\mathfrak{R}}(\jmath_{\mathcal{H}}(u_0)) + \int_\Omega \frac{\mathfrak{I}(x)|u_0|^{1-\eta(x)}}{\eta(x)-1}dx\\
\leq& \liminf_{n\to\infty} \mathcal{E}(u_n) = m,
\end{aligned}$$

that is,

$$m = \mathcal{E}(u_0) = \inf_X \mathcal{E}(u) < 0. \tag{48}$$

This concludes that $u_0 \in X$ is a global minimum for $\mathcal{E}$. □

Now, we present the proof of Theorem 1.2.

Proof. We claim that the global minimizer $u_0 \in X$ is a positive weak solution to problem (6).
We know that $u_0 \geq 0$, $u_0 \neq 0$ since $m = \mathcal{E}(u_0) < 0 = \mathcal{E}(0)$. Let $\varphi \in X$ with $\varphi \geq 0$, and $t > 0$. Then

$$\begin{aligned}
0 \leq& \liminf_{t\to 0} \frac{\mathcal{E}(u_0 + t\varphi) - \mathcal{E}(u_0)}{t}\\
\leq& \int_\Omega \mathfrak{R}(\jmath_{\mathcal{H}}(u_0)) \left(|\nabla u_0|^{p(x)-2} + \nu(x)|\nabla u_0|^{q(x)-2}\right) \nabla u_0 \cdot \nabla\varphi dx - \limsup_{t\to 0} \int_\Omega \mathfrak{I}(x) \frac{[(u_0 + t\varphi)^{1-\eta(x)} - u_0^{1-\eta(x)}]}{1-\eta(x)}dx,
\end{aligned}$$

which gives

$$\begin{aligned}
&\limsup_{t\to 0} \int_\Omega \mathfrak{I}(x) \frac{[(u_0 + t\varphi)^{1-\eta(x)} - u_0^{1-\eta(x)}]}{1-\eta(x)}dx\\
&\qquad \leq \int_\Omega \mathfrak{R}(\jmath_{\mathcal{H}}(u_0)) \left(|\nabla u_0|^{p(x)-2}\nabla u_0 + \nu(x)|\nabla u_0|^{q(x)-2}\nabla u_0\right) \cdot \nabla\varphi dx.
\end{aligned} \tag{49}$$

There exists $\varepsilon \in (0, 1)$ (the Mean Value Theorem) such that

$$\int_\Omega \mathfrak{I}(x) \frac{[(u_0 + t\varphi)^{1-\eta(x)} - u_0^{1-\eta(x)}]}{1-\eta(x)}dx = \int_\Omega \mathfrak{I}(x)(u_0 + t\varepsilon\varphi)^{-\eta(x)}\varphi dx. \tag{50}$$

Next, using Fatou's lemma provides

$$\begin{aligned}
&\limsup_{t\to 0}\int_\Omega \Im(x)\frac{[(u_0+t\varphi)^{1-\eta(x)}-u_0^{1-\eta(x)}]}{1-\eta(x)}dx\\
&\geq \liminf_{t\to 0}\int_\Omega \Im(x)\frac{[(u_0+t\varphi)^{1-\eta(x)}-u_0^{1-\eta(x)}]}{1-\eta(x)}dx\\
&=\liminf_{t\to 0}\int_\Omega \Im(x)(u_0+t\varepsilon\varphi)^{-\eta(x)}\varphi dx\\
&\geq \int_\Omega \Im(x)u_0^{-\eta(x)}\varphi dx \geq 0.
\end{aligned} \tag{51}$$

Considering (49) and (51) together, we have

$$\Re(J_{\mathcal{H}}(u_0))\int_\Omega B(u_0,p,q,\nu)\cdot\nabla\varphi dx-\int_\Omega \Im(x)u_0^{-\eta(x)}\varphi dx\geq 0, \tag{52}$$

for all $\varphi\in X$ with $\varphi\geq 0$. However, since $u_0\geq 0$ and $u_0\neq 0$, by the strong maximum principle

$$u_0(x)>0,\ \text{ almost everywhere in } \Omega.$$

Next, we prove that $u_0\in X$ satisfies (42). To do so, for a given $\epsilon>0$, define $\mathcal{E}^*:[-\epsilon,\epsilon]\to(-\infty,\infty)$ by $\mathcal{E}^*(t)=\mathcal{E}(u_0+tu_0)$. Due to Lemma 4.4, $\mathcal{E}^*$ assumes its minimum at $t=0$. Therefore

$$\frac{d}{dt}\mathcal{E}^*(t)|_{t=0}=\frac{d}{dt}\mathcal{E}(u_0+tu_0)|_{t=0}=0,$$

which means

$$\Re(J_{\mathcal{H}}(u_0))\int_\Omega B(u_0,p,q,\nu)\,dx=\int_\Omega \Im(x)u_0^{1-\eta(x)}dx \tag{53}$$

Let $\phi\in X$ and $\varepsilon>0$. Set $\varphi=(u_0+\varepsilon\phi)^+=\max\{0,u_0+\varepsilon\phi\}$. In the similar way (as we did in the proof of Theorem 1.1), we split Ω into the same sets $\Omega_\geq$ and $\Omega_<$. Then, replacing φ with $(u_0+\varepsilon\phi)$ in (52), using (53), and applying the similar steps, we obtain

$$\begin{aligned}
0\leq&\Re(J_{\mathcal{H}}(u_0))\int_{\Omega_\geq} B(u_0,p,q,\nu)\cdot\nabla(u_0+\varepsilon\phi)dx-\int_{\Omega_\geq}\Im(x)u_0^{-\eta(x)}(u_0+\varepsilon\phi)dx\\
=&\Re(J_{\mathcal{H}}(u_0))\left(\int_\Omega-\int_{\Omega_<}\right)B(u_0,p,q,\nu)\cdot\nabla(u_0+\varepsilon\phi)dx-\left(\int_\Omega-\int_{\Omega_<}\right)\Im(x)u_0^{-\eta(x)}(u_0+\varepsilon\phi)dx\\
\leq&\varepsilon\left[\Re(J_{\mathcal{H}}(u_0))\int_\Omega B(u_0,p,q,\nu)\cdot\nabla\phi dx-\int_\Omega \Im(x)u_0^{-\eta(x)}\phi dx\right]\\
&-\varepsilon\left[\Re(J_{\mathcal{H}}(u_0))\int_{\Omega_<} B(u_0,p,q,\nu)\cdot\nabla\phi dx+\int_{\Omega_<}\Im(x)u_0^{-\eta(x)}\phi dx\right].
\end{aligned} \tag{54}$$

Dividing (54) by ε, when $\varepsilon\to 0$, it reads

$$\Re(J_{\mathcal{H}}(u_0))\int_\Omega B(u_0,p,q,\nu)\cdot\nabla\phi dx-\int_\Omega \Im(x)u_0^{-\eta(x)}\phi dx\geq 0. \tag{55}$$

Since $\phi\in X$ is arbitrary

$$\Re(J_{\mathcal{H}}(u_0))\int_\Omega B(u_0,p,q,\nu)\cdot\nabla\phi dx=\int_\Omega \Im(x)u_0^{-\eta(x)}\phi dx, \tag{56}$$

that is, $u_0\in X$ is a positive weak solution for equation (6). As well, since $\mathcal{E}(u_0)<0$, this solution has a negative energy level.
The uniqueness of the solution follows from Lemma 3.4. □

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