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HESITANT FUZZY RELATION INEQUALITIES AND EQUATIONS OVER THE POWER SET QUANTALE OF A SEMIGROUP

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Наслов:	Неодлучне фази релацијске неједначине и једначине над кванталом подскупова полугрупе
Резиме:	Основни циљ докторске дисертације јесте развој новог алгебарског оквира за неодлучне фази скупове заснованог на кванталу подскупова полугрупе. У истраживању се користе алгебарске и теоријско-уређајне методе, конструктивни и алгоритамски приступи, као и методе апстракције и генерализације. У оквиру овог приступа карактеришу се неодлучне фази релације еквиваленције и квази-уређења, као и одговарајуће класе еквиваленције и партиције. Затим, проучавају се неодлучне фази релацијске неједнаџхине и једначине које укључују један или два непозната неодлучна фази скупа и развијају се алгоритми за одређивање њихових највећих решења. Истраживање се даље проширује на слабо линеарне системе, укључујући хомогене системе, са релацијама на истом скупу, и хетерогене системе, са релацијама између два различита скупа. На крају, развијају се ефикасне процедуре за одређивање највећих решења хомогених слабо линеарних система. Добијени резултати пружају општи теоријски оквир за проширивање класичних и фази-релацијских метода на контекст теорије неодлучних фази скупова, уз очување својства резидуације.
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Introduction

Since the seminal work of Zadeh [139] introducing fuzzy sets as a mathematical framework for modeling uncertainty, vagueness and imprecision, fuzzy set theory has evolved into a rich and diverse field with numerous extensions designed to capture different facets of uncertainty. Classical fuzzy sets describe uncertainty by assigning to each element a single membership degree in the unit interval $[0, 1]$. They have been successfully applied in a wide range of fields where knowledge is imprecise or qualitative, such as control systems, decision making, image processing and pattern recognition, artificial intelligence, expert systems, and medical diagnosis [54, 68, 83, 108, 123]. However, despite this representation has proven effective in many applications, it may be overly restrictive in situations where decision-makers are unable to specify a unique membership value due to incomplete information, cognitive limitations, or conflicting evaluations.

Over time, various generalizations of fuzzy sets have been proposed to better capture different forms of uncertainty. Among them, interval-valued fuzzy sets extend the classical fuzzy set framework by allowing membership degrees to be represented by intervals rather than single values. In the mid-1970s, Jahn [64] and Sambuc [104] independently introduced interval-valued fuzzy sets by replacing single membership degrees with intervals. Gehrke et al. [48] later provided a solid theoretical foundation for interval-valued fuzzy sets, showing how they can be naturally studied within the framework of lattice theory.

In 1976, Zadeh [141] proposed type-2 fuzzy sets, where the membership function is represented as a fuzzy set rather than a single crisp value, thus enabling a more nuanced representation of uncertainty. In the same year, Mizumoto and Tanaka [84] investigated the algebraic structures of fuzzy grades in type-2 fuzzy sets. The basic operations and fundamental concepts of type-2 fuzzy sets were further studied in [69, 80], while [81] provides an overview of type-2 fuzzy sets and systems, including their applications. Among other important approaches to representing and handling uncertainty are rough sets by Pawlak [88] and intuitionistic fuzzy sets by Atanassov [4].

The theory of fuzzy sets has provided a foundation for the development of various relational structures, including fuzzy equivalence relations and fuzzy quasi-orders, which are essential in modeling similarity, clustering, and hierarchical structures under uncertainty. Zadeh originally introduced the notions of fuzzy equivalence relations and quasi-orders in [140]. Fuzzy equivalence relations, also referred to as indistinguishability operators in the literature, have been extensively studied from both algebraic and relational perspectives [17, 98]. Bělohlávek [11] further developed these ideas and established a formal algebraic and relational framework. Later, in 2005, Bělohlávek and Vychodil [12] introduced fuzzy equational logic, providing a method for reasoning about fuzzy equality and relations. Ćirić et al. [24] conducted a comprehensive analysis of fuzzy equivalence relations and the corresponding equivalence classes and partitions, while Ignjatović et al. [63] explored the properties of fuzzy quasi-orders. More recently, Ghodousian et al. [49] investigated the structural and optimization aspects of fuzzy relation equations, including the characterization of their feasible solution sets and feasibility conditions.

In many real-world applications, especially those involving multiple experts or incomplete information, it is more appropriate to assign several possible membership values to an element rather than a single one. While classical fuzzy set theory allows gradual membership through a single membership degree, it is limited in situations where hesitation arises from multiple plausible evaluations. To address this limitation, Torra [125] introduced hesitant fuzzy sets (HFSs) as a natural generalization of fuzzy sets that explicitly accounts for hesitation by allowing multiple plausible membership values. In an HFS, the membership degree of an element is represented by a set of possible values in $[0, 1]$, called a hesitant fuzzy element (HFE), reflecting the fact that several membership degrees may be considered plausible. This approach is particularly suitable for modeling human judgments, where hesitation often arises from uncertainty, lack of confidence, or disagreement among experts. Various operations, aggregation operators, distance and similarity measures, and ranking methods for hesitant fuzzy elements have been developed [132, 134]. These theoretical advances have enabled the incorporation of hesitant fuzzy information into established methodologies, particularly in multi-criteria and group decision-making frameworks, where decision makers may hesitate among multiple possible evaluations [133]. In terms of applications, hesitant fuzzy sets have been successfully applied in a wide range of domains, including decision support systems, supplier selection, risk assessment, medical diagnosis, pattern recognition, and social network analysis (see, e.g., [46, 79, 130, 135, 138]).

This concept was further developed in subsequent works. Xu and Xia [134] in-

troduced hesitant fuzzy elements as the basic building blocks of hesitant fuzzy sets. Rodríguez et al. [100] further extended the hesitant fuzzy approach to linguistic information by introducing hesitant fuzzy linguistic term sets, which allow experts to express hesitation among several linguistic terms. Dehmiry et al. [34] extended this approach to hesitant $\mathcal{L}$-fuzzy sets by replacing the unit interval with a lattice structure. In addition, Dehmiry and Mashinchi extended hesitant fuzzy relations to hesitant $\mathcal{L}$-fuzzy relations in [33], further increasing the expressive power of this framework. More recently, Talafha et al. [120] proposed a new structure of hesitant fuzzy relations based on a hesitant fuzzy Cartesian product and a lattice-based framework, with an application to academic performance assessment. Other recent studies have explored hesitant fuzzy structures from different perspectives. For instance, sup-hesitant fuzzy ideals and p-ideals have been studied in the setting of BCK/BCI-algebras by Takallo et al. [119], while Tilahun [124] investigated hesitant fuzzy subalgebras, ideals, and congruences on autometrized algebras, as well as characteristic equations and level subsets of hesitant fuzzy sets and relations. In addition, hesitant fuzzy soft sets have been investigated through new numerical characteristics such as scored-energy, with applications to clustering and decision-making [3].

For comparison with other extensions of fuzzy set theory, it is useful to briefly highlight the main differences between fuzzy sets, HFSs, and type-2 fuzzy sets (T2FSs). Although both HFSs and T2FSs provide ways of representing uncertainty beyond a single membership degree, they do so through different mathematical structures. The main distinctions among these three types of fuzzy sets are summarized in Table 1.

Table 1: Comparison of FSs, HFSs, and T2FSs

Property	FSs	HFSs	T2FSs
Membership value	Single value in $[0,1]$	Set of values in $[0,1]$	Fuzzy set over $[0,1]$
Mathematical form	$f : X \to [0,1]$	$h : X \to \mathcal{P}([0,1])$	$A : X \to FS([0,1])$
Uncertainty	Vagueness	Hesitation	Membership uncertainty
Secondary membership	No	No	Yes
Complexity	Low	Medium	High
Example	0.7	$\{0.4, 0.6, 0.8\}$	$\{0.2/0.4, 1/0.6\}$

A central issue in the theory of hesitant fuzzy sets is the handling of hesitant information through appropriate operations on hesitant fuzzy elements. One of the foundational approaches to defining operations on HFSs is based on lifting classical fuzzy set operations to hesitant fuzzy elements [125, 126]. In this framework, binary operations such as union, intersection and complement, as well as algebraic sum and product, are defined by applying the corresponding fuzzy operation to every pair of values across the relevant hesitant fuzzy elements. This lifted operation methodology ensures that the resulting hesitant fuzzy element represents all possible outcomes of combining the original hesitations, which offers high-resolution representation of uncertainty compared to approaches that reduce hesitant information immediately via aggregation. Beyond pointwise operations, researchers have investigated the algebraic structure of hesitant fuzzy sets. Bustince et al. showed that hesitant fuzzy sets equipped with Torra's operations do not satisfy the law of absorption [18]. In [65, 92], it was shown how hesitant fuzzy sets can be endowed with a lattice structure, and even with the structure of a residuated lattice. These algebraic structures provide a theoretical basis for approximate reasoning and systematic decision-making under hesitation. While lifted operations preserve all possible membership interactions, decision-making contexts often require condensed representations of hesitant information. Aggregation operators designed for HFSs compute representative aggregated values from a hesitant set of membership degrees. These operators generalize classical aggregation methods to the hesitant context and support the synthesis of multi-attribute or multi-expert information. Another line of development involves defining ranking functions, distance measures, and similarity indices for hesitant fuzzy elements. These tools facilitate ordering and comparison between HFEs, which is essential when decision alternatives must be evaluated or clustered.

Although residuated structures have been considered in some works [92], they are typically developed under additional assumptions or restricted to specific subclasses of hesitant fuzzy sets. Recently, Merino et al. [82] proposed an order-oriented framework for hesitant fuzzy elements, showing that several classical orders fail to induce lattice structures, while the symmetric order yields scores satisfying key normative criteria for scoring functions.

Various types of lattice-valued fuzzy sets and relations have been studied in the literature. However, most existing approaches are formulated within classical residuated lattices or other fixed algebraic structures, where uncertainty is typically represented by a single membership degree. The main flow of fuzzy relation equations is that they frequently do not have solutions (meaning no solution exists for the given data). In contrast, hesitant fuzzy information involves sets of possible values,

which requires an algebraic framework that can naturally handle such multi-valued information while still supporting the study of relation inequalities and equations. For that reason, hesitant fuzzy relation equations provide a much more flexible solution space because the system can calculate a range of solutions when shifting from a single value to sets of values. This motivates the use of power set quantales as the underlying membership structure, since they provide a natural setting for extending fuzzy relational methods to the hesitant context, while preserving the residuation property needed for solving relation inequalities and equations.

In addition, previous studies on hesitant fuzzy equations have primarily focused on solving specific systems using numerical or parametric methods. For example, Babakordi and Allahviranloo [6] introduced methods for solving hesitant fuzzy systems of the form $AX = B$, defining norms of hesitant fuzzy vectors and introducing the concepts of hesitant fuzzy zero, "almost equal", "less than" and "equal" of two hesitant fuzzy numbers. The solution of the system is obtained via a minimization problem. Furthermore, Babakordi and Taghi-Nezhad [8] studied partial and half hesitant fuzzy equations and classified solution types. Further developments include fully hesitant parametric fuzzy equations [118].

While the previously mentioned contributions offer practical computational techniques, they are primarily focused on particular hesitant fuzzy equations rather than on developing a general algebraic framework for reasoning over relational structures. A challenging aspect of this area is the construction of an algebraic structure for hesitant fuzzy sets that is both residuated and supports efficient solving of equations using residuals. Such a structure would generalize classical fuzzy logic while preserving the expressive power of hesitant evaluations.

This dissertation develops a novel framework for hesitant fuzzy sets and relations based on power set quantales induced by semigroups. The proposed framework provides a residuated algebraic setting for the systematic study of hesitant fuzzy relational structures, with particular emphasis on residuation and relational properties. It allows hesitant fuzzy relation inequalities and equations to be studied within a unified quantale-based framework, without requiring an additional specialization of hesitant fuzzy sets to obtain a residuated algebraic structure.

We begin with a semigroup $\langle S, * \rangle$, where $*$ denotes a binary associative operation on S. From this semigroup, we construct its power set $\mathcal{P}(S)$, and define a binary operation on subsets $A, B \subseteq S$ by

$$A \otimes B = \{a * b \mid a \in A \text{ and } b \in B\}.$$

This construction yields an algebraic structure called a *quantale*. We refer to this

structure as the *power set quantale of a semigroup* S, and we develop the theory of hesitant fuzzy sets and hesitant fuzzy relations over such quantales.

Although inspired by fuzzy set theory, this approach is not merely a replacement of the underlying space of membership values by a power set structure. In the present framework, hesitant fuzzy sets assign to each element a value from the power set quantale of a semigroup S. This leads to a distinct algebraic setting in which the power set quantale operations are induced by the set-theoretic structure and the operation of the underlying semigroup. One of the advantages of this setting is that the join operation is given by set-theoretic union, so that all membership values contained in the operands are preserved. This may result in a larger number of membership values to be handled, but ensures that no membership information is lost.

The resulting quantale-based framework provides a natural setting for residuation and the study of hesitant fuzzy relational structures, while laying the foundation for investigations of relation inequalities and equations. In particular, the residuation property enables the characterization and computation of greatest solutions of such inequalities and equations.

While quantale-based approaches and hesitant fuzzy sets have been investigated independently throughout the literature, their combination in the form considered in this dissertation has not been previously addressed. Within this setting, we introduce and study hesitant fuzzy equivalence relations and hesitant fuzzy quasi-orders and investigate their connections with hesitant fuzzy partitions and the corresponding classes of equivalence. These results extend the theory of hesitant fuzzy relations to a broader algebraic setting and open new directions for research on uncertainty modeling in non-standard logical frameworks.

The results of the dissertation provide an algebraic framework for modeling relations in situations where disagreement among experts, measurement variability, or insufficient information may lead to many degrees of similarity. In such settings, hesitant fuzzy relations allow similarity evaluations to be represented as sets of possible value. The characterization of hesitant fuzzy equivalence offers a mathematical foundation for creating hesitant fuzzy partitions, which may be treated as clusters of related items. This paradigm is effective in making similarity evaluations in uncertain fields including medical diagnosis, document classification, and image recognition. Another important applications include multi-expert decision making. Many decision-making processes involve specialists evaluating alternatives based on several factors. Their assessments often yield many evaluation values rather than a single score. Hesitant fuzzy relations can model evaluations without premature

aggregation. Using hesitant fuzzy equivalence relations, experts can detect indistinguishable alternatives, whereas hesitant fuzzy quasi-orders can reflect preference or dominance relations. Such structures are crucial in multi-criteria decision analysis, where options must be ranked or classified despite evaluation uncertainty.

The main aim of this dissertation is to introduce and analyze a new algebraic framework for hesitant fuzzy sets based on the power set quantale of a semigroup. Within this framework, we develop a general theoretical setting that enables the extension of classical and fuzzy relational methods within the HFS theory, while preserving the residuation property.

The dissertation is organized into five chapters. Chapter 1 presents the basic concepts and results used throughout the dissertation. It begins with fundamental notions concerning sets, binary relations, and functions, followed by an introduction to semigroups, monoids, and semirings. Basic concepts related to ordered sets and lattices are also presented. Particular attention is given to quantales and related algebraic structures, especially commutative quantales and power set quantales, which provide the algebraic foundation for the framework developed in this dissertation.

Chapter 2 is devoted to hesitant fuzzy sets and hesitant fuzzy relations over the power set quantale of a semigroup. Hesitant fuzzy sets are introduced in this algebraic setting, together with additional operations and properties that are used throughout the dissertation. The chapter then develops the corresponding theory of hesitant fuzzy relations, including relations with specific properties and residuals on hesitant fuzzy relations. Several additional properties of the residuals are also established, providing the necessary foundation for the study of hesitant fuzzy relational systems in the subsequent chapters.

In Chapter 3, special classes of hesitant fuzzy relations over the power set quantale of a semigroup are studied. Hesitant fuzzy equivalences and their equivalence classes are considered, together with hesitant fuzzy partitions and semi-partitions. The chapter also investigates hesitant fuzzy quasi-orders and the corresponding equivalence relations induced by them, including related constructions such as aftersets and foresets. Several illustrative applications of these classes of hesitant fuzzy relations are presented. Finally, uniform hesitant fuzzy relations are introduced and their fundamental properties are examined.

Chapter 4 includes the study of systems of hesitant fuzzy relation inequalities and equations, with one side containing the composition of an unknown hesitant fuzzy set and given hesitant fuzzy relations, while the other side contains either the same unknown hesitant fuzzy set or another one. In particular, our research includes the characterization of solution sets of these systems and the development

of algorithms for computing their greatest solutions. Moreover, the solution sets are characterized in terms of closure and opening systems on the quantale of hesitant fuzzy sets, together with the corresponding closure and opening operators.

Chapter 5 extends the investigation to more general nonlinear systems of hesitant fuzzy relation inequalities and equations, referred to as hesitant weakly linear systems. Two types of such systems are considered within the proposed framework: homotypical systems, involving hesitant fuzzy relations on a single set, and heterotypical systems, involving hesitant fuzzy relations on two possibly different sets. The chapter also establishes effective procedures for computing greatest solutions of homotypical hesitant weakly linear systems.

Chapter 1

Basic concepts and results

This chapter presents the basic concepts and results that provide the algebraic and order-theoretic background for the dissertation. It recalls the necessary notions and properties of the algebraic structures considered in this thesis, together with several known results that will be used in the subsequent chapters.

The chapter is organized into four sections. In Section 1.1, we recall some basic notions and notation concerning sets, binary relations, and functions that will be used throughout the dissertation. Section 1.2 introduces the basic algebraic structures of semigroups, monoids, and semirings, together with the notation and properties needed in the subsequent chapters. In Section 1.3, we recall fundamental notions concerning ordered sets and lattices, including the order-theoretic and lattice-theoretic structures that will be used in the development of the theory.

Section 1.4 is devoted to quantales and related structures. We first recall some standard properties of commutative quantales. We then present the construction of power set quantales associated with semigroups and give several examples illustrating this construction, including examples based on different underlying semigroups. Finally, we recall residuated lattices and related structures, which provide an important algebraic framework for fuzzy logic.

1.1 Sets, binary relations and functions

Throughout this dissertation, we use the standard notions and notation of set theory and number theory. The terminology in this section is based on the works of Bělohávek [11], Bělohávek and Vychodil [12], and Clark [22].

We use the notion of a set in the sense of the intuitive set theory, that is, as a collection of objects called elements or members of the set. Every set is naturally equipped with the identity relation $=$, which provides basic information about the

equality of its elements. The symbols $=$, $\subseteq$, $\cap$, and $\cup$, denote, respectively, equality, inclusion, intersection, and union. The set of natural numbers excluding zero is denoted by $\mathbb{N}$, while $\mathbb{R}$ denotes the set of real numbers. We also use the notation $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

A family of sets indexed by a set I is denoted by $\{A_i\}_{i\in I}$ or, equivalently, by $\{A_i \mid i \in I\}$. If I is finite, we usually write $A_1, A_2, \ldots, A_n$ or $\{A_i\}_{i=1}^n$.

Definition 1.1. Let $\{A_i\}_{i\in I}$ be a family of non-empty sets, where I is not necessarily finite. The *Cartesian product* (*direct product*) of $\{A_i\}_{i\in I}$ is defined by,

$$\prod_{i\in I} A_i = \{a : I \to \bigcup_{i\in I} A_i \mid a(i) \in A_i \text{ for all } i \in I\}. \tag{1.1}$$

Thus, an element of the Cartesian product can be viewed as a tuple whose i-th component belongs to A_i. For simplicity, an element $a \in \prod_{i\in I} A_i$ will be denoted by $(a_i)_{i\in I}$, or simply by (a_i) when the index set is clear. Here, a_i denotes the *i-th coordinate* of a.

If $I = \{1, \ldots, n\}$ is a finite set, then the Cartesian product is called an n-ary Cartesian product and is usually written as

$$\prod_{i\in I} A_i = A_1 \times A_2 \times \cdots \times A_n = \{(a_1, \ldots, a_n) \mid a_i \in A_i,\ i = 1, \ldots, n\}.$$

In this case, each element $(a_1, \ldots, a_n)$ is called an ordered *n-tuple*. For $n = 2$, it is referred to as an *ordered pair*.

If $A_i = A$ for every $i \in I$, then the Cartesian product $\prod_{i\in I} A_i$ is denoted by A^I and is called the *Cartesian power* of A. In particular, if I has n elements, we write A^n instead of $\prod_{i\in I} A_i$. By convention, we define $A^0 = \{\emptyset\}$.

Definition 1.2. Let A and B be non-empty sets. A *binary relation from A to B* is defined as a subset of the Cartesian product $A \times B$.

The operations of equality, inclusion, union, and intersection of binary relations from A to B are defined in the same way as for subsets of $A \times B$.

In the case when $A_1 = A_2 = A$, a binary relation from A to A is simply called a *binary relation on A*. For a binary relation R on A, we write aRb whenever $(a, b) \in R$. The following special types of binary relations on A are frequently used.

- the *empty relation*, usually denoted by $\emptyset$;
- the *identity relation* $\Delta_A = \{(a, a) \mid a \in A\}$;
- the *universal relation* $\nabla_A = \{(a, b) \mid a, b \in A\}$.

Let A and B be non-empty sets and let R be a binary relation from A to B. The *inverse relation* of R, denoted by R^{-1}, is the binary relation R^{-1} from B to A defined by

$$R^{-1} = \{(b, a) \in B \times A \mid (a, b) \in R\}.$$

The set of all first components of the ordered pairs belonging to a binary relation R is called the *domain* of R, while the set of all second components is called the *image* (or *range* or *codomain*) of R. These sets are defined by

$$\begin{aligned} \mathrm{Dom}(R) &= \{a \in A \mid (\exists b \in B)\,(a, b) \in R\}, \\ \mathrm{Im}(R) &= \{b \in B \mid (\exists a \in A)\,(a, b) \in R\}. \end{aligned}$$

For the inverse relation, we have $\mathrm{Dom}(R^{-1}) = \mathrm{Im}(R)$ and $\mathrm{Im}(R^{-1}) = \mathrm{Dom}(R)$.

Definition 1.3. Let A, B, C be non-empty sets, and let $R \subseteq A \times B$ and $T \subseteq B \times C$ be binary relations. The composition $R \circ T \subseteq A \times C$ of R and T is defined by

$$R \circ T = \{(a, c) \in A \times C \mid (\exists b \in B)\ ((a, b) \in R \text{ and } (b, c) \in T)\}. \tag{1.2}$$

We next recall some basic notions concerning functions.

Definition 1.4. Let A and B be non-empty sets. A *function* (or *mapping*) from A to B is a binary relation $f \subseteq A \times B$ such that:

(1) for every $a \in A$ there exists $b \in B$ with $(a, b) \in f$,

(2) if $(a, b_1) \in f$ and $(a, b_2) \in f$, then $b_1 = b_2$.

If f is a function from A to B, we use the notation $f : A \to B$ instead of $f \subseteq A \times B$, and write $f(x) = y$ instead of $(x, y) \in f$.

Definition 1.5. A function $f : A \to B$ is called *injective* if $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$ for all $a_1, a_2 \in A$. It is called *surjective* if for every $b \in B$ there exists $a \in A$ such that $b = f(a)$. A function is called *bijective* if it is both injective and surjective.

Definition 1.6. Given functions $f : A \to B$ and $g : B \to C$, their composition is the function $g \circ f : A \to C$ defined by $(g \circ f)(a) = g(f(a))$ for all $a \in A$.

We now recall some basic properties of binary relations. Let A be a non-empty set and let R be a binary relation on A. The relation R is said to be:

- *reflexive* if $(a, a) \in R$ for every $a \in A$, equivalently, if $\Delta_A \subseteq R$,

- *symmetric* if $(a, b) \in R$ implies $(b, a) \in R$ for all $a, b \in A$, that is, if $R^{-1} \subseteq R$,
- *antisymmetric* if $(a, b) \in R$ and $(b, a) \in R$ together imply $a = b$ for all $a, b \in A$, equivalently, if $R \cap R^{-1} \subseteq \Delta_A$,
- *transitive* if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$, that is, if $R \circ R \subseteq R$.

A binary relation R on a non-empty set A is called a *quasi-order* if it is reflexive and transitive. It is called a *partial order* (or simply an *order*) if it is reflexive, antisymmetric, and transitive, and an *equivalence relation* if it is reflexive, symmetric, and transitive. The term quasi-order is also commonly referred to as *preorder* in the literature.

For a binary relation R on A and $a \in A$, we define the following subsets of A:

$$aR = \{b \in A \mid (a, b) \in R\}, \quad Ra = \{b \in A \mid (b, a) \in R\}.$$

Definition 1.7. Let A be a non-empty set and let E be an equivalence relation on A. For each $a \in A$, the set $E_a = \{b \in A \mid (a, b) \in E\}$ is called the *equivalence class* of a with respect to E. The collection of all equivalence classes, $A/E = \{E_a \mid a \in A\}$ is called the *factor set* (or *quotient set*) of A.

Equivalence relations on a set induce partitions of that set. We recall the relevant definitions and properties below.

Definition 1.8. Let A be a non-empty set and let $\{A_i\}_{i \in I}$ be a family of subsets of A. The family $\{A_i\}_{i \in I}$ is called a *semi-partition* of A if $A_i \neq \emptyset$ and $A_i \cap A_j = \emptyset$ for all $i, j \in I$ with $i \neq j$. It is called a *cover* of A if $\bigcup_{i \in I} A_i = A$. A semi-partition of A that is also a cover of A is called a *partition* of A.

Let E be an equivalence relation on a non-empty set A. Observe that, every element of A belongs to exactly one equivalence class of E. Let $\alpha : A \to A/E$ be the function defined by $\alpha(a) = R_a$, for all $a \in A$. This function is known as the *natural map* (or the *canonical map*) from A to A/E. Furthermore, the mapping α can also be regarded as a binary relation between A and A/E, given by

$$\alpha(a, E_b) = E(a, b), \quad \text{for all } a, b \in A.$$

Definition 1.9. Let A and B be non-empty sets and let $\alpha : A \to B$ be any function. Then $\ker(\alpha) = \{(a, b) \in A^2 \mid f(a) = f(b)\}$ is called the *kernel* of α.

It is straightforward to verify that $\ker(\alpha)$ is an equivalence relation on A.

1.2 Semigroups, monoids and semirings

In this section, we recall the basic concepts and properties of semigroups and monoids that will be used throughout the dissertation. The presentation follows the standard terminology and notation from the literature. The main references used in this section are Bogdanović and Ćirić [16], Gondran and Minoux [53], and Golan [52].

Let S be a non-empty set and let $n \in \mathbb{N}$. Any mapping $f : S^n \to S$ is called an *n-ary operation* on S. In the special case when $n = 2$, the operation f is called a *binary operation* on S, while for $n = 1$ it is called a *unary operation* on S.

A binary operation $*$ on S is said to be:

- *associative* if it satisfies the *associative law*,

$$(a * b) * c = a * (b * c) \quad \text{for all } a, b, c \in S$$

- *commutative* if it satisfies the *commutative law*,

$$a * b = b * a \quad \text{for all } a, b \in S,$$

- *idempotent* if it satisfies the *idempotent law*,

$$a * a = a \quad \text{for all } a \in S.$$

Definition 1.10. An algebra $\langle S, * \rangle$ is called a *semigroup* if S is a non-empty set and $* : S \times S \to S$ is an associative binary operation on S.

A semigroup is said to be *commutative* if its binary operation is commutative, and *idempotent* if its binary operation is idempotent.

Let $\langle S, * \rangle$ be a semigroup. An element $1_* \in S$ is called a *left identity* if $1_* * a = a$ for every $a \in S$, and it is called a *right identity* if $a * 1_* = a$ for every $a \in S$. An element that is both a left and a right identity is called an *identity element* of S. A semigroup has an *identity element* if and only if it has both a left identity and a right identity. Moreover, whenever an identity element exists, it is unique.

Definition 1.11. An algebra $\langle S, *, 1_* \rangle$ is called a *monoid* if

- $\langle S, * \rangle$ is a semigroup,
- 1_* is the identity element of S, that is, $1_* * a = a * 1_* = a$ for all $a \in S$.

A monoid $\langle S, *, 1_* \rangle$ is called *commutative* if the semigroup $\langle S, * \rangle$ is commutative, and *idempotent* if the semigroup $\langle S, * \rangle$ is idempotent.

Definition 1.12. Let $\langle S, *, 1_* \rangle$ be a monoid. A subset $H \subseteq S$ is called a *submonoid* of $\langle S, *, 1_* \rangle$ if

- $1_* \in H$,
- $a * b \in H$ for all $a, b \in H$.

Notice that every submonoid of a monoid is itself a monoid. For a subset $H \subseteq S$, the *submonoid generated by* H, denoted by $\langle H \rangle$, is the smallest submonoid of S containing H. Equivalently, it is the intersection of all submonoids of S containing H.

Definition 1.13. A submonoid of S is called *finitely generated* if it is generated by a finite subset of S. A monoid $\langle S, *, 1_* \rangle$ is called *locally finite* if every finitely generated submonoid of S is finite.

Clearly, every commutative and idempotent monoid is locally finite. For more details on finitely generated commutative monoids, see [102].

Definition 1.14. An algebra $\langle S, +, *, 0, 1_* \rangle$ is called a *semiring* if the following conditions are satisfied:

(S1) $\langle S, +, 0 \rangle$ is a commutative monoid with identity element 0,

(S2) $\langle S, *, 1_* \rangle$ is a monoid with identity element 1_*,

(S3) the operations $+$ and $*$ satisfy the *left* and *right distributive laws*, that is, for all $a, b, c \in S$,

$$a * (b + c) = a * b + a * c, \quad (a + b) * c = a * c + b * c,$$

(S4) 0 is an *absorbing element*, that is, for all $a \in S$

$$a * 0 = 0 * a = 0.$$

A semiring S is called *without zero divisors* if $x * y = 0$ implies $x = 0$ or $y = 0$, for all $x, y \in S$. A semiring S is called *commutative* if $\langle S, *, 1_* \rangle$ is a commutative monoid, and *additively idempotent* if $\langle S, +, 0 \rangle$ is an idempotent monoid.

The most familiar example of a semiring is the semiring $\langle \mathbb{N}_0, +, \cdot, 0, 1 \rangle$ of nonnegative integers with the usual addition and multiplication. Clearly, every ring with identity, as well as every field, is a semiring. In particular, the integers, rational numbers, real numbers, and complex numbers form semirings with respect to the usual addition and multiplication.

Definition 1.15. Let $\mathcal{S} = \langle S, +, *, 0_S, 1_* \rangle$ be a semiring. A *left $\mathcal{S}$-semimodule* is a commutative monoid $\langle M, \oplus, 0_M \rangle$ equipped with an external multiplication $S \times M \to M$, denoted by $(\lambda, x) \mapsto \lambda x$ and called *left scalar multiplication*, satisfying the following axioms for all $\lambda, \mu \in S$ and all $a, b \in M$:

$$(\lambda * \mu)a = \lambda(\mu a), \tag{1.3}$$
$$\lambda(a \oplus b) = \lambda a \oplus \lambda b, \tag{1.4}$$
$$(\lambda + \mu)a = \lambda a \oplus \mu a, \tag{1.5}$$
$$1_* a = a, \tag{1.6}$$
$$\lambda 0_M = 0_S a = 0_M. \tag{1.7}$$

A *right $\mathcal{S}$-semimodule* is defined analogously, with an external multiplication $M \times S \to M$, denoted by $(x, \lambda) \mapsto x\lambda$ and called *right scalar multiplication*, satisfying the dual versions of conditions (1.3)–(1.7).

An *$\mathcal{S}$-bisemimodule* is a commutative monoid $\langle M, \oplus, 0_M \rangle$ that is simultaneously a left and a right $\mathcal{S}$-semimodule, and for all $\lambda, \mu \in S$ and $a \in M$,

$$(\lambda a)\mu = \lambda(\mu a). \tag{1.8}$$

If the left and right scalar multiplications coincide, i.e., if $\lambda x = x\lambda$, for all $\lambda \in S$ and $a \in M$, then $\langle M, \oplus, 0_M \rangle$ is a *strong $\mathcal{S}$-bisemimodule*.

1.3 Ordered sets and lattices

In this section, we recall the basic notions and results concerning ordered sets, ordered algebraic structures, lattices, and fixed points that will be used throughout the dissertation. The terminology and notation are based on the standard references Birkhoff [14], Blyth [15], Davey and Priestley [29], and Roman [101].

Definition 1.16. Let A be a non-empty set. A pair $\langle A, \leqslant \rangle$, where $\leqslant$ is a partial order on A, is called a *partially ordered set* (or, briefly, an *ordered set* or a *poset*).

Throughout this section, the symbol $\leqslant$ denotes a partial order on a set A. For simplicity, we shall often refer to A as an ordered set, with the understanding that the underlying order relation is denoted by $\leqslant$. The relation $<$ induced by $\leqslant$ is defined by

$$a < b \quad \text{if and only if} \quad a \leqslant b \text{ and } a \neq b.$$

A partial order $\leqslant$ on A is called *linear* if, for all $a, b \in A$, either $a \leqslant b$ or $b \leqslant a$. In this case, A is called a *linearly ordered set* or a *chain*.

The inverse relations of $\leqslant$ and $<$ are denoted by $\geqslant$ and $>$, respectively. The ordered set $\langle A, \geqslant \rangle$ is called the *dual ordered set* of $\langle A, \leqslant \rangle$.

Principle of Duality. Every statement concerning an ordered set has a corresponding dual statement obtained by replacing the order relation $\leqslant$ with the dual order $\geqslant$. Consequently, every theorem valid for an ordered set remains valid for its dual ordered set after this replacement.

In the previous section, we recalled the notions of semigroups and monoids. We now introduce their ordered counterparts.

Definition 1.17. A (partially) *ordered semigroup* is a structure $\langle S, *, \leqslant \rangle$, where $\langle S, * \rangle$ is a semigroup and $\langle S, \leqslant \rangle$ is a (partially) ordered set, such that the order is compatible with multiplication, i.e., for all $a, b, c \in S$,

$$a \leqslant b \quad \Rightarrow \quad c * a \leqslant c * b \quad \text{and} \quad a * c \leqslant b * c.$$

If S additionally contains an identity element 1_*, then the structure $\langle S, *, 1_*, \leqslant \rangle$ is called an (ordered) *monoid*.

We next recall some basic types of mappings on ordered sets that will be used in the sequel.

Definition 1.18. Let $\langle A, \leqslant_1 \rangle$ and $\langle B, \leqslant_2 \rangle$ be ordered sets and let $\varphi : A \to B$. The function φ is called

- *isotone* (or *order-preserving*) if $a \leqslant_1 b$ implies $\varphi(a) \leqslant_2 \varphi(b)$,
- *antitone* if $a \leqslant_1 b$ implies $\varphi(a) \geqslant_2 \varphi(b)$,

for all $a, b \in A$. A bijection $\varphi : A \to B$ is called an *order isomorphism* if both φ and its inverse φ^{-1} are isotone.

Definition 1.19. Let $\langle A, \leqslant \rangle$ be an ordered set and let $\varphi : A \to A$. The function φ is said to be:

- *extensive* if $a \leqslant \varphi(a)$,
- *intensive* if $\varphi(a) \leqslant a$,
- *idempotent* if $\varphi(\varphi(a)) = \varphi(a)$,

for all $a \in A$.

An isotone, extensive, and idempotent function φ is called a *closure operator* on $\langle A, \leqslant \rangle$, and an element $a \in A$ satisfying $\varphi(a) = a$ is called φ-*closed*. Dually, an isotone, intensive, and idempotent function φ is called an *opening operator*, and an element $a \in A$ satisfying $\varphi(a) = a$ is called φ-*open element*.

Definition 1.20. Let E be an arbitrary set. a subset $\mathcal{C}$ of $\langle \mathcal{P}(E), \subseteq \rangle$ is called

- a *closure system* on E if it is closed under arbitrary intersections and contains E,
- an *opening system* (or *dual closure system*) if it is closed under arbitrary unions and contains $\emptyset$.

When φ is a closure operator on $\mathcal{P}(E)$, the φ-closed elements are called the *closed sets* of φ. Similarly, when φ is an opening operator on $\mathcal{P}(E)$, the φ-open elements are called the *open sets* of φ. For more details on closure and opening operators and systems, see [19].

We now turn to some basic lattice-theoretic notions.

Definition 1.21. Let $\langle A, \leqslant \rangle$ be an ordered set and let $a \in A$. For every $x \in A$, the element a is called

- *minimal* if $x \leqslant a$ implies $x = a$,
- the *least element* of A if $a \leqslant x$,
- *maximal* if $a \leqslant x$ implies $x = a$,
- the *greatest element* of A if $x \leqslant a$.

Observe that the notions of a *maximal element* and the *greatest element* are defined by duality.

Definition 1.22. Let M be a non-empty subset of an ordered set $\langle A, \leqslant \rangle$. An element $a \in A$ is called

- a *lower bound* of M if $a \leqslant x$ for every $x \in M$,
- the *greatest lower bound* or the *infimum* of M if a is the greatest element among all lower bounds of M. Equivalently, a is a lower bound of M and every lower bound b of M satisfies $b \leqslant a$,
- an *upper bound* of M if $x \leqslant a$ for every $x \in M$,
- the *least upper bound* or the *supremum* of M if a is the least element among all upper bounds of M. Equivalently, a is an upper bound of M and every upper bound b of M satisfies $a \leqslant b$.

Whenever they exist, the supremum and the infimum of a set M are denoted by $\bigvee M$ and $\bigwedge M$, respectively. If $M = \{a_i\}_{i \in I}$, we also use the notation

$$\bigvee_{i \in I} a_i \quad \text{and} \quad \bigwedge_{i \in I} a_i,$$

for the supremum and the infimum of M, respectively.

Definition 1.23. An ordered set $\langle L, \leqslant \rangle$ is called a *join semilattice* if every pair of elements has a supremum. Dually, it is called a *meet semilattice* if every pair of elements has an infimum.

The supremum and infimum of a pair of elements $a, b \in L$ are denoted by $a \vee b$ and $a \wedge b$, respectively. The corresponding binary operations,

$$\vee : L \times L \to L \quad \text{and} \quad \wedge : L \times L \to L,$$

are called the *join* and *meet* operations, respectively.

Remark 1.1. Every finite subset of a join semilattice has a supremum, and every finite subset of a meet semilattice has an infimum. This property does not necessarily extend to infinite subsets.

Definition 1.24. A join semilattice (respectively, a meet semilattice) is called *complete* if every subset of L has a supremum (respectively, an infimum).

Definition 1.25. An ordered set $\langle L, \leqslant \rangle$ is called a *lattice* if it is both a join semilattice and a meet semilattice. Equivalently, every pair of elements of L has both a supremum and an infimum.

Equivalently, a lattice can be characterized algebraically as follows.

Theorem 1.1. *Let L be a lattice. Then $\langle L, \vee, \wedge \rangle$ is an algebraic structure with two binary operations such that, for all $a, b, c \in L$, the following identities hold:*

(L1) $a \wedge a = a, \quad a \vee a = a$ (idempotency)

(L2) $a \wedge b = b \wedge a, \quad a \vee b = b \vee a$ (commutativity)

(L3) $(a \wedge b) \wedge c = a \wedge (b \wedge c), \quad (a \vee b) \vee c = a \vee (b \vee c)$ (associativity)

(L4) $a \wedge (a \vee b) = a, \quad a \vee (a \wedge b) = a$ (absorption)

Conversely, if L is an algebraic structure with two binary operations $\vee$ and $\wedge$ satisfying conditions (L1)-(L4)*, then L is a lattice with respect to the partial order $\leq$ defined by*

$$a \leq b \Leftrightarrow a \vee b = b \quad (\textit{or, equivalently, } a \leq b \Leftrightarrow a \wedge b = a).$$

In what follows, we identify a lattice $\langle L, \vee, \wedge \rangle$ with its underlying set L whenever no confusion can arise.

Definition 1.26. Let L be a lattice. A non-empty subset $I \subseteq L$ is called an *ideal* if the following conditions hold:

(1) if $a \in I$ and $x \leq a$, then $x \in I$,

(2) if $a, b \in I$, then $a \vee b \in I$.

Dually, a non-empty subset $D \subseteq L$ is called a *dual ideal* (or *filter*) whenever:

(1) if $a \in D$ and $a \leq x$, then $x \in D$,

(2) if $a, b \in D$, then $a \wedge b \in D$.

The ideal generated by an element $x \in L$, called the *principal ideal*, is defined by

$$\downarrow x = \{a \in L \mid a \leq x\}.$$

Dually, the *principal dual ideal* (or *principal filter*) generated by x is

$$\uparrow x = \{a \in L \mid x \leq a\}.$$

The least element of a lattice L, if it exists, is denoted by 0, while the greatest element, if it exists, is denoted by 1. A lattice containing both elements 0 and 1 is called a *bounded lattice*. Every complete lattice is bounded. A subset $K \subseteq L$ of a complete lattice L is said to be a *complete sublattice* of L if, for every non-empty subset of K, the corresponding infimum and supremum in L are elements of K.

A particularly important class of lattices consists of *distributive lattices*. A lattice L is called distributive if, for all $a, b, c \in L$, the following identities hold:

(L5) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

(L5') $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Definition 1.27. Let L be a bounded lattice with least element 0 and greatest element 1. An element $b \in L$ is said to be a *complement* of $a \in L$ if

$$a \wedge b = 0 \quad \text{and} \quad a \vee b = 1.$$

The complement relation is symmetric, that is, if b is a complement of a, then a is also a complement of b. Moreover, if L is a distributive lattice, then every element of L has at most one complement. The complement of an element $a \in L$, whenever it exists, is denoted by a'.

A bounded distributive lattice in which every element has a complement is called a *Boolean algebra*. The mapping $' : L \to L$, defined by $a \mapsto a'$, is a unary operation on L, called the *complement operation*. Consequently, a Boolean algebra may be regarded as the algebraic structure $(L, \wedge, \vee, ', 0, 1)$, consisting of the binary operations $\wedge$ and $\vee$, the unary operation $'$, and the constants 0 and 1. For further details on Boolean algebras, see [51].

In what follows, we present some important examples of lattices.

Example 1.1. Let X be a non-empty set. The power set $\mathcal{P}(X)$ equipped with the inclusion relation $\subseteq$ is a lattice, where the meet and join operations are given by

$$A \wedge B = A \cap B, \qquad A \vee B = A \cup B,$$

for all $A, B \in \mathcal{P}(X)$. This lattice is bounded, with the least and greatest elements given by $0 = \emptyset$ and $1 = X$. Since the meet and join operations satisfy the distributive laws (L5) and (L5'), and every element $A \in \mathcal{P}(X)$ has a complement $A' = X \setminus A$, the lattice $\mathcal{P}(X)$ is a Boolean algebra.

Example 1.2. Let A be a non-empty set, and let $\mathcal{E}(A)$ denote the set of all equivalence relations on A. Equipped with the inclusion relation, $\mathcal{E}(A)$ forms a complete lattice.

Let $\{R_i \mid i \in I\} \subseteq \mathcal{E}(A)$ be an arbitrary family of equivalence relations. The infimum of this family is given by their intersection:

$$\bigwedge_{i \in I} R_i = \bigcap_{i \in I} R_i.$$

Indeed, the intersection of equivalence relations is again an equivalence relation.

The supremum cannot, in general, be obtained by taking the union of the relations, since the union of equivalence relations need not be transitive. Instead, the supremum is given by the smallest equivalence relation containing their union, that is, by the *transitive closure* of the union:

$$\bigvee_{i \in I} R_i = \left(\bigcup_{i \in I} R_i \right)^* .$$

More explicitly, for $a, b \in A$, we have $(a, b) \in \bigvee_{i \in I} R_i$ if and only if there exist elements $c_1, \ldots, c_n \in A$ and indices $i_1, \ldots, i_{n+1} \in I$ such that

$$a \, R_{i_1} \, c_1 \, R_{i_2} \, c_2 \cdots R_{i_n} \, c_n \, R_{i_{n+1}} \, b.$$

Thus, the join of a family of equivalence relations is obtained by adding the necessary transitive connections to their union.

We finally recall some fundamental notions concerning fixed points of functions on ordered sets.

Definition 1.28. Let L be an ordered set and let $f : L \to L$ be a mapping. An element $x \in L$ is called:

- a *fixed point* of f if $f(x) = x$,
- a *pre-fixed* point of f if $f(x) \leq x$,
- a *post-fixed* point of f if $x \leq f(x)$.

The sets of fixed, pre-fixed, and post-fixed points of f are denoted by $\mathrm{Fix}(f)$, $\mathrm{Pre}(f)$, and $\mathrm{Post}(f)$, respectively. If these sets have least and greatest elements, they are denoted by $\mathrm{MinFix}(f)$, $\mathrm{MinPre}(f)$, $\mathrm{MinPost}(f)$ and $\mathrm{MaxFix}(f)$, $\mathrm{MaxPre}(f)$, $\mathrm{MaxPost}(f)$, respectively.

The existence of fixed points for isotone mappings on complete lattices is guaranteed by the following result.

Theorem 1.2. *Let L be a complete lattice and let $f : L \to L$ be an isotone mapping. Then f has a fixed point.*

A stronger result is provided by Knaster-Tarski fixed point theorem.

Theorem 1.3 (Knaster-Tarski)**.** *Let L be a complete lattice and let $f : L \to L$ be an isotone mapping. Then $\mathrm{Fix}(f)$ is a complete lattice. Moreover,*

$$\mathrm{MaxFix}(f) = \mathrm{MaxPost}(f) = \bigvee \mathrm{Post}(f), \tag{1.9}$$

$$\mathrm{MinFix}(f) = \mathrm{MinPre}(f) = \bigwedge \mathrm{Pre}(f). \tag{1.10}$$

Consequently, each of the sets $\mathrm{Fix}(f)$, $\mathrm{Pre}(f)$, and $\mathrm{Post}(f)$ is a complete lattice. For further details on the Knaster-Tarski fixed point theorem and its applications, we refer to [29, 101].

1.4 Quantales and related structures

The results of this dissertation are established within the framework of quantales and their associated algebraic structures. In particular, the power set quantale of a semigroup serves as the underlying algebraic structure throughout the dissertation.

Therefore, this section recalls the basic definitions and properties of quantales and several related algebraic structures that will be be used in the subsequent chapters.

The notion of a quantale was introduced by Mulvey [85] and later extensively studied by Rosenthal [103] (see Galatos et al. [47]). Since then, quantales have been studied from various algebraic and categorical perspectives, with numerous generalizations and extensions being developed [72, 137, 142]. They have found applications in a wide range of areas, including the semantics of linear logic developed by Girard [50]. For some representative examples of applications in other areas, we refer the reader to Abramsky and Vickers [1], Resende [99], and Yetter [136]. The basic definitions and properties presented in this section are based on the works of Roman [101], Rosenthal [103], Galatos et al. [47], and Šlesinger [117].

Definition 1.29. A *quantale* is an algebra $\mathcal{Q} = \langle Q, \vee, \wedge, \otimes, 0, 1\rangle$ satisfying the following conditions:

(Q1) $\langle Q, \vee, \wedge, 0, 1\rangle$ is a complete lattice with the least element 0 and the greatest element 1,

(Q2) $\langle Q, \otimes\rangle$ is a semigroup, which does not have to be commutative,

(Q3) for every $a \in Q$ and every family $\{b_i\}_{i\in I} \subseteq Q$ the following infinite distributive laws hold:

$$a \otimes \Big(\bigvee_{i\in I} b_i\Big) = \bigvee_{i\in I}(a \otimes b_i), \qquad \Big(\bigvee_{i\in I} b_i\Big) \otimes a = \bigvee_{i\in I}(b_i \otimes a). \tag{1.11}$$

The least element 0 is the *zero* of the semigroup $\langle Q, \otimes\rangle$, i.e., $a \otimes 0 = 0 \otimes a = 0$ for every $a \in Q$. The binary operation $\otimes$ is called the multiplication. If $\otimes$ is commutative on Q, then $\mathcal{Q}$ is called a *commutative quantale*. If the semigroup $\langle Q, \otimes\rangle$ has an identity element, that is, if there exists an element $1_\otimes \in Q$ such that $\langle Q, \otimes, 1_\otimes\rangle$ is a monoid, then $\mathcal{Q}$ is called a *unital quantale* and $1_\otimes$ is called the *multiplicative identity* of $\mathcal{Q}$.

A quantale $\mathcal{Q}$ is called *integral* if its multiplicative identity coincides with the greatest element of the underlying complete lattice, that is, $1_\otimes = 1$. Under the standard terminology, a commutative integral quantale is precisely a *complete residuated lattice*.

If $\mathcal{Q} = \langle Q, \vee, \wedge, \otimes, 0, 1\rangle$ is a unital quantale with the multiplicative identity $1_\otimes$, then $\mathcal{Q}^* = \langle Q, \vee, \otimes, 0, 1_\otimes\rangle$ is a semiring, called the *semiring reduct* of $\mathcal{Q}$. A detailed study of algebras arising from Łukasiewicz's logic and their corresponding semiring reducts can be found in the article by Di Nola and Gerla [37].

A quantale $\mathcal{Q}$ admits two binary operations $\backslash$ and $/$ on its carrier Q, defined as:

$$a\backslash b = \bigvee\{x \in Q \mid a \otimes x \leqslant b\}, \tag{1.12}$$
$$b/a = \bigvee\{x \in Q \mid x \otimes a \leqslant b\}, \tag{1.13}$$

for any $a, b \in Q$. We call $a\backslash b$ the *right residual of* b *by* a, and b/a the *left residual of* b *by* a. The left residual b/a is isotone in its first argument and antitone in its second argument, while the right residual $a\backslash b$ is antitone in its first argument and isotone in its second argument. The following *residuation property* holds:

$$a \otimes b \leqslant c \quad \Leftrightarrow \quad a \leqslant c/b \quad \Leftrightarrow \quad b \leqslant a\backslash c, \tag{1.14}$$

for each $a, b, c \in Q$. Moreover, for all $a, b, c \in Q$, the following implications hold:

$$b \leqslant c \quad \Rightarrow \quad a \otimes b \leqslant a \otimes c, \quad b \otimes a \leqslant c \otimes a, \quad a\backslash b \leqslant a\backslash c, \quad b/a \leqslant c/a. \tag{1.15}$$

If $\mathcal{Q}$ is commutative, then $a\backslash b = b/a$, for all $a, b \in Q$. In this case, the common value is denoted by $\rightarrow$ and is called the *residuum* of a with respect to b. The operation $\rightarrow$ is antitone in its first argument and isotone in its second argument. The residuation property then takes the form:

$$a \otimes b \leqslant c \quad \Leftrightarrow \quad a \leqslant b \rightarrow c, \tag{1.16}$$

for all $a, b, c \in Q$. Equivalently,

$$b \rightarrow c \text{ is the greatest element of } \{x \in Q \mid x \otimes b \leqslant c\}. \tag{1.17}$$

Furthermore, in this commutative setting, we define a new binary operation on Q, called the *biresiduum*, by:

$$a \leftrightarrow b \equiv (a \rightarrow b) \wedge (b \rightarrow a),$$

for all $a, b \in Q$. Note that the multiplicative identity $1_\otimes$ satisfies $1_\otimes \otimes a = a$ for all $a \in Q$. By the residuation property (1.16), it follows that $1_\otimes \leqslant a \rightarrow a$. Consequently, for all $a \in Q$, we obtain:

$$1_\otimes \leqslant a \leftrightarrow a. \tag{1.18}$$

Finally, the unary operation of *negation* is defined by:

$$\neg a \equiv a \to 0, \tag{1.19}$$

for all $a \in Q$. It follows directly from the definition that negation is antitone, that is, for all $a, b \in Q$

$$a \leqslant b \quad \Rightarrow \quad \neg b \leqslant \neg a. \tag{1.20}$$

In the sequel, we consider an example of a quantale, known as a power set quantale, presented in the work of Ignjatović and Ćirić [59]. Kruml and Paseka [72] referred to this example as the most general example that leads to any quantale. This example is particularly significant in the present context, as it serves as a starting point for the main results developed in this dissertation.

Example 1.3. Let $\langle S, * \rangle$ be a semigroup, and let $\mathcal{P}(S)$ denote the powerset of S. Then, $\langle \mathcal{P}(S), \cup, \cap \rangle$ is a complete lattice with respect to the inclusion order $\subseteq$. Define a multiplication $\otimes$ on $\mathcal{P}(S)$ by

$$A \otimes B = \{a * b \mid a \in A \text{ and } b \in B\}, \tag{1.21}$$

for all $A, B \in \mathcal{P}(S)$. From the definition of $\otimes$, it follows that multiplication is isotone in both arguments, that is, for all $A, B, C \in \mathcal{P}(S)$,

$$A \subseteq B \quad \Rightarrow \quad A \otimes C \subseteq A \otimes B, \quad \text{and} \quad C \otimes A \subseteq C \otimes B. \tag{1.22}$$

Moreover, $\otimes$ distributes over arbitrary unions. In particular, for any $B \in \mathcal{P}(S)$ and any family $\{A_i\}_{i \in I} \subseteq \mathcal{P}(S)$, the following infinite distributive laws hold:

$$\left(\bigcup_{i \in I} A_i \right) \otimes B = \bigcup_{i \in I} (A_i \otimes B), \quad B \otimes \left(\bigcup_{i \in I} A_i \right) = \bigcup_{i \in I} (B \otimes A_i). \tag{1.23}$$

Then $\mathcal{P}(\mathcal{S}) = \langle \mathcal{P}(S), \cup, \cap, \otimes, \emptyset, S \rangle$ forms a quantale, known as the *power set quantale of* S. It is clear that $\mathcal{P}(\mathcal{S})$ is a commutative quantale (resp. unital quantale) if $\langle S, * \rangle$ is a commutative semigroup (resp. $\langle S, *, 1_* \rangle$ is a commutative monoid). In the latter case, the identity for the multiplication $\otimes$ is given by $1_\otimes = \{1_*\}$.

The right residual $A \backslash B$ of B by A and the left residual B / A of B by A are

respectively defined as:

$$A\backslash B = \{c \in S \mid A \otimes \{c\} \subseteq B\}, \tag{1.24}$$
$$B/A = \{c \in S \mid \{c\} \otimes A \subseteq B\}, \tag{1.25}$$

for every $A, B \in \mathcal{P}(S)$ (see [59]). We call $\backslash$ and $/$ the *right and left residuals* on $\mathcal{P}(\mathcal{S})$. The *residuation property* is satisfied for every $A, B, C \in \mathcal{P}(S)$:

$$A \otimes B \subseteq C \quad \Leftrightarrow \quad A \subseteq C/B \quad \Leftrightarrow \quad B \subseteq A\backslash C. \tag{1.26}$$

If $\mathcal{P}(\mathcal{S})$ is a commutative power set quantale, the right and left residuals coincide. Hence, we denote both residuals by the same symbol $\rightarrow$, where

$$A \rightarrow B = A\backslash B = B/A, \quad \text{for all } A, B \in \mathcal{P}(S).$$

We refer to $\rightarrow$ as the *residuum operation* on $\mathcal{P}(\mathcal{S})$. In this case, the residuation property takes the form

$$A \otimes B \subseteq C \quad \Leftrightarrow \quad A \subseteq B \rightarrow C, \tag{1.27}$$

for all $A, B, C \in \mathcal{P}(S)$. The *biresiduum operation* $\leftrightarrow$ is then defined by

$$A \leftrightarrow B = (A \rightarrow B) \cap (B \rightarrow A), \tag{1.28}$$

for all $A, B \in \mathcal{P}(S)$.

Remark 1.2. The following alternative definitions for the residuals hold:

$$\bigcup\{C \in \mathcal{P}(S) \mid B \otimes C \subseteq A\} = \{c \in S \mid B \otimes \{c\} \subseteq A\},$$
$$\bigcup\{C \in \mathcal{P}(S) \mid C \otimes B \subseteq A\} = \{c \in S \mid \{c\} \otimes B \subseteq A\},$$

for all $A, B \in \mathcal{P}(S)$. By definition of $\otimes$ in (1.21), for $B, C \in \mathcal{P}(S)$, we have:

$$B \otimes C = \{\, b * c \mid b \in B, c \in C \,\} = \bigcup_{c \in C} (B \otimes \{c\}).$$

Hence, for any $C \subseteq S$,

$$B \otimes C \subseteq A \quad \Leftrightarrow \quad B \otimes \{c\} \subseteq A, \quad \text{for all } c \in C.$$

Thus, the largest such set C consists exactly of all elements $c \in S$ satisfying

$B \otimes \{c\} \subseteq A$, which proves the claim. The second equality follows analogously. However,for notational simplicity, we retain the original definitions (1.24) and (1.25) of the residuals in power set quantales.

For notational simplicity, the power set quantale $\mathcal{P}(S)$ is denoted by $\mathcal{Q}$ throughout the dissertation, unless otherwise specified.

In the next remark, we point out the connection between the residuum operation in $\mathcal{Q} = \mathcal{P}(\mathcal{S})$ and the residuum operation in S, in the case when S is the real unit interval and the multiplication operation on S is a left-continuous or continuous t-norm.

Remark 1.3. Let $\mathcal{Q} = \mathcal{P}(\mathcal{S})$ be the power set quantale of a semigroup S. Then, for arbitrary $a, b \in S$, we have that

$$\{a\}\backslash\{b\} = \{x \in S \mid \{a\} \otimes \{x\} \subseteq \{b\}\} = \{x \in S \mid a * x = b\},$$
$$\{b\}/\{a\} = \{x \in S \mid \{x\} \otimes \{a\} \subseteq \{b\}\} = \{x \in S \mid x * a = b\}.$$

In other words, we have that $\{a\}\backslash\{b\}$ is the set of all solutions to the equation $a * x = b$ in S, whereas $\{b\}/\{a\}$ is the set of all solutions to the equation $x * a = b$ in S.

In what follows, let $S = [0, 1]$, let $* = \otimes$ be an arbitrary left-continuous t-norm on $[0, 1]$, and let $\to$ be the residuum operation (residual implication) on $[0, 1]$ corresponding to $\otimes$. Using the same symbol $\to$ for the residuum operation on $[0, 1]$ and for the residuum operation in the power set quantale $\mathcal{P}([0, 1])$ will not cause confusion, since it will always be clear whether the operation is applied to subsets of $[0, 1]$ or to individual elements of $[0, 1]$. As known, for arbitrary $a, b \in S$ we have that $a \to b$ is the greatest solution of the inequality $a \otimes x \leqslant b$, and according to [23, Lemma 2], if the set $\{a\}\backslash\{b\} = \{b\}/\{a\}$ is non-empty, i.e., if the equation $a \otimes x = b$ is solvable in $[0, 1]$, then $a \to b$ is also the greatest solution of this equation, that is, $a \to b$ is the greatest element of the set $\{a\}\backslash\{b\}$.

If the set $\{a\}\backslash\{b\}$ is empty, that is, if the equation $a \otimes x = b$ is not solvable, then the relationship between $\{a\}\backslash\{b\}$ and $a \to b$ is not so clear. However, if $\otimes$ is a continuous t-norm, this relationship can be made considerably clearer. Recall that a continuous t-norm $\otimes$ satisfies the *divisibility condition*

$$a \otimes (a \to b) = a \wedge b, \tag{1.29}$$

for all $a, b \in [0, 1]$, whose equivalent form is

$$b \leqslant a \quad \Rightarrow \quad (\exists x \in [0, 1])\ a \otimes x = b, \tag{1.30}$$

for all $a, b \in [0,1]$ (cf. [11, Theorem 2.36 and Lemma 2.90] or [12, Lemma 1.39]). Now, according to (1.30) and the fact that the interval $[0,1]$ is linearly ordered, for arbitrary $a, b \in [0,1]$ we have that

$$b \leqslant a \quad \Leftrightarrow \quad \{a\}\backslash\{b\} \neq \emptyset \qquad \text{and} \qquad a < b \quad \Leftrightarrow \quad \{a\}\backslash\{b\} = \emptyset,$$

and therefore

$$a \to b = \begin{cases} \top(\{a\}\backslash\{b\}) & \text{if } \{a\}\backslash\{b\} \neq \emptyset, \\ 1 & \text{if } \{a\}\backslash\{b\} = \emptyset, \end{cases}$$

where $\top X$ denotes the greatest element of a set X.
It is interesting to observe that for $a \neq b$, the equation $a \otimes x = b$ has no solution in $[0,1]$ if and only if all elements of $[0,1]$ are solutions to the inequality $a \otimes x \leqslant b$.

1.4.1 Properties of commutative quantales

In this subsection, we recall some standard properties of multiplication and residuation in quantales that will be used throughout this dissertation. The formulations are adapted to the commutative setting considered here; see [59, 103, 117].

The following proposition states that the multiplication $\otimes$ is isotone in both arguments, while the residuum operation $\to$ is isotone in its second argument and antitone in its first argument.

Proposition 1.1. *Let $\mathcal{Q}$ be a commutative quantale. For all $a, b, c \in Q$, the following assertions hold:*

$$a \leqslant b \quad \Rightarrow \quad a \otimes c \leqslant b \otimes c, \tag{1.31}$$

$$a \leqslant b \quad \Rightarrow \quad c \to a \leqslant c \to b, \tag{1.32}$$

$$a \leqslant b \quad \Rightarrow \quad b \to c \leqslant a \to c. \tag{1.33}$$

In particular, since $b \leqslant b \vee c$ and $c \leqslant b \vee c$, Proposition 1.1 yields:

$$a \to b \leqslant a \to (b \vee c), \qquad a \to c \leqslant a \to (b \vee c). \tag{1.34}$$

Proposition 1.2. *Let $\mathcal{Q}$ be a commutative quantale. Then, for every $a, b, c \in Q$,*

the following assertions hold:

$$a \otimes (a \to b) \leqslant b, \tag{1.35}$$
$$b \leqslant a \to (a \otimes b), \tag{1.36}$$
$$a \to b \leqslant (a \otimes c) \to (b \otimes c), \tag{1.37}$$
$$(a \to b) \otimes c \leqslant a \to (b \otimes c), \tag{1.38}$$
$$(a \to b) \otimes (b \to c) \leqslant (a \to c), \tag{1.39}$$
$$c \to (b \to a) = (b \otimes c) \to a. \tag{1.40}$$

Observe that, in a commutative quantale $\mathcal{Q}$, the definition of the biresiduum together with (1.39), yields, for all $a, b, c \in Q$

$$(a \leftrightarrow b) \otimes (b \leftrightarrow c) \leq (a \leftrightarrow c). \tag{1.41}$$

Proposition 1.3. *Let $\mathcal{Q}$ be a commutative quantale. Then, for all $a \in Q$,*

$$a \to 1 = 1, \tag{1.42}$$
$$0 \to a = 1. \tag{1.43}$$

If, in addition, $\mathcal{Q}$ is unital, with $1_\otimes$ denoting its multiplicative identity, then

$$1_\otimes \to a = a. \tag{1.44}$$

Proposition 1.4. *Let $\mathcal{Q}$ be a commutative quantale. Then for any $a \in Q$ and any families $\{a_i\}_{i\in I}, \{b_i\}_{i\in I} \subseteq Q$, the following assertions hold:*

$$a \otimes \left(\bigwedge_{i\in I} b_i \right) \leqslant \bigwedge_{i\in I} (a \otimes b_i), \tag{1.45}$$
$$\bigwedge_{i\in I} (a \to b_i) = a \to \left(\bigwedge_{i\in I} b_i \right), \tag{1.46}$$
$$\bigwedge_{i\in I} (a_i \to b_i) \leqslant \bigwedge_{i\in I} a_i \to \bigwedge_{i\in I} b_i, \tag{1.47}$$
$$\bigwedge_{i\in I} (a_i \to b_i) \leqslant \bigvee_{i\in I} a_i \to \bigvee_{i\in I} b_i, \tag{1.48}$$
$$\bigvee_{i\in I} (a \to b_i) \leqslant a \to \left(\bigvee_{i\in I} b_i \right), \tag{1.49}$$
$$\left(\bigvee_{i\in I} b_i \right) \to a = \bigwedge_{i\in I} (b_i \to a). \tag{1.50}$$

1.4.2 Examples of power set quantales

We next present several examples of power set quantales that will be used to illustrate the general framework developed in this work.

Example 1.4. Let $S = [0,1]$ be the real unit interval equipped with a t-norm $*$. Then, the structure $\langle S, *, 1\rangle$ is a commutative monoid. According to Example 1.3, it follows that the structure $\langle \mathcal{P}(S), \cup, \cap, \otimes, \emptyset, S\rangle$ forms a commutative unital power set quantale, where $\otimes$ is defined by (1.21), and where the identity element is given by $1_\otimes = \{1\}$. The residuals are calculated by (1.24) and (1.25), and they are equal as $\otimes$ is a commutative operation on $\mathcal{P}(S)$. In other words, $A\backslash B = B/A$, for every $A, B \in \mathcal{P}(S)$. We further adapt the following conventions:

- If $*$ is the *Gödel t-norm*, specified as $a * b = \min\{a, b\}$, for every $a, b \in S$, $\mathcal{P}(S)$ is called the *Gödel power set quantale*;
- If $*$ is the *Łukasiewicz t-norm*, specified as $a * b = \max\{0, a + b - 1\}$, for every $a, b \in S$, $\mathcal{P}(S)$ is called the *Łukasiewicz power set quantale*;
- If $*$ is the *product t-norm*, specified as $a * b = a \cdot b$ (the usual multiplication), for every $a, b \in S$, $\mathcal{P}(S)$ is called the *product power set quantale*.

For example, let $A = \{0.3, 0.5, 0.6\}$ and $B = \{0.1, 0.3, 0.7\}$ be two subsets of S. In the Gödel power set quantale, the corresponding operations yield

$$A \otimes B = \{0.1, 0.3, 0.5, 0.6\}$$
$$A \to B = A\backslash B = B/A = \{0.1, 0.3\}.$$

For the product power set quantale, we obtain

$$A \otimes B = \{0.03, 0.05, 0.06, 0.09, 0.15, 0.18, 0.21, 0.35, 0.42\},$$
$$A \to B = A\backslash B = B/A = \emptyset.$$

In the Łukasiewicz power set quantale, we have

$$A \otimes B = \{0, 0.2, 0.3\},$$
$$A \to B = A\backslash B = B/A = \emptyset.$$

Example 1.5. Consider the set $\mathbb{N}$ of natural numbers equipped with the minimum operation $\wedge$. Then, the ordered pair $\langle \mathbb{N}, \wedge\rangle$ forms a commutative semigroup. According to Example 1.3, the structure $\langle \mathcal{P}(\mathbb{N}), \cup, \cap, \otimes, \emptyset, \mathbb{N}\rangle$ forms a commutative power set quantale, where $\otimes$ is defined as (1.21), which takes the form

$A \otimes B = \{a \wedge b \mid a \in A \text{ and } b \in B\}$, for every $A, B \in \mathcal{P}(\mathbb{N})$. For example, if $A = \{1,2,3,4,5,6\}$ and $B = \{4,5,6,7,8\}$, then $A \otimes B = \{1,2,3,4,5,6\}$ and $A \to B = B \backslash A = A/B = \{1,2,3,4,5,6\}$.

Example 1.6. Let $S = \{(a_1, a_2, a_3, a_4) \mid a_1, a_2, a_3, a_4 \in \mathbb{Z}\}$ be the set of all four-dimensional vectors with integer coordinates. Define a binary operation $\oplus$ on S by componentwise addition:

$$(a_1, a_2, a_3, a_4) \oplus (b_1, b_2, b_3, b_4) = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$$

Since componentwise addition of vectors is associative, $\langle S, \oplus \rangle$ is a semigroup. Moreover, the neutral element for $\oplus$ is the vector $(0,0,0,0)$, hence, $\langle S, \oplus \rangle$ forms a commutative monoid. Consequently, the structure $\langle \mathcal{P}(S), \cup, \cap, \otimes, \emptyset, S \rangle$ forms a unital power set quantale, where $\otimes$ is defined by

$$A \otimes B = \{a \oplus b \mid a \in A, b \in B\},$$

for all $A, B \subseteq S$. As an illustration, let $A, B \subseteq S$ be defined by

$$A = \{(1,2,1,-1), (1,0,2,1)\}, \quad B = \{(1,0,0,1), (1,-2,1,3)\}.$$

By applying the quantale multiplication and the corresponding residuum operation, we obtain

$$A \otimes B = \{(2,2,1,0), (2,0,2,2), (2,-2,3,4)\},$$
$$A \to B = \{0,-2,-1,2\},$$
$$B \to A = \{0,2,1,-2\}.$$

The vectors in S may be interpreted as evaluations with respect to four criteria. Since different experts may assign different evaluations to the same object, a subset of S can be used to represent the collection of all evaluations assigned to that object. In this way, the power set quantale provides an algebraic structure for representing and manipulating hesitant information arising from multiple expert opinions.

Example 1.7. Now we present an example of a non-commutative power set quantale. Let

$$S = \{123, 132, 213, 231, 312, 321\}$$

be the set of all permutations of $M = \{1,2,3\}$, linearly ordered by the lexicographic

order $\leqslant$. Then, the structure $\langle S, \leq, \mathbf{0}, \mathbf{1}\rangle$ constitutes a complete lattice structure with least and greatest elements denoted by $\mathbf{0} = 123$ and $\mathbf{1} = 321$, respectively.

Consider the composition $\circ$ of permutations as the usual composition of mappings. Consequently, $\langle S, \circ\rangle$ is a noncommutative semigroup (for example, $132 \circ 312 = 213$ and $312 \circ 132 = 321$). According to Example 1.3, $\langle \mathcal{P}(S), \cup, \cap, \otimes, \emptyset, S\rangle$ is a power set quantale, where $\otimes$ is defined by (1.21), that is, $A \otimes B = \{p_1 \circ p_2 \mid p_1 \in A \text{ and } p_2 \in B\}$, for every $A, B \in \mathcal{P}(S)$. Consider the particular example where $A = \{123, 132, 213\}$ and $B = \{213, 312\}$. According to Example 1.3 we get the residuals $B \backslash A = \{213, 231\}$ and $A/B = \emptyset$.

1.4.3 Residuated lattices and related structures

Residuated lattices play an important role in the algebraic foundations of fuzzy logic and are closely related to commutative integral quantales. We therefore recall their definition. The introductory chapters of [109, 112] provide useful overviews of the most important truth structures and their corresponding logics.

Definition 1.30. A *residuated lattice* is an algebra $\mathcal{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ where:

(L1) $\langle L, \wedge, \vee, 0, 1\rangle$ is a lattice with the least element 0 and the greatest element 1,

(L2) $\langle L, \otimes, 1\rangle$ is a commutative monoid with the unit 1,

(L3) operations $\otimes$ and $\rightarrow$ form an *adjoint pair*, i.e., the *residuation property*

$$a \otimes b \leqslant c \quad \text{iff} \quad a \leqslant b \rightarrow c, \tag{1.51}$$

holds for all $a, b, c \in L$, where $\leqslant$ denotes lattice ordering.

If, in addition, $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, then $\mathcal{L}$ is called a *complete residuated lattice*.

A residuated lattice L is called a *Heyting algebra* if $a \otimes b = a \wedge b$ for all $a, b \in L$. If, in addition, L is a complete lattice, then it is called a complete Heyting algebra. If the partial order $\leqslant$ on L is linear, then L is called a linearly ordered Heyting algebra.

Example 1.8. Let $L = [0, 1]$ be the real unit interval equipped with its natural order. Then $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, where

$$a \wedge b = \min(a, b), \qquad a \vee b = \max(a, b).$$

The following three choices of the operations $\otimes$ and $\rightarrow$ give the most widely studied

complete residuated lattices on $[0,1]$, which serve as standard truth-value structures in fuzzy logic:

1. *Łukasiewicz structure*:

$$a \otimes b = \max(a + b - 1, 0), \qquad a \to b = \min(1 - a + b, 1),$$

2. *Gödel structure*:

$$a \otimes b = \min(a, b), \qquad a \to b = \begin{cases} 1, & a \leqslant b, \\ b, & a > b, \end{cases}$$

3. *Goguen (product) structure*:

$$a \otimes b = a \cdot b, \qquad a \to b = \begin{cases} 1, & a \leqslant b, \\ b/a, & a > b. \end{cases}$$

In each case, $\langle L, \wedge, \vee, \otimes, \to, 0, 1 \rangle$ is a complete residuated lattice.

We next recall the notion of a *Brouwer lattice*, also known as a relatively pseudocomplemented lattice [2].

Definition 1.31. Let L be a lattice and let $a, b \in L$. If the set $\{x \in L \mid a \wedge x \leqslant b\}$ has a greatest element, this element is called the *relative pseudocomplement* of a with respect to b, and is denoted by $a \to b$. If the relative pseudocomplement $a \to b$ exists for every pair $a, b \in L$, then L is called a *Brouwerian lattice* (or *relatively pseudo-complemented lattice*).

Equivalently, Blyth [15] defined a Brouwerian lattice as a lattice L in which, for every $a \in L$, the mapping $x \mapsto a \wedge x$ preserves arbitrary joins, that is,

$$a \wedge \Big(\bigvee_{i \in I} b_i \Big) = \bigvee_{i \in I} (a \wedge b_i), \tag{1.52}$$

for every family $\{b_i\}_{i \in I} \subseteq L$. It is well known that every Brouwerian lattice is distributive.

Chapter 2

Hesitant fuzzy sets and relations over the power set quantale of a semigroup

The study of hesitant fuzzy sets has led to numerous theoretical developments [9, 10, 43, 44] and applications in various fields. These applications include decision making [45, 76–78, 129], feature selection [41], clustering analysis [20], and decision support systems [96].

Various definitions of hesitant fuzzy sets have been proposed in the literature. Let X denote a non-empty universal set. In the original approach introduced by Torra [125], a *hesitant fuzzy set* on X is defined as a mapping h that assigns to each element $x \in X$ a subset of the unit interval $[0, 1]$, where X is assumed to be finite. Each such subset is called a *hesitant fuzzy element* and consists of the possible membership degrees of x. In particular, when $h(x)$ is a nonempty closed interval, the corresponding hesitant fuzzy set reduces to an intuitionistic fuzzy set.

In the same paper [125], Torra proposed an alternative formulation based on sets of membership functions. More precisely, given a collection of n membership functions, $M = \{\mu_1, \ldots, \mu_n\}$, the corresponding hesitant fuzzy set $h_M : X \to \mathcal{P}([0, 1])$ is defined as:

$$h_M(x) = \bigcup_{\mu \in M} \{\mu(x)\}.$$

Subsequently, Pei and Yi [92] introduced an equivalent representation by defining a hesitant fuzzy set h on X as a collection of ordered pairs $h = \{\langle x, h(x)\rangle \mid x \in X\}$, where $h(x)$ is a set of distinct values from $[0, 1]$, representing the possible membership degrees of x.

Motivated by these approaches, we develop a generalized framework for hesitant fuzzy sets in which membership values are taken from the power set quantale of an arbitrary semigroup. The proposed framework extends the well-known definitions introduced by Torra [125] and subsequently studied by Torra and Narukawa [126] and Pei and Yi [92]. The algebraic properties of quantales are used to establish the main results presented in this chapter.

This chapter is structured as follows. In Section 2.1, we introduce and study hesitant fuzzy sets with membership values in the power set quantale of a semigroup. We establish their basic algebraic structure and properties, and introduce the corresponding operations, residuals, and related notions. In Section 2.2, we introduce and study hesitant fuzzy relations with membership values in the power set quantale of a semigroup. We first consider hesitant fuzzy relations with specific properties. We then introduce residuals of hesitant fuzzy relations and establish their additional properties.

2.1 Hesitant fuzzy sets over the power set quantale

This section introduces hesitant fuzzy sets with membership values in the power set quantale of an arbitrary semigroup.

Let $\mathcal{S} = \langle S, * \rangle$ be a semigroup and let $Q = \mathcal{P}(S)$ denote its power set. As illustrated in Example 1.3, the algebraic structure $\mathcal{Q} = \langle Q, \cup, \cap, \otimes, \emptyset, S \rangle$, where $\otimes$ is the multiplication operation defined by (1.21), forms the power set quantale of $\mathcal{S}$. If $\mathcal{S} = \langle S, *, 1_* \rangle$ is a monoid with identity element 1_*, then the induced power set quantale $\mathcal{Q}$ is unital, with the multiplicative identity $1_\otimes = \{1_*\}$. The notation introduced above will be used throughout the dissertation, and X will denote a non-empty set unless explicitly stated otherwise.

Definition 2.1. Let X be a non-empty set and let $\mathcal{Q}$ be the power set quantale of a monoid $\mathcal{S}$. A *hesitant fuzzy set* h *on* X *over* $\mathcal{Q}$, or simply a *hesitant fuzzy set on* X, is a function $h : X \to \mathcal{Q}$.

Some important examples of hesitant fuzzy sets on X are:

1. the *empty hesitant fuzzy set* $h_\emptyset$, defined by $h_\emptyset(x) = \emptyset$, for each $x \in X$,
2. the *full hesitant fuzzy set* h_S, defined by $h_S(x) = S$, for each $x \in X$.

For a hesitant fuzzy set $h : X \to \mathcal{Q}$, each $h(x)$ represents the membership value of x in h. The collection of all hesitant fuzzy sets on X with membership values in the quantale $\mathcal{Q}$ is denoted by Q^X.

Table 2.1: Comparison of the proposed quantale-based approach with Torra's model

Feature	Quantale-Based Approach	Approach by Torra
Values $h(x)$	Subsets of a monoid S	Subsets of $[0, 1]$
Underlying structure	Power set quantale	Not a lattice; the absorption laws are not satisfied
Join and meet operations	Set-theoretic union and intersection in $\mathcal{P}(S)$	Operations based on t-norms and t-conorms
Main focus	Axiomatic and algebraic approach based on quantales	Semantic framework based on fuzzy logic

As stated in Definition 2.1, this approach models hesitant fuzzy sets as functions of the form $h : X \to \mathcal{Q}$, where the values are subsets of a monoid S. The union and intersection of two hesitant fuzzy sets h_1 and h_2 on X are defined pointwise as follows:

$$(h_1 \cup h_2)(x) = h_1(x) \cup h_2(x), \quad (h_1 \cap h_2)(x) = h_1(x) \cap h_2(x),$$

for all $x \in X$. These operations correspond to the standard quantale structure $\mathcal{Q}$, with union and intersection serving as the join and meet operations, respectively, which stands in contrast to the different algebraic framework proposed by Torra [125] and Pei and Yi [92]. This formulation is significant for two reasons. First, in some sense, it generalizes the well-known definitions of a hesitant fuzzy set proposed by Torra. Second, the algebraic richness of quantales offers a robust foundation for structuring hesitant fuzzy sets, which in turn facilitates the development of the results established in [122]. In Table 2.1 we present a brief comparative overview between our approach and that in [125]. This comparison highlights the fundamental structural and operational distinctions between the two frameworks.

Let h be a hesitant fuzzy set on X. The *crisp part* of h, denoted by $c(h)$, is a classical (crisp) subset of X described by

$$c(h) = \{x \in X \mid 1_{\otimes} \subseteq h(x)\}. \tag{2.1}$$

The height of h, represented by $||h||$, is given by

$$||h|| = \bigcup_{x \in X} h(x). \tag{2.2}$$

The *kernel* of a hesitant fuzzy set h on X is the crisp equivalence relation on X,

given by $\ker h = \{(x, y) \in X \times X \mid h(x) = h(y)\}$.

Let h_1 and h_2 be two hesitant fuzzy sets on X. The equality and inclusion of hesitant fuzzy sets are defined pointwise as follows

$$h_1 = h_2 \quad \Leftrightarrow \quad h_1(x) = h_2(x), \quad \forall x \in X, \tag{2.3}$$

$$h_1 \subseteq h_2 \quad \Leftrightarrow \quad h_1(x) \subseteq h_2(x), \quad \forall x \in X. \tag{2.4}$$

The structure $\langle Q^X, \cup, \cap, h_\emptyset, h_S \rangle$ forms a complete lattice with respect to the partial order $\subseteq$, where $h_\emptyset$ and h_S are the least and greatest elements, respectively. The meet (intersection) $\bigcap_{i \in I} h_i$ and join (union) $\bigcup_{i \in I} h_i$ of an arbitrary family $\{h_i\}_{i \in I}$ of hesitant fuzzy sets on X are functions from X into $\mathcal{Q}$ defined by:

$$\left(\bigcap_{i \in I} h_i\right)(x) = \bigcap_{i \in I} h_i(x), \quad \left(\bigcup_{i \in I} h_i\right)(x) = \bigcup_{i \in I} h_i(x), \quad \forall x \in X. \tag{2.5}$$

Definition 2.2. Let $h_1, h_2 \in Q^X$. The *product* of h_1 and h_2 is the hesitant fuzzy set $h_1 \otimes h_2$ defined by

$$(h_1 \otimes h_2)(x) = h_1(x) \otimes h_2(x) = \{a * b \mid a \in h_1(x),\, b \in h_2(x)\}, \tag{2.6}$$

for all $x \in X$.

For simplicity of notation, we use the same symbol $\otimes$ for the multiplication in $\mathcal{Q}$ and for the induced multiplication on Q^X. No confusion should arise from the context.

Proposition 2.1. *Let $\langle S, * \rangle$ be a semigroup and let $Q = \mathcal{P}(S)$ be its power set. Then $\langle Q^X, \otimes \rangle$ is a partially ordered semigroup with respect to the inclusion $\subseteq$.*

Proof. Since $\langle S, * \rangle$ is a semigroup, the induced operation $\otimes$ on Q^X is associative. More precisely, for all $h_1, h_2, h_3 \in Q^X$ and $x \in X$, associativity of $*$ implies:

$$\begin{aligned}
(h_1 \otimes (h_2 \otimes h_3))(x) &= h_1(x) \otimes (h_2(x) \otimes h_3(x)) \\
&= h_1(x) \otimes \{\beta * \gamma \mid \beta \in h_2(x), \gamma \in h_3(x)\} \\
&= \{\alpha * (\beta * \gamma) \mid \alpha \in h_1(x), \beta \in h_2(x), \gamma \in h_3(x)\} \\
&= \{(\alpha * \beta) * \gamma \mid \alpha \in h_1(x), \beta \in h_2(x), \gamma \in h_3(x)\} \\
&= h_1(x) \otimes (h_2(x) \otimes h_3(x)) \\
&= ((h_1 \otimes h_2) \otimes h_3)(x).
\end{aligned}$$

Moreover, the multiplication $\otimes$ on Q^X is isotone with respect to the inclusion order in each argument, that is, for all $h_1, h_2, h_3 \in Q^X$

$$h_1 \subseteq h_2 \quad \Rightarrow \quad h_1 \otimes h_3 \subseteq h_2 \otimes h_3 \quad \text{and} \quad h_3 \otimes h_1 \subseteq h_3 \otimes h_2.$$

Hence, the structure $\langle Q^X, \otimes \rangle$, equipped with the inclusion order $\subseteq$, is a partially ordered semigroup. □

In addition, the infinite distributive laws (1.23) in $\mathcal{Q}$, imply that the multiplication $\otimes$ induced on Q^X distributes over arbitrary joins. More precisely, for every family $\{h_i\}_{i \in I} \subseteq Q^X$ and every $g \in Q^X$,

$$\left(\bigcup_{i \in I} h_i \right) \otimes g = \bigcup_{i \in I} (h_i \otimes g), \quad g \otimes \left(\bigcup_{i \in I} h_i \right) = \bigcup_{i \in I} (g \otimes h_i). \tag{2.7}$$

These equalities hold pointwise for every $x \in X$.

Corollary 2.1. *The structure $\mathcal{Q}(X) = \langle Q^X, \cup, \cap, \otimes, h_\emptyset, h_S \rangle$ is a quantale.*

Proof. The result follows from the complete lattice structure of $\langle Q^X, \cup, \cap, h_\emptyset, h_S \rangle$, Proposition 2.1, and the infinite distributive laws (2.7) □

Remark 2.1. If $\langle S, *, 1_* \rangle$ is a monoid with identity element 1_*, then the induced quantale $\mathcal{Q}(X)$ is unital. Its multiplicative identity is the constant hesitant fuzzy set $h_{1_\otimes} : X \to \mathcal{Q}$ defined by $h_{1_\otimes}(x) = 1_\otimes$ for all $x \in X$, where $1_\otimes = \{1_*\}$ is the multiplicative identity of the underlying quantale $\mathcal{Q}$.

In the quantale $\mathcal{Q}(X)$, the residuals are defined as follows. For arbitrary $h_1, h_2 \in Q^X$, the *right residual* of h_1 by h_2 is the hesitant fuzzy set $h_2 \backslash\!\!\otimes h_1 \in Q^X$ defined pointwise by

$$(h_2 \backslash\!\!\otimes h_1)(x) = h_2(x) \backslash h_1(x) = \{c \in S \mid h_2(x) \otimes c \subseteq h_1(x)\}, \tag{2.8}$$

and the *left* $\otimes$*-residual* of h_1 by h_2 is the hesitant fuzzy set $h_1 \otimes\!\!/ h_2 \in Q^X$ defined by

$$(h_1 \otimes\!\!/ h_2)(x) = h_1(x) / h_2(x) = \{c \in S \mid c \otimes h_2(x) \subseteq h_1(x)\}, \tag{2.9}$$

for all $x \in X$. These operations satisfy the residuation property, that is, for all $h, h_1, h_2 \in Q^X$:

$$h_1 \otimes h_2 \subseteq h \quad \Leftrightarrow \quad h_1 \subseteq h \otimes\!\!/ h_2 \quad \Leftrightarrow \quad h_2 \subseteq h_1 \backslash\!\!\otimes h. \tag{2.10}$$

If $\mathcal{Q}$ is a commutative power set quantale, the right and left residuals coincide and and we use the notation:

$$(h_1 \mathbin{\mathrel{\reflectbox{$\twoheadrightarrow$}}\!\!\!\!\otimes} h_2)(x) = (h_2 \mathbin{\otimes\!\!\!\!\mathrel{\reflectbox{$\twoheadrightarrow$}}} h_1)(x) = (h_1 \mathbin{\mathrel{\reflectbox{$\twoheadrightarrow$}}\!\!\!\!\otimes} h_2)(x) = h_1(x) \to h_2(x), \tag{2.11}$$

for all $x \in X$. The operation $\mathbin{\mathrel{\reflectbox{$\twoheadrightarrow$}}\!\!\!\!\otimes}$ is called *residuum*. Hence, the residuation property is defined as follows

$$h_1 \otimes h_2 \subseteq h_3 \quad \Leftrightarrow \quad h_1 \subseteq h_2 \mathbin{\mathrel{\reflectbox{$\twoheadrightarrow$}}\!\!\!\!\otimes} h_3, \tag{2.12}$$

where $h_1, h_2, h_3 \in \mathcal{Q}(X)$ are arbitrary hesitant fuzzy sets on X. The commutativity of $\otimes$ allows the introduction of the concept of *biresiduum* in a standard way, i.e.,

$$(h_1 \mathbin{\twoheadleftarrow\!\!\!\!\otimes\!\!\!\!\twoheadrightarrow} h_2)(x) \equiv (h_1 \mathbin{\mathrel{\reflectbox{$\twoheadrightarrow$}}\!\!\!\!\otimes} h_2)(x) \cap (h_2 \mathbin{\mathrel{\reflectbox{$\twoheadrightarrow$}}\!\!\!\!\otimes} h_1)(x), \tag{2.13}$$

for all $h_1, h_2 \in \mathcal{Q}(X)$ and all $x \in X$. Recall that the term biresiduum was introduced in the Chapter 1. Here, we extend it to the commutative quantale of hesitant fuzzy sets.

Remark 2.2. Observe that this setting provides all the structural properties needed in the sequel. In particular, since S is a monoid, the quantale $\mathcal{Q}$ is unital. The existence of the unit element in $\mathcal{Q}$ ensures the existence of the corresponding neutral element $1_{\otimes}$ in $\mathcal{Q}(X)$, which is used in the definition of a normalized hesitant fuzzy set, ensuring their well-defined normalization property. Moreover, since $\mathcal{Q}$ is a complete residuated quantale, the induced structure $\mathcal{Q}(X)$ is also residuated, so that left and right residuals exist. These properties together ensure consistency of the induced hesitant fuzzy relational framework in the sense that all induced operations are well-defined and compatible, and support all subsequent constructions in the paper.

Throughout the dissertation, hesitant fuzzy sets are represented as row vectors. In the following example, we apply the residuation property to compute the greatest solution to hesitant fuzzy inequalities of the forms $h \otimes u \subseteq g$ and $w \otimes h \subseteq g$, where u and w are unknown hesitant fuzzy sets on X.

Example 2.1. Let $X = \{x_1, x_2, x_3\}$, let $\mathcal{Q}$ be the product power set quantale such that $S = [0, 1]$ and $a * b = a \cdot b$ for all $a, b \in S$. Consider the hesitant fuzzy sets h and g on X, defined as follows:

$$h = \begin{bmatrix} \{0.1, 0.5\} & \{0\} & \{0.5, 1\} \end{bmatrix}, \quad g = \begin{bmatrix} \{0.5\} & \{0, 0.5\} & \{0.5, 1\} \end{bmatrix}.$$

We now consider the following hesitant fuzzy inequalities on $\mathcal{Q}$:

$$h \otimes u \subseteq g, \qquad w \otimes h \subseteq g, \tag{2.14}$$

where u and w are unknown hesitant fuzzy sets on X. Observe that $\otimes$ is commutative here. Consequently, the left and right residuals coincide and, by residuation property (2.12), we obtain:

$$\begin{aligned}
u(x_1) = w(x_1) = h(x_1) \to g(x_1) &= \{0.1, 0.5\} \to \{0.5\} \\
&= \{c \in S \mid \{c\} \otimes \{0.1, 0.5\} \subseteq \{0.5\}\} = \emptyset, \\
u(x_2) = w(x_2) = h(x_2) \to g(x_2) &= \{0\} \to \{0, 0.5\} \\
&= \{c \in S \mid \{c\} \otimes \{0\} \subseteq \{0, 0.5\}\} = [0, 1], \\
u(x_3) = w(x_3) = h(x_3) \to g(x_3) &= \{0.5, 1\} \to \{0.5, 1\} \\
&= \{c \in S \mid \{c\} \otimes \{0.5, 1\} \subseteq \{0.5, 1\}\} = \{1\}.
\end{aligned}$$

Hence, the greatest solution to the inequalities (2.14) is:

$$u = w = \begin{bmatrix} \emptyset & [0,1] & \{1\} \end{bmatrix}.$$

2.1.1 Additional operations and properties of hesitant fuzzy sets

In this section, we introduce some notions used throughout the remainder of the dissertation. Observe that $\langle Q, \cup, \emptyset \rangle$ is a commutative monoid, $\langle Q, \otimes, 1_\otimes \rangle$ is a monoid, the multiplication $\otimes$ distributes over arbitrary joins, and $\emptyset \otimes A = A \otimes \emptyset = \emptyset$. Therefore, the structure $\mathcal{Q}^* = \langle Q, \cup, \otimes, \emptyset, 1_\otimes \rangle$ forms a semiring, called the semiring reduct of the quantale $\mathcal{Q} = \langle Q, \cup, \cap, \otimes, \emptyset, S \rangle$.

For any $\lambda \in Q$ and $h \in Q^X$ we introduce the *left scalar multiplication* $\lambda \otimes h \in Q^X$ and the *right scalar multiplication* $h \otimes \lambda \in Q^X$, respectively, as follows:

$$\begin{aligned}
(\lambda \otimes h)(x) &= \lambda \otimes h(x) = \{a * b \mid a \in \lambda \text{ and } b \in h(x)\}, \\
(h \otimes \lambda)(x) &= h(x) \otimes \lambda = \{a * b \mid a \in h(x) \text{ and } b \in \lambda\},
\end{aligned}$$

for all $x \in X$. In the case where multiplication $\otimes$ is commutative, the left and right scalar multiplication coincide, i.e., $\lambda \otimes h = h \otimes \lambda$. Consequently, the monoid $\langle Q^X, \cup, h_\emptyset \rangle$ forms a strong $\mathcal{Q}^*$-bisemimodule. In this setting, the scalar multiplication refers to either the left or right scalar multiplication, but we use the left notation $(\lambda, h) \mapsto \lambda \otimes h$ to represent it. The lattice $\mathcal{Q}(X)$ with this scalar multiplication is

denoted by $\mathcal{Q}_{\otimes}(X)$ and referred to as the $\mathcal{Q}$-lattice of hesitant fuzzy sets on X. In addition, we say that a hesitant fuzzy set $h \in Q^X$ is a linear combination of hesitant fuzzy sets $h_i \in Q^X (i \in I)$ if h can be represented as follows:

$$h = \bigcup_{i \in I} \lambda_i \otimes h_i. \tag{2.15}$$

A family $\{h_i\}_{i \in I} \subseteq Q^X$ is referred to as a *complete* $\mathcal{Q}$*-sublattice* of $\mathcal{Q}_{\otimes}(X)$ if it is closed under scalar multiplication and arbitrary joins and meets.

Let $\lambda \in Q$ be an arbitrary scalar and let $h \in Q^X$. The *right s-residual* of h by λ, denoted by $\lambda \backslash_s h$, and the *left s-residual* of h by λ, denoted by $h /_s \lambda$, are hesitant fuzzy sets on X defined, respectively, for all $x \in X$:

$$(\lambda \backslash_s h)(x) = \lambda \backslash h(x), \qquad (h /_s \lambda)(x) = h(x) / \lambda. \tag{2.16}$$

Functions $h \mapsto \lambda \backslash_s h$ and $h \mapsto h /_s \lambda$ are called respectively the *right scalar residuation* and the *left scalar residuation*, respectively. Clearly, for all $x \in X$, we have:

$$(\lambda \backslash_s h)(x) = \{c \in S \mid \lambda \otimes \{c\} \subseteq h(x)\}, \qquad (h /_s \lambda)(x) = \{c \in S \mid \{c\} \otimes \lambda \subseteq h(x)\}. \tag{2.17}$$

The following residuation properties hold for all $\lambda \in Q$ and $h, h' \in Q^X$:

$$\lambda \otimes h' \subseteq h \;\Leftrightarrow\; h' \subseteq \lambda \backslash_s h, \qquad h' \otimes \lambda \subseteq h \;\Leftrightarrow\; h' \subseteq h /_s \lambda. \tag{2.18}$$

If $\mathcal{Q}$ is a commutative quantale, the right and left s-residuals coincide and we adopt the following notation:

$$\lambda \xrightarrow{s} h = \lambda \backslash_s h = h /_s \lambda.$$

The following notions are inspired by their counterparts in classical fuzzy set theory [63] and are adapted here to the hesitant fuzzy setting. Throughout the remainder of this section, we assume that the underlying monoid $\mathcal{S}$ is commutative. Consequently, the power set quantale $\mathcal{Q}$ of S is also commutative and admits a unique residuum operation. Given two hesitant fuzzy sets $h_1, h_2 \in Q^X$, we define the following functions:

$$I(h_1, h_2) = \bigcap_{x \in X} h_1(x) \to h_2(x), \tag{2.19}$$

$$E(h_1, h_2) = \bigcap_{x \in X} h_1(x) \leftrightarrow h_2(x), \tag{2.20}$$

$$O(h_1, h_2) = \bigcup_{x \in X} h_1(x) \otimes h_2(x). \tag{2.21}$$

The resulting functions $I(h_1, h_2)$, $E(h_1, h_2)$ and $O(h_1, h_2)$ are referred to as the *hesitant implication*, *hesitant biimplication*, and *hesitant multiplication*, respectively. Intuitively, $I(h_1, h_2)$ measures the degree to which h_1 implies h_2, $E(h_1, h_2)$ quantifies their mutual agreement or similarity, and $O(h_1, h_2)$ represents the mutual effect of h_1 and h_2 under multiplication.

Assume that $\mathcal{C}$ is a closure system in $\mathcal{Q}(X)$. Let $\mathcal{H} = \{h_i\}_{i \in I}$ be a family of hesitant fuzzy sets on X. The *$\mathcal{C}$-closure of $\mathcal{H}$* is defined by:

$$\mathcal{C}_{\mathcal{H}} = \bigcap \{g \in \mathcal{C} \mid h_i \subseteq g, \text{ for every } i \in I\}. \tag{2.22}$$

It is the least element of $\mathcal{C}$ containing all members of $\mathcal{H}$. If $\mathcal{H}$ consists of a single hesitant fuzzy set h, then its $\mathcal{C}$-closure is denoted by $\mathcal{C}_h$ and referred to as the *$\mathcal{C}$-closure of h*. Moreover, it is easy to verify that a $\mathcal{C}$-closure of $\mathcal{H} = \{h_i\}_{i \in I}$ coincides with the $\mathcal{C}$-closure $\mathcal{C}_h$ of their union, where $h = \bigcup_{i \in I} h_i$.

Analogously, let $\mathcal{C}$ be an opening system in $\mathcal{Q}(X)$. For a family $\mathcal{H} = \{h_i\}_{i \in I}$ of hesitant fuzzy sets on X, the $\mathcal{C}$-opening of $\mathcal{H}$ is defined by:

$$\mathcal{C}_{\mathcal{H}} = \bigcup \{g \in \mathcal{C} \mid g \subseteq h_i, \text{ for every } i \in I\}. \tag{2.23}$$

It is the greatest element of $\mathcal{C}$ contained in all members of $\mathcal{H}$. If $\mathcal{H}$ consists of a single hesitant fuzzy set h on X, its $\mathcal{C}$-opening is denoted by $\mathcal{C}_h$ and called the $\mathcal{C}$-opening of h. Furthermore, the $\mathcal{C}$-opening of $\mathcal{H} = \{h_i\}_{i \in I}$ coincides with the $\mathcal{C}$-opening $\mathcal{C}_h$ of their intersection, where $h = \bigcap_{i \in I} h_i$.

If $\mathcal{C}$ is a closure system in $\mathcal{Q}(X)$, it necessarily contains the full hesitant fuzzy set h_S on X. Consequently, for every $x \in X$, the collection $\{h \in \mathcal{C} \mid 1_\otimes \subseteq h(x)\}$ is non-empty, as it contains at least this maximal element. Hence, we defiine:

$$\mathcal{C}_x = \bigcap \{h \in \mathcal{C} \mid 1_\otimes \subseteq h(x)\} \in \mathcal{C}. \tag{2.24}$$

The hesitant fuzzy set $\mathcal{C}_x$ is the $\mathcal{C}$-closure of the singleton crisp set $x \in X$ and is

referred to as the *principal element* of $\mathcal{C}$ generated by x. The collection

$$\mathcal{P}(\mathcal{C}) = \{\mathcal{C}_x \mid x \in X\} \tag{2.25}$$

is called the *principal part* of $\mathcal{C}$.

Given a closure operator φ on the lattice $\mathcal{Q}(X)$, the family of all hesitant fuzzy sets $h \in Q^X$ that are fixed points of φ, i.e., those satisfying $\varphi(h) = h$, forms a closure system. Conversely, for any closure system $\mathcal{C} \subseteq \mathcal{Q}(X)$, there exists a unique closure operator φ such that $\mathcal{C}$ consists precisely of the φ-closed elements. Similarly, if φ is an opening operator on $\mathcal{Q}(X)$, then the collection of all φ-open elements forms an opening system in $\mathcal{Q}(X)$. Conversely, every opening system in $\mathcal{Q}(X)$ arises as the set of fixed points of a unique opening operator. The concepts related to closures and openings of fuzzy sets are thoroughly discussed by Ignjatović et al. [63].

2.2 Hesitant fuzzy relations over the power set quantale

Throughout the remainder of the dissertation, we assume that the underlying monoid $\mathcal{S}$ is commutative. Consequently, the power set quantale $\mathcal{Q}$ of $\mathcal{S}$ is also commutative and admits a unique residuum operation.

In this section, we introduce hesitant fuzzy relations on the universal set X over the power set quantale $\mathcal{Q}$, following the basic definitions given by Tasić et al. [122].

Definition 2.3. Let X and Y be non-empty sets and let $\mathcal{Q}$ be the power set quantale of a monoid $\mathcal{S}$. A *hesitant fuzzy relation R between X and Y over $\mathcal{Q}$*, or simply a *hesitant fuzzy relation R between X and Y* is a function $R : X \times Y \to \mathcal{Q}$.

Observe that every hesitant fuzzy relation between X and Y can be equivalently viewed as a hesitant fuzzy set on $X \times Y$. In the special case where $X = Y$, we simply say that R is a hesitant fuzzy relation on X. The collection of all hesitant fuzzy relations between X and Y with membership values in the quantale $\mathcal{Q}$ is denoted by $Q^{X\times Y}$.

The standard notions of equality, inclusion, joins, meets and ordering of hesitant fuzzy relations are defined analogously to those for hesitant fuzzy sets. The *identity hesitant fuzzy relation* Δ_X on X is defined as:

$$\Delta_X(x_1, x_2) = \begin{cases} 1_\otimes & \text{if } x_1 = x_2, \\ \emptyset & \text{if } x_1 \neq x_2, \end{cases} \tag{2.26}$$

for all $x_1, x_2 \in X$. The *zero hesitant fuzzy relation* $\mathbf{0}$ and the *full hesitant fuzzy relation* $\mathbf{I}$ on X are defined by $\mathbf{0}(x_1, x_2) = \emptyset$ and $\mathbf{I}(x_1, x_2) = S$, respectively, for all $x_1, x_2 \in X$. The set of all hesitant fuzzy relations on X is denoted by $Q^{X\times X}$.

Definition 2.4. Let X, Y, Z be non-empty sets and let $R \in Q^{X\times Y}$ and $T \in Q^{Y\times Z}$ be hesitant fuzzy relations. The *composition* of R and T is the hesitant fuzzy relation $R \circ T \in Q^{X\times Z}$ defined by

$$(R \circ T)(x, z) = \bigcup_{y\in Y} R(x, y) \otimes T(y, z), \tag{2.27}$$

for all $x \in X$ and $z \in Z$.

For a hesitant fuzzy relation $R \in Q^{X\times X}$, we define the *n-th power* R^n of R inductively as follows: $R^0 = \Delta_X$ and $R^{n+1} = R^n \circ R$, for each $n \in \mathbb{N}_0$, where $\mathbb{N}_0$ represents the set of non-negative integers.

Let $R \in Q^{X\times Y}$ be a hesitant fuzzy relation. The *inverse of* R is the hesitant fuzzy relation $R^{-1} \in Q^{Y\times X}$ defined by

$$R^{-1}(y, x) = R(x, y), \tag{2.28}$$

for all $x \in X$ and $y \in Y$. Moreover, $(R^{-1})^{-1} = R$.

Definition 2.5. Let X, Y be non-empty sets, let $R \in Q^{X\times Y}$ be a hesitant fuzzy relation, and let $h \in Q^X$ and $g \in Q^Y$ be hesitant fuzzy sets. The compositions $h \circ R \in Q^Y$ and $R \circ g \in Q^X$ are hesitant fuzzy sets, defined by

$$(h \circ R)(y) = \bigcup_{x\in X} h(x) \otimes R(x, y), \quad (R \circ g)(x) = \bigcup_{y\in Y} R(x, y) \otimes g(y), \tag{2.29}$$

for all $x \in X$ and $y \in Y$.

If $\mathcal{Q}$ is a commutative power set quantale, the following identities hold for all $h \in Q^X$, $g \in Q^Y$ and $R \in Q^{X\times Y}$:

$$(R^{-1} \circ h)(y) = (h \circ R)(y), \quad (g \circ R^{-1})(x) = (R \circ g)(x), \tag{2.30}$$

for every $y \in Y$ and $x \in X$.

The composition $h_1 \circ h_2$ of two hesitant fuzzy sets $h_1, h_2 \in Q^X$ is defined by:

$$(h_1 \circ h_2)(x) = \bigcup_{x\in X} h_1(x) \otimes h_2(x), \tag{2.31}$$

for all $x \in X$. The value $h_1 \circ h_2$ represents the *degree of overlap* between h_1 and h_2.

The associativity and monotonicity of $\otimes$ directly yield the following properties of the composition of hesitant fuzzy relations and hesitant fuzzy sets.

Proposition 2.2. *Let $R, T \in Q^{X\times X}$ be hesitant fuzzy relations and let $h, g \in Q^X$ be hesitant fuzzy sets. Then*

$$(h \circ R) \circ T = h \circ (R \circ T), \quad (h \circ R) \circ g = h \circ (R \circ g) \tag{2.32}$$

$$(R \circ T) \circ h = R \circ (T \circ h),$$

$$R \subseteq T \quad \Rightarrow \quad R \circ h \subseteq T \circ h, \quad h \circ R \subseteq h \circ T. \tag{2.33}$$

Proposition 2.3. *Let $R \in Q^{X\times X}$ be hesitant fuzzy relation, let $h \in Q^X$ be hesitant fuzzy set, and let $\{R_i\}_{i\in I} \subseteq Q^{X\times X}$ and $\{h_i\}_{i\in I} \subseteq Q^X$ be be families of hesitant fuzzy relations and hesitant fuzzy sets, respectively. Then the following assertions are satisfied:*

$$h \circ \left(\bigcup_{i\in I} R_i\right) = \bigcup_{i\in I}(h \circ R_i), \qquad \left(\bigcup_{i\in I} R_i\right) \circ h = \bigcup_{i\in I}(R_i \circ h), \tag{2.34}$$

$$h \circ \left(\bigcap_{i\in I} R_i\right) \subseteq \bigcap_{i\in I}(h \circ R_i), \qquad \left(\bigcap_{i\in I} R_i\right) \circ h \subseteq \bigcap_{i\in I}(R_i \circ h), \tag{2.35}$$

$$\left(\bigcup_{i\in I} h_i\right) \circ R = \bigcup_{i\in I}(h_i \circ R), \qquad R \circ \left(\bigcup_{i\in I} h_i\right) = \bigcup_{i\in I}(R \circ h_i), \tag{2.36}$$

$$\left(\bigcap_{i\in I} h_i\right) \circ R \subseteq \bigcap_{i\in I}(h_i \circ R), \qquad R \circ \left(\bigcap_{i\in I} h_i\right) \subseteq \bigcap_{i\in I}(R \circ h_i). \tag{2.37}$$

The assertions of Proposition 2.3 follow from the definition of composition and the properties of the quantale multiplication $\otimes$.

Proposition 2.4. *Let X, Y, Z and D be non-empty sets and let $R \in Q^{X\times Y}$, $T, T_1, T_2 \in Q^{Y\times Z}$, $V \in Q^{Z\times D}$, and $\{T_i\}_{i\in I} \subseteq Q^{Y\times Z}$. Then the following assertions are satisfied:*

$$(R \circ T)^{-1} = T^{-1} \circ R^{-1}, \tag{2.38}$$

$$R \circ (T \circ V) = (R \circ T) \circ V, \tag{2.39}$$

$$T_1 \subseteq T_2 \quad \Rightarrow \quad T_1^{-1} \subseteq T_2^{-1}, \tag{2.40}$$

$$T_1 \subseteq T_2 \quad \Rightarrow \quad R \circ T_1 \subseteq R \circ T_2, \quad T_1 \circ V \subseteq T_2 \circ V, \tag{2.41}$$

$$R \circ \left(\bigcup_{i\in I} T_i\right) = \bigcup_{i\in I} (R \circ T_i), \qquad \left(\bigcup_{i\in I} T_i\right) \circ V = \bigcup_{i\in I}(T_i \circ V). \tag{2.42}$$

Proof. We first prove the equality in (2.38). By the commutativity of $\otimes$, for arbitrary

$x \in X$ and $z \in Z$, we have:

$$\Big((R \circ T)^{-1}\Big)(z,x) = (R \circ T)(x,z) = \bigcup_{y \in Y} R(x,y) \otimes T(y,z)$$
$$= \bigcup_{y \in Y} T^{-1}(z,y) \otimes R^{-1}(y,x) = (T^{-1} \circ R^{-1})(z,x).$$

Next, we prove the equality in (2.39). Applying the definition of the composition of hesitant fuzzy relations given in (2.27) and using the associativity of $\otimes$ and its distributivity over arbitrary joins in $\mathcal{Q}$, we obtain, for arbitrary $x \in X$, $d \in D$:

$$\begin{aligned}\Big(R \circ (T \circ V)\Big)(x,d) &= \bigcup_{y \in Y} R(x,y) \otimes (T \circ V)(y,d) \\ &= \bigcup_{y \in Y} R(x,y) \otimes \left(\bigcup_{z \in Z} T(y,z) \otimes V(z,d) \right) \\ &= \bigcup_{z \in Z} \left(\bigcup_{y \in Y} R(x,y) \otimes T(y,z) \right) \otimes V(z,d) \\ &= \bigcup_{z \in Z} (R \circ T)(x,z) \otimes V(z,d) \\ &= \Big((R \circ T) \circ V\Big)(x,d).\end{aligned}$$

To prove the implication in (2.40), suppose that $T_1 \subseteq T_2$. Then, for all $x \in X$ and $y \in Y$, we have

$$T_1^{-1}(y,x) = T_1(x,y) \subseteq T_2(x,y) = T_2^{-1}(y,x).$$

Next, consider the implications in (2.41). Suppose that $T_1(y,z) \subseteq T_2(y,z)$ for all $y \in Y$, $z \in Z$. By the isotonicity of $\otimes$, for all $x \in X$ and $z \in Z$, we have:

$$\begin{aligned}(R \circ T_1)(x,z) &= \bigcup_{y \in Y} R(x,y) \otimes T_1(y,z) \subseteq \bigcup_{y \in Y} R(x,y) \otimes T_2(y,z) \\ &= (R \circ T_2)(x,z).\end{aligned}$$

The remaining inclusion in (2.41) can be proved analogously.

The equalities in (2.42) follow directly from the distributivity of $\otimes$ over arbitrary joins in $\mathcal{Q}$. Indeed, for arbitrary $x \in X$ and $z \in Z$, we have:

$$\begin{aligned}R \circ \left(\bigcup_{i \in I} T_i \right)(x,z) &= \bigcup_{y \in Y} R(x,y) \otimes \left(\bigcup_{i \in I} T_i(y,z) \right) = \bigcup_{i \in I} \bigcup_{y \in Y} R(x,y) \otimes T_i(y,z) \\ &= \bigcup_{i \in I} (R \circ T_i)(x,z).\end{aligned}$$

The remaining equality can be proved analogously. □

The next proposition characterizes the structure formed by all hesitant fuzzy relations on a non-empty set X.

Proposition 2.5. *The algebra $\mathcal{R}(X) = \langle Q^{X\times X}, \cup, \cap, \circ, \mathbf{0}, \mathbf{I}\rangle$ is a unital quantale in which the least element is the zero hesitant fuzzy relation $\mathbf{0}$, the greatest element is the full hesitant fuzzy relation $\mathbf{I}$, and the identity element is the identity hesitant fuzzy relation Δ_X.*

Proof. It is straightforward to verify that the structure $\langle Q^{X\times X}, \cup, \cap, \mathbf{0}, \mathbf{I}\rangle$ is a complete lattice, where $\mathbf{0}$ and $\mathbf{I}$ are its least and greatest elements, respectively. Since (2.39) holds, the algebraic structure $\langle Q^{X\times X}, \circ\rangle$ is a semigroup, and, together with the identity relation Δ_X, it forms a monoid. Finally, the infinite distributivity laws (2.42) also hold, which completes the proof of the proposition. □

If X and Y are finite sets of cardinalities n and m, respectively, then every $R \in Q^{X\times Y}$ can be interpreted as an $n \times m$ hesitant fuzzy matrix over $\mathcal{Q}$, i.e., a matrix whose entries are subsets of S. For hesitant fuzzy relations $R \in Q^{X\times Y}$ and $T \in Q^{Y\times Z}$, where $|Z| = l$, the composition $R \circ T$ corresponds to the product of the associated hesitant fuzzy matrices and is represented by an $n \times l$ matrix. Similarly, if $h \in Q^X$ and $g \in Q^Y$, then $h \circ R$ can be treated as the product of a $1 \times n$ row vector and $n \times m$ matrix, while $R \circ g$ corresponds to the product of an $n \times m$ matrix and an $m \times 1$ column vector. Moreover, for $h, f \in Q^X$, the composition $h \circ f$ can be interpreted as the product of a row vector and a column vector.

Let X and Y be non-empty sets, let $R \in Q^{X\times Y}$ be a hesitant fuzzy relation, and let $h \in Q^X$, $g \in Q^Y$ be hesitant fuzzy sets. The *left $\circ$-residual* of g by R is a hesitant fuzzy set $g \not\phi R \in Q^X$ defined by:

$$(g \not\phi R)(x) = \bigcap_{y\in Y} R(x, y) \to g(y), \tag{2.43}$$

for all $x \in X$, and the *right $\circ$-residual* of h by R is a hesitant fuzzy set $R \not\backslash h \in Q^Y$ defined by:

$$(R \not\backslash h)(y) = \bigcap_{x\in X} R(x, y) \to h(x), \tag{2.44}$$

for all $y \in Y$. Hence, the following residuation properties hold:

$$h \circ R \subseteq g \Leftrightarrow h \subseteq g \not\phi R, \quad R \circ g \subseteq h \Leftrightarrow g \subseteq R \not\backslash h. \tag{2.45}$$

Observe that, if $R \in Q^{X\times X}$ is a symmetric hesitant fuzzy relation (see Section 2.2.1) and h is a hesitant fuzzy set on the same universe X, then

$$h \not\phi R = R \not\backslash h. \tag{2.46}$$

Additionally, let $h, g \in Q^{X\times X}$ be hesitant fuzzy sets. The *left* $\circ$*-residual* of h by g is a hesitant fuzzy relation $h \not\phi g \in Q^{X\times X}$, defined by:

$$(h \not\phi g)(x, y) = g(y) \to h(x), \tag{2.47}$$

for arbitrary $x, y \in X$. The *right* $\circ$*-residual* of h by g is a hesitant fuzzy relation $g \not\backslash h \in Q^{X\times X}$, defined by:

$$(g \not\backslash h)(x, y) = g(x) \to h(y), \tag{2.48}$$

for arbitrary $x, y \in X$. In that case, the following residuation properties hold for arbitrary $h, g \in Q^X$ and $R \in Q^{X\times X}$:

$$g \circ R \subseteq h \Leftrightarrow R \subseteq g \not\backslash h, \quad R \circ g \subseteq h \Leftrightarrow R \subseteq h \not\phi g. \tag{2.49}$$

Proposition 2.6. *Let $h \in Q^X$ and $g \in Q^Y$ be hesitant fuzzy sets, let $R \in Q^{X\times Y}$ and be hesitant fuzzy relations, and let $\{R_i\}_{i\in I} \subseteq Q^{X\times Y}$ be a family of hesitant fuzzy relations. Then the following assertions are satisfied:*

$$(g \not\phi R) \circ R \subseteq g, \quad R \circ (R \not\backslash h) \subseteq h, \tag{2.50}$$

$$g \not\phi \Big(\bigcup_{i\in I} R_i\Big) = \bigcap_{i\in I} (g \not\phi R_i), \quad \Big(\bigcup_{i\in I} R_i\Big) \not\backslash h = \bigcap_{i\in I} (R_i \not\backslash h). \tag{2.51}$$

Proof. The inclusions in (2.50) follow directly from (1.35), while the identities in (2.51) follow from (1.50). □

Proposition 2.7. *Let $h \in Q^X$ be a hesitant fuzzy set and let $R, T \in Q^{X\times X}$ be hesitant fuzzy relations.*

$$R \not\backslash (T \not\backslash h) = (T \circ R) \not\backslash h, \quad (h \not\phi R) \not\phi T = h \not\phi (T \circ R).$$

Proof. Using the definitions of $\circ$-residuals and the quantale properties (1.40) and

(1.50) we obtain, for every $x \in X$, the following sequence of equalities:

$$\begin{aligned}
\Big((h \not\backslash R) \not/ T\Big)(x) &= \bigcap_{a\in X}\Big(T(x,a) \to (h \not\backslash R)(a)\Big) = \bigcap_{a\in X} T(x,a) \to \Big(\bigcap_{b\in X} R(a,b) \to h(b)\Big) \\
&= \bigcap_{a\in X}\bigcap_{b\in X}\Big(T(x,a) \to \Big(R(a,b) \to h(b)\Big)\Big) \\
&= \bigcap_{b\in X}\bigcap_{a\in X}\Big(\Big(T(x,a) \otimes R(a,b)\Big) \to h(b)\Big) \\
&= \bigcap_{b\in X}\Big(\Big(\bigcup_{a\in X} T(x,a) \otimes R(a,b)\Big) \to h(b)\Big) \\
&= \bigcap_{b\in X}(T \circ R)(x,b) \to h(b) \\
&= \Big(h \not\backslash (T \circ R)\Big)(x).
\end{aligned}$$

□

Example 2.2. Let $X = \{x_1, x_2\}$ be the universe and let $\mathcal{Q}$ be a Gödel power set quantale. Observe that this quantale is commutative. Consider the hesitant fuzzy relation R on X and the hesitant fuzzy set h on X defined by:

$$R = \begin{bmatrix} \{0.1\} & \{0.1, 0.6\} \\ \{0.1, 0.6\} & \{0.2, 1\} \end{bmatrix}, \quad h = \begin{bmatrix} \{0.1, 0.3\} & \{0.1, 0.7\} \end{bmatrix}.$$

According to (2.43), the left residual of h by R is computed as follows:

$$\begin{aligned}
(h \not\backslash R)(x_1) &= \Big(R(x_1,x_1) \to h(x_1)\Big) \cap \Big(R(x_1,x_2) \to h(x_2)\Big) \\
&= \Big(\{0.1\} \to \{0.1, 0.3\}\Big) \cap \Big(\{0.1, 0.6\} \to \{0.1, 0.7\}\Big) \\
&= [0.1, 1] \cap \{0.1\} = \{0.1\}, \\
(h \not\backslash R)(x_2) &= \Big(R(x_2,x_1) \to h(x_1)\Big) \cap \Big(R(x_2,x_2) \to h(x_2)\Big) \\
&= \Big(\{0.1, 0.6\} \to \{0.1, 0.3\}\Big) \cap \Big(\{0.2, 1\} \to \{0.1, 0.7\}\Big) \\
&= \{0.1\} \cap \{0.1\} = \{0.1\}.
\end{aligned}$$

Similarly, using (2.44) the right residual of h by R is computed as follows:

$$(R \not/ h)(x_1) = (R \not/ h)(x_2) = \{0.1\}.$$

Therefore, the left and right residuals of h by R coincide:

$$h \not\backslash R = R \not/ h = \begin{bmatrix} \{0.1\} & \{0.1\} \end{bmatrix}.$$

2.2.1 Hesitant fuzzy relations with specific properties

Let $R \in Q^{X\times X}$ be a hesitant fuzzy relation. We say that R is:

(R) *Reflexive*, if $\Delta_X \subseteq R$, i.e., $1_\otimes \subseteq R(x,x)$, for each $x \in X$;

(S) *Symmetric*, if $R = R^{-1}$, i.e., $R(x,y) = R(y,x)$, for each $x, y \in X$;

(T) *Transitive*, if $R \circ R \subseteq R$, i.e.,

$$R(x,y) \otimes R(y,z) \subseteq R(x,z), \quad \text{for each } x,y,z \in X. \tag{2.52}$$

If R is reflexive, then $R \subseteq R \circ R$. Moreover, if R is both reflexive and transitive, then $R \circ R = R$. A reflexive and transitive hesitant fuzzy relation on X is called a hesitant *fuzzy quasi-order on* X. A reflexive, symmetric and transitive hesitant fuzzy relation on X is called a *hesitant fuzzy equivalence relation on* X (or simply a *hesitant fuzzy equivalence on* X). The collections of all hesitant fuzzy quasi-orders and all hesitant fuzzy equivalences on X are denoted by $\mathcal{F}(X)$ and $\mathcal{E}(X)$, respectively.

Notice that the lattice $\mathcal{R}(X)$ of hesitant fuzzy relations on X can be identified with the lattice of hesitant fuzzy sets on $X \times X$. Consequently, the notions of closure and opening systems in $\mathcal{R}(X)$, as well as the corresponding $\mathcal{C}$-closure and $\mathcal{C}$-opening for $\mathcal{C} \subseteq Q^{X\times X}$ are defined analogously to those for hesitant fuzzy sets.

The collection $\mathcal{F}(X)$ of all hesitant fuzzy quasi-orders on X, as well as the collection $\mathcal{E}(X)$ of all hesitant fuzzy equivalence relations on X, form closure systems within the lattice $\mathcal{R}(X)$. The closure of a family of hesitant fuzzy relations (or a single hesitant fuzzy relation) with respect to $\mathcal{F}(X)$ is referred to as the *hesitant fuzzy quasi-order closure*, or simply the *$\mathcal{F}$-closure*, while the closure corresponding to $\mathcal{E}(X)$ is known as the *fuzzy equivalence closure*, or *$\mathcal{E}$-closure* for short. The collections of all reflexive, symmetric, and transitive fuzzy relations within $\mathcal{R}(X)$ also constitute closure systems.

Accordingly, the *reflexive*, *symmetric*, and *transitive closures* can be defined for a single hesitant fuzzy relation or any family of hesitant fuzzy relations. Let $R \in Q^{X\times X}$ be a hesitant fuzzy relation. The *reflexive closure of* R is defined as the smallest reflexive hesitant fuzzy relation $R^r \in Q^{X\times X}$ containing R. It is given by

$$R^r = R \cup \Delta_X. \tag{2.53}$$

Similarly, the *symmetric closure* of R is defined as the smallest symmetric hesitant

fuzzy relation $R^s \in Q^{X\times X}$ containing R, and it is given by

$$R^s = R \cup R^{-1}. \tag{2.54}$$

Finally, the *transitive closure* of R is the smallest transitive hesitant fuzzy relation $R^t \in Q^{X\times X}$ containing R. It can be represented as

$$R^t = \bigcup_{n\in\mathbb{N}} R^n. \tag{2.55}$$

If X is finite, the construction of the transitive closure terminates after finitely many steps.

Using the closure operations described above, an arbitrary hesitant fuzzy relation $R \in Q^{X\times X}$ can be extended to a reflexive, symmetric, and transitive hesitant fuzzy relation on X. The resulting *reflexive-transitive closure* R^f of R is given by

$$R^f = (R^r)^t = (R^t)^r = \bigcup_{n\in\mathbb{N}_0} R^n. \tag{2.56}$$

Similarly, *reflexive-symmetric-transitive closure* R^e of R can be expressed as:

$$R^e = ((R^r)^s)^t = ((R^s)^r)^t = ((R^s)^t)^r. \tag{2.57}$$

Example 2.3. To illustrate the concepts introduced above, we consider Gödel power set quantale $\mathcal{Q}$ and a hesitant fuzzy relation R on $X = \{x_1, x_2, x_3\}$ over $\mathcal{Q}$, given by:

$$R = \begin{bmatrix} \{0.1\} & \{0.3\} & \{0.5\} \\ \{1\} & \{0.2\} & \{0.7\} \\ \{0.8\} & \{0.3, 1\} & \{0.5\} \end{bmatrix}.$$

The reflexive closure R^r of R is:

$$R^r = \begin{bmatrix} \{0.1, 1\} & \{0.3\} & \{0.5\} \\ \{1\} & \{0.2, 1\} & \{0.7\} \\ \{0.8\} & \{0.3, 1\} & \{0.5, 1\} \end{bmatrix}.$$

To obtain the reflexive-transitive closure of R, we proceed by computing the transi-

tive closure of R^r according to the procedure given in (2.56):

$$(R^r)^2 = \begin{bmatrix} \{0.1, 0.3, 0.5, 1\} & \{0.1, 0.2, 0.3, 0.5\} & \{0.1, 0.3, 0.5\} \\ \{0.1, 0.2, 0.7, 1\} & \{0.2, 0.3, 0.7, 1\} & \{0.2, 0.5, 0.7\} \\ \{0.1, 0.3, 0.5, 0.8, 1\} & \{0.2, 0.3, 0.5, 1\} & \{0.3, 0.5, 0.7, 1\} \end{bmatrix}.$$

Since $((R^r)^2 \circ (R^r)^2)(x_1, x_1) = \{0.1, 0.2, 0.3, 0.5, 1\} \not\subseteq (R^r)^2(x_1, x_1)$, it follows that $(R^r)^2$ is not transitive. Next, we compute $(R^r)^3$:

$$(R^r)^3 = \begin{bmatrix} \{0.1, 0.2, 0.3, 0.5, 1\} & \{0.1, 0.2, 0.3, 0.5\} & \{0.1, 0.2, 0.3, 0.5\} \\ \{0.1, 0.2, 0.3, 0.5, 0.7, 1\} & \{0.1, 0.2, 0.3, 0.5, 0.7, 1\} & \{0.1, 0.2, 0.3, 0.5, 0.7\} \\ \{0.1, 0.2, 0.3, 0.5, 0.7, 0.8, 1\} & \{0.1, 0.2, 0.3, 0.5, 0.7, 1\} & \{0.1, 0.2, 0.3, 0.5, 0.7, 1\} \end{bmatrix}.$$

A direct computation shows that $(R^r)^3 \circ (R^r)^3 = (R^r)^3$, so the hesitant fuzzy relation $(R^r)^3$ is transitive. Since it is also reflexive and contains R, it follows that $(R^r)^3$ is the reflexive-transitive closure of R.

In the following, we introduce fundamental concepts associated with hesitant fuzzy relations, which extends the corresponding notions from classical and fuzzy relational theory and play an important role in the analysis of their structural properties.

Let $R \in Q^{X \times X}$ be a hesitant fuzzy relation. For each $x \in X$, the *R-afterset* of x is the hesitant fuzzy set $xR \in Q^X$ defined by $xR(y) = R(x, y)$, and the *R-foreset* of x is the hesitant fuzzy set $Rx \in Q^X$ defined by $Rx(y) = R(y, x)$, for any $y \in X$. The collection of all R-aftersets is denoted by X/F, while the collection of all R-foresets is denoted by $X \backslash F$. Under the matrix representation of R, R-aftersets and R-foresets are represented by the row and column vectors of the corresponding hesitant fuzzy matrix, respectively.

If E is a hesitant fuzzy equivalence relation, we denote E_x instead of $xE = Ex$ and we refer to E_x as the *hesitant fuzzy equivalence class of* E, or just an *E-class*, determined by the element x. Hence, $E_x(a) = E(x, a)$ for all $a \in X$. The collection of all equivalence classes of E is referred to as the *factor set of* X with respect to E, and is denoted by X/E, i.e.,

$$X/E = \{E_x \mid x \in X\}. \tag{2.58}$$

Lemma 2.1. *Let h be a hesitant fuzzy set on X. The left and right $\circ$-residuals of h with respect to itself, denoted by $h \not/ h$ and $h \not\backslash h$, respectively, are both hesitant fuzzy*

quasi-orders on X.

Proof. For any hesitant fuzzy set h on X, it is straightforward to verify that $1_{\otimes} \subseteq (h \not\phi h)(x,x)$ and $1_{\otimes} \subseteq (h \not\backslash h)(x,x)$ for all $x \in X$. Consequently, by property (1.39), both hesitant fuzzy relations $h \not\phi h$ and $h \not\backslash h$, defined in (2.47) and (2.48) as the left and right $\circ$-residuals of h with respect to itself, are hesitant fuzzy quasi-orders on X. □

Proposition 2.8. *Let F_1 and F_2 be two hesitant fuzzy quasi-orders on X. Then the following equivalence holds:*

$$F_1 \subseteq F_2 \quad \Leftrightarrow \quad F_1 \circ F_2 = F_2 \circ F_1 = F_2. \tag{2.59}$$

Proof. Assume that $F_1 \subseteq F_2$. Since F_2 is a hesitant fuzzy quasi-order, it is transitive, and hence, $F_2 \circ F_2 \subseteq F_2$. Then, for all $x, y \in X$:

$$\begin{aligned} F_1 \circ F_2(x,y) &= \bigcup_{z \in X} F_1(x,z) \otimes F_2(z,y) \\ &\subseteq \bigcup_{z \in X} F_2(x,z) \otimes F_2(z,y) \\ &= (F_2 \circ F_2)(x,y) \\ &= F_2(x,y). \end{aligned}$$

Therefore, $F_1 \circ F_2 \subseteq F_2$. On the other hand, since F_1 is a hesitant fuzzy quasi-order, then F_1 is reflexive, so $1_{\otimes} \subseteq F_1(x,x)$ for all $x \in X$. Consequently, for all $x, y \in X$:

$$F_2(x,y) \subseteq F_1(x,x) \otimes F_2(x,y) \subseteq \bigcup_{z \in X} F_1(x,z) \otimes F_2(z,y) = (F_1 \circ F_2)(x,y).$$

This proves the reverse inclusion $F_2 \subseteq F_1 \circ F_2$, and hence $F_1 \circ F_2 = F_2$. The equality $F_2 \circ F_1 = F_2$ follows by an analogous argument.

Conversely, let $F_1 \circ F_2 = F_2 \circ F_1 = F_2$. Then for arbitrary $x, y \in X$ we have that

$$\begin{aligned} F_1(x,y) &= F_1(x,y) \otimes 1_{\otimes} \subseteq F_1(x,y) \otimes F_2(y,y) \\ &\subseteq \bigcup_{z \in X} F_1(x,z) \otimes F_2(z,y) = (F_1 \circ F_2)(x,y) = F_2(x,y), \end{aligned}$$

and therefore, $F_1 \subseteq F_2$. This completes the proof of the proposition. □

The following lemma establishes two useful properties that will be used in several proofs in the sequel.

Lemma 2.2. *Let $h, g \in Q^X$. If $1_\otimes \subseteq h(a)$, for some $a \in X$, then, for every $x \in X$,*

$$g(x) \subseteq h(a) \otimes g(x), \tag{2.60}$$

$$h(a) \to g(x) \subseteq g(x). \tag{2.61}$$

Proof. Since $1_\otimes \subseteq h(a)$ and $1_\otimes \otimes g(x) = g(x)$ for each $x \in X$, the monotonicity of $\otimes$ yields

$$1_\otimes \otimes g(x) \subseteq h(a) \otimes g(x),$$

which proves (2.60). Furthermore, (2.61) follows from the property (1.44) and the antitonicity of $\to$ in the first argument. □

Proposition 2.9. *Let $R \in Q^{X\times X}$ be an arbitrary hesitant fuzzy relation. The following properties hold:*

(1) *Functions $h \mapsto h \circ R$, $h \mapsto h \not\circ R$, $h \mapsto R \circ h$ and $h \mapsto R \not\circ h$ are isotone self-maps of the lattice $\mathcal{Q}(X)$.*

(2) *If R is reflexive, the functions $h \mapsto h \circ R$ and $h \mapsto R \circ h$ are extensive, while $h \mapsto h \not\circ R$ and $h \mapsto R \not\circ h$ are intensive.*

(3) *If R is a hesitant fuzzy quasi-order, then each of the functions $h \mapsto h \circ R$, $h \mapsto h \not\circ R$, $h \mapsto R \circ h$ and $h \mapsto R \not\circ h$ is idempotent.*

Proof. (1) Let $h_1, h_2 \in Q^{X\times X}$ be hesitant fuzzy sets satisfying $h_1 \subseteq h_2$. First, we show that $h \mapsto h \circ R$ is isotone. By the isotonicity of $\otimes$, the following holds for all $x \in X$:

$$(h_1 \circ R)(x) = \bigcup_{y\in X} h_1(y) \otimes R(y, x) \subseteq \bigcup_{y\in X} h_2(y) \otimes R(y, x) = (h_2 \circ R)(x).$$

Hence, $h \mapsto h \circ R$ is isotone. Next, we consider $h \mapsto h \not\circ R$. Since the left $\circ$-residual $\not\circ$ is isotone in its second argument and $h_1 \subseteq h_2$, it follows that for all $x \in X$:

$$(h_1 \not\circ R)(x) = \bigcap_{y\in X} R(x, y) \to h_1(y) \subseteq \bigcap_{y\in X} R(x, y) \to h_2(y) = (h_2 \not\circ R)(x).$$

Therefore, $h \mapsto h \not\circ R$ is isotone. The isotonicity of the functions $h \mapsto R \circ h$ and $h \mapsto R \not\circ h$ follows by analogous reasoning.

(2) Let $R \in Q^{X\times X}$ be a reflexive hesitant fuzzy relation, that is, $1_\otimes \subseteq R(x, x)$ for all $x \in X$. We first show that $h \mapsto h \circ R$ is extensive. It follows from Lemma

2.2 that, for every $x \in X$,

$$h(x) \subseteq h(x) \otimes R(x,x) \subseteq \bigcup_{y \in X} h(y) \otimes R(y,x) = (h \circ R)(x).$$

Hence, $h \mapsto h \circ R$ is extensive. Next, we consider the mapping $h \mapsto h \not\phi R$. Since $1_\otimes \subseteq R(x,x)$, Lemma 2.2 yields that for every hesitant fuzzy set $h \in Q^X$,

$$(h \not\phi R)(x) = \bigcap_{y \in X} R(x,y) \to h(y) \subseteq R(x,x) \to h(x) \subseteq h(x),$$

for all $x \in X$. Thus, the mapping $h \mapsto h \not\phi R$ is intensive. The remaining assertions follow by analogous arguments.

(3) Assume that R is a hesitant fuzzy quasi-order. First, we show that $h \mapsto h \circ R$ is idempotent, that is, $(h \circ R) \circ R = h \circ R$. Since $h \mapsto h \circ R$ is extensive, we immediately have $(h \circ R) \circ R \supseteq h \circ R$. For the reverse inclusion, we use the transitivity of R and the isotonicity of $\otimes$. For every $x \in X$, we obtain,

$$\begin{aligned}
\Big((h \circ R) \circ R\Big)(x) &= \bigcup_{y \in X} (h \circ R)(y) \otimes R(y,x) \\
&= \bigcup_{y \in X} \left(\bigcup_{z \in X} h(z) \otimes R(z,y) \right) \otimes R(y,x) \\
&= \bigcup_{z \in X} h(z) \otimes \left(\bigcup_{y \in X} R(z,y) \otimes R(y,x) \right) \\
&\subseteq \bigcup_{z \in X} h(z) \otimes R(z,x) \\
&= (h \circ R)(x).
\end{aligned}$$

Thus, $(h \circ R) \circ R = (h \circ R)$ for every $h \in Q^X$ and the function $h \mapsto h \circ R$ is idempotent. Next, we show the idempotency of $h \mapsto h \not\phi R$. Since it is intensive, we immediately obtain $(h \not\phi R) \supseteq (h \not\phi R) \not\phi R$. For the converse, recall that the transitivity of R yields $R(x,y) \otimes R(y,z) \subseteq R(x,z)$, for all $x, y, z \in X$. By the antitonicity of the left residuum in its first argument, this implies:

$$\Big(R(x,y) \otimes R(y,z)\Big) \to h(z) \supseteq R(x,z) \to h(z),$$

for all $x, y, z \in X$. By properties (1.46) and (1.40), together with the transitivity of

R and antitonicity of the residuum in its first argument, we obtain:

$$\begin{aligned}\Big((h \not\leftarrow R) \not\leftarrow R\Big)(x) &= \bigcap_{y\in X} R(x,y) \to \left(\bigcap_{z\in X} R(y,z) \to h(z)\right)\\ &= \bigcap_{y\in X}\bigcap_{z\in X}\Big(R(x,y)\otimes R(y,z)\Big) \to h(z)\\ &\supseteq \bigcap_{y\in X}\left(\bigcap_{z\in X} R(x,z)\to h(z)\right)\\ &= (h \not\leftarrow R)(x),\end{aligned}$$

for all $x \in X$. Therefore, $(h \not\leftarrow R) \not\leftarrow R = h \not\leftarrow R$, and the function $h \mapsto h \not\leftarrow R$ is idempotent. The remaining assertions follow by similar arguments. □

2.2.2 Residuals of hesitant fuzzy relations

In this subsection, we introduce residuals of hesitant fuzzy relations and establish several results concerning their properties.

Theorem 2.1. *Let $\mathcal{Q} = \langle Q, \cup, \cap, \otimes, \emptyset, S\rangle$ be a commutative unital powerset quantale of S, as defined in Example 1.3 satisfying property (1.31). The following conditions are equivalent:*

(1) *There exists a binary operation $\to$ such that $(\otimes, \to)$ form an adjoint pair.*

(2) *For all $x, y \in Q$, the set $\{z \in Q \mid x \otimes z \subseteq y\}$ has a greatest element.*

(3) *For any two hesitant fuzzy relations $V, W \in Q^{X\times X}$, the right $\circ$-residual $V \backslash\!\!\!\circ W$ exists and satisfies $V \circ U \subseteq W \Leftrightarrow U \subseteq V \backslash\!\!\!\circ W$.*

(4) *For any two hesitant fuzzy relations $V, W \in Q^{X\times Y}$, the left $\circ$-residual $W \not/\!\!\!\circ V$ exists and satisfies $U \circ V \subseteq W \Leftrightarrow U \subseteq W \not/\!\!\!\circ V$.*

Proof. Recall that the structure $\langle Q, \cup, \cap, \emptyset, S\rangle$, introduced in Example 1.3 is a complete lattice with $\emptyset$ as the least element and S as the greatest element.

(1) $\Leftrightarrow$ (2) This equivalence is a straightforward consequence of the basic properties of $\mathcal{Q}$. It follows from (1.35) that $x \otimes (x \to y) \subseteq y$ for all $x, y \in Q$. Hence,

$$x \to y \in \{z \in Q \mid x \otimes z \subseteq y\}.$$

Moreover, by residuation property (1.16), $x \otimes z \subseteq y \Leftrightarrow z \subseteq x \to y$. Hence, $x \to y$ is the greatest element of the set $\{z \in Q \mid x \otimes z \subseteq y\}$.

(1) $\Rightarrow$ (3) Define a hesitant fuzzy relation $R \in Q^{X\times X}$ by:

$$R(x,z) = \bigcap_{y\in X} V(y,x) \to W(y,z),$$

for all $x, z \in X$. By properties (1.45) and (1.35), we obtain:

$$\begin{aligned}(V\circ R)(x,y) &= \bigcup_{z\in X} V(x,z)\otimes R(z,y) = \bigcup_{y\in X} V(x,z)\otimes\left(\bigcap_{t\in X} V(t,z)\to W(t,y)\right)\\ &\subseteq \bigcup_{z\in X}\bigcap_{t\in X}\Big(V(x,z)\otimes\big(V(t,z)\to W(t,y)\big)\Big)\\ &\subseteq \bigcup_{z\in X}\Big(V(x,z)\otimes\big(V(x,z)\to W(x,y)\big)\Big)\\ &\subseteq W(x,y).\end{aligned}$$

for all $x, y \in X$. Therefore, $V \circ R \subseteq W$. Next, we prove that R is the greatest hesitant fuzzy relation satisfying this property. Let $T \in Q^{X\times X}$ be a hesitant fuzzy relation such that $V \circ T \subseteq W$. For arbitrary $x, y, z \in X$, we obtain:

$$V(y,x)\otimes T(x,z) \subseteq \bigcup_{t\in X} V(y,t)\otimes T(t,z) = (V\circ T)(y,z) \subseteq W(y,z).$$

By the residuation property (1.27), it follows that $T(x,z) \subseteq V(y,x) \to W(y,z)$. Consequently,

$$T(x,z) \subseteq \bigcap_{y\in X} V(y,x)\to W(y,z) = R(x,z),$$

for all $x, y, z \in X$, and hence, $T \subseteq R$. Therefore, R is the greatest solution to the inequality $V \circ U \subseteq W$, where $U \in Q^{X\times X}$ is unknown hesitant fuzzy relation. Hence, by the definition of the right $\circ$-residual, we obtain $V \backslash\!\!\circ W = U$.

Using a similar argument, we obtain 1. $\Rightarrow$ 4.

(3) $\Rightarrow$ (2) Let $a, b \in Q$ and let $X = \{X\}$. Define hesitant fuzzy relations $V, W \in Q^{X\times X}$ by $V(x,x) = a$ and $W(x,x) = b$, respectively, and let $R = V \backslash\!\!\circ W$. Denote $R(x,x) = u$. Then we have:

$$a\otimes u = V(x,x)\otimes R(x,x) = (V\circ R)(x,x) \subseteq W(x,x) = b.$$

Hence, $u \in \{z \in Q \mid x \otimes z \subseteq y\}$. Next, we show that u is the greatest element with this property. Let $v \in Q$ be such that $a \otimes v \subseteq b$ and define a hesitant fuzzy relation T on X by $T(x,x) = v$. Then, $V \circ T \subseteq W$, and consequently, $T \subseteq R$, which implies

$v \subseteq u$. Therefore, the set $\{z \in Q \mid x \otimes z \subseteq y\}$ has the greatest element u.

Using a similar argument, we obtain (4) $\Rightarrow$ (2). □

In the following, we introduce the right and left $\circ$-residuals on hesitant fuzzy relations, as a consequence of Theorem 2.1. Let X and Y be non-empty sets, and let $R \in Q^{X\times X}$, $T \in Q^{Y\times Y}$ and $V \in Q^{X\times Y}$ be hesitant fuzzy relations. The *right $\circ$-residual of V by R* is a hesitant fuzzy relation $R \backslash V \in Q^{X\times Y}$ defined by

$$(R \backslash V)(x, y) = \bigcap_{a\in X} R(a, x) \to V(a, y), \tag{2.62}$$

for every $x \in X$, $y \in Y$, while the *left $\circ$-residual of V by T* is a hesitant fuzzy relation $V / T \in Q^{X\times Y}$ defined by

$$(V / T)(x, y) = \bigcap_{b\in Y} T(y, b) \to V(x, b), \tag{2.63}$$

for every $x \in X$, $y \in Y$. Verifying the following residuation properties for an unknown hesitant fuzzy relation $U \in Q^{X\times Y}$ is straightforward.

$$R \circ U \subseteq V \quad \text{iff} \quad U \subseteq R \backslash V \tag{2.64}$$

$$U \circ T \subseteq V \quad \text{iff} \quad U \subseteq V / T \tag{2.65}$$

$$\tag{2.66}$$

Theorem 2.2. *Let V be a hesitant fuzzy relation on X. Then the following statements hold:*

(1) *V / V and $V \backslash V$ are hesitant fuzzy quasi-orders.*

(2) *V / V is the greatest solution to the equation $V \circ U = V$, where $U \in Q^{X\times X}$ is an unknown hesitant fuzzy relation.*

(3) *$V \backslash V$ is the greatest solution to the equation $U \circ V = V$, where $U \in Q^{X\times X}$ is an unknown hesitant fuzzy relation.*

Proof. (1) By property (1.39), we have:

$$\begin{aligned}(V / V)(x, y) \otimes (V / V)(y, z) &= \bigcap_{a\in X} V(a, x) \to V(a, y) \otimes \bigcap_{b\in X} V(b, y) \to V(b, z) \\ &\subseteq \Big(V(t, x) \to V(t, y)\Big) \otimes \Big(V(t, y) \to V(t, z)\Big) \\ &\subseteq V(t, x) \to V(t, z).\end{aligned}$$

for every $x, y, z, t \in X$. Hence:

$$(V \not\phi V)(x, y) \otimes (V \not\phi V)(y, z) \subseteq \bigcap_{t \in X} V(t, x) \to V(t, z) = (V \not\phi V)(x, z),$$

for all $X, y, z \in X$, and consequently, $V \not\phi V$ is transitive.

To prove reflexivity, observe that $1_{\otimes} \otimes (V \not\phi V)(x, x) = (V \not\phi V)(x, x)$ for each $x \in X$. By the residuation property (1.26), it follows that

$$1_{\otimes} \subseteq (V \not\phi V)(x, x) \to (V \not\phi V)(x, x),$$

which immediately implies $1_{\otimes} \subseteq (V \not\phi V)(x, x)$.

Analogously, one can verify that $V \not\backslash V$ also defines a hesitant fuzzy quasi-order.

(2) According to the definition of the right residual, $V \not\phi V$ represents the greatest solution U to the inequality $V \circ U \subseteq V$. Furthermore, the reflexivity of $V \not\phi V$ ensures that, for any $x, z \in X$, the following holds:

$$\begin{aligned} V(x, z) &= V(x, z) \otimes (V \not\phi V)(z, z) \subseteq \bigcup_{y \in X} V(x, y) \otimes (V \not\phi V)(y, z) \\ &= \Big(V \circ (V \not\phi V)\Big)(x, z). \end{aligned}$$

Therefore, $V \subseteq V \circ (V \not\phi V)$. Thus, $V \not\phi V$ satisfies the equation $V \circ U = V$. Moreover, since any solution to $V \circ U = V$ also satisfies $V \circ U \subseteq V$, it follows that $V \not\phi V$ is the largest solution to $V \circ U = V$.

The statement (3) is proved analogously. □

Theorem 2.3. *Let $V \in Q^{X \times X}$ be a hesitant fuzzy relation. Then the following conditions are equivalent:*

(1) *V is a hesitant fuzzy quasi-order,*

(2) *$V \not\phi V = V$,*

(3) *$V \not\backslash V = V$.*

Proof. (1) $\Leftrightarrow$ (2) If $V \not\phi V = V$, then, by Theorem 2.2, V is a hesitant fuzzy quasi-order. Conversely, assume that V is a hesitant fuzzy quasi-order. Since V is reflexive, it follows that $V \not\phi V \subseteq V \circ (V \not\phi V) = V$. On the other hand, both V and $V \not\phi V$ are solutions to the equation $V \circ U = V$, and since $V \not\phi V$ is the greatest among them, we obtain $V \subseteq V \not\phi V$. Hence, $V \not\phi V = V$.

The equivalence between (1) and (3) can be shown in a similar way. □

2.2.3 Additional Properties of Residuals

In this subsection, we establish further properties of residuals on hesitant fuzzy relations. In addition, we introduce several operations induced by right and left $\circ$-residuals, which will be used in the subsequent results.

By the monotonicity properties of the residual operations (1.15), the right $\circ$-residual ($\backslash\!\!\!\diamond$) is antitone in the first argument and isotone in the second, while the left $\circ$-residual ($\not\diamond$) is isotone in the first argument and antitone in the second. Consequently, for all $R \in Q^{X\times Y}$, $V, W \in Q^{X\times Z}$ and $T \in Q^{Y\times Z}$, the following implications hold:

$$V \subseteq W \quad \Rightarrow \quad V \not\diamond T \subseteq W \not\diamond T, \quad R \backslash\!\!\!\diamond V \subseteq R \backslash\!\!\!\diamond W. \tag{2.67}$$

By property (1.46), for $R \in Q^{X\times Y}$, $\{V_i\}_{i\in I} \subseteq Q^{X\times Z}$ and $T \in Q^{Y\times Z}$, the following equalities hold:

$$\left(\bigcap_{i\in I} V_i\right) \not\diamond T = \bigcap_{i\in I}(V_i \not\diamond T), \qquad R \backslash\!\!\!\diamond \left(\bigcap_{i\in I} V_i\right) = \bigcap_{i\in I}(R \backslash\!\!\!\diamond V_i). \tag{2.68}$$

Proposition 2.10. *Let $R, T \in Q^{X\times X}$. The following properties hold:*

$$R \subseteq (R \circ T) \not\diamond T, \qquad R \subseteq T \backslash\!\!\!\diamond (T \circ R), \tag{2.69}$$

$$T \subseteq \left((R \circ T)^{-1} \not\diamond R^{-1}\right)^{-1}, \qquad T \subseteq \left(R^{-1} \backslash\!\!\!\diamond (T \circ R)^{-1}\right)^{-1}, \tag{2.70}$$

$$(T \not\diamond R) \circ R \subseteq T, \qquad R \circ (R \backslash\!\!\!\diamond T) \subseteq T, \tag{2.71}$$

$$R^{-1} \circ (T^{-1} \not\diamond R)^{-1} \subseteq T, \qquad (R \backslash\!\!\!\diamond T^{-1})^{-1} \circ R^{-1} \subseteq T. \tag{2.72}$$

Proof. We start by verifying the inequalities in (2.69). The proof is an immediate consequence of properties (1.36) and (1.34). For arbitrary $x, z \in X$, we have:

$$\begin{aligned}
\Big((R \circ T) \not\diamond T\Big)(x,z) &= \bigcap_{y\in X} T(z,y) \to (R \circ T)(x,y) \\
&= \bigcap_{y\in X} T(z,y) \to \left(\bigcup_{t\in X} R(x,t) \otimes T(t,y)\right) \\
&= \bigcap_{y\in X} T(z,y) \to \left(\Big(R(x,z) \otimes T(z,y)\Big) \cup \bigcup_{t\neq z} R(x,t) \otimes T(t,y)\right) \\
&\supseteq \bigcap_{y\in X} T(z,y) \to \Big(T(z,y) \otimes R(x,z)\Big) \supseteq R(x,z),
\end{aligned}$$

$$
\begin{aligned}
\Big(T \mathbin{\backslash\!\!\!\circ} (T \circ R)\Big)(x,z) &= \bigcap_{y\in X} T(y,x) \to (T\circ R)(y,z) \\
&= \bigcap_{y\in X} T(y,x) \to \left(\bigcup_{t\in X} (T(y,t)\otimes R(t,z)\right) \\
&= \bigcap_{y\in X} T(y,x) \to \left(\Big(T(y,x)\otimes R(x,z)\Big) \cup \bigcup_{t\neq x} T(y,t)\otimes R(t,z)\right) \\
&\supseteq \bigcap_{y\in X} T(y,x) \to \Big(T(y,x)\otimes R(x,z)\Big) \supseteq R(x,z).
\end{aligned}
$$

The inequalities in (2.70) follow directly from (2.69). For arbitrary $x, z \in X$, we have:

$$
\begin{aligned}
\Big((R\circ T)^{-1} \mathbin{\phi\!\!\!/} R^{-1}\Big)^{-1}(x,z) &= \Big((R\circ T)^{-1} \mathbin{\phi\!\!\!/} R^{-1}\Big)(z,x) \\
&= \bigcap_{y\in X} R(y,x) \to (R\circ T)(y,z) \\
&= \Big(R \mathbin{\backslash\!\!\!\circ} (R\circ T)\Big)(x,z) \supseteq T(x,z),
\end{aligned}
$$

$$
\begin{aligned}
\Big(R^{-1} \mathbin{\backslash\!\!\!\circ} (T\circ R)^{-1}\Big)^{-1}(x,z) &= \Big(R^{-1} \mathbin{\backslash\!\!\!\circ} (T\circ R)^{-1}\Big)(z,x) \\
&= \bigcap_{y\in X} R(z,y) \to (T\circ R)(x,y) \\
&= \Big((T\circ R) \mathbin{\phi\!\!\!/} R\Big)(x,z) \supseteq T(x,z).
\end{aligned}
$$

The inequalities in (2.71) are an immediate consequence of the isotonicity of the multiplication $\otimes$ together with property (1.35). Therefore, for all $x, z \in X$, we have:

$$
\begin{aligned}
\Big((T \mathbin{\phi\!\!\!/} R)\circ R\Big)(x,z) &= \bigcup_{y\in X} (T \mathbin{\phi\!\!\!/} R)(x,y)\otimes R(y,z) \\
&= \bigcup_{y\in X} \left(\bigcap_{t\in X} R(y,t) \to T(x,t)\right) \otimes R(y,z) \\
&= \bigcup_{y\in X} \left(\bigcap_{t\neq z} R(y,t) \to T(x,t) \cap \Big(R(y,z) \to T(x,z)\Big)\right) \otimes R(y,z) \\
&\subseteq \bigcup_{y\in X} R(y,z) \otimes \Big(R(y,z) \to T(x,z)\Big) \subseteq T(x,z),
\end{aligned}
$$

$$\begin{aligned}\Big(R \circ (R \backslash T)\Big)(x,z) &= \bigcup_{y\in X} R(x,y) \otimes (R \backslash T)(y,z) \\ &= \bigcup_{y\in X} R(x,y) \otimes \left(\bigcap_{t\in X} R(t,y) \to T(t,z) \right) \\ &= \bigcup_{y\in X} R(x,y) \otimes \left(\bigcap_{t\neq x} R(t,y) \to T(t,z) \cap \Big(R(x,y) \to T(x,z)\Big) \right) \\ &\subseteq \bigcup_{y\in X} R(x,y) \otimes \Big(R(x,y) \to T(x,z)\Big) \subseteq T(x,z).\end{aligned}$$

Analogously to the previous case, the inequalities in (2.72) follow from the isotonicity of the multiplication $\otimes$ and property (1.35).

$$\begin{aligned}\Big(R^{-1} \circ (T^{-1} \not\backslash R)^{-1}\Big)(x,z) &= \bigcup_{y\in X} R^{-1}(x,y) \otimes (T^{-1} \not\backslash R)^{-1}(y,z) \\ &= \bigcup_{y\in X} R(y,x) \otimes (T^{-1} \not\backslash R)(z,y) \\ &= \bigcup_{y\in X} R(y,x) \otimes \left(\bigcap_{t\in X} R(y,t) \to T^{-1}(z,t) \right) \\ &\subseteq \bigcup_{y\in X} R(y,x) \otimes \Big(R(y,x) \to T^{-1}(z,x)\Big) \subseteq T(x,z),\end{aligned}$$

$$\begin{aligned}\Big((R \backslash T^{-1})^{-1} \circ R^{-1}\Big)(x,z) &= \bigcup_{y\in X} (R \backslash T^{-1})^{-1}(x,y) \otimes R^{-1}(y,z) \\ &= \bigcup_{y\in X} (R \backslash T^{-1})(y,x) \otimes R(z,y) \\ &= \bigcup_{y\in X} \left(\bigcap_{t\in X} R(t,y) \to T^{-1}(t,x) \right) \otimes R(z,y) \\ &\subseteq \bigcup_{y\in X} R(z,y) \otimes \Big(R(z,y) \to T^{-1}(z,x)\Big) \subseteq T(x,z).\end{aligned}$$

□

Theorem 2.4. *Let $U, V \in Q^{X\times X}$ be hesitant fuzzy relations, and define*

$$\Phi = \{\, R \in Q^{X\times X} \mid R \circ U = V \,\}.$$

Then $\Phi \neq \emptyset$ iff $V \not\backslash U \in \Phi$. In that case, $V \not\backslash U$ is the greatest element of Φ.

Proof. We prove only the nontrivial direction. Assume that $\Phi \neq \emptyset$. Then there exists $R \in Q^{X\times X}$ such that $R \circ U = V$. By property (2.69), we obtain:

$$R \subseteq (R \circ U) \not\backslash U.$$

Since $R \circ U = V$, it follows that $R \subseteq V \not/ U$. Moreover, by the isotonicity of $\circ$ (2.41),

$$R \circ U \subseteq (V \not/ U) \circ U.$$

Hence, $V \subseteq (V \not/ U) \circ U$. On the other hand, by property (2.71),

$$(V \not/ U) \circ U \subseteq V.$$

Combining the last two inclusions, we obtain $(V \not/ U) \circ U = V$. Hence, $(V \not/ U) \in \Phi$.

Finally, we prove that $V \not/ U$ is the greatest element of Φ. For arbitrary $R \in \Phi$, $R \circ U = V$ holds. By property (2.69), it follows that $R \subseteq (R \circ U) \not/ U$. Hence, $R \subseteq V \not/ U$ holds for all $R \in \Phi$. □

Theorem 2.5. *Let $U, V \in Q^{X \times X}$ be hesitant fuzzy relations on X, and define*

$$\Psi = \{ R \in Q^{X \times X} \mid U \circ R = V \}.$$

Then $\Psi \neq \emptyset$ iff $U \backslash V \in \Psi$. In that case, $U \backslash V$ is the greatest element of Ψ.

Proof. The proof is analogous to the previous one. Again, we prove only the non-trivial direction. Assume that $\Psi \neq \emptyset$. Then there exists $R \in Q^{X \times X}$ such that $U \circ R = V$. By property (2.69), we obtain:

$$R \subseteq U \backslash (U \circ R).$$

Since $U \circ R = V$, it follows that $R \subseteq U \backslash V$. Moreover, by property (2.41), we have:

$$U \circ R \subseteq U \circ (U \backslash V).$$

Hence, $V \subseteq U \circ (U \backslash V)$. On the other hand, using property (2.71), we obtain $U \circ (U \backslash V) \subseteq V$. The last two inclusions imply that $U \circ (U \backslash V) = V$. Hence, $(U \backslash V) \in \Psi$.

Finally, we prove that $U \backslash V$ is the greatest element of Ψ. Consider arbitrary $R \in \Psi$. Then $U \circ R = V$. By property (2.69), it follows that $R \subseteq U \backslash (U \circ R)$. Hence, $R \subseteq U \backslash V$ holds for all $R \in \Psi$. □

The following Example illustrates Theorems 2.4 and 2.5.

Example 2.4. Let $X = \{x_1, x_2, x_3\}$. Consider hesitant fuzzy relations $U, R \in Q^{X \times X}$

over the Gödel power set quantale $\mathcal{Q}$, given by:

$$U = \begin{bmatrix} \{0.1, 0.3\} & \{1\} & \{0.2, 0.4\} \\ \{0.3, 0.5\} & \{0.2\} & \{0.1\} \\ \{0.2, 0.4\} & \{0\} & \{1\} \end{bmatrix}, \quad R = \begin{bmatrix} \{0.2\} & \{0.1, 0.3\} & \{0\} \\ \{0.4\} & \{1\} & \{0.5\} \\ \{0.3\} & \{0.1, 0.2\} & \{0.6\} \end{bmatrix}.$$

By the definition of composition given in (2.27), we obtain:

$$W = R \circ U = \begin{bmatrix} \{0, 0.1, 0.2, 0.3\} & \{0, 0.1, 0.2\} & \{0, 0.1, 0.2\} \\ \{0.1, 0.2, 0.3, 0.4, 0.5\} & \{0, 0.2, 0.4\} & \{0.1, 0.2, 0.4, 0.5\} \\ \{0.1, 0.2, 0.3, 0.4\} & \{0, 0.1, 0.2, 0.3\} & \{0.1, 0.2, 0.3, 0.6\} \end{bmatrix},$$

$$V = U \circ R = \begin{bmatrix} \{0.1, 0.2, 0.3, 0.4\} & \{0.1, 0.2, 0.3, 1\} & \{0, 0.2, 0.4, 0.5\} \\ \{0.1, 0.2\} & \{0.1, 0.2, 0.3\} & \{0, 0.1, 0.2\} \\ \{0, 0.2, 0.3\} & \{0, 0.1, 0.2, 0.3\} & \{0, 0.6\} \end{bmatrix}.$$

Following Theorems 2.4 and 2.5, define the sets $\Phi = \{R \in Q^{X \times X} \mid R \circ U = W\}$ and $\Psi = \{R \in Q^{X \times X} \mid U \circ R = V\}$. Clearly, the given hesitant fuzzy relation R belongs to both sets Φ and Ψ. We next verify whether $U \backslash V \in \Psi$. By the definition of the right $\circ$-residual in (2.62), we obtain:

$$U \backslash V = \begin{bmatrix} \{0.2\} & \{0.1, 0.2, 0.3\} & \{0\} \\ \{0.1, 0.2, 0.3, 0.4\} & \{0.1, 0.2, 0.3, 1\} & \{0, 0.2, 0.4, 0.5\} \\ \{0.2, 0.3\} & \{0.1, 0.2, 0.3\} & \{0.6\} \end{bmatrix}.$$

Therefore, $U \circ (U \backslash V) = V$ and consequently $U \backslash V \in \Psi$. Observe that $R \subseteq U \backslash V$.

In what follows, we verify whether $W \not/ U \in \Phi$. By the definition of the left $\circ$-residual in (2.63), we obtain:

$$W \not/ U = \begin{bmatrix} \{0, 0.1, 0.2\} & \{0, 0.1, 0.2, 0.3\} & \{0, 0.1, 0.2\} \\ \{0.2, 0.4\} & \{0.2, 0.3, 0.4\} \cup [0.5, 1] & \{0.1, 0.2, 0.4, 0.5\} \\ \{0.1, 0.2, 0.3\} & \{0.1, 0.2, 0.3, 0.4\} & \{0.1, 0.2, 0.3, 0.6\} \end{bmatrix}.$$

Hence, $(W \not/ U) \circ U = W$ and consequently, $W \not/ U \in \Phi$. Observe that $R \subseteq W \not/ U$.

Based on the left and right residuals and the composition of hesitant fuzzy re-

lations, we introduce and study several new operations on hesitant fuzzy relations. In particular, for any hesitant fuzzy relation $V \in Q^{X\times X}$, we define the following hesitant fuzzy relations on X:

$$V\backslash\backslash V = (V \mathbin{\backslash\!\!\!\circ} V) \cap (V \mathbin{\backslash\!\!\!\circ} V)^{-1},$$
$$V//V = (V \mathbin{/\!\!\!\circ} V) \cap (V \mathbin{/\!\!\!\circ} V)^{-1}.$$

Hence, for all $x, z \in X$ we have:

$$(V\backslash\backslash V)(x,z) = \bigcap_{y\in X} (V(x,y) \leftrightarrow V(z,y)), \tag{2.73}$$
$$(V//V)(x,z) = \bigcap_{y\in X} (V(y,x) \leftrightarrow V(y,z)). \tag{2.74}$$

According to Theorem 2.2, the relations $V\backslash\backslash V$ and $V//V$ are hesitant fuzzy quasi-orders. Therefore, $V\backslash\backslash V$ and $V//V$ represent the natural hesitant fuzzy equivalences of $V \mathbin{\backslash\!\!\!\circ} V$ and $V \mathbin{/\!\!\!\circ} V$, respectively.

In addition, given hesitant fuzzy relations W and V on X, we define new hesitant fuzzy relations $V \mathbin{\oslash\!\!\!\backslash} W$, $W \mathbin{\oslash\!\!\!/} V \in Q^{X\times X}$ by

$$W \mathbin{\oslash\!\!\!\backslash} V = (W \circ V) \mathbin{\backslash\!\!\!\circ} (W \circ V), \quad W \mathbin{\oslash\!\!\!/} V = (W \circ V) \mathbin{/\!\!\!\circ} (W \circ V). \tag{2.75}$$

Then, for arbitrary $x, z \in X$, we have:

$$(W \mathbin{\oslash\!\!\!\backslash} V)(x,z) = \bigcap_{y\in X} (W \circ V)(z,y) \to (W \circ V)(x,y), \tag{2.76}$$
$$(W \mathbin{\oslash\!\!\!/} V)(x,z) = \bigcap_{y\in X} (W \circ V)(y,x) \to (W \circ V)(y,z). \tag{2.77}$$

By Theorem 2.2, the hesitant fuzzy relations $W \mathbin{\oslash\!\!\!\backslash} V$ and $W \mathbin{\oslash\!\!\!/} V$ are hesitant fuzzy quasi-orders, for any hesitant fuzzy relations W and V on X.

We further introduce the hesitant fuzzy relations $V \hat{\mathbin{\oslash\!\!\!\backslash}} W$ and $W \hat{\mathbin{\oslash\!\!\!/}} V$ on X as follows:

$$W \hat{\mathbin{\oslash\!\!\!\backslash}} V = (W \circ V)\backslash\backslash(W \circ V) = (W \mathbin{\oslash\!\!\!\backslash} V) \cap (W \mathbin{\oslash\!\!\!\backslash} V)^{-1}, \tag{2.78}$$
$$W \hat{\mathbin{\oslash\!\!\!/}} V = (W \circ V)//(W \circ V) = (W \mathbin{\oslash\!\!\!/} V) \cap (W \mathbin{\oslash\!\!\!/} V)^{-1}. \tag{2.79}$$

The hesitant fuzzy relations $V \hat{\mathbin{\oslash\!\!\!\backslash}} W$ and $W \hat{\mathbin{\oslash\!\!\!/}} V$ are, respectively, the natural hesitant

fuzzy equivalences of $W \oslash V$ and $W \oslash V$. Therefore, for arbitrary $x, z \in X$ we have:

$$(W \hat{\oslash} V)(x,z) = \bigcap_{y \in X} ((W \circ V)(x,y) \leftrightarrow (W \circ V)(z,y)), \tag{2.80}$$

$$(W \hat{\oslash} V)(x,z) = \bigcap_{y \in X} ((W \circ V)(y,z) \leftrightarrow (W \circ V)(y,x)). \tag{2.81}$$

The above results establish several operations of hesitant fuzzy relations induced by the left and right residuals associated with the composition of two arbitrary hesitant fuzzy relations on X. These operations yield hesitant fuzzy quasi-orders and natural hesitant fuzzy equivalences. The resulting constructions will be used in the subsequent development of the algebraic framework for hesitant fuzzy relational structures.

Chapter 3

Special classes of hesitant fuzzy relations over the power set quantale of a semigroup

Relations provide a general mathematical framework for describing and analyzing relationships between elements of sets. Binary relations, in particular, have been extensively studied and have found applications in various areas of mathematics and computer science, as discussed by Lambek [75] and Toth [127].

In the classical setting, a binary relation describes the presence or absence of a relationship between elements. Zadeh [139] generalized this notion by introducing the concept of a fuzzy relation, in which each pair of elements is assigned a membership grade representing the degree to which they are related. In his subsequent work [140], Zadeh further developed this concept by introducing the notions of fuzzy equivalence and fuzzy ordering. Since then, fuzzy relations have been extensively investigated, with particular attention to their various properties and special classes, including fuzzy equivalences, fuzzy partitions, and uniform fuzzy relations [24, 25, 35, 58, 71, 86, 87]. Applications of fuzzy relations in various fields can be found, for example, in [5, 13, 32, 55, 73].

A further extension of this framework is given by hesitant fuzzy relations, which allow several possible membership values to be associated with a pair of elements. Hesitant fuzzy relations and their properties have subsequently been investigated in various contexts [21, 33, 56]. In particular, Qiao [97] explored their application in three-way decision-making through the construction of decision evaluation functions.

In this chapter, we introduce and study several important classes of hesitant fuzzy relations over the power set quantale of a semigroup that will be used throughout the remainder of the dissertation. Section 3.1 is devoted to hesitant fuzzy equivalences,

hesitant fuzzy equivalence classes, and hesitant fuzzy partitions over the power set quantale of a semigroup. Section 3.2 focuses on hesitant fuzzy quasi-orders, while Section 3.3 introduces and investigates uniform hesitant fuzzy relations in the developed framework. Section 3.4 presents several illustrative applications of the developed theory. The results presented in Sections 3.1, 3.2, and 3.4 were previously published in [122].

3.1 Hesitant fuzzy equivalences and equivalence classes

The current section provides the characterization of hesitant fuzzy equivalences and their relationship to hesitant fuzzy partitions and hesitant fuzzy semi-partitions.

Let $\mathcal{E}(X)$ be a set of all hesitant fuzzy equivalences on X, i.e., $\mathcal{E}(X)$ is a subset of $\mathcal{R}(X)$. The set $\mathcal{E}(X)$ admits a partial order on $\mathcal{E}(X)$ induced by the $\subseteq$ ordering on $\mathcal{R}(X)$. Moreover, the following characterization holds.

Theorem 3.1. *The structure $\langle \mathcal{E}(X), \cup, \cap \rangle$ forms a complete lattice, in which the meet $\cap$ coincides with the usual intersection of hesitant fuzzy relations, while the join $\cup$ of two hesitant fuzzy equivalence relations E_1 and E_2 is defined as follows:*

$$E_1 \cup E_2 = \bigcap \{H \in \mathcal{E}(X) \mid E_1 \subseteq H \text{ and } E_2 \subseteq H\}.$$

Valverde [128] established that the meet of a family of fuzzy equivalence relations is itself a fuzzy equivalence relation (Proposition 3.1). Motivated by this result, we formulate the following statement as a direct consequence of the preceding theorem.

Corollary 3.1. *Given a family $\{E_i\}_{i \in I}$ of hesitant fuzzy equivalences on X, the relation*

$$E(x, y) = \bigcap_{j \in J} E_j(x, y)$$

is itself a hesitant fuzzy equivalence on X.

Before stating the next results, we introduce several specific types of hesitant fuzzy sets that will be used in the sequel.

Definition 3.1. A hesitant fuzzy set h on X is said to be *normalized* if there exists $x \in X$ such that $1_{\otimes} \subseteq h(x)$.

Definition 3.2. A hesitant fuzzy set h on X is said to be *extensional with respect to (WRT)* a given hesitant fuzzy equivalence E if

$$h(x) \otimes E(x, y) \subseteq h(y), \tag{3.1}$$

and it is said to be *indistinguishable WRT* to E if

$$h(x) \otimes h(y) \subseteq E(x, y), \tag{3.2}$$

for every $x, y \in X$. Moreover, if h is *normalized*, extensional, and indistinguishable WRT to E, h is called a *hesitant fuzzy point* WRT to E.

The following lemma provides two basic properties of a normalized hesitant fuzzy set with respect to the operations $\otimes$ and $\rightarrow$, used in later proofs.

Lemma 3.1. *A normalized hesitant fuzzy set h on X is indistinguishable WRT to $E \in \mathcal{E}(X)$ if and only if it is included in some equivalence class of E.*

Proof. Assume that h is a normalized hesitant fuzzy set on X, i.e. $1_{\otimes} \subseteq h(a)$, for some $a \in X$. According to (2.60), $h(x) \subseteq h(a) \otimes h(x)$ for every $x \in X$. Next, suppose that h is indistinguishable WRT to E, that is (3.2) holds. Then,

$$h(x) \subseteq h(a) \otimes h(x) \subseteq E(a, x) = E_a(x),$$

for all $x \in X$, thus $h \subseteq E_a$. For the converse direction, assume that $h \subseteq E_a$ for some $a \in X$. It follows that $h(x) \subseteq E(a, x)$ and $h(y) \subseteq E(a, y)$, for all $x, y \in X$. By the monotonocity of $\otimes$ and as E is symmetric, we get:

$$h(x) \otimes h(y) \subseteq E(a, x) \otimes E(a, y) \subseteq E(x, y),$$

for every $x, y \in X$, hence h is indistinguishable WRT to E. □

The following lemma establishes a connection between hesitant fuzzy points and equivalence classes induced by a hesitant fuzzy equivalence relation.

Lemma 3.2. *A hesitant fuzzy set h on X is a hesitant fuzzy point WRT to $E \in \mathcal{E}(X)$ if and only if h is an equivalence class of E.*

Proof. Assume that h is a hesitant fuzzy point WRT to a given $E \in \mathcal{E}(X)$. Since h is normalized, then according to Lemma 3.1, we have $h \subseteq E_a$ for $a \in X$ such that $1_{\otimes} \subseteq h(a)$. Moreover, from (2.60) and the fact that h is extensive WRT to E, it

follows:

$$E_a(x) \subseteq h(a) \otimes E_a(x) = h(a) \otimes E(a,x) \subseteq h(x),$$

for every $x \in X$, meaning that $E_a \subseteq h$. Hence, we showed that $E_a = h$. To prove the converse implication, assume that $h = E_a$ for some $a \in X$. Since E_a is an equivalence class, $1_\otimes \subseteq E(a,a) = h(a)$, meaning that h is normalized. Moreover, as E is transitive, it holds:

$$h(x) \otimes E(x,y) = E(a,x) \otimes E(x,y) \subseteq E(a,y) = h(y),$$

for every $x, y \in X$, so h is extensive WRT to E. Finally, by using the fact that E is symmetric and transitive, the following holds:

$$h(x) \otimes h(y) = E(a,x) \otimes E(a,y) \subseteq E(x,y),$$

for all $x, y \in X$. Thus, h is indistinguishable WRT to E. □

The properties of hesitant fuzzy equivalences stated below are standard in related settings; however, for completeness, we provide their proofs.

Lemma 3.3. *The following assertions hold for an arbitrary $E \in \mathcal{E}(X)$:*

(a) $||E_x \otimes E_y|| = E(x,y)$,

(b) $1_\otimes \subseteq E(x,y) \Leftrightarrow E_x = E_y$,

for every $x, y \in X$.

Proof. (a) Let $E \in \mathcal{E}(X)$ be arbitrary. Then, for every $x, y \in X$,

$$||E_x \otimes E_y|| = \bigcup_{a \in X} (E_x(a) \otimes E_y(a)) = \bigcup_{a \in X} (E(x,a) \otimes E(a,y)) = E \circ E(x,y) = E(x,y),$$

where the last equality holds due to reflexivity and transitivity of E.

(b) Let $E_x = E_y$. Since E is an equivalence relation, it follows $E(y,y) = E_y(y) = E_x(y) = E(x,y)$. Therefore, as $1_\otimes \subseteq E(y,y)$, we also have $1_\otimes \subseteq E(x,y)$. Conversely, suppose that $1_\otimes \subseteq E(x,y)$. For every $a \in X$, due to the transitivity of E and (2.60):

$$E_y(a) = E(y,a) \subseteq E(x,y) \otimes E(y,a) \subseteq E(x,a) = E_x(a).$$

Analogously, we prove that $E_x(a) \subseteq E_y(a)$. Hence, $E_x = E_y$. □

In order to construct hesitant fuzzy equivalence relation from a given hesitant fuzzy set h on X, we define the relation E_h by:

$$E_h(x, y) = h(x) \leftrightarrow h(y) \quad \text{for every } x, y \in X. \tag{3.3}$$

In what follows, we show that E_h is a hesitant fuzzy equivalence on X. First, for every $x \in X$, $1_\otimes \otimes h(x) = h(x)$, and by the residuation property (2.11), it follows that $1_\otimes \subseteq h(x) \to h(x)$. Hence,

$$1_\otimes \in h(x) \leftrightarrow h(x) = E_h(x, x),$$

for every $x \in X$. Consequently, E_h is reflexive. Symmetry follows from the commutativity of biresiduum, i.e., for every $x, y \in X$, we have:

$$E_h(x, y) = h(x) \leftrightarrow h(y) = h(y) \leftrightarrow h(x) = E_h(y, x).$$

Finally, the transitivity of E_h follows directly from poperty (1.41). Therefore, E_h is a hesitant fuzzy equivalence on X.

Example 3.1. This example illustrates how formula (3.3) can be used to construct the corresponding hesitant fuzzy equivalences E_h^G, E_h^L and E_h^P in a Gödel, Łukasiewicz, and product power set quantales, respectively, which are defined in Example (1.4).

Let $X = \{x, y, z\}$ be the underlying universe in each of the three cases. First, consider a hesitant fuzzy set h on X over the Gödel power set quantale given by $h(x) = h(y) = \{0.1, 0.6\}$ and $h(z) = \{0.3, 1\}$. The corresponding equivalence is:

$$E_h^G = \begin{bmatrix} \{0.1\} \cup [0.6, 1] & \{0.1\} \cup [0.6, 1] & \emptyset \\ \{0.1\} \cup [0.6, 1] & \{0.1\} \cup [0.6, 1] & \emptyset \\ \emptyset & \emptyset & \{0.3, 1\} \end{bmatrix}.$$

Next, we consider a hesitant fuzzy set h on X over the Łukasiewicz power set quantale, defined by $h(x) = \{0, 0.9\}$, $h(y) = \{0.6, 1\}$ and $h(z) = \{0, 0.4, 0.7\}$. In this case:

$$E_h^L = \begin{bmatrix} \{0, 0.1, 1\} & \emptyset & \{0\} \\ \emptyset & \{1\} & \emptyset \\ \{0\} & \emptyset & [0, 0.3] \cup \{1\} \end{bmatrix}.$$

Finally, consider a hesitant fuzzy set h on X over the product power set quantale, defined by $h(x) = \{0, 0.7\}$, $h(y) = \{0.2, 0.7, 1\}$ and $h(z) = \{0.2, 0.6\}$. The resulting equivalence is:

$$E_h^P = \begin{bmatrix} \{0,1\} & \emptyset & \emptyset \\ \emptyset & \{1\} & \emptyset \\ \emptyset & \emptyset & \{1\} \end{bmatrix}.$$

Example 3.2. Let $\mathcal{Q} = \langle \mathcal{P}(\mathbb{N}_0), \cup, \cap, \otimes, \emptyset, \mathbb{N}_0 \rangle$ be the power set quantale of the semigroup $\langle \mathbb{N}_0, \cdot \rangle$, in which

$$A \otimes B = \{a \cdot b \mid a \in A, b \in B\},$$

for every $A, B \subseteq \mathbb{N}_0$. Let $X = \{x, y\}$ and take h to be the hesitant fuzzy set on X over $\mathcal{Q}$, given by

$$h = \begin{bmatrix} \{0,4,8,12\} & \{0,8,16,24\} \end{bmatrix}.$$

Applying formula (3.3), we obtain the relation

$$E_h = \begin{bmatrix} \{0,1\} & \{0\} \\ \{0\} & \{0,1\} \end{bmatrix}.$$

It is straightforward to verify that E_h is reflexive and symmetric. Moreover, $E_h \circ E_h = E_h$, and therefore E_h is transitive. Consequently, E_h is a hesitant fuzzy equivalence on X over $\mathcal{Q}$.

The previous example illustrates the construction of hesitant fuzzy equivalences generated by a single hesitant fuzzy set. We now extend this idea to families of hesitant fuzzy sets. Consider a family of hesitant fuzzy sets $\mathcal{H}$ on X. According to Corollary 3.1, the hesitant fuzzy relation on X given by:

$$E_{\mathcal{H}}(x, y) = \bigcap_{h \in \mathcal{H}} E_h(x, y), \tag{3.4}$$

for all $x, y \in X$, is a hesitant fuzzy equivalence on X.

Remark 3.1. Recall that the collection of fuzzy sets defined on the same universe forms a residuated lattice where the greatest element coincides with the unit element, denoted by 1. In contrast, in the hesitant fuzzy setting the unit element does not,

in general, coincide with the greatest element.

In the classical fuzzy setting, a fuzzy equivalence can be obtained from a given fuzzy set in several ways. In addition to the construction via biresiddum considered above, the following two fuzzy relations are also known to be fuzzy equivalence relations:

$$E'_f(x,y) = \begin{cases} f(x) \otimes f(y) & \text{if } x \neq y, \\ 1 & \text{if } x = y, \end{cases} \qquad E''_f(x,y) = \begin{cases} f(x) \otimes f(y) & \text{if } f(x) \neq f(y), \\ 1 & \text{if } f(x) = f(y), \end{cases} \tag{3.5}$$

for every $x, y \in X$ (see [24]).

We can extend the definitions in (3.5) to the hesitant fuzzy setting and define the relations $E'_h, E''_h \in Q^{X\times X}$ as follows

$$E'_h(x,y) = \begin{cases} h(x) \otimes h(y) & \text{if } x \neq y, \\ T & \text{if } x = y, \end{cases} \qquad E''_h(x,y) = \begin{cases} h(x) \otimes h(y) & \text{if } h(x) \neq h(y), \\ T & \text{if } h(x) = h(y), \end{cases} \tag{3.6}$$

for every $x, y \in X$, where $T \subseteq S$ is an arbitrary subset containing the multiplicative identity 1_* of S (that is, $1_\otimes \subseteq T$). However, E'_h and E''_h need not be hesitant fuzzy equivalences on X for any such T. The following example illustrates this observation.

Example 3.3. Let $X = \{x, y, z\}$, let $T \subseteq S = [0, 1]$ be an arbitrary subset such that $1 \in T$, and let h be a hesitant fuzzy set on X over the Gödel power set quantale $\mathcal{Q}$ given by

$$h = \begin{bmatrix} \{0.3\} & \{0.4, 0.5\} & \{0.1\} \end{bmatrix}. \tag{3.7}$$

Using (3.6), we obtain that E'_h is given by

$$E'_h = \begin{bmatrix} T & \{0.3\} & \{0.1\} \\ \{0.3\} & T & \{0.1\} \\ \{0.1\} & \{0.1\} & T \end{bmatrix}.$$

Since $a \wedge b \in \{a, b\} \subseteq T$, for all $a, b \in T$, we have that $T \otimes T = T \wedge T = \{a \wedge b \mid a, b \in T\} \subseteq T$. On the other hand, since $1 \in T$, we have that $a = a \wedge 1 \in T \wedge T$, for each $a \in T$, so we get that $T \subseteq T \wedge T = T \otimes T$. Therefore, $T \otimes T = T$.

Now, if $0.3 \notin T$ or $0.1 \notin T$, then

$$\begin{aligned} E'_h \circ E'_h(x,x) &= T \otimes T \cup \{0.3\} \otimes \{0.3\} \cup \{0.1\} \otimes \{0.1\} \nsubseteq T \\ &= T \cup \{0.3\} \cup \{0.1\} \nsubseteq T = E'_h(x,x). \end{aligned}$$

Otherwise, if $0.3 \in T$ and $0.1 \in T$, then

$$E'_h \circ E'_h(x,y) = T \otimes \{0.3\} \cup \{0.3\} \otimes T \cup \{0.1\} \nsubseteq \{0.3\} = E'_h(x,y).$$

Therefore, in both cases we obtain that E'_h is not transitive, and consequently, it is not a hesitant fuzzy equivalence relation on X.

On the other hand, since $h(x) \neq h(y)$, $h(y) \neq h(z)$, and $h(z) \neq h(x)$, it follows that $E'_h = E''_h$, and hence E''_h is not a hesitant fuzzy equivalence relation on X either.

3.1.1 Hesitant fuzzy partitions and semi-partitions

We now turn to the notion of partitions induced by equivalence relations. A family of crisp subsets $\mathcal{X} = \{X_i\}_{i \in I}$ of a set X is said to be a *semi-partition* of X if $X_i \neq \emptyset$ and $X_i \cap X_j = \emptyset$, for all $i, j \in I$. The family $\mathcal{X} = \{X_i\}_{i \in I}$ satisfying $\bigcup_{i \in I} X_i = X$ is referred to as a *cover* of X. A semi-partition of X which is also a cover of X is termed a partition of X. In the crisp case, the collection of all equivalence classes of an equivalence relation on X is a partition of X and every element of X belongs to one and only one equivalence class X_i.

A precise definition of partitions induced by fuzzy equivalence relations can be found in the work of Murali [86]. Ćirić et al. in [24] defined fuzzy partitions and fuzzy semi-partitions following the approaches of De Baets [31] and Ovchinnikov [87]. According to them, a fuzzy partition of X is a collection of all distinct equivalence classes generated by a fuzzy equivalence relation on X. A *fuzzy semi-partition* is a collection of equivalence classes (not neccessarily all) of some fuzzy equivalence on X. In this subsection, we proceed to define analogous concepts in the context of hesitant fuzzy equivalences.

Definition 3.3. Let $\mathcal{X}$ be a family of normalized hesitant fuzzy sets on X. The family $\mathcal{X}$ is called a *hesitant fuzzy partition* of X if it coincides with the collection of all mutually distinct equivalence classes induced by a hesitant fuzzy equivalence on X.

Definition 3.4. A subset of a hesitant fuzzy partition of X is called a *hesitant fuzzy semi-partition* of X.

Lemma 3.4. *A hesitant fuzzy set on X is normalized iff it contains an equivalence class of a hesitant fuzzy equivalence on X.*

Proof. ($\Rightarrow$) Let h be a normalized hesitant fuzzy set on X. Hence, there exists $a \in X$ such that $1_\otimes \in h(a)$. Let define the hesitant fuzzy equivalence E on X via biresiduum (3.3),

$$E(x, y) = h(x) \leftrightarrow h(y), \qquad \text{for every } x, y \in X.$$

Consider the equivalence class E_a. By definition of biresiduum and property (2.61) of Lemma 2.2, we obtain:

$$E_a(x) = E(a, x) = h(a) \leftrightarrow h(x) \subseteq h(a) \rightarrow h(x) \subseteq h(x),$$

for all $x \in X$. Consequently, $E_a \subseteq h$.

($\Leftarrow$) Consider a hesitant fuzzy set h on X that contains an equivalence class E_a of some hesitant fuzzy equivalence E on X. Observe that $1_\otimes \subseteq E_a(a)$ for every $a \in X$. Since $E_a \subseteq h$, h is normalized. □

Example 3.4. Observe that equivalence classes of an arbitrary hesitant fuzzy equivalence are normalized hesitant fuzzy sets. For illustration, consider the hesitant fuzzy equivalence $E = E_h^G$ given in Example 3.1. Denote by E_x the equivalence class induced by $x \in X$, i.e., $E_x(y) = E(x, y)$ for every $y \in X$. In particular, since $E_x = E_y$, and by reflexivity of E, we obtain:

$$1_\otimes \subseteq E_x(x) = E_y(x).$$

The same reasoning applies to the class E_y. Moreover, E_z is also a normalized hesitant fuzzy set since $1_\otimes \subseteq E_z(z)$.

To illustrate the converse, consider the normalized hesitant fuzzy set h on $X = \{x, y, z\}$ over the Gödel power set quantale, given by:

$$h = \begin{bmatrix} \{0.3, 0.5, 1\} & \{0.2, 0.3\} & \{0.3, 0.6\} \end{bmatrix}.$$

Using formula (3.3), we obtain the following hesitant fuzzy equivalence E on X:

$$E = \begin{bmatrix} \{1\} & \emptyset & \{0.3\} \\ \emptyset & \{0.2\} \cup [0.3, 1] & \emptyset \\ \{0.3\} & \emptyset & \{0.3\} \cup [0.6, 1] \end{bmatrix}.$$

Thus, h contains the equivalence class E_x.

Remark 3.2. Observe that, in the classical fuzzy setting, a normalized fuzzy subset coincides exactly with equivalence class of a fuzzy equivalence on the same universe (see Theorem 3.1 in [24]). However, in the hesitant fuzzy setting, a normalized hesitant fuzzy set only contains an equivalence class of a hesitant fuzzy equivalence relation. This observation is illustrated in Example 3.4.

One of the structural reasons for this difference is the lack of integrality of the underlying structure. In the classical setting of complete residuated lattices, the structure is integral, meaning that the multiplicative identity coincides with the greatest element, $1 = \top$. Therefore, the requirement that a normalized fuzzy set contains the multiplicative identity is equivalent to requiring that its membership value reaches the greatest element.

In contrast, in the power set quantale $\mathcal{Q} = \langle \mathcal{P}(S), \cup, \cap, \otimes, \emptyset, S \rangle$, the multiplicative identity $1_\otimes$ and the greatest element $\top = S$ are generally different. Hence, normalization only requires that the multiplicative identity $1_\otimes$ is contained in a hesitant fuzzy element, while the corresponding hesitant fuzzy element may contain additional elements of S and therefore need not coincide with the greatest element $\top$.

The lack of integrality of the underlying quantale has another important consequence: the relations defined in (3.6) need not be transitive. Indeed, for classical fuzzy relations defined by (3.5), transitivity follows from the fact that the product of two elements a and b of a complete residuated lattice is bounded above by their meet, that is,

$$a \otimes b \leqslant a \wedge b,$$

which is a direct consequence of the fact that the multiplicative identity is the greatest element. However, in a non-integral quantale, this inequality does not hold in general.

Example 3.5. This example shows that, in contrast to the classical fuzzy setting, a hesitant fuzzy set does not necessarily coincide with the equivalence classes of the induced hesitant fuzzy equivalence relation. Consider a semigroup $\langle \mathbb{Z}_4, +_4 \rangle$, where $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ is and $+_4$ denotes addition modulo 4. Let $\mathcal{Q} = \langle \mathbb{Z}_4, \cup, \cap, \otimes, \emptyset, \mathbb{Z}_4 \rangle$ be the power set quantale induced by this semigroup. Further, let h be the hesitant fuzzy set on $X = \{x, y, z\}$ over $\mathcal{Q}$ given by

$$h = \begin{bmatrix} \{0,2\} & \{1,3\} & \{0,1\} \end{bmatrix}. \tag{3.8}$$

The corresponding hesitant fuzzy equivalence relation on X induced by h using formula (3.3) is given by

$$E_h = \begin{bmatrix} \{0,2\} & \{1,3\} & \emptyset \\ \{1,3\} & \{0,2\} & \emptyset \\ \emptyset & \emptyset & \{0\} \end{bmatrix}.$$

It is immediately clear that h contains the equivalence class E_x. However, h does not coincide with the equivalence classes of E_x.

We now state a lemma characterizing all hesitant fuzzy equivalence relations WRT to which a hesitant fuzzy set is extensional.

Lemma 3.5. *The collection of all $E \in \mathcal{E}(X)$ WRT to which a heistant fuzzy set h is extensional, forms the principal ideal generated by E_h, defined in (3.3).*

Proof. Let $E \in \mathcal{E}(X)$ and $h \in \mathcal{Q}(X)$. To establish the claim, it is enough to show that h is extensional w.r.t. E iff $E \subseteq E_h$. Recall that $E_h(x,y) = h(x) \leftrightarrow h(y)$ for every $x, y \in X$ (see (3.3)).

We first prove that, if h is extensional WRT to E, then $E \subseteq E_h$. By extensionality, for all $x, y \in X$:

$$h(x) \otimes E(x,y) \subseteq h(y), \quad h(y) \otimes E(y,x) \subseteq h(x). \tag{3.9}$$

Since E is symmetric, the second inclusion is equivalent with $h(y) \otimes E(x,y) \subseteq h(x)$. Applying the residuation property to (3.9) we obtain:

$$E(x,y) \subseteq h(x) \to h(y), \quad E(x,y) \subseteq h(y) \to h(x).$$

Hence,

$$E(x,y) \subseteq (h(x) \to h(y)) \cap (h(y) \to h(x)) = h(x) \leftrightarrow h(y).$$

Therefore, $E(x,y) \subseteq E_h(x,y)$ for all $x, y \in X$, i.e., $E \subseteq E_h$.

Conversely, assume that $E \subseteq E_h$, i.e., $E(x,y) \subseteq E_h(x,y)$ for every $x, y \in X$. By definition of E_h we have:

$$E(x,y) \subseteq h(x) \leftrightarrow h(y) \subseteq h(x) \to h(y),$$

and hence,

$$E(x,y) \subseteq h(x) \to h(y). \tag{3.10}$$

By applying residuation property on (3.10), we obtain:

$$h(x) \otimes E(x,y) \subseteq h(y),$$

which shows that h is extensional with respect to E. □

The following theorem gives several equivalent conditions for a family of normalized hesitant fuzzy sets to form a hesitant fuzzy semi-partition of X.

Theorem 3.2. *Given a family $\mathcal{H}$ of normalized hesitant fuzzy sets on X, the following statements are equivalent:*

(i) *$\mathcal{H}$ forms a hesitant fuzzy semi-partition of X;*

(ii) *Given any $h, g \in \mathcal{H}$:*

$$\bigcup_{x\in X} h(x) \otimes g(x) \subseteq \bigcap_{y\in X} h(y) \leftrightarrow g(y); \tag{3.11}$$

(iii) *Given any $x, y \in X$:*

$$\bigcup_{h\in\mathcal{H}} h(x) \otimes h(y) \subseteq \bigcap_{g\in\mathcal{H}} g(x) \leftrightarrow g(y). \tag{3.12}$$

Proof. (i) $\Rightarrow$ (ii). Take $\mathcal{H}$ to be a hesitant fuzzy semi-partition of X. Hence, $\mathcal{H}$ consists of equivalence classes determined by a hesitant fuzzy equivalence E on X. Then, for arbitrary $h, g \in \mathcal{H}$, there exist $a, b \in X$ satisfying $h = E_a$ and $g = E_b$. According to Lemma 3.3,

$$E(a,b) = ||E_a \otimes E_b|| = ||h \otimes g|| = \bigcup_{x\in X} h(x) \otimes g(x).$$

Next, choose an arbitrary $y \in X$ and set $E_y = f$. According to Lemma 3.2, f is a hesitant fuzzy point WRT to E, and specifically, f is extensional WRT to E. Then, by Lemma 3.5, $E \subseteq E_f$. However, we further have:

$$\begin{aligned} E(a,b) \subseteq E_f(a,b) &= f(a) \leftrightarrow f(b) = E_y(a) \leftrightarrow E_y(b) = E_a(y) \leftrightarrow E_b(y) \\ &= h(y) \leftrightarrow g(y), \end{aligned}$$

which yields:

$$E(a,b) \subseteq \bigcap_{y \in X} h(y) \leftrightarrow g(y).$$

Thus, (ii) holds.

(ii) $\Rightarrow$ (iii). Assume that (3.11) holds. Notice that this property can be writen in the equivalent form:

$$h(x) \otimes g(x) \subseteq h(y) \leftrightarrow g(y), \tag{3.13}$$

for every $h, g \in \mathcal{H}$, $x, y \in X$. According to definition of biresiduum, for arbitrary $h, g \in \mathcal{H}$ and $x, y \in X$, we have:

$$h(x) \otimes g(x) \subseteq h(y) \rightarrow g(y), \quad h(y) \otimes g(y) \subseteq h(x) \rightarrow g(x).$$

Residuation property (2.11), together with the commutativity of $\otimes$ implies that

$$\begin{aligned} h(x) \otimes g(x) \subseteq h(y) \rightarrow g(y) \quad &\Leftrightarrow \quad h(x) \otimes g(x) \otimes h(y) \subseteq g(y) \\ &\Leftrightarrow \quad h(x) \otimes h(y) \otimes g(x) \subseteq g(y) \\ &\Leftrightarrow \quad h(x) \otimes h(y) \subseteq g(x) \rightarrow g(y). \end{aligned}$$

Analogously, it can be shown:

$$h(y) \otimes g(y) \subseteq h(x) \rightarrow g(x) \quad \Leftrightarrow \quad h(x) \otimes h(y) \subseteq g(y) \rightarrow g(x).$$

Hence,

$$h(x) \otimes h(y) \subseteq g(x) \leftrightarrow g(y), \tag{3.14}$$

for every $h, g \in \mathcal{H}$ and $x, y \in X$. Hence, (iii) follows.

(iii) $\Rightarrow$ (i). Suppose that condition (iii) is satisfied. It can be equivalently written as (3.14), for all $h, g \in \mathcal{H}$, $x, y \in X$. Now, choose an arbitrary $h \in \mathcal{H}$. For any $x, y \in X$, it holds that:

$$h(x) \otimes h(y) \subseteq \bigcap_{g \in \mathcal{H}} g(x) \leftrightarrow g(y). \tag{3.15}$$

Recall that, according to Corollary 3.1, the hesitant fuzzy relation $E_{\mathcal{H}}$ on X given by (3.4), is a hesitant fuzzy equivalence on X. By (3.15), for any $x, y \in X$, we

obtain:

$$h(x) \otimes h(y) \subseteq E_{\mathcal{H}}(x, y),$$

which means that h is indistinguishable with respect to $E_{\mathcal{H}}$. By the definition of $E_{\mathcal{H}}$, we have $E_{\mathcal{H}} \subseteq E_h$. According to Lemma 3.5, this fact is equivalent to the one that h is extensional WRT to $E_{\mathcal{H}}$. Finally, h is normalized by the assumption, thus h is a hesitant fuzzy point WRT to $E_{\mathcal{H}}$. Thus, by Lemma 3.2, h Corresponds to an equivalence class of $E_{\mathcal{H}}$. As h was chosen arbitrarily, it means that each $h \in \mathcal{H}$ is an equivalence class of $E_{\mathcal{H}}$, thus $\mathcal{H}$ is a hesitant fuzzy semi-partition of X, therefore (i) follows. □

Theorem 3.2 not only provides algebraic conditions under which a family of normalized hesitant fuzzy sets on X to constitute a hesitant fuzzy semi-partition of X, but also offers computational insights. Rather than providing full computational details, we only outline the main aspects of the computational analysis. To do so, assume that the set X, the monoid of values S and the family of normalized hesitant fuzzy sets $\mathcal{H}$ on X are all finite. Denote $|X| = n_X$, $|S| = n_S$, and $|\mathcal{H}| = n_{\mathcal{H}}$. We begin by analyzing the computation costs for performing the operations $\otimes$ and $\leftrightarrow$ on hesitant fuzzy sets on X.

Let c_* denote the computational cost for performing the operation $*$ in the monoid $\mathcal{S}$. Then, we have $c_{\otimes} = O(n_S^2 \cdot c_*)$ as, according to (1.21), we perform $*$ on each pair of elements of S in the worst case, and subsequently add the computed element in the set, which does not exceed this complexity bound. Moreover, since we work within the commutative power set quantale, the operation $\leftrightarrow$ between two hesitant fuzzy sets on X can be computed from either of the residuals (1.24) or (1.25). In both cases, the computation reduces to performing $\otimes$ between a hesitant fuzzy set and a one-element hesitant fuzzy set $\{c\}$, for each $c \in S$. Consequently, we also obtain $c_{\leftrightarrow} = O(n_S^2 \cdot c_*)$.

According to Theorem 3.2, $\mathcal{H}$ constitutes a hesitant fuzzy semi-partition on X if and only if (3.11) holds. Let $h, g \in \mathcal{H}$. For the left-hand side of $\subseteq$ in (3.11), we compute $h(x) \otimes g(x)$ for each element $x \in X$, each at cost of $c_{\otimes} = O(n_S^2 \cdot c_*)$. Hence, the complexity for computing all hesitant fuzzy sets on the left-hand side of $\subseteq$ is $O(n_X \cdot n_S^2 \cdot c_*)$. The subsequent union $\bigcup_{x \in X} h(x) \otimes g(x)$ can be computed linearly in the size of each hesitant fuzzy set $h(x) \otimes g(x)$, giving a complexity of $O(n_X \cdot n_S^2)$. Therefore, the total complexity of the left-hand size of (3.11) remains $O(n_X \cdot n_S^2 \cdot c_*)$. A similar argument applies to the right-hand side.

Consequently, the total complexity to compute (3.11) for given $h, g \in \mathcal{H}$, is

$O(n_X \cdot n_S^2 \cdot c_*)$. We compute (3.11) for each pair $h, g \in \mathcal{H}$, making the total complexity to compute (3.11) equal to $O(n_{\mathcal{H}}^2 \cdot n_X \cdot n_S^2 \cdot c_*)$. Therefore, the total complexity to determine whether $\mathcal{H}$ is a hesitant fuzzy semi-partition on X is

$$O(n_{\mathcal{H}}^2 \cdot n_X \cdot n_S^2 \cdot c_*), \tag{3.16}$$

which is polynomial in sizes of the input parameters.

On the other hand, Theorem 3.2 also implies that $\mathcal{H}$ forms a hesitant fuzzy semi-partition on X if and only if (3.12) holds. By analogous reasoning, the total complexity to compute (3.12) is

$$O(n_{\mathcal{H}} \cdot n_X^2 \cdot n_S^2 \cdot c_*), \tag{3.17}$$

which is also the total complexity to determine whether $\mathcal{H}$ is a hesitant fuzzy semi-partition on X. Note that this complexity is different from the one obtained using (3.11), but remains polynomial in the sizes of the input parameters.

Corollary 3.2. *Given a hesitant fuzzy semi-partition $\mathcal{H}$ of X, for every $h, g \in \mathcal{H}$, it holds that:*

$$\bigcup_{x \in X} h(x) \otimes g(x) = \bigcap_{y \in X} h(y) \leftrightarrow g(y).$$

Proof. Choose arbitrary $h, g \in \mathcal{H}$. It follows that h coincides with an equivalence class of a hesitant fuzzy equivalence, thus h is normalized, meaning that there exists an element $a \in X$ for which $1_\otimes \subseteq h(a)$. Thus, by applying (2.61) and (2.60) successively, we get:

$$\begin{aligned}\bigcap_{y \in X} h(y) \leftrightarrow g(y) &\subseteq h(a) \leftrightarrow g(a) \subseteq h(a) \rightarrow g(a) \subseteq g(a) \subseteq h(a) \otimes g(a) \\ &\subseteq \bigcup_{x \in X} h(x) \otimes g(x).\end{aligned}$$

Combining the previous inclusion with Theorem 3.2, we obtain that the statement holds. □

Example 3.6. Let $X = \{x, y, z\}$, and let $\mathcal{Q}$ be the Gödel power set quantale, defined in Example 1.4. Consider a hesitant fuzzy set h on X over $\mathcal{Q}$, given by:

$$h = \begin{bmatrix} \{0.1, 0.6\} & \{0.3, 1\} & \{0.1, 0.7\} \end{bmatrix}.$$

Using formula (3.3), we construct the corresponding hesitant fuzzy equivalence E_h

on X:

$$E_h = \begin{bmatrix} \{0.1\} \cup [0.6, 1] & \emptyset & \{0.1\} \\ \emptyset & \{0.3, 1\} & \emptyset \\ \{0.1\} & \emptyset & \{0.1\} \cup [0.7, 1] \end{bmatrix}.$$

Now, consider the hesitant fuzzy semi-partition $\mathcal{X} = \{h, g\}$ of X, where:

$$h = \begin{bmatrix} \{0.1\} \cup [0.6, 1] & \emptyset & \{0.1\} \end{bmatrix}, \qquad g = \begin{bmatrix} \{0.1\} & \emptyset & \{0.1\} \cup [0.7, 1] \end{bmatrix}.$$

It can be verified that the part (ii) of Theorem 3.2 holds. Furthermore, the condition in Corollary 3.2 also holds, namely,

$$\bigcup_{x \in X} h(x) \otimes g(x) = \bigcap_{x \in X} h(x) \leftrightarrow g(x) = \{0.1\}.$$

However, note that $\mathcal{X}$ does not satisfy condition (H3) of Theorem 3.3 stated below. More precisely, for $y \in X$, we have $h(y) = g(y) = \emptyset$, and

$$1_\otimes \not\subseteq h(y), \qquad 1_\otimes \not\subseteq g(y).$$

It shows that $\mathcal{X}$ does not form a hesitant fuzzy partition.

Example 3.7. Let $\mathcal{Q} = \langle \mathcal{P}(\mathbb{Z}), \cup, \cap, \otimes, \emptyset, \mathbb{Z} \rangle$ be the power set quantale of the semigroup $\langle \mathbb{Z}, + \rangle$, where

$$A \otimes B = \{a + b \mid a \in A, b \in B\},$$

for every $A, B \subseteq \mathbb{Z}$. Let $X = \{x, y, z, w\}$ and let h be the hesitant fuzzy set on X over $\mathcal{Q}$, given by

$$h = \begin{bmatrix} \{0, 2\} & \{1, 3\} & \{0, 2\} & \{1, 3\} \end{bmatrix}.$$

Let us observe that $h(y) = h(w) = \{1, 3\} = \{0, 2\} \otimes \{1\} = h(x) \otimes \{1\} = h(z) \otimes \{1\}$. Thus, we have $\{0, 2\} \to \{0, 2\} = \{0\}$, $\{1, 3\} \to \{1, 3\} = \{0\}$, $\{0, 2\} \to \{1, 3\} = \{1\}$ and $\{1, 3\} \to \{0, 2\} = \{-1\}$. By definition, $A \leftrightarrow B = (A \to B) \cap (B \to A)$, for

every $A, B \in \mathcal{P}(\mathbb{Z})$, so we get:

$$E_h = \begin{bmatrix} \{0\} & \emptyset & \{0\} & \emptyset \\ \emptyset & \{0\} & \emptyset & \{0\} \\ \{0\} & \emptyset & \{0\} & \emptyset \\ \emptyset & \{0\} & \emptyset & \{0\} \end{bmatrix}.$$

We further observe that E_h has two different hesitant fuzzy equivalence classes:

$$(E_h)_x = (E_h)_z = \begin{bmatrix} \{0\} & \emptyset & \{0\} & \emptyset \end{bmatrix}, \quad \text{and} \quad (E_h)_y = (E_h)_w = \begin{bmatrix} \emptyset & \{0\} & \emptyset & \{0\} \end{bmatrix}.$$

In other words, the residuum detects the translation by 1. The biresiduum then detects whether the translation works in both directions simultaneously, which happens only when the two sets are equal.
It can be verified that the parts (ii) and (iii) of Theorem 3.2 are satisfied. Furthermore, the condition in Corollary 3.2 also holds. For example:

$$\bigcup_{t \in X} E_x(t) \otimes E_x(t) = \bigcap_{t \in X} E_x(t) \leftrightarrow E_x(t) = \{0\}.$$

Corollary 3.3. *Let $\mathcal{X}$ be a hesitant fuzzy semi-partition of X. For each $h, g \in \mathcal{X}$, the following holds:*

(H1) *If $h \neq g$, then $1_\otimes \nsubseteq ||h \otimes g||$;*

(H2) *$g(a) = ||h \otimes g||$, for all $a \in c(h)$.*

Proof. (H1) Let $h, g \in \mathcal{X}$ such that $h \neq g$. According to Corollary 3.2 and definition (2.2), we obtain:

$$||h \otimes g|| = \bigcap_{y \in X} h(y) \leftrightarrow g(y).$$

Assume that $1_\otimes \subseteq ||h \otimes g||$. Consequently, we get $1_\otimes \subseteq h(x) \leftrightarrow g(x)$, for every $x \in X$. However, according to residuation property, we obtain:

$$1_\otimes \subseteq h(x) \leftrightarrow g(x) \subseteq h(x) \rightarrow g(x) \quad \text{iff} \quad h(x) \subseteq g(x),$$

and dually:

$$1_\otimes \subseteq h(x) \leftrightarrow g(x) \subseteq g(x) \rightarrow h(x) \quad \text{iff} \quad g(x) \subseteq h(x),$$

for each $x \in X$. In other words, we obtained $h = g$, which is a contradiction with the initial assumption. Thus, the statements holds.

(H2) Follows directly from the proof of Corollary 3.2. □

We proceed to examine fuzzy partitions, as characterized in the subsequent theorem.

Theorem 3.3. *Let $\mathcal{H}$ be a family of normalized hesitant fuzzy sets on X. The following statements are equivalent:*

(i) *$\mathcal{H}$ is a hesitant fuzzy partition of X,*

(ii) *$\mathcal{H}$ is a hesitant fuzzy semi-partition of X and*

(H3) *for all $x \in X$ there is some $h \in \mathcal{H}$ with $1_\otimes \subseteq h(x)$,*

(iii) *$\mathcal{H}$ satisfies:*

(H4) *Every $x \in X$ uniquely determines $h_x \in \mathcal{H}$ satisfying $1_\otimes \subseteq h_x(x)$,*

(H5) *$h_x(y) = ||h_x \otimes h_y||$, for every $x, y \in X$.*

Proof. (i) ⇒ (ii). A hesitant fuzzy partition $\mathcal{H}$ induces a hesitant fuzzy equivalence E on X such that each $f \in \mathcal{H}$ represents an equivalence class of E. Therefore, $E_x \in \mathcal{H}$ and $1_\otimes \subseteq E_x(x)$, for each $x \in X$. Hence, the condition (ii) follows.

(ii) ⇒ (iii). Assume that (ii) holds. According to (H3), for every $x \in X$ there is some $h \in \mathcal{H}$ with $1_\otimes \subseteq h(x)$. Assume that there exists $g \in \mathcal{H}$ satisfying $g \neq f$ and $1_\otimes \subseteq g(x)$. Then, from the monotonicity of $\otimes$, we get

$$1_\otimes \subseteq h(x) \otimes g(x) \subseteq \bigcup_{u \in X} h(y) \otimes g(y) = ||h \otimes g||,$$

which contradicts (H1). Thus, (H4) is valid. On the other hand, (H5) is reformulation of (H2).

(iii) ⇒ (i). Assume that (iii) holds. Following the notations from (H4) and (H5), let us define a hesitant fuzzy relation $E \in \mathcal{R}(X)$ on X by $E(x, y) = ||h_x \otimes h_y||$, for every $x, y \in X$. From (H4) and (H5), we get $1_\otimes \subseteq h_x(x) = ||h_x \otimes h_x|| = E(x, x)$, for every $x \in X$, thus E is reflexive. Moreover, it is obvious that E is symmetric. In the end, for every $x, y, z \in X$, we have:

$$\begin{aligned} E(x, y) \otimes E(y, z) &= ||h_x \otimes h_y|| \otimes ||h_y \otimes h_z|| = h_x(y) \otimes h_z(y) \\ &\subseteq \bigcup_{t \in X} h_x(t) \otimes h_z(t) = ||h_x \otimes h_z|| = E(x, z), \end{aligned}$$

meaning that E is transitive. In other words, E is a hesitant fuzzy equivalence on X. According to (H5), $E(x,y) = h_x(y)$ for every $x, y \in X$. Hence, $\mathcal{H}$ contains all equivalence classes of E, i.e., it is a hesitant fuzzy partition on X. □

The conditions $(H3)$ and $(H4)$ admit the alternative formulations $(H3')$ and $(H4')$, respectively, given below:

(H3') $\{c(h)\}_{h\in\mathcal{C}}$ forms a cover of X,

(H4') $\{c(h)\}_{h\in\mathcal{C}}$ forms a partition of X.

Theorem 3.3 also provides a basis for analyzing the computational complexity of determing whether a family of normalized hesitant fuzzy sets on X forms a hesitant fuzzy partition of X. Indeed, assume again that the sets X, S and $\mathcal{H}$ are all finite, and denote $|X| = n_X$, $|S| = n_S$, and $|\mathcal{H}| = n_{\mathcal{H}}$.

If the condition (ii) is used, we must first determine whether $\mathcal{H}$ is a hesitant fuzzy semi-partition of X, which can be verified in (3.16) or (3.17) complexity. Afterwads, we need to check whether (H3) is satisfied, which is done in $O(n_X \cdot n_{\mathcal{H}} \cdot n_S)$ complexity. However, this bound does not exceed the one given either by (3.16) or (3.17). Hence, the overall complexity to determine the condition (ii) remains the same as the complexity of determining whether $\mathcal{H}$ is a hesitant fuzzy semi-partition of X.

On the other hand, if the condition (iii) is considered, both the conditions (H4) and (H5) must be verified. Regarding the condition (H4), we easily see that it can be performed also in $O(n_X \cdot n_{\mathcal{H}} \cdot n_S)$ complexity. For the condition (H5), for each pair $x, y \in X$, we need to calculate $(h_x \otimes h_y)(z)$, which requires $O(n_X \cdot n_S^2 \cdot c_*)$ operations, and then compute the union over all $z \in X$, which can be done in $O(n_X \cdot n_S^2)$ complexity. We do this for each pair $x, y \in X$, leading to the conclusion that (H5) is performed in $O(n_X^3 \cdot n_S^2 \cdot c_*)$ complexity. Thus, the overall complexity for the condition (iii) is $O(n_X \cdot n_{\mathcal{H}} \cdot n_S + n_X^3 \cdot n_S^2 \cdot c_*)$.

A final note on the complexity issue is that, in the case when $n_{\mathcal{H}} = n_X = n$ (which belongs to the case (iii) that for every $x \in X$ uniquely determines $h_x \in \mathcal{H}$ satisfying $1_\otimes \subseteq h_x(x)$), then all final complexities for determing whether a family of normalized hesitant fuzzy sets on X forms a hesitant fuzzy (semi-)partition of X are equal to $O(n^3 \cdot n_S^2 \cdot c_*)$. We emphasize that the above complexity considerations are applicable only when the involved sets are finite. Theorem 3.2 and Theorem 3.3, however, are stated in a general mathematical form and remain valid for arbitrary (possibly infinite) families.

3.2 Hesitant fuzzy quasi-orders

In this section, we examine hesitant fuzzy quasi-orders on a finite set X, with membership values in the power set quantale of a semigroup, and characterize the natural hesitant fuzzy equivalencies induced by them. We establish necessary and sufficient conditions for the existence of a bijection between the collection X/F of all aftersets of an arbitrary hesitant fuzzy quasi-order F and the family X/E of all equivalence classes of its associated natural hesitant fuzzy equivalence E. We also establish necessary and sufficient conditions for the existence of a bijection between the collections X/F and $X\backslash F$ of all aftersets and foresets of F, respectively.

Lemma 3.6. *Let F be a hesitant fuzzy quasi-order on X. Then, for all $x, y \in X$ the following conditions are equivalent:*

(1) $1_{\otimes} \subseteq F(x, y)$,

(2) $xF \supseteq yF$,

(3) $Fx \subseteq Fy$.

Proof. (1) $\Leftrightarrow$ (2) Assume that $1_{\otimes} \subseteq F(x, y)$. In that case, according to (2.52) and (2.60), for every $a \in X$ we obtain:

$$xF(a) = F(x, a) \supseteq F(x, y) \otimes F(y, a) \supseteq F(y, a) = yF(a).$$

Therefore, we conclude $xF \supseteq yF$.

Conversely, assume that $xF \supseteq yF$. Then

$$F(x, y) = xF(y) \supseteq yF(y) = F(y, y).$$

Reflexivity of F implies $1_{\otimes} \subseteq F(y, y)$, i.e., $1_{\otimes} \subseteq F(x, y)$.

Similarly, we show that (1) $\Leftrightarrow$ (3). □

Assume that F is a hesitant fuzzy quasi-order on a finite set X with n elements, represented by an $n \times n$ hesitant fuzzy matrix. s noted in Section 3.1, in this representation, F-aftersets correspond to the row vectors, whereas F-foresets correspond to the column vectors of this matrix.

Before proceeding, recall that every hesitant fuzzy equivalence relation E on X induces a partition of X into equivalence classes E_x, for $x \in X$, forming the factor set X/E defined in (2.58). This follows directly from the definition of a hesitant fuzzy partition, introduced in Section 3.1.

We now turn our attention to construct hesitant fuzzy equivalence relations on X from a hesitant fuzzy quasi-order F on X, as described in Theorems 3.4 and 3.7. Each of these equivalence relations induces a partition of X into equivalence classes, and hence a corresponding factor set X/E. On these factor sets, we define the induced relations $\tilde{E}$ and $\tilde{F}$ on X/E. This construction allows us to work on a reduced domain while preserving the relevant properties.

Theorem 3.4. *For a given hesitant fuzzy quasi-order F on X, the following assertions hold:*

(a) *The mapping*

$$E(x,y) = F(x,y) \cap F(y,x), \qquad \text{for all } x,y \in X, \tag{3.18}$$

defines a hesitant fuzzy equivalence on X.

(b) *For any $x, x', y, y' \in X$ such that $E_x = E_{x'}$ and $E_y = E_{y'}$ it holds*

$$F(x,y) = F(x',y'), \quad E(x,y) = E(x',y').$$

(c) *The hesitant fuzzy relation*

$$\tilde{E}(E_x, E_y) = E(x,y),$$

for all $x, y \in X$ is a hesitant fuzzy equality on X/E.

(d) *The hesitant fuzzy relation*

$$\tilde{F}(E_x, E_y) = F(x,y),$$

for every $x, y \in X$ is a hesitant fuzzy quasi-order on X/E.

Proof. (a) Let $E(x,y) = F(x,y) \cap F(y,x)$ for arbitrary $x, y \in X$. Reflexivity of E follows directly from reflexivity of F, since $1_{\otimes} \subseteq F(x,x)$ for every $x \in X$. Symmetry holds by construction:

$$E(y,x) = F(y,x) \cap F(x,y) = E(x,y).$$

Transitivity is inherited from transitivity of F. Therefore, E constitutes a hesitant fuzzy equivalence on X.

(b) From $E_x = E_{x'}$, we obtain $1_\otimes \subseteq E(x, x')$, that is,

$$\begin{aligned} 1_\otimes \subseteq F(x,x') \cap F(x',x) \quad &\Rightarrow \quad 1_\otimes \subseteq F(x,x'), 1_\otimes \subseteq F(x',x) \\ &\Rightarrow \quad xF \supseteq x'F \text{ and } x'F \supseteq xF \\ &\Rightarrow \quad xF = x'F. \end{aligned}$$

Similarly, since $E_y = E_{y'}$, then $Fy = Fy'$. Hence, the following holds

$$F(x,y) = xF(y) = x'F(y) = F(x',y) = Fy(x') = Fy'(x') = F(x',y'). \quad (3.19)$$

Analogously, $F(y,x) = F(y',x')$ and finally $E(x,y) = E(x',y')$.

(c) Let $\tilde{E}(E_x, E_y) = E(x,y)$ for every $x, y \in X$. According to item (b), $\tilde{E}$ is a well-defined hesitant fuzzy relation on X/E. Since E is a hesitant fuzzy equivalence, $\tilde{E}$ inherits reflexivity, symmetry, and transitivity, and hence is itself a hesitant fuzzy equivalence on X/E. To prove that it is also a hesitant fuzzy equality on X/E, assume that $1_\otimes \subseteq \tilde{E}(E_x, E_y) = E(x,y)$, for some $x, y \in X$. Hence, for each $z \in X$, we obtain:

$$E_x(z) = E(x,z) \subseteq E(y,x) \otimes E(x,z) \subseteq E(y,z) = E_y(z).$$

In addition,

$$E_y(z) = E(y,z) \subseteq E(x,y) \otimes E(y,z) \subseteq E(x,z) = E_x(z).$$

Therefore, $E_x = E_y$, thus we have proved that $\tilde{E}$ is a hesitant fuzzy equivalence on X/E.

(d) Again, according to item (b), $\tilde{F}$ is well-defined on X/E, and it is a hesitant fuzzy quasi-order on X/E. □

The hesitant fuzzy equivalence E on X, defined as in Theorem 3.4 given a hesitant fuzzy quasi-order F on a set X, is represented by E_F. We reffer to it as the *natural hesitant fuzzy equivalence of* F.

According to the following result, the i-th and j-th row vectors of the matrix representation of E_F are identical if and only if the i-th and j-th column vectors are equal, and vice versa. Additionally, the same result shows that there are bijective mappings between the collection of all aftersets X/F of a given hesitant fuzzy quasi-order F and the collection of all equivalence classes X/E_F of E_F, as well as between the collection of all aftersets X/F and all foresets $X\backslash F$ of F.

Theorem 3.5. *Let $E = E_F$ denote the natural hesitant fuzzy equivalence associated with F, with F being a hesitant fuzzy quasi-order on X.*

(a) *The following conditions are equivalent for any $x, y \in X$:*

(i) $1_\otimes \subseteq E(x, y)$,

(ii) $E_x = E_y$,

(iii) $xF = yF$,

(iv) $Fx = Fy$.

(b) *Functions $xF \mapsto E_x$ of X/F to X/E, and $xF \mapsto Fx$ of X/F to $X\backslash F$, are bijective functions.*

Proof. (a) For any $x, y \in X$, we have the following implications:

(i) $\Rightarrow$ (ii) This implication was proved in the proof of Theorem 3.4, part (c).

(ii) $\Rightarrow$ (i) Let $E_x = E_y$. Then $E(x, y) = E_x(y) = E_y(y) = E(y, y)$. Since $1_\otimes \subseteq E(y, y)$, we also have $1_\otimes \subseteq E(x, y)$.

(i) $\Rightarrow$ (iii) Let $1_\otimes \subseteq E(x, y)$, i.e. $1_\otimes \subseteq F(x, y) \cap F(y, x)$. It follows that $1_\otimes \subseteq F(x, y)$ and $1_\otimes \subseteq F(y, x)$. Then for every $z \in X$ the following holds

$$yF(z) = F(y, z) \subseteq F(x, y) \otimes F(y, z) \subseteq F(x, z) = xF(z).$$

Hence, $yF(z) \subseteq xF(z)$. Also, for every $z \in X$ we have

$$xF(z) = F(x, z) \subseteq F(y, x) \otimes F(x, z) \subseteq F(y, z) = yF(z).$$

So, $xF(z) \subseteq yF(z)$. Finally, we get $xF(z) = yF(z)$, for all $z \in X$.

(iii) $\Rightarrow$ (i) Assume $xF = yF$. Then

$$F(x, y) = xF(y) = yF(y) = F(y, y) \supseteq 1_\otimes.$$

Similarly,

$$F(y, x) = yF(x) = yF(y) = F(y, y) \supseteq 1_\otimes.$$

Hence, $1_\otimes \subseteq F(x, y) \cap F(y, x)$ and $1_\otimes \subseteq E(x, y)$.

The equivalence (i) $\Leftrightarrow$ (iv) can be shown in the same way as (i) $\Leftrightarrow$ (iii).

It follows from part (a) that the functions defined in (b) are both injective and surjective. □

The next theorem presents three important properties of a hesitant fuzzy quasi-order F and the natural hesitant fuzzy equivalence E_F induced by it.

Theorem 3.6. *For a given set X, let F be a hesitant fuzzy quasi-order on X. Then:*

(a) *E_F is the greatest hesitant fuzzy equivalence on X contained in F.*

(b) *For any $x, y \in X$*

$$F(x,y) = \bigcap_{a\in X} aF(x) \to aF(y) = \bigcap_{a\in X} Fa(y) \to Fa(x), \tag{3.20}$$

(c) *For any $x, y \in X$*

$$E_F(x,y) = \bigcap_{a\in X} aF(x) \leftrightarrow aF(y) = \bigcap_{a\in X} Fa(x) \leftrightarrow Fa(y). \tag{3.21}$$

Proof. (a) Assume E is an arbitrary hesitant fuzzy equivalence on X included in F. Thus:

$$E = E \cap E^{-1} \subseteq F \cap F^{-1} = E_F.$$

(b) Since F is a transitive relation, due to the residuation property (2.11), the following sequence of equivalences holds for all $x, y, a \in X$

$$\begin{aligned} F(x,y) \otimes F(y,a) \subseteq F(x,a) \quad &\Leftrightarrow \quad F(x,y) \subseteq F(y,a) \to F(x,a) \\ &\Leftrightarrow \quad F(x,y) \subseteq \bigcap_{a\in X} F(y,a) \to F(x,a) \\ &\Leftrightarrow \quad F(x,y) \subseteq \bigcap_{a\in X} Fa(y) \to Fa(x). \end{aligned}$$

Let now consider the opposite direction,

$$\bigcap_{a\in X} Fa(y) \to Fa(x) = \bigcap_{a\in X} F(y,a) \to F(x,a) \subseteq F(y,y) \to F(x,y).$$

Since F is a reflexive hesitant fuzzy relation, then $1_\otimes \subseteq F(y,y)$. Taking into account that the residuum is an antitone function by the first argument, we obtain $F(y,y) \to F(x,y) \subseteq F(x,y)$, hence

$$\bigcap_{a\in X} Fa(y) \to Fa(x) \subseteq F(x,y).$$

Finally,

$$F(x,y) = \bigcap_{a\in X} Fa(y) \to Fa(x).$$

The equality $F(x,y) = \bigcap_{a\in X} aF(x) \to aF(y)$ can be proved analogously.

(c) This follows directly from assertion (b) of this theorem and assertion (a) of Theorem 3.4. □

The following theorem introduces an additional hesitant fuzzy equivalence induced by a given hesitant fuzzy quasi-order and describes the relationship between them.

Theorem 3.7. *Let F be a hesitant fuzzy quasi-order on a set X. The following conditions hold.*

(a) *The hesitant fuzzy relation E on X given by*

$$E(x,y) = F(x,y) \otimes F(y,x), \quad \text{for every } x,y \in X,$$

is a hesitant fuzzy equivalence on X.

(b) *For all $x, x', y, y' \in X$ such that $E_x = E_{x'}$ and $E_y = E_{y'}$ the next equalities hold*

$$F(x,y) = F(x',y'), \quad E(x,y) = E(x',y').$$

(c) *The hesitant fuzzy relation $\tilde{E}$ on X/E defined by*

$$\tilde{E}(E_x, E_y) = E(x,y), \quad \text{for every } x,y \in X,$$

is a hesitant fuzzy equality on X/E.

(d) *The hesitant fuzzy relation $\tilde{F}$ on X/E,*

$$\tilde{F}(E_x, E_y) = F(x,y), \quad \text{for every } x,y \in X,$$

is a hesitant fuzzy E-order on X/E.

Proof. (a) Since F is a hesitant fuzzy quasi-order, then F is reflexive and transitive. Hence, by the associativity and commutativity of $\otimes$, for every $x, y, z \in X$ the

following is true:

$$1_\otimes \subseteq F(x,x) \quad \Rightarrow \quad 1_\otimes \subseteq F(x,x) \otimes F(x,x) \quad \Rightarrow \quad 1_\otimes \subseteq E(x,x),$$

$$E(x,y) = F(x,y) \otimes F(y,x) = F(y,x) \otimes F(x,y) = E(y,x),$$

$$\begin{aligned} E(x,y) \otimes E(y,z) &= (F(x,y) \otimes F(y,x)) \otimes (F(y,z) \otimes F(z,y)) \\ &= (F(x,y) \otimes F(y,z)) \otimes (F(z,y) \otimes F(y,x)) \\ &\subseteq F(x,z) \otimes F(z,x) \\ &= E(x,z). \end{aligned}$$

Therefore, E is a hesitant fuzzy equivalence on X.

(b) Assume $E_x = E_{x'}$ for arbitrary $x, x' \in X$, i.e., assume $E(x,y) = E(x',y)$ for all $y \in X$. In particular, by taking $y = x$, we obtain $1_\otimes \subseteq E(x,x) = E(x',x) = E(x,x')$. Therefore,

$$1_\otimes \subseteq E(x,x') = F(x,x') \otimes F(x',x).$$

As a direct consequence, $1_\otimes \subseteq F(x,x')$ and $1_\otimes \subseteq F(x',x)$. Then, by the Lemma 3.6, $xF \supseteq x'F$ and $x'F \supseteq xF$, and hence $xF = x'F$. Similarly, if $E_y = E_{y'}$, then $Fy = Fy'$. Therefore,

$$F(x,y) = xF(y) = x'F(y) = F(x',y) = Fy(x') = Fy'(x') = F(x',y').$$

The equality $F(y,x) = F(y',x')$ can be proved analogously. Finally,

$$E(x,y) = F(x,y) \otimes F(y,x) = F(x',y') \otimes F(y',x') = E(x',y').$$

(c) This part is a direct consequence of the definition of the relation $\tilde{E}$ and the statement in part (b).

(d) Based on (b), $\tilde{F}$ is well-defined on X/E. It is clear that $\tilde{F}$ is a hesitant fuzzy quasi-order. Consequently, for every $x, y \in X$,

$$\tilde{F}(E_x, E_y) \otimes \tilde{F}(E_y, E_x) = F(x,y) \otimes F(y,x) = E(x,y) = \tilde{E}(E_x, E_y).$$

Therefore, $\tilde{F}$ is $\tilde{E}$-antisymmetric. □

Example 3.8. Let $\mathcal{Q}$ be the Łukasiewicz power set quantale over the semigroup

$\langle [0,1], * \rangle$, as introduced in Example 1.4. Recall that $*$ is given by $a * b = \max\{0, a + b - 1\}$ for all $a, b \in [0,1]$. Hence, by the definition of $\otimes$ (see (1.21)), we have:

$$h_1(x) \otimes h_2(x) = \{\max\{0, a + b - 1\} \mid a \in h_1(x) \text{ and } b \in h_2(x)\},$$

for all $h_1, h_2 \in Q^X$ and all $x \in X$. Consider the hesitant fuzzy relation R on $X = \{x, y\}$ over $\mathcal{Q}$, given by:

$$R = \begin{bmatrix} \{0,1\} & \{0,0.4\} \\ \{0,0.7\} & \{0,0.1,1\} \end{bmatrix}.$$

A direct verification shows that:

$$R \circ R(x,x) = \{0,0.1,1\} \not\subseteq R(x,x),$$

and thus R is not a hesitant fuzzy quasi-order. Applying the procedure for computing the transitive closure, given in (2.55), we obtain a hesitant fuzzy quasi-order F, given by $F = R^2$:

$$F = \begin{bmatrix} \{0,0.1,1\} & \{0,0.4\} \\ \{0,0.7\} & \{0,0.1,1\} \end{bmatrix}.$$

It can be verified that $F \circ F = F$, which implies that F is reflexive and transitive. Consequently, F is a hesitant fuzzy quasi-order.

Next, we construct the hesitant fuzzy equivalences on X induced by F. By applying Theorem 3.4, we obtain the natural hesitant fuzzy equivalence E_F:

$$E_F = \begin{bmatrix} \{0,0.1,1\} & \{0\} \\ \{0\} & \{0,0.1,1\} \end{bmatrix}.$$

On the other hand, using Theorem 3.7, we obtain the hesitant fuzzy equivalence E:

$$E = \begin{bmatrix} \{0,0.1,1\} & \{0,0.1\} \\ \{0,0.1\} & \{0,0.1,1\} \end{bmatrix}.$$

This example shows that, in general, the hesitant fuzzy equivalences derived from Theorems 3.4 and 3.7 do not coincide.

3.3 Uniform hesitant fuzzy relations

In this chapter, we introduce the concept of uniform hesitant fuzzy relations by adapting the notion of uniform fuzzy relations to the hesitant fuzzy framework.

Definition 3.5. Let X and Y be non-empty sets, and let $E_X \in Q^{X\times X}$ and $E_Y \in Q^{Y\times Y}$ be hesitant fuzzy equivalence relations. A hesitant fuzzy relation $R \in Q^{X\times Y}$ is said to be *extensional with respect to* E_X if it satisfies:

$$R(x_1, y) \otimes E_X(x_1, x_2) \subseteq R(x_2, y), \tag{EX}$$

for all $x_1, x_2 \in X$ and $y \in Y$. Similarly, R is said to be *extensional with respect to* E_Y if it satisfies:

$$R(x, y_1) \otimes E_Y(y_1, y_2) \subseteq R(x, y_2), \tag{EY}$$

for all $x \in X$ and $y_1, y_2 \in Y$. If R satisfies both conditions and, moreover,

$$R(x, y_1) \otimes R(x, y_2) \subseteq E_Y(y_1, y_2), \tag{PHF}$$

for all $x \in X$ and $y_1, y_2 \in Y$, then R is said to be a *partial hesitant fuzzy function with respect to* E_X *and* E_Y.

Using the residuation property together with symmetry of E_X and E_Y, conditions (EX) and (EY) can be equivalently expressed as follows:

$$E_X(x_1, x_2) \subseteq R(x_1, y) \leftrightarrow R(x_2, y), \text{ for all } x_1, x_2 \in X \text{ and } y \in Y, \tag{EX'}$$
$$E_Y(y_1, y_2) \subseteq R(x, y_1) \leftrightarrow R(x, y_2), \text{ for all } x \in X \text{ and } y_1, y_2 \in Y. \tag{EY'}$$

For any hesitant fuzzy relation $R \in Q^{X\times Y}$, we can define a hesitant fuzzy equivalence relations $E_X^R \in Q^{X\times X}$ and $E_Y^R \in Q^{Y\times Y}$ by:

$$E_X^R(x_1, x_2) = \bigcap_{y\in Y} R(x_1, y) \leftrightarrow R(x_2, y), \text{ for all } x_1, x_2 \in X, \tag{3.22}$$
$$E_Y^R(y_1, y_2) = \bigcap_{x\in X} R(x, y_1) \leftrightarrow R(x, y_2), \text{ for all } y_1, y_2 \in Y. \tag{3.23}$$

The reflexivity and transitivity of E_X^R and E_Y^R follow from the corresponding quantale properties given in (1.18) and (1.39), respectively. The symmetry follows from the symmetry of the biresiduum. The relations E_X^R and E_Y^R are called *hesitant fuzzy equivalences on* X *and* Y *induced by* R. In particular, E_X^R is called the *kernel* of R

and E_Y^R the *co-kernel* of R. By (EX') and (EY'), E_X^R and E_Y^R are the greatest hesitant fuzzy equivalence relations on X and Y, respectively, such that R is extensional with respect to them.

Definition 3.6. A hesitant fuzzy relation $R \in Q^{X\times Y}$ is called a *partial hesitant fuzzy function* if it is a partial hesitant fuzzy function with respect to the induced hesitant fuzzy equivalence relations E_X^R and E_Y^R.

The following theorem establishes several equivalent characterizations of partial hesitant fuzzy functions.

Theorem 3.8. *Let X and Y be non-empty sets and let $R \in Q^{X\times Y}$ be a hesitant fuzzy relation. The following statements are equivalent:*

(a) *R is a partial hesitant fuzzy function.*

(b) *There exist hesitant fuzzy equivalences $E_X \in Q^{X\times X}$ and $E_Y \in Q^{Y\times Y}$ such that R is a partial hesitant fuzzy function with respect to E_X and E_Y.*

(c) $R(x, y_1) \otimes R(x, y_2) \subseteq E_Y^R(y_1, y_2)$, *for all $x \in X$ and $y_1, y_2 \in Y$.*

(d) $R(x_1, y) \otimes R(x_2, y) \subseteq E_X^R(x_1, x_2)$, *for all $x_1, x_2 \in X$ and $y \in Y$.*

(e) $R(x_1, y_1) \otimes R(x_2, y_1) \otimes R(x_1, y_2) \subseteq R(x_2, y_2)$, *for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.*

(f) *R^{-1} is a partial hesitant fuzzy function.*

Proof. (a) $\Rightarrow$ (b) This implication follows by the definition of a partial hesitant fuzzy function.

(b) $\Rightarrow$ (c) Let R be a partial hesitant fuzzy function with respect to some hesitant fuzzy equivalences $E_X \in Q^{X\times X}$ and $E_Y \in Q^{Y\times Y}$. By condition (PHF) and the fact that E_Y^R is the greatest hesitant fuzzy equivalence on Y with respect to which R is extensional, we obtain:

$$R(x, y_1) \otimes R(x, y_2) \subseteq E_Y(y_1, y_2) \subseteq E_Y^R(y_1, y_2),$$

for all $x \in X$ and $y_1, y_2 \in Y$. Hence, (c) holds.

(c) $\Rightarrow$ (a) By the construction of E_X^R and E_Y^R, the relation R is extensional with respect to them. Together with (c), this yields that R is a partial hesitant fuzzy function.

(c) $\Rightarrow$ (e) Let $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ be arbitrary. Then,

$$\begin{aligned} R(x_1, y_1) \otimes R(x_1, y_2) \subseteq E_Y^R(y_1, y_2) \subseteq R(x_2, y_1) &\leftrightarrow R(x_2, y_2) \\ &\subseteq R(x_2, y_1) \rightarrow R(x_2, y_2). \end{aligned}$$

Hence, (e) follows immediately from the residuation property (1.27).

(e) $\Rightarrow$ (c) From (e), for arbitrary $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, the following holds:

$$R(x_1, y_1) \otimes R(x_2, y_1) \otimes R(x_1, y_2) \subseteq R(x_2, y_2),$$
$$R(x_1, y_2) \otimes R(x_2, y_2) \otimes R(x_1, y_1) \subseteq R(x_2, y_1).$$

Next, using the associativity and commutativity of $\otimes$ together with the residuation property (1.27), we obtain

$$R(x_1, y_1) \otimes R(x_1, y_2) \subseteq R(x_2, y_1) \to R(x_2, y_2),$$
$$R(x_1, y_1) \otimes R(x_1, y_2) \subseteq R(x_2, y_2) \to R(x_2, y_1).$$

Hence,

$$R(x_1, y_1) \otimes R(x_1, y_2) \subseteq R(x_2, y_1) \leftrightarrow R(x_2, y_2),$$

and, finally,

$$R(x_1, y_1) \otimes R(x_1, y_2) \subseteq \bigcap_{x_2 \in X} R(x_2, y_1) \leftrightarrow R(x_2, y_2) = E_Y^R(y_1, y_2).$$

Therefore, (c) holds.

The implications (d) $\Rightarrow$ (e) and (e) $\Rightarrow$ (d) can be shown analogously.

(a) $\Leftrightarrow$ (f) By the equivalence (a) $\Leftrightarrow$ (e), the inverse hesitant fuzzy relation $R^{-1} \in Q^{Y \times X}$ is a partial hesitant fuzzy function if and only if

$$R^{-1}(y_1, x_1) \otimes R^{-1}(y_1, x_2) \otimes R^{-1}(y_2, x_1) \subseteq R^{-1}(y_2, x_2),$$

for all $y_1, y_2 \in Y$ and $x_1, x_2 \in X$. Since this inequality is equivalent to (e), and hence to (a), we conclude that (a) $\Leftrightarrow$ (f). $\square$

Remark 3.3. Property (e) provides a useful characterization of partial hesitant fuzzy functions, since it enables one to verify this property for a given hesitant fuzzy relation $R \in Q^{X \times Y}$ without computing the induced equivalences E_X^R and E_Y^R.

Definition 3.7. Let X and Y be non-empty sets and let $R \in Q^{X \times Y}$ be a hesitant fuzzy relation.

The relation R is called a *hesitant fuzzy function* if, for every $x \in X$, there exists $y \in Y$ such that $1_\otimes \subseteq R(x, y)$.

The relation R is called *surjective* if, for every $y \in Y$, there exists $x \in X$ such that $1_\otimes \subseteq R(x, y)$. Equivalently, R is surjective if and only if its inverse relation

R^{-1} is a hesitant fuzzy function.

If both R and R^{-1} are hesitant fuzzy functions, then R is called a *surjective hesitant fuzzy function.*

Remark 3.4. Let $R \in Q^{X\times Y}$ be a hesitant fuzzy relation. If the family $\{R_x\}_{x\in X}$ of R-aftersets forms a hesitant fuzzy semi-partition of Y, then R is a hesitant fuzzy function. Indeed, by the definition of a hesitant fuzzy semi-partition, for every $x \in X$ there exists $y \in Y$ such that $1_\otimes \subseteq R_x(y)$. Since $R_x(y) = R(x, y)$, it follows that R is a hesitant fuzzy function.

Definition 3.8. Let X and Y be non-empty sets and let $R \in Q^{X\times Y}$ be a hesitant fuzzy relation. A function $\psi : X \to Y$ is called a *crisp description* of R if

$$1_\otimes \subseteq R(x, \psi(x)),$$

for every $x \in X$. The set of all crisp descriptions of R is denoted by $CR(R)$.

It follows immediately from Definition 3.8 that R is a hesitant fuzzy function if and only if it admits at least one crisp description.

Definition 3.9. Let X and Y be non-empty sets, and let $E_X \in Q^{X\times X}$ and $E_Y \in Q^{Y\times Y}$ be hesitant fuzzy equivalence relations. A hesitant fuzzy function $R \in Q^{X\times Y}$ is called a *perfect hesitant fuzzy function with respect to E_X and E_Y* if it is a partial hesitant fuzzy function with respect to E_X and E_Y.

In particular, if $E_X = E_X^R$ and $E_Y = E_Y^R$, then R is simply called a *perfect hesitant fuzzy function.* The following theorem provides several equivalent characterizations of perfect hesitant fuzzy functions.

Theorem 3.9. *Let X and Y be non-empty sets and let $R \in Q^{X\times Y}$ be a hesitant fuzzy relation. The following statements are equivalent:*

(a) *R is a perfect hesitant fuzzy function.*

(b) *There exist hesitant fuzzy equivalences E_X on X and E_Y on Y such that R is a perfect hesitant fuzzy function WRT. E_X and E_Y.*

(c) *$\{R_x\}_{x\in X}$ is a hesitant fuzzy semi-partition of Y.*

(d) *R is a hesitant fuzzy function and for all $x_1, x_2 \in X$ we have*

$$E_X^R(x_1, x_2) = \bigcup_{y\in Y} R(x_1, y) \otimes R(x_2, y). \tag{3.24}$$

(e) *R is a hesitant fuzzy function and for all $\psi \in CR(R)$, $x \in X$ and $y \in Y$ we have*

$$R(x,y) = E_Y^R(\psi(x), y). \tag{3.25}$$

Proof. (a) $\Leftrightarrow$ (b) This is a direct consequence of Theorem 3.8.

(a) $\Rightarrow$ (e) Since R is a hesitant fuzzy function, there exists a function $\psi : X \to Y$ such that $1_\otimes \subseteq R(x, \psi(x))$ for all $x \in X$. Let $x \in X$, $y \in Y$ and $\psi \in CR(R)$. Then we have

$$\begin{aligned} E_Y^R(\psi(x), y) &= \bigcap_{x' \in X} R(x', \psi(x)) \leftrightarrow R(x', y) \subseteq R(x, \psi(x)) \leftrightarrow R(x, y) \\ &\subseteq R(x, \psi(x)) \to R(x, y) \subseteq R(x, y). \end{aligned}$$

For the opposite inequality, observe that by the condition (c) of Theorem 3.8 the following holds for all $x \in X$ and $y \in Y$:

$$R(x,y) \subseteq R(x, \psi(x)) \otimes R(x,y) \subseteq E_Y^R(\psi(x), y).$$

Combining both inequalities, we obtain $E_Y^R(\psi(x), y) = R(x,y)$.

(e) $\Rightarrow$ (c) To simplify the notation, set $E = E_Y^R$. By condition (e), for every $x \in X$, $y \in Y$ and $\psi \in CR(R)$, we have

$$R_x(y) = R(x,y) = E(\psi(x), y) = E_{\psi(x)}(y).$$

Therefore, since $R_x = E_{\psi(x)}$ for every $x \in X$, and the family of hesitant fuzzy equivalence classes induced by E forms a hesitant fuzzy semi-partition of Y, it follows that $\{R_x\}_{x \in X}$ is also a hesitant fuzzy semi-partition of Y.

(c) $\Rightarrow$ (d) By Remark 3.4, if $\{R_x\}_{x \in X}$ is a hesitant fuzzy semi-partition of Y, then R is a hesitant fuzzy function. Moreover, by Corollary 5.14 in [122], for all $x_1, x_2 \in X$,

$$E_X^R(x_1, x_2) = \bigcap_{y \in Y} R(x_1, y) \leftrightarrow R(x_2, y) = \bigcup_{y' \in Y} R(x_1, y') \otimes R(x_2, y').$$

(d) $\Rightarrow$ (a) Assume that (d) holds. Then R is a hesitant fuzzy function and for all $x_1, x_2 \in X$ we have

$$E_X^R(x_1, x_2) = \bigcup_{y \in Y} R(x_1, y) \otimes R(x_2, y). \tag{3.26}$$

Hence, for all $x_1, x_2 \in X$ and $y \in Y$, we obtain

$$R(x_1, y) \otimes R(x_2, y) \subseteq E_X^R(x_1, x_2).$$

The equivalence (a) $\Leftrightarrow$ (d) in Theorem 3.8, implies that R is a partial hesitant fuzzy function. Hence, by Definition 3.9 R is a perfect hesitant fuzzy function. □

Definition 3.10. Let X and Y be non-empty sets, and let $E \in Q^{Y\times Y}$ be a hesitant fuzzy equivalence relation. A function $\psi : X \to Y$ is called *E-surjective* if, for every $y \in Y$, there exists $x \in X$ such that $1_\otimes \subseteq E(\psi(x), y)$.

The above condition is equivalent to requiring that the mapping $\psi \circ E^\sharp$ is a surjective function of X onto Y/E, where $E^\sharp : Y \to Y/E$ is defined by $E^\sharp(y) = E_y$, for each $y \in Y$. Observe that, ψ is E-surjective if every equivalence class determined by $ker(E)$ has a non-empty intersection with $\mathrm{Im}(\psi)$.

Definition 3.11. Let X and Y be non-empty sets, and let $R \in Q^{X\times Y}$ be a partial hesitant fuzzy function. If R is a surjective hesitant fuzzy function, then R is called a *uniform hesitant fuzzy relation.*

Equivalently, a uniform hesitant fuzzy relation can be regarded as a perfect hesitant fuzzy function that additionally satisfies the surjectivity condition. When a uniform hesitant fuzzy relation is crisp, it is called a *uniform relation.* The following theorem presents several characterizations of uniform hesitant fuzzy relations.

Theorem 3.10. *Let X and Y be non-empty sets and let $R \in Q^{X\times Y}$ be a hesitant fuzzy relation. The following conditions are equivalent:*

(a) *R is a uniform hesitant fuzzy relation.*

(b) *R^{-1} is a uniform hesitant fuzzy relation.*

(c) *R is a surjective hesitant fuzzy function and*

$$R \circ R^{-1} \circ R = R. \tag{3.27}$$

(d) *$\{xR\}_{x\in X}$ is a hesitant fuzzy partition of Y.*

(e) *$\{Ry\}_{y\in Y}$ is a hesitant fuzzy partition of X.*

(f) *R is a surjective hesitant fuzzy function and*

$$E_X^R = R \circ R^{-1}. \tag{3.28}$$

(g) *R is a surjective hesitant fuzzy function and*

$$E_Y^R = R^{-1} \circ R. \tag{3.29}$$

(h) *R is a hesitant fuzzy function and for all $\psi \in CR(R)$ and $x \in X$, $y \in Y$, we have that ψ is E_Y^R-surjective and*

$$R(x, y) = E_Y^R(\psi(x), y). \tag{3.30}$$

(i) *R is a hesitant fuzzy function and for all $\psi \in CR(R)$, $x_1, x_2 \in X$, we have that ψ is E_Y^R-surjective and*

$$R(x_1, \psi(x_2)) = E_X^R(x_1, x_2). \tag{3.31}$$

Proof. According Theorem 3.9, (f) $\Leftrightarrow$ (a) and (g) $\Leftrightarrow$ (b).

(a) $\Leftrightarrow$ (b) By Theorem 3.8, R is a partial hesitant fuzzy function if and only if R^{-1} is a partial hesitant fuzzy function. Furthermore, by definition, the hesitant fuzzy function R is surjective if and only if its inverse R^{-1} is surjective. Since a uniform hesitant fuzzy relation is, by definition, a surjective hesitant fuzzy function, the stated equivalence follows.

(a) $\Rightarrow$ (c) Since R is a uniform hesitant fuzzy relation, it is a surjective hesitant fuzzy function, i.e., for all $y \in Y$ there exists $x \in X$ such that $1_\otimes \subseteq R(x, y)$. By the definition of composition, for all $x_2 \in X$ and $y_2 \in Y$ we have

$$\begin{aligned} R \circ R^{-1} \circ R(x_2, y_2) &= \bigcup_{y_1 \in Y} \bigcup_{x_1 \in X} (R(x_2, y_1) \otimes R^{-1}(y_1, x_1) \otimes R(x_1, y_2)) \\ &= \bigcup_{y_1 \in Y} \bigcup_{x_1 \in X} (R(x_2, y_1) \otimes R(x_1, y_1) \otimes R(x_1, y_2)). \end{aligned}$$

By condition (e) of Theorem 3.8 and the commutativity of $\otimes$, it follows that $R(x_2, y_1) \otimes R(x_1, y_1) \otimes R(x_1, y_2) \subseteq R(x_2, y_2)$. Consequently,

$$R \circ R^{-1} \circ R(x_2, y_2) \subseteq R(x_2, y_2).$$

For the converse inequality, let $x_2 \in X$ and $y_2 \in Y$ be arbitrary. By surjectivity,

there exists $x_1 \in X$ such that $1_{\otimes} \subseteq R(x_1, y_2)$. Hence,

$$\begin{aligned} R(x_2, y_2) &\subseteq R(x_2, y_2) \otimes R(x_1, y_2) \otimes R(x_1, y_2) \\ &\subseteq \bigcup_{y_1 \in Y} \bigcup_{x_1 \in X} (R(x_2, y_1) \otimes R(x_1, y_1) \otimes R(x_1, y_2)) \\ &= R \circ R^{-1} \circ R(x_2, y_2). \end{aligned}$$

Since both inequalities hold for all $x_2 \in X$ and $y_2 \in Y$, it follows that $R \circ R^{-1} \circ R = R$.

(c) $\Rightarrow$ (a) Assume that R is a surjective hesitant fuzzy function and that $R \circ R^{-1} \circ R = R$. Obviously,

$$R \circ R^{-1} \circ R \subseteq R,$$

which implies that for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$,

$$R(x_2, y_1) \otimes R(x_1, y_1) \otimes R(x_1, y_2) \subseteq R(x_2, y_2).$$

Hence, R satisfies condition (e) of Theorem 3.8 and therefore it is a partial hesitant fuzzy function. Since R is surjective, it follows that R is a uniform hesitant fuzzy relation.

(a) $\Leftrightarrow$ (d) In accordance with Theorem 3.9, the family $\{xR\}_{x \in X}$ forms a hesitant fuzzy partition of Y if and only if R is a perfect hesitant fuzzy function. On the other hand, the same family forms a fuzzy cover of Y if and only if R is surjective. Consequently, it follows that statements (a) and (d) are equivalent.

(b) $\Leftrightarrow$ (e) This equivalence is proved similarly to the previous case.

(f) $\Leftrightarrow$ (h) According to Theorem 3.9, condition (3.28) is equivalent to condition (3.30) which holds for all $\psi \in CR(R)$, $x \in X$, and $y \in Y$. Furthermore, by (3.30), R is surjective if and only if every $\psi \in CR(R)$ is E_Y^R-surjective.

(f) $\Rightarrow$ (i) Assume, by (f), that R is a surjective hesitant fuzzy function and $E_X^R = R \circ R^{-1}$. Then, by Lemma 2.2, for all $\psi \in CR(R)$ and $x_1, x_2 \in X$:

$$\begin{aligned} E_X^R(x_1, x_2) = \bigcap_{y \in Y} R(x_1, y) \leftrightarrow R(x_2, y) &\subseteq R(x_1, \psi(x_2)) \leftrightarrow R(x_2, \psi(x_2)) \\ &\subseteq R(x_2, \psi(x_2)) \rightarrow R(x_1, \psi(x_2)) \subseteq R(x_1, \psi(x_2)) \\ &\subseteq R(x_1, \psi(x_2)) \otimes R(x_2, \psi(x_2)) \subseteq \bigcup_{y \in Y} R(x_1, y) \otimes R(x_2, y) \\ &= R \circ R^{-1} = E_X^R(x_1, x_2). \end{aligned}$$

Hence, $E_X^R(x_1, x_2) \subseteq R(x_1, \psi(x_2)) \subseteq E_X^R(x_1, x_2)$, that is, (3.31) holds. By the previously established equivalence (f) $\Leftrightarrow$ (h), every $\psi \in CR(R)$ is E_Y^R-surjective which completes the proof.

(i) $\Rightarrow$ (e) For simplicity, set $E_X^R = E$. Consider an arbitrary $\psi \in CR(R)$. Since ψ is E_Y^R-surjective, for any $y \in Y$ there exists $x \in X$ such that $1_\otimes \subseteq E_Y^R(y, \psi(x))$. Consequently, we obtain $Ry = R\psi(x)$. On the other hand, by (i), for every $a \in X$ we obtain

$$R\psi(x)(a) = R(a, \psi(x)) = E(a, x) = E_x(a),$$

whence $R\psi(x) = E_x$. Therefore,

$$\{Ry\}_{y \in Y} = \{R\psi(x)\}_{x \in X} = \{E_x\}_{x \in X},$$

which shows that $\{Ry\}_{y \in Y}$ forms a hesitant fuzzy partition of X. □

Corollary 3.4. *Let X and Y be non-empty sets and let $R \in Q^{X \times Y}$ be a uniform hesitant fuzzy relation. Then for all $\psi \in CR(R)$ and all $x_1, x_2 \in X$ the following holds:*

$$E_X^R(x_1, x_2) = E_Y^R(\psi(x_1), \psi(x_2)). \tag{3.32}$$

Proof. The result follows directly from conditions (h) and (i) in Theorem 3.10. □

Theorem 3.11. *Let X and Y be non-empty sets, let $R \in Q^{X \times Y}$ be a uniform hesitant fuzzy relation. For simplicity, denote $E = E_X^R$ and $G = E_Y^R$. Define a function $\tilde{R}$ by*

$$\tilde{R}(E_x) = G_{\psi(x)} \text{ for any } x \in X \text{ and } \psi \in CR(R).$$

Then $\tilde{R}$ is well defined, i.e., independent of the choice of $x \in X$ and $\psi \in CR(R)$. Moreover, $\tilde{R}$ is a bijection from X/E onto Y/G.

Proof. First, we show that $\tilde{R}$ is well defined. Assume that $E_x = E_{x'}$ for some $x, x' \in X$. Since $E_x = E_{x'}$. Then, by Theorem 3.5, $1_\otimes \subseteq E(x, x')$. Furthermore, Corollary 3.4 yields

$$E(x, x') = G(\psi(x), \psi(x')),$$

for every $\psi \in CR(R)$ and all $x, x' \in X$. Hence, $1_{\otimes} \subseteq G(\psi(x), \psi(x'))$, which implies

$$G_{\psi(x)} = G_{\psi(x')}.$$

Next, we show that $\tilde{R}$ does not depend on the choice of $\psi \in CR(R)$. Let $\psi, \varphi \in CR(R)$ and let $x \in X$ be arbitrary. By Corollary 3.4, we have

$$E(x, x) = G(\psi(x), \varphi(x)).$$

Since E is reflexive, $1_{\otimes} \subseteq E(x, x)$, and therefore $1_{\otimes} \subseteq G(\psi(x), \varphi(x))$. Hence, by Theorem 3.5, it follows that

$$G_{\psi(x)} = G_{\varphi(x)}.$$

Consequently, $\tilde{R}$ is independent of the choice of ψ.

To prove injectivity of $\tilde{R}$, consider $E_x, E_{x'} \in X/E$ such that

$$\tilde{R}(E_x) = \tilde{R}(E_{x'}).$$

By the definition of $\tilde{R}$, this implies $G_{\psi(x)} = G_{\psi(x')}$. Since G is a hesitant fuzzy equivalence on Y, it follows from Theorem 3.5 that

$$1_{\otimes} \subseteq G(\psi(x), \psi(x')).$$

Applying Corollary 3.4, we obtain $1_{\otimes} \subseteq E(x, x')$, which yields

$$E_x = E_{x'}.$$

To prove surjectivity of $\tilde{R}$, consider arbitrary $G_y \in Y/G$. Since R is a uniform hesitant fuzzy relation, then R is surjective, i.e., there exists $x \in X$ such that $1_{\otimes} \subseteq R(x, y)$. Furthermore, condition (h) of Theorem 3.10 implies

$$R(x, y) = G(\psi(x), y),$$

for every $\psi \in CR(R)$. Hence, $1_{\otimes} \subseteq G(\psi(x), y)$, and by Theorem 3.5, we obtain

$$G_{\psi(x)} = G_y.$$

Consequently,

$$\tilde{R}(E_x) = G_{\psi(x)} = G_y,$$

which proves that $\tilde{R}$ is surjective. Hence, $\tilde{R}$ is a bijection from X/E onto Y/G.

□

3.4 Illustrative applications

In this section, we illustrate the applicability of the theoretical results developed in the previous sections through concrete examples over the power set quantale. The aim is to demonstrate how the developed theory leads to interpretable structures in illustrative examples.

Consider the power set quantale induced by the additive semigroup of non-negative integers. Let $S = \langle \mathbb{N}_0, + \rangle$ be the additive semigroup of non-negative integers, and let $\mathcal{Q} = \langle \mathcal{P}(\mathbb{N}_0), \cup, \cap, \otimes, \emptyset, \mathbb{N}_0 \rangle$ be its power set quantale, as constructed in Example 1.3, where $A \otimes B = \{a + b \mid a \in A, b \in B\}$ for all $A, B \in \mathcal{P}(\mathbb{N}_0)$, and the identity element is $1_\otimes = \{0\}$. Since S is commutative, the left and right residuals coincide and we write

$$A \to B = \{c \in \mathbb{N}_0 \mid A \otimes \{c\} \subseteq B\} = \{c \in \mathbb{N}_0 \mid a + c \in B \text{ for all } a \in A\}.$$

This setting is well suited to modeling accumulation of uncertain, discretely-estimated quantities. For example, when managing a portfolio of projects, several independent auditors may each report a distinct point estimate of a project's budget overrun, rather than a continuous range of possible values; every additional procurement step then accumulates these overruns rather than replacing them. The resulting hesitant fuzzy element is precisely the finite set of these distinct reported values.

For simplicity of notation, we identify the quantale $\mathcal{Q}$ with its underlying set $\mathcal{P}(\mathbb{N}_0)$ and, in the following examples, use the notation $\mathcal{P}(\mathbb{N}_0)$ for both. We now illustrate how the results of Sections 3.1 and 3.2 lead to equivalence-based clustering and preference grouping in the hesitant fuzzy setting over the power set quantale $\mathcal{P}(\mathbb{N}_0)$, within a shared budget-management scenario.

Example 3.9 (Application to clustering over $\mathcal{P}(\mathbb{N}_0)$)**.** Let $X = \{p_1, p_2, p_3, p_4\}$ represent four ongoing projects. Two auditors independently report point estimates of each project's budget overrun (in thousands of euros), yielding a hesitant fuzzy set

h on X over $\mathcal{P}(\mathbb{N}_0)$:

$$h(p_1) = h(p_2) = \{0, 3\}, \qquad h(p_3) = h(p_4) = \{0, 6\}.$$

Using formula (3.3), we compute the induced hesitant fuzzy equivalence E_h. For instance,

$$\begin{aligned}
\{0,3\} \to \{0,6\} &= \{c \in \mathbb{N}_0 \mid c \in \{0,6\} \text{ and } c+3 \in \{0,6\}\} = \emptyset,\\
\{0,6\} \to \{0,3\} &= \{c \in \mathbb{N}_0 \mid c \in \{0,3\} \text{ and } c+6 \in \{0,3\}\} = \emptyset,\\
\{0,3\} \to \{0,3\} &= \{c \in \mathbb{N}_0 \mid c \in \{0,3\} \text{ and } c+3 \in \{0,3\}\} = \{0\},\\
\{0,6\} \to \{0,6\} &= \{c \in \mathbb{N}_0 \mid c \in \{0,6\} \text{ and } c+6 \in \{0,6\}\} = \{0\}.
\end{aligned}$$

Hence,

$$E_h = \begin{bmatrix} 1_\otimes & 1_\otimes & \emptyset & \emptyset \\ 1_\otimes & 1_\otimes & \emptyset & \emptyset \\ \emptyset & \emptyset & 1_\otimes & 1_\otimes \\ \emptyset & \emptyset & 1_\otimes & 1_\otimes. \end{bmatrix}$$

Consequently, E_h produces exactly two equivalence classes, E_{p_1} and E_{p_3}, given by

$$E_{p_1} = \begin{bmatrix} 1_\otimes & 1_\otimes & \emptyset & \emptyset \end{bmatrix}, \qquad E_{p_3} = \begin{bmatrix} \emptyset & \emptyset & 1_\otimes & 1_\otimes \end{bmatrix},$$

which can naturally be interpreted as two clusters of projects with comparable budget-risk profiles: $\{p_1, p_2\}$ and $\{p_3, p_4\}$.

We now verify that the family of equivalence classes $\{E_{p_1}, E_{p_3}\}$ satisfies the condition (H1) of Corollary 3.3. Since $E_{p_1} \neq E_{p_3}$, condition (H1) requires $1_\otimes \notin \|E_{p_1} \otimes E_{p_3}\|$. Indeed,

$$\|E_{p_1} \otimes E_{p_3}\| = \bigcup_{z \in X} E_{p_1}(z) \otimes E_{p_3}(z) = \emptyset,$$

which strictly excludes $1_\otimes = \{0\}$, confirming that the two risk clusters are provably distinct with respect to the induced hesitant fuzzy equivalence relation, rather than merely numerically distant.

Condition (iii) of Theorem 3.3 allows us to verify the hesitant fuzzy partition property by checking conditions (H4)-(H5), instead of directly verifying that the relation induced by $\{E_{p_1}, E_{p_3}\}$ is reflexive, symmetric, and transitive. This makes the

complexity analysis following (3.16)-(3.17) applicable in practice: for each $x \in X$, there exists a *unique* $h_x \in \{E_{p_1}, E_{p_3}\}$ with $1_\otimes \subseteq h_x(x)$ (condition (H4)), and one directly checks $h_x(y) = \|h_x \otimes h_y\|$ for every $x, y \in X$ (condition (H5)), e.g.,

$$E_{p_1}(p_1) = \{0\} = \|E_{p_1} \otimes E_{p_1}\|, \qquad E_{p_1}(p_3) = \emptyset = \|E_{p_1} \otimes E_{p_3}\|.$$

Both conditions hold, confirming that $\{E_{p_1}, E_{p_3}\}$ is indeed a hesitant fuzzy partition of X.

Now, we compare the obtained hesitant fuzzy clustering with the result that would be obtained after aggregating the hesitant information into single numerical values. If the two auditors' estimates were collapsed to a single averaged number (e.g., 1.5 and 3, in thousands of euros), a standard threshold-based clustering could plausibly merge projects whose averages lie close together, depending on the chosen threshold. In contrast, working directly with E_h over $\mathcal{P}(\mathbb{N}_0)$ yields a *provable* $\emptyset$-separation between the two risk clusters, since no discrete estimate reported for one project ever coincides, even after accumulation, with an estimate reported for the other: this is a structural, not merely a numerical, guarantee.

Having grouped the projects by their comparable budget-overrun profiles, we now turn to a related but distinct question: how a committee's pairwise preference judgments over the same portfolio induce a natural ranking structure.

Example 3.10 (Application to preference clustering via a hesitant fuzzy quasi-order)**.** Let $X = \{p_1, p_2, p_3, p_4\}$ represent four candidate projects evaluated by several budget committee experts. Let F be a hesitant fuzzy relation on X over $\mathcal{Q} = \mathcal{P}(\mathbb{N}_0)$ representing the *margin of budgetary preference*: $F(x, y)$ is the set of additional funding values under which x is still preferred to (or tied with) y, given by the matrix

$$F = \begin{bmatrix} 1_\otimes & 1_\otimes & \{0,6,9\} & \{0,6,9\} \\ 1_\otimes & 1_\otimes & \{0,6,9\} & \{0,6,9\} \\ \emptyset & \emptyset & 1_\otimes & 1_\otimes \\ \emptyset & \emptyset & 1_\otimes & 1_\otimes \end{bmatrix}.$$

It can be directly verified that F is reflexive and transitive (i.e., a hesitant fuzzy quasi-order on X): in particular, the only non-trivial products are of the form $1_\otimes \otimes \{0,6,9\} = \{0,6,9\}$ and $\{0,6,9\} \otimes 1_\otimes = \{0,6,9\}$, both of which are contained in the corresponding entry of F, while every product involving $\emptyset$ trivially satisfies transitivity. Note that the tied pairs are deliberately kept at the trivial margin

$1_\otimes = \{0\}$: enriching a *mutual* (two-directional) relation beyond the identity would force $F(p_1, p_1)$ to absorb the self-composition $F(p_1, p_2) \otimes F(p_2, p_1)$, which grows without bound under a non-idempotent operation such as addition; only the one-directional (strict dominance) entries can safely carry a richer margin.

By Theorem 3.4(a), the hesitant fuzzy relation $E_F(x, y) = F(x, y) \cap F(y, x)$ is a hesitant fuzzy equivalence on X. Computing E_F yields

$$E_F(p_1, p_2) = 1_\otimes \cap 1_\otimes = 1_\otimes, \qquad E_F(p_3, p_4) = 1_\otimes \cap 1_\otimes = 1_\otimes,$$
$$E_F(p_1, p_3) = \{0, 6, 9\} \cap \emptyset = \emptyset, \qquad E_F(p_2, p_4) = \{0, 6, 9\} \cap \emptyset = \emptyset,$$

so E_F produces exactly two equivalence classes, $\{p_1, p_2\}$ and $\{p_3, p_4\}$, interpreted as two *ranks of equally-preferred projects*: the committee considers p_1 and p_2 mutually substitutable (each is preferred to the other with only the trivial margin $1_\otimes = \{0\}$, i.e., no genuine advantage), and likewise for p_3, p_4, while both members of the first rank are strictly preferred to both members of the second, with a non-trivial margin $\{0, 6, 9\}$ (thousand euros) and no reverse preference ($\emptyset$).

By Theorem 3.6(a), E_F is the *greatest* hesitant fuzzy equivalence contained in F; hence no rougher tie can be justified without contradicting the observed preference $F(p_1, p_3) = \{0, 6, 9\} \neq \emptyset$, and no finer distinction can be justified within $\{p_1, p_2\}$, since $F(p_1, p_2) = F(p_2, p_1) = 1_\otimes = \{0\}$. This provides the budget committee with a provably optimal (finest-possible) grouping of projects into indistinguishable preference ranks, directly usable for multi-criteria funding decisions.

When we combine hesitant fuzzy sets with the power set quantale of a general semigroup, we gain a framework capable of modeling multi-expert, highly complex budgeting systems, without being restricted to numerical aggregation.

The obtained results provide an algebraic framework for representing and analyzing relations between objects in situations where evaluations are uncertain, imprecise, or provided by multiple sources. One possible area of application is data clustering and pattern recognition. In many practical problems, objects must be grouped according to similarity measures that cannot be expressed by a single numerical value. Instead, several possible similarity degrees may arise due to disagreement among experts, measurement variability, or incomplete information. In such situations, hesitant fuzzy relations allow similarity assessments to be represented as sets of possible values. The established characterization of hesitant fuzzy equivalence relations then provides a mathematical basis for constructing hesitant fuzzy partitions, which can be interpreted as clusters of mutually similar objects. This framework can be useful in domains such as medical diagnosis, document classifica-

tion, and image recognition, where similarity judgments are inherently uncertain.

Another important application concerns multi-expert decision making. In many decision processes, several experts evaluate alternatives with respect to different criteria. Their assessments often differ, producing sets of possible evaluation values rather than a single precise score. Hesitant fuzzy relations can model these evaluations without forcing premature aggregation. Then, hesitant fuzzy equivalence relations may be used to identify groups of alternatives that are considered essentially indistinguishable by the experts, while hesitant fuzzy quasi-orders can represent preference or dominance relations between alternatives. Such structures are particularly relevant in multi-criteria decision analysis, where alternatives must be ranked or grouped despite uncertainty in the evaluation process.

The applications considered in this section illustrate the potential of the developed framework for modeling similarity and preference relations in environments characterized by hesitation and uncertainty. The results also open up possibilities for further research on algorithmic methods and practical implementations in areas such as data analysis, decision support systems, and intelligent information processing.

Chapter 4

Hesitant fuzzy relation inequalities and equations

Fuzzy relation equations and inequalities have been extensively studied since the pioneering work of Sanchez [106] and have found applications in various areas, including medical research, fuzzy control, knowledge engineering, system identification, and decision-making [30, 38, 66, 107]. Their study has subsequently been extended from the Gödel structure to more general frameworks, including Brouwerian lattices and, more generally, complete residuated lattices [36, 62, 67, 70, 93].

In the hesitant fuzzy setting, several types of hesitant fuzzy equations have been investigated, including fully, dual, partial, and half hesitant fuzzy equations, with applications such as the determination of market equilibrium prices [6–8, 118].

The theory of fuzzy relation equations and inequalities therefore provides the main motivation for our study of hesitant fuzzy relation inequalities and equations over the power set quantale of a semigroup. In this chapter, we consider *hesitant fuzzy relation inequalities* (HRIs) on X of the form

$$w \circ R \subseteq w, \quad R \circ w \subseteq w, \quad w \subseteq w \not\phi R, \quad w \subseteq R \not\backslash w, \tag{4.1}$$

as well as *hesitant fuzzy relation equations* (HREs) on X of the form

$$w \circ R = w, \quad R \circ w = w, \quad w = w \not\phi R, \quad w = R \not\backslash w. \tag{4.2}$$

Let Φ denote an arbitrary hesitant fuzzy relation inequality or equation of the form (4.1) or (4.2). We denote the set of all solutions to Φ by $\langle \Phi \rangle$.

A family $\{\Phi_i\}_{i \in I}$ of hesitant fuzzy relation inequalities (resp. equations) is called a *system of hesitant fuzzy relation inequalities* (resp. *equations*). A hesitant fuzzy set $h \in Q^X$ is a solution to the system $\{\Phi_i\}_{i \in I}$ if it is a solution to each hesitant fuzzy

relation inequality or equation from the system. The set of all solutions to the system $\{\Phi_i\}_{i\in I}$ is denoted by $\langle \Phi_i \mid i \in I\rangle$. In particular, for a finite set $I = \{1, \ldots, n\}$, we write $\langle \Phi_1, \ldots, \Phi_n\rangle$ instead of $\langle \Phi_i \mid i \in I\rangle$.

Chapter 4 is devoted to hesitant fuzzy relation inequalities and equations. Section 4.1 deals with relation inequalities involving one unknown hesitant fuzzy set, while Section 4.2 considers the corresponding relation equations. Section 4.3 provides algebraic characterizations of the corresponding solution sets. The results presented in Sections 4.1–4.3 are reported in the paper by Tasić et al. [121], which has been accepted for publication. Finally, Section 4.4 investigates hesitant fuzzy relation inequalities and equations involving two unknown hesitant fuzzy sets.

4.1 Hesitant fuzzy relation inequalities with one unknown hesitant fuzzy set

In this section, the specific classes of hesitant fuzzy relation inequalities are formally introduced and analyzed. Throughout the section, $R \in Q^{X\times X}$ denotes a given hesitant fuzzy relation, while $w \in Q^X$ is an *unknown* hesitant fuzzy set.

Theorem 4.1. *Let $\{R_i\}_{i\in I}$ be a family of hesitant fuzzy relations on X, let $R = \bigcup_{i\in I} R_i$ and $F = R^f$. Then*

$$\begin{aligned}\langle w \circ R_i \subseteq w \mid i \in I\rangle &= \langle w \circ R \subseteq w\rangle = \langle w \circ F \subseteq w\rangle = \langle w \circ F = w\rangle \\ &= \langle w \subseteq w \not\phi R_i \mid i \in I\rangle = \langle w \subseteq w \not\phi R\rangle \\ &= \langle w \subseteq w \not\phi F\rangle = \langle w = w \not\phi F\rangle.\end{aligned} \tag{4.3}$$

In addition, these sets consist of all hesitant fuzzy sets on X that are linear combinations of F-aftersets.

Proof. Due to the residuation property (2.45), the following equalities hold:

$$\langle w \circ R_i \subseteq w \mid i \in I\rangle = \langle w \subseteq w \not\phi R_i \mid i \in I\rangle, \quad \langle w \circ R \subseteq w\rangle = \langle w \subseteq w \not\phi R\rangle,$$
$$\langle w \circ F \subseteq w\rangle = \langle w \subseteq w \not\phi F\rangle.$$

Let $h \in \langle w \circ R_i \subseteq w \mid i \in I\rangle$, i.e., $h \circ R_i \subseteq h$. Then, by property (2.34), we obtain:

$$h \circ R = h \circ \Big(\bigcup_{i\in I} R_i\Big) = \bigcup_{i\in I}(h \circ R_i) \subseteq h.$$

Hence, $\langle w \circ R_i \subseteq w \mid i \in I\rangle \subseteq \langle w \circ R \subseteq w\rangle$. Conversely, let $h \in \langle w \circ R \subseteq w\rangle$, that is, assume that $h \circ R \subseteq h$. Since $R_i \subseteq R$ for all $i \in I$, property (2.33) implies that

$h \circ R_i \subseteq h \circ R \subseteq h$ for all $i \in I$. Consequently, we obtain $\langle w \circ R \subseteq w \rangle \subseteq \langle w \circ R_i \subseteq w \mid i \in I \rangle$. Therefore, the equality $\langle w \circ R_i \subseteq w \mid i \in I \rangle = \langle w \circ R \subseteq w \rangle$ holds.

Next, let $h \in \langle w \circ R \subseteq w \rangle$. Then, since $h \circ R^0 \subseteq h$, it follows by induction that $h \circ R^n \subseteq h$ for all $n \in \mathbb{N}_0$. Consequently, the following holds:

$$h \circ F = h \circ \Big(\bigcup_{n \in \mathbb{N}_0} R^n \Big) = \bigcup_{n \in \mathbb{N}_0} (h \circ R^n) \subseteq h.$$

Hence, $\langle w \circ R \subseteq w \rangle \subseteq \langle w \circ F \subseteq w \rangle$. Conversely, suppoose that $h \in \langle w \circ F \subseteq w \rangle$, that is, $h \circ F \subseteq h$. If $R \subseteq F$, it follows that $h \circ R \subseteq h \circ F \subseteq h$, and, consequently, $\langle w \circ F \subseteq w \rangle \subseteq \langle w \circ R \subseteq w \rangle$. Therefore, the equality $\langle w \circ R \subseteq w \rangle = \langle w \circ F \subseteq w \rangle$ holds.

Assume now $h \in \langle w \circ F \subseteq w \rangle$. For any $h \in Q^X$, the reflexivity of F implies that $h \subseteq h \circ F$. Together with $h \circ F \subseteq h$, this yields $h \circ F = h$, and, consequently, $\langle w \circ F \subseteq w \rangle \subseteq \langle w \circ F = w \rangle$. The opposite implication is evident, so we obtain $\langle w \circ F \subseteq w \rangle = \langle w \circ F = w \rangle$. Similarly, since $h \mapsto h \not\circ F$ is intensive (Proposition 2.9), we have $\langle w \subseteq w \not\circ F \rangle = \langle w = w \not\circ F \rangle$. Thus, all the equations in (4.3) have been proved.

We now show that the set $\langle w \circ F = w \rangle$ consists of all hesitant fuzzy sets on X that can be represented as a linear combination of F-aftersets. For any $h \in \langle w \circ F = w \rangle$ and any $y \in X$, we have:

$$h(y) = (h \circ F)(y) = \bigcup_{x \in X} h(x) \otimes F(x, y) = \bigcup_{x \in X} h(x) \otimes xF(y) = \Big(\bigcup_{x \in X} h(x) \otimes xF \Big)(y).$$

Consequently:

$$h = \bigcup_{x \in X} h(x) \otimes xF = \bigcup_{x \in X} \lambda_x \otimes xF,$$

where $\lambda_x = h(x)$, for each $x \in X$. Hence, h is represented as a linear combination of F-aftersets. Conversely, assume that h can be represented as a linear combination of F-aftersets, i.e., there are scalars $\lambda_x \in Q$, $x \in X$, such that:

$$h = \bigcup_{x \in X} \lambda_x \otimes xF.$$

Then it follows:

$$\begin{aligned}(h \circ F)(y) &= \bigcup_{z\in X} h(z) \otimes F(z,y) = \bigcup_{z\in X} \Big(\bigcup_{x\in X} \lambda_x \otimes xF\Big)(z) \otimes F(z,y) \\ &= \bigcup_{z\in X} \Big(\bigcup_{x\in X} \lambda_x \otimes F(x,z)\Big) \otimes F(z,y) = \bigcup_{x\in X} \lambda_x \otimes \Big(\bigcup_{z\in X} F(x,z) \otimes F(z,y)\Big) \\ &= \bigcup_{x\in X} \lambda_x \otimes F(x,y) = \Big(\bigcup_{x\in X} \lambda_x \otimes xF\Big)(y) = h(y),\end{aligned}$$

for any $y \in X$. Therefore, $h \in \langle w \circ F = w\rangle$, which completes the proof. □

By a similar argument, we obtain the following theorem.

Theorem 4.2. *Let $\{R_i\}_{i\in I} \subseteq Q^{X\times X}$ be a family of hesitant fuzzy relations, let $R = \bigcup_{i\in I} R_i$ and let $F = R^f$. Then*

$$\begin{aligned}\langle R_i \circ w \subseteq w \mid i \in I\rangle &= \langle R \circ w \subseteq w\rangle = \langle F \circ w \subseteq w\rangle = \langle F \circ w = w\rangle \\ &= \langle w \subseteq R_i \backslash\!\!\!\diamond w \mid i \in I\rangle = \langle w \subseteq R \backslash\!\!\!\diamond w\rangle \\ &= \langle w \subseteq F \backslash\!\!\!\diamond w\rangle = \langle w = F \backslash\!\!\!\diamond w\rangle.\end{aligned} \tag{4.4}$$

In addition, these sets consist of all hesitant fuzzy sets on X that are linear combinations of F-foresets.

The following result addresses hesitant fuzzy relation inequalities that involve the equivalence closure of the corresponding hesitant fuzzy relation.

Theorem 4.3. *Let $\{R_i\}_{i\in I}$ be a family of hesitant fuzzy relations on X, let $R = \bigcup_{i\in I} R_i$ and $E = R^e$. Then*

$$\begin{aligned}&\langle w \circ R_i \subseteq w, R_i \circ w \subseteq w \mid i \in I\rangle = \langle w \circ R \subseteq w, R \circ w \subseteq w\rangle \\ &= \langle w \circ E \subseteq w\rangle = \langle w \circ E = w\rangle = \langle w \subseteq w \not\phi R_i, w \subseteq R_i \backslash\!\!\!\diamond w \mid i \in I\rangle \\ &= \langle w \subseteq w \not\phi R, w \subseteq R \backslash\!\!\!\diamond w\rangle = \langle w \subseteq w \not\phi E\rangle = \langle w = w \not\phi E\rangle.\end{aligned} \tag{4.5}$$

In addition, these sets consist of all hesitant fuzzy sets on X that are linear combinations of E-classes.

Proof. The following equailities are direct consequences of Theorem 4.1 and Theorem 4.2:

$$\begin{aligned}\langle w \circ R_i \subseteq w, R_i \circ w \subseteq w \mid i \in I\rangle &= \langle w \circ R \subseteq w, R \circ w \subseteq w\rangle \\ &= \langle w \subseteq w \not\phi R_i, w \subseteq R_i \backslash\!\!\!\diamond w \mid i \in I\rangle \\ &= \langle w \subseteq w \not\phi R, w \subseteq R \backslash\!\!\!\diamond w\rangle.\end{aligned}$$

It follows immediately that $\langle w \circ E \subseteq w \rangle = \langle w \subseteq w \not\circ E \rangle$.

Assume now that $h \in \langle w \circ E \subseteq w \rangle$, i.e., $h \circ E \subseteq h$. Since $h \mapsto h \circ E$ is extensive, we immediately obtain $h \subseteq h \circ E$, and hence $h \circ E = h$. Conversely, if $h \in \langle w \circ E \subseteq w \rangle$ then clearly $h \circ E \subseteq h$. Therefore, $\langle w \circ E \subseteq w \rangle = \langle w \circ E = w \rangle$.

The equality $\langle w \subseteq w \not\circ E \rangle = \langle w = w \not\circ E \rangle$ can be proved in a similar way.

Next, let $h \in \langle w \circ E \subseteq w \rangle$. Since $R \subseteq E$, we obtain $h \circ R \subseteq h \circ E \subseteq h$. Hence, $\langle w \circ E \subseteq w \rangle \subseteq \langle w \circ R \subseteq w, R \circ w \subseteq w \rangle$. Conversely, assume that $h \in \langle w \circ R \subseteq w, R \circ w \subseteq w \rangle$. By properties (2.34) and (2.30), we obtain:

$$h \circ R^s = h \circ (R \cup R^{-1}) = (h \circ R) \cup (h \circ R^{-1}) = (h \circ R) \cup (R \circ h) \subseteq h.$$

By induction, $h \circ (R^s)^n \subseteq h$ holds for each $n \in \mathbb{N}_0$. Therefore:

$$h \circ E = h \circ (R^s)^f = h \circ \bigcup_{n \in \mathbb{N}_0} (R^s)^n = \bigcup_{n \in \mathbb{N}_0} h \circ (R^s)^n \subseteq h,$$

which implies $\langle w \circ R \subseteq w, R \circ w \subseteq w \rangle \subseteq \langle w \circ E \subseteq w \rangle$. Hence, $\langle w \circ R \subseteq w, R \circ w \subseteq w \rangle = \langle w \circ E = w \rangle$.

Using the same approach as in the proof of Theorem 4.1, we obtain that $\langle w \circ E = w \rangle$ coincides with the family of all hesitant fuzzy sets on X that can be expressed as linear combinations of E-classes. □

In view of Theorems 4.1-4.3, it suffices to consider the equations $w \circ F = F$, $F \circ w = F$ and $w \circ E = w$ instead of the systems of inequalities (4.3), (4.4) and the system of equations (4.5), respectively.

Theorem 4.4. *Let $\{R_i\}_{i \in I}$ be a family of hesitant fuzzy relations on X. Then the following holds:*

(1) *if $h \in \langle w \circ R_i \subseteq w \mid i \in I \rangle$, then $h^\neg \in \langle R_i \circ w \subseteq w \mid i \in I \rangle$,*

(2) *if $h \in \langle R_i \circ w \subseteq w \mid i \in I \rangle$, then $h^\neg \in \langle w \circ R_i \subseteq w \mid i \in I \rangle$,*

(3) *if $h \in \langle w \circ R_i \subseteq w, R_i \circ w \subseteq w \mid i \in I \rangle$, then $h^\neg \in \langle w \circ R_i \subseteq w, R_i \circ w \subseteq w \mid i \in I \rangle$.*

Proof. (1) Let h be a hesitant fuzzy set on X such that $h \in \langle w \circ R_i \subseteq w \mid i \in I \rangle$ and let $R = \bigcup_{i \in I} R_i$. By Theorem 4.1, it follows that $h \in \langle w \circ R \subseteq w \rangle$. Using properties

(1.50) and (1.43), together with the commutativity of $\otimes$, we obtain:

$$\begin{aligned}(h \circ R)^{\neg}(x) = (h \circ R)(x) \to \emptyset &= \left(\bigcup_{y \in X} \Big(h(y) \otimes R(y,x) \Big) \right) \to \emptyset \\ &= \bigcap_{y \in X} \Big(\big(h(y) \otimes R(y,x) \big) \to \emptyset \Big) = \bigcap_{y \in X} \Big(\big(R(y,x) \otimes h(y) \big) \to \emptyset \Big) \\ &= \bigcap_{y \in X} \Big(R(y,x) \to \big(h(y) \to \emptyset \big) \Big) = \bigcap_{y \in X} \Big(R(y,x) \to h^{\neg}(y) \Big) \\ &= (R \backslash h^{\neg})(x),\end{aligned}$$

for all $x \in X$. Therefore, the equality $(h \circ R)^{\neg} = R \backslash h^{\neg}$ holds. Next, by properties (1.39) and (2.45) we obtain:

$$h \circ R \subseteq h \quad \Rightarrow \quad h^{\neg} \subseteq (h \circ R)^{\neg} \quad \Leftrightarrow \quad h^{\neg} \subseteq R \backslash h^{\neg} \quad \Leftrightarrow \quad R \circ h^{\neg} \subseteq h^{\neg}.$$

Consequently, $h \in \langle w \circ R \subseteq w \rangle$ implies $h^{\neg} \in \langle R \circ w \subseteq w \rangle$, and hence, by Theorem 4.2, $h^{\neg} \in \langle R_i \circ w \subseteq w \mid i \in I \rangle$.

The assertion (2) can be proved using the similar approach, while (3) directly follows from (1) and (2). □

Let R be a hesitant fuzzy relation on X and let h be a hesitant fuzzy set on X. For the purposes of this paper, we define left and right R-closed hesitant fuzzy sets. We say that h is *left R-closed* if, for all $x, y \in X$, $h(x) \otimes R(x,y) \subseteq h(y)$, which is equivalent to $h \circ R \subseteq h$. Similarly, h is *right R-closed* if $R(x,y) \otimes h(y) \subseteq h(x)$, for all $x, y \in X$, or, equivalently, $R \circ h \subseteq h$. A hesitant fuzzy set that is both left and right R-closed is termed *double R-closed*. Note that, if R is symmetric, then $h \circ R = R \circ h$ and left and right R-closedness coincide; in this case we simply speak of *R-closed* hesitant fuzzy sets. The families of left, right, and double R-closed hesitant fuzzy sets on X are denoted by $\Gamma_l(R)$, $\Gamma_r(R)$, and $\Gamma_d(R)$, respectively. Moreover, following the standard fuzzy set theory terminology (cf. [24, 70]), if E is a hesitant fuzzy equivalence, then *E-closed* hesitant fuzzy sets are said to be *extensional with respect to E*.

Motivated by the work of Lai and Zhang [74] we formulate and prove the following lemma, which will be used to prove the next theorem.

Lemma 4.1. *Let R be a hesitant fuzzy relation on X and let $\Gamma_l(R)$ denote the family of left R-closed hesitant fuzzy sets on X. Then, for any family $\{h_i\}_{i \in I} \subseteq \Gamma_l(R)$ of hesitant fuzzy sets on X, any hesitant fuzzy set $g \in \Gamma_l(R)$ on X, and any scalar $\lambda \in Q$, the following conditions hold:*

(1) $h_\emptyset, h_S \in \Gamma_l(R)$,

(2) $\bigcup_{i\in I}\{h_i\} \in \Gamma_l(R)$,

(3) $\bigcap_{i\in I}\{h_i\} \in \Gamma_l(R)$,

(4) $\lambda \otimes g \in \Gamma_l(R)$,

(5) $\lambda \xrightarrow{s} g \in \Gamma_l(R)$.

Proof. (1) Clearly, $h_\emptyset \otimes F(x,y) = h_\emptyset$ and $h_S \otimes R(x,y) \subseteq h_S$ for all $x, y \in X$, which implies $h_\emptyset, h_S \in \Gamma_l(R)$.

(2) Given that $\{h_i\}_{i\in I} \subseteq \Gamma_l(R)$, we have:

$$h_i(x) \otimes R(x,y) \subseteq h_i(y),$$

for all $i \in I$ and $x, y \in X$. This implies that, for all $x, y \in X$:

$$\Big(\bigcup_{i\in I} h_i(x)\Big) \otimes R(x,y) = \bigcup_{i\in I}\Big(h_i(x) \otimes R(x,y)\Big) \subseteq \bigcup_{i\in I} h_i(y).$$

Consequently, $\bigcup_{i\in I} h_i \in \Gamma_l(R)$.

(3) The statement can be proved in the same way as statement (2).

(4) Given that $g \in \Gamma_l(R)$, we have:

$$g(x) \otimes R(x,y) \subseteq g(y),$$

for all $x, y \in X$. Next, according to property (1.3) of scalar multiplication, for any $\lambda \in Q$ the following holds:

$$\Big(\lambda \otimes g(x)\Big) \otimes R(x,y) = \lambda \otimes \Big(g(x) \otimes R(x,y)\Big) \subseteq \lambda \otimes g(y),$$

for all $x, y \in X$. Therefore, $\lambda \otimes g \in \Gamma_l(R)$.

(5) Given that $g \in \Gamma_l(R)$, the following holds:

$$g(x) \otimes R(x,y) \subseteq g(y) \quad \Leftrightarrow \quad R(x,y) \subseteq g(x) \to g(y),$$

for all $x, y \in X$. According to property (1.50), we have:

$$\Big(\lambda \to g(x)\Big) \otimes \Big(g(x) \to g(y)\Big) \subseteq \Big(\lambda \to g(y)\Big),$$

for all $x, y \in X$. Then, by the residuation property (2.12), we obtain:

$$\Big(g(x) \to g(y)\Big) \subseteq \Big(\lambda \to g(x)\Big) \to \Big(\lambda \to g(y)\Big),$$

for all $x, y \in X$. Hence, the following holds:

$$R(x, y) \subseteq g(x) \to g(y) \subseteq \Big(\lambda \to g(x))\Big) \to \Big(\lambda \to g(y)\Big),$$

for all $x, y \in X$. Therefore:

$$\Big(\lambda \to g(x)\Big) \otimes R(x, y) \subseteq \Big(\lambda \to g(y)\Big),$$

for all $x, y \in X$. Consequently, $\lambda \xrightarrow{s} g \in \Gamma_l(R)$. □

Remark 4.1. The conditions (1)-(5) established for $\Gamma_l(R)$ also hold for $\Gamma_r(R)$ and $\Gamma_d(R)$.

The following theorem is devoted to a detailed characterization of the solutions to the considered inequalities using closures and aftersets (foresets).

Theorem 4.5. *Let $\{R_i\}_{i\in I}$ be a family of hesitant fuzzy relations on X. Define $R = \bigcup_{i\in I} R_i$, $F = R^f$ and $E = R^e$. Then:*

(1) *$\langle w \circ R_i \subseteq w \mid i \in I\rangle$, $\langle R_i \circ w \subseteq R_i \mid i \in I\rangle$, $\langle w \circ R_i \subseteq w, R_i \circ w \subseteq w \mid i \in I\rangle$ form both closure and opening systems in $\mathcal{Q}(X)$.*

(2) *The principal part of $\langle w \circ R_i \subseteq w \mid i \in I\rangle$ consists of all F-aftersets.*

(3) *The principal part of $\langle R_i \circ w \subseteq w \mid i \in I\rangle$ consists of all F-foresets.*

(4) *The principal part of $\langle w \circ R_i \subseteq w, R_i \circ w \subseteq w \mid i \in I\rangle$ consists of all E-classes.*

Proof. (1) The assertion follows directly from Lemma 4.1.

(2) To simplify the notation, set $\mathcal{C} = \langle w \circ R_i \subseteq w \mid i \in I\rangle$. By Theorem 4.1, we have $\mathcal{C} = \langle w \circ F = w\rangle$. Let $\mathcal{P}(\mathcal{C}) = \{C_x \mid x \in X\}$ denote the principal part of $\mathcal{C}$. According to definition (2.24), $\mathcal{C}_x = \bigcap\{g \in \mathcal{C} \mid 1_\otimes \subseteq g(x)\}$ for each $x \in X$. Fix an arbitrary $x \in X$ and let xF denote the F-afterset of x. Then, using the reflexivity and transitivity of F, we obtain:

$$\begin{aligned}\Big(xF \circ F\Big)(y) &= \bigcup_{z\in X} xF(z) \otimes F(z, y) = \bigcup_{z\in X} F(x, z) \otimes F(z, y)\\ &= \Big(F \circ F\Big)(x, y) = F(x, y) = xF(y),\end{aligned}$$

for all $y \in X$. Hence, $xF \in \mathcal{C}$. Next, we show that $\mathcal{C}_x = xF$ for all $x \in X$. Due to the reflexivity of F we have $1_\otimes \subseteq xF(x)$ for all $x \in X$, and, consequently, $xF \in \{g \in \mathcal{C} \mid 1_\otimes \subseteq g\}$. Therefore:

$$\mathcal{C}_x = \bigcap \Big\{ g \in \mathcal{C} \mid 1_\otimes \subseteq g(x) \Big\} \subseteq xF,$$

for all $x \in X$. Consider now an arbitrary $g \in \mathcal{C}$ such that $1_\otimes \subseteq g(x)$ for some $x \in X$. Then, the following holds:

$$(xF)(y) = F(x,y) \subseteq g(x) \otimes F(x,y) \subseteq \bigcup_{z \in X} g(z) \otimes F(z,y) = (g \circ F)(y) = g(y),$$

for all $y \in X$. Hence, we obtain $xF \subseteq g$ and it follows that:

$$xF \subseteq \bigcap \{g \in \mathcal{C} \mid 1_\otimes \subseteq g(x)\} = \mathcal{C}_x,$$

for all $x \in X$. Therefore, $\mathcal{C}_x = xF$ for all $x \in X$ and $\mathcal{P}(\mathcal{C}) = \{\mathcal{C}_x \mid x \in X\} = \{xF \mid x \in X\}$.

(3) According to Theorem 4.2 it holds that $\langle R_i \circ w \subseteq w \mid i \in I \rangle = \langle F \circ w = w \rangle$. As in the previous case, we introduce the notation $\mathcal{C} = \langle R_i \circ w \subseteq w \mid i \in I \rangle = \langle F \circ w = w \rangle$ to simplify expressions. The principal part of $\mathcal{C}$ is denoted by $\mathcal{P}(C)$, as usual. Consider an arbitrary $x \in X$. For each $y \in X$ the following holds:

$$\begin{aligned} \Big(F \circ Fx\Big)(y) &= \bigcup_{z \in X} F(y,z) \otimes Fx(z) = \bigcup_{z \in X} F(y,z) \otimes F(z,x) \\ &= F \circ F(y,x) = F(y,x) = Fx(y). \end{aligned}$$

Therefore, $Fx \in \mathcal{C}$. Next, we show that $\mathcal{C}_x = Fx$ for all $x \in X$. Since F is reflexive, then $1_\otimes \subseteq Fx(x)$ for all $x \in X$ and, consequently, we obtain:

$$\mathcal{C}_x = \bigcap \Big\{ g \in \mathcal{C} \mid 1_\otimes \subseteq g(x) \Big\} \subseteq Fx.$$

To prove the opposite inclusion, consider an arbitrary $g \in \mathcal{C}$ such that $1_\otimes \subseteq g(x)$. Then for all $y \in X$ the following holds:

$$(Fx)(y) = F(y,x) \subseteq F(y,x) \otimes g(x) \subseteq \bigcup_{z \in X} F(y,z) \otimes g(z) = (F \circ g)(y) = g(y).$$

It follows that $Fx \subseteq g$, i.e., $Fx \subseteq \mathcal{C}_x$, for all $x \in X$. Consequently, $\mathcal{C}_x = Fx$ for all $x \in X$.

(4) According to Theorem 4.3 it holds that $\langle w \circ R_i \subseteq w, R_i \circ w \subseteq w \mid i \in I \rangle =$

$\langle w \circ E = w \rangle$. Let $x \in X$ be an arbitrary element and let $\mathcal{C} = \langle w \circ R_i \subseteq w, R_i \circ w \subseteq w \mid i \in I \rangle = \langle w \circ E = w \rangle$. Then, for all $y \in X$ the following holds:

$$\begin{aligned}(E_x \circ E)(y) &= \bigcup_{z \in X} E_x(z) \otimes E(z,y) = \bigcup_{z \in X} E(x,z) \otimes E(z,y) \\ &= E \circ E(x,y) = E(x,y) = E_x(y).\end{aligned}$$

Consequently, $E_x \in \mathcal{C}$. Since E is reflexive, then $1_\otimes \subseteq E_x(x)$. By definition, $\mathcal{C}_x = \bigcap\{g \in \mathcal{C} \mid 1_\otimes \subseteq g(x)\}$, and, hence, $\mathcal{C}_x \subseteq E_x$ holds for all $x \in X$.

To prove the opposite inclusion, assume that $g \in \mathcal{C}$ satisfies $1_\otimes \subseteq g(x)$. Then, for all $y \in X$ we have:

$$E_x(y) = E(x,y) \subseteq g(x) \otimes E(x,y) \subseteq \bigcup_{z \in X} g(z) \otimes E(z,y) = (g \circ E)(y) = g(y).$$

Therefore, $E_x \subseteq g$, i.e., $E_x \subseteq \mathcal{C}_x$ for all $x \in X$. Finally, we obtain $\mathcal{C}_x = E_x$ for all $x \in X$. □

Consider a family of hesitant fuzzy relations $\{R_i\}_{i \in I}$ on X and define $R = \bigcup_{i \in I} R_i$, $F = R^f$ and $E = R^e$. Next, we define mappings $g \mapsto g^{lc}$, $g \mapsto g^{rc}$, and $g \mapsto g^{dc}$ to denote closure operators on $\mathcal{Q}(X)$ that correspond to the following closure systems, respectively:

$$\langle w \circ R_i \subseteq w \mid i \in I \rangle,\ \langle R_i \circ w \subseteq w \mid i \in I \rangle,\ \langle w \circ R_i \subseteq w, R_i \circ w \subseteq w \mid i \in I \rangle. \tag{4.6}$$

Analogously, the mappings $g \mapsto g^{lo}$, $g \mapsto g^{ro}$, and $g \mapsto g^{do}$ denote the corresponding opening operators. That is, $g \mapsto g^{lc}$, $g \mapsto g^{rc}$, and $g \mapsto g^{dc}$ represent the least solutions to the systems described in (4.6), respectively, that include the hesitant fuzzy set g, while $g \mapsto g^{lo}$, $g \mapsto g^{ro}$, and $g \mapsto g^{do}$ correspond to the greatest solutions to the same systems that are contained within g. The following proposition provides a method for computing the least and greatest solutions described above.

Proposition 4.1. *Let $\{R_i\}_{i \in I}$ be a family of hesitant fuzzy relations on X, let $R = \bigcup_{i \in I} R_i$, $F = R^f$ and $E = R^e$ and let g be an arbitrary hesitant fuzzy set on X. Then:*

$$\begin{aligned} &g^{lc} = g \circ F, \quad g^{rc} = F \circ g, \quad g^{dc} = g \circ E, \\ &g^{lo} = g/F, \quad g^{ro} = F \backslash g, \quad g^{do} = g/E. \end{aligned} \tag{4.7}$$

Proof. According to Proposition 2.9, the functions $g \mapsto g \circ F$, $g \mapsto F \circ g$, and $g \mapsto g \circ E$ are closure operators on $\mathcal{Q}(X)$, while $g \mapsto g \not/ F$, $g \mapsto F \not\backslash g$, and $g \mapsto g \not/$

E are opening operators on $\mathcal{Q}(X)$.

Since F is reflexive and transitive, we have $g \circ F \circ F = g \circ F$ for each $g \in \mathcal{Q}(X)$. Consequently, $g \circ F \in \langle w \circ F = w \rangle = \langle w \circ R_i \subseteq w \mid i \in I \rangle$. Moreover, if $h \in \langle w \circ R_i \subseteq w \mid i \in I \rangle$ such that $g \subseteq h$, then $g \circ F \subseteq h \circ F = h$. Hence, $g \circ F$ represents the least solution to the system $w \circ R_i \subseteq w$, for $i \in I$, that contains g. That means $g \circ F = g^{lc}$.

The remaining equalities in (4.7) can be proved analogously. □

The Proposition 4.1 provides effective methods for computing the greatest solutions to the systems (4.7) that are contained in a given hesitant fuzzy set g on X, as well as the least solutions that contain g. A method for determining the greatest solution to the system $w \circ R_i \subseteq w$ $(i \in I)$ contained in a given hesitant fuzzy set g is presented in Algorithm 1. Analogous algorithms can be formulated for the other systems of inequalities that we have considered.

Algorithm 1 Computing the greatest solution to the system of inequalities $w \circ R_i \subseteq w$ $(i \in I)$ contained in g

Input: A family $\{R_i\}_{i \in I}$ of hesitant fuzzy relations on a finite set X (with finite index set I) and a hesitant fuzzy set g on X.
Output: The greatest solution to the system of inequalities $w \circ R_i \subseteq w$ $(i \in I)$ contained in g.

1: Compute: $R = \bigcup_{i \in I} R_i$
2: Compute the reflexive-transitive closure: $F \leftarrow R^f$
3: Compute the left $\circ$-residual: $L \leftarrow g \not\phi F$
4: **return** L

4.2 Hesitant fuzzy relation equations with one unknown hesitant fuzzy set

In this section, we investigate hesitant fuzzy relation equations with one unknown hesitant fuzzy set and present a detailed analysis of their solution sets.

Theorem 4.6. *Let $\{R_i\}_{i \in I}$ be a family of hesitant fuzzy relations on X and let $R = \bigcup_{i \in I} R_i$. Then the following holds:*

$$\langle w \circ R_i = w \mid i \in I \rangle \subseteq \langle w \circ R = w \rangle, \quad \langle R_i \circ w = w \mid i \in I \rangle \subseteq \langle R \circ w = w \rangle. \tag{4.8}$$

Proof. Assume that $g \in \langle w \circ R_i = w \mid i \in I \rangle$, i.e., $g \circ R_i = g$ for all $i \in I$. Then, by

applying property (2.34), we obtain:

$$g \circ R = g \circ \Big(\bigcup_{i\in I} R_i\Big) = \bigcup_{i\in I}(g \circ R_i) = g.$$

Therefore, $g \in \langle w \circ R = w\rangle$ and thus the first inclusion in (4.8) is proved. The second inclusion can be proved similarly. □

The following example illustrates that the inclusions presented in (4.8) can be strict.

Example 4.1. Let $X = \{x, y, z\}$, and let $\mathcal{Q}$ be a Gödel power set quantale. Consider the hesitant fuzzy relations $R_1, R_2 \in \mathcal{R}(X)$ and the hesitant fuzzy set $g \in \mathcal{Q}(X)$, given by:

$$R_1 = \begin{bmatrix} \{0.1, 0.2\} & [0, 1] & \{0.3\} \\ \{0.4\} & \emptyset & \{0.5, 0.6\} \\ \{0.1, 0.2\} & [0, 1] & \emptyset \end{bmatrix}, \quad R_2 = \begin{bmatrix} \{0.1\} & \{0.5\} & \{0.2\} \\ \{0.2\} & \emptyset & [0, 1] \\ \{0.3\} & \{0.4\} & \emptyset \end{bmatrix}, \quad g = \begin{bmatrix} [0, 1] \\ [0, 1] \\ [0, 1] \end{bmatrix}.$$

In that case, we have:

$$R = R_1 \cup R_2 = \begin{bmatrix} \{0.1, 0.2\} & [0, 1] & \{0.2, 0.3\} \\ \{0.2, 0.4\} & \emptyset & [0, 1] \\ \{0.1, 0.2, 0.3\} & [0, 1] & \emptyset \end{bmatrix}.$$

Consequently, $R \circ g = g$ and $g \in \langle R \circ w = w\rangle$. However,

$$R_2 \circ g = \begin{bmatrix} [0, 0.5] & [0, 1] & [0, 0.4] \end{bmatrix} \neq g,$$

whence it follows that $g \notin \langle R_i \circ w = w \mid i \in I\rangle$.

Theorem 4.7. *Let R be a hesitant fuzzy relation on X. The following holds:*

$$\langle w \circ R = w\rangle = \langle w \circ R^t = w\rangle, \quad \langle R \circ w = w\rangle = \langle R^t \circ w = w\rangle. \tag{4.9}$$

Proof. Assume that $g \in \langle w \circ R = w\rangle$, i.e., $g \circ R = g$. We show by mathematical induction that $g \circ R^n = g$ for all $n \in \mathbb{N}$. The base case $n = 1$ holds by the assumption $g \circ R = g$. If $g \circ R^n = g$, for some $n \in \mathbb{N}$, then then the associativity of composition

implies:

$$g \circ R^{n+1} = (g \circ R^n) \circ R = g \circ R = g.$$

Hence, by induction, the equality $g \circ R^n = g$ holds for all $n \in \mathbb{N}$. Consequently, we have:

$$g \circ R^t = g \circ \Big(\bigcup_{n\in\mathbb{N}} R^n\Big) = \bigcup_{n\in\mathbb{N}} g \circ R^n = g,$$

that is, $g \in \langle w \circ R^t = w\rangle$. It follows that $\langle w \circ R = w\rangle \subseteq \langle w \circ R^t = w\rangle$.

Conversely, let $g \in \langle w \circ R^t = w\rangle$. Then $g \circ R \subseteq g \circ R^t = g$, and we obtain:

$$g \circ R^2 = (g \circ R) \circ R \subseteq g \circ R.$$

Assume that $g \circ R^n \subseteq g \circ R^{n-1}$ for some $n \in \mathbb{N}$. Then:

$$g \circ R^{n+1} = (g \circ R^n) \circ R \subseteq g \circ R^{n-1} \circ R = g \circ R^n.$$

By induction, it follows that $g \circ R^n \subseteq g \circ R^{n-1}$ for all $n \in \mathbb{N}$, so $\{(g \circ R^n)\}_{n\in\mathbb{N}}$ is a decreasing sequence. Hence, its join is the first element, giving:

$$g = g \circ R^t = g \circ \Big(\bigcup_{n\in\mathbb{N}} R^n\Big) = \bigcup_{n\in\mathbb{N}} (g \circ R^n) = g \circ R.$$

Consequently, $g \in \langle w \circ R = w\rangle$, which implies the inclusion $\langle w \circ R^t = w\rangle \subseteq \langle w \circ R = w\rangle$. Together with the previously proved reverse inclusion, it follows that $\langle w \circ R = w\rangle = \langle w \circ R^t = w\rangle$. The other equality in (4.9) can be proved analogously. □

Theorem 4.8. *Let $\{R_i\}_{i\in I}$ be a family of hesitant fuzzy relations on X. Then*

$$\langle w \circ R_i = w \mid i \in I\rangle,\ \langle R_i \circ w = w \mid i \in I\rangle,\ \langle w \circ R_i = w, R_i \circ w = w \mid i \in I\rangle \tag{4.10}$$

are opening systems in $\mathcal{Q}(X)$.

Proof. To simplify notation we set $\mathcal{C} = \langle w \circ R_i = w \mid i \in I\rangle$. It follows directly that $\emptyset \in \mathcal{C}$. Next, let $\{g_j\}_{j\in J}$ be an arbitrary family of hesitant fuzzy sets on X such that $\{g_j\}_{j\in J} \subseteq \mathcal{C}$. Then, for every $i \in I$ we have $g_j \circ R_i = g_j$ for all $j \in J$. Consequently, by property (2.36):

$$\Big(\bigcup_{j\in J} g_j\Big) \circ R_i = \bigcup_{j\in J} (g_j \circ R_i) = \bigcup_{j\in J} g_j, \tag{4.11}$$

which implies that $\bigcup_{j\in J} g_j \in \mathcal{C}$. Hence, $\mathcal{C} = \langle w \circ R_i = w \mid i \in I\rangle$ is an opening system in $\mathcal{Q}(X)$.

The proof for the remaining systems in (4.10) is analogous and therefore omitted. □

The following example demonstrates that the systems given in (4.10) do not always form closure systems.

Example 4.2. Let $X = \{x, y, z\}$, and let $\mathcal{Q}$ be the Gödel power set quantale. Consider the hesitant fuzzy relations R_1 and R_2 on X, given in Example 4.1, and the full hesitant fuzzy set $h_S = [[0,1][0,1][0,1]]$ on X. Then, we obtain:

$$h_S \circ R_2 = \begin{bmatrix} [0,0.3] & [0,0.5] & [0,1] \end{bmatrix} \neq h_S.$$

It follows that h_S is not a solution to the systems $\langle w \circ R_2 = w\rangle$ and $\langle w \circ R_1 = w, w \circ R_2 = w\rangle$, defined in (4.10). Consequently, these systems do not constitute closure systems.

In addition to the hesitant fuzzy relation inequalities specified in (4.1), the term hesitant fuzzy relation inequality can more generally refer to any expression of the form:

$$w \subseteq \varphi(w), \tag{4.12}$$

where w is unknown hesitant fuzzy set taking values in Q^X and $\varphi : Q^X \mapsto Q^X$ is a function. Any hesitant fuzzy set $g \in Q^X$ satisfying the inequality $g \subseteq \varphi(g)$ is considered as a solution to that inequality. In order to establish a connection between the solution sets of the inequalities given in (4.1) and those corresponding to the more general expression in (4.12), we define the following functions:

$$\begin{aligned} \varphi^l(g) &= \bigcap_{i\in I}(g \circ R_i \cap g \not\backslash R_i), \\ \varphi^r(g) &= \bigcap_{i\in I}(R_i \circ g \cap R_i \not/ g), \\ \varphi^d(g) &= \bigcap_{i\in I}(g \circ R_i \cap g \not\backslash R_i \cap R_i \circ g \cap R_i \not/ g) = (\varphi^l \cap \varphi^r)(g). \end{aligned} \tag{4.13}$$

The following theorem shows that the functions φ^l, φ^r and φ^d are isotone, and it characterizes the relationship between the sets of solutions to the inequalities defined in (4.1) and those given in (4.12).

Theorem 4.9. *Let $\{R_i\}_{i\in I}$ be a family of hesitant fuzzy relations on X. Let $\varphi^l, \varphi^r, \varphi^d$ be functions on $\mathcal{Q}(X)$, defined for any $g \in Q^X$ as in (4.13). Then, the functions $\varphi^l, \varphi^r, \varphi^d$ are isotone and the following holds:*

$$\begin{aligned}
&\langle w \circ R_i = w \mid i \in I\rangle = \langle w \subseteq \varphi^l(w)\rangle,\\
&\langle R_i \circ w = w \mid i \in I\rangle = \langle w \subseteq \varphi^r(w)\rangle,\\
&\langle w \circ R_i = w, R_i \circ w = w \mid i \in I\rangle = \langle w \subseteq \varphi^d(w)\rangle.
\end{aligned} \tag{4.14}$$

Proof. The isotonicity of the functions φ^l, φ^r and φ^d follows directly from Proposition 2.9.

Consider the first equality in (4.14). Based on the previous results, we establish the following sequence of equalities:

$$\begin{aligned}
\langle w \circ R_i = w \mid i \in I\rangle &= \langle w \circ R_i \subseteq w, w \subseteq w \circ R_i \mid i \in I\rangle\\
&= \langle w \subseteq w \not\phi R_i, w \subseteq w \circ R_i \mid i \in I\rangle\\
&= \langle w \subseteq (w \not\phi R_i \cap w \circ R_i) \mid i \in I\rangle\\
&= \left\langle w \subseteq \bigcap_{i\in I}(w \not\phi R_i \cap w \circ R_i)\right\rangle\\
&= \langle w \subseteq \varphi^l(w)\rangle.
\end{aligned}$$

This completes the proof of the first equality in (4.14), and the remaining equalities follow by analogous arguments. □

We proved that functions φ^l, φ^r and φ^d are isotone and it can be easily shown that the sets of all solutions to the inequalities $g \subseteq \varphi^l(g)$, $g \subseteq \varphi^r(g)$ and $g \subseteq \varphi^d(g)$ are complete sublattices of Q^X, containing the greatest post fixed points of the corresponding functions. Considering that all the conditions of Knaster-Tarski fixed point theorem are fulfilled, we have that the greatest fixed points of φ^l, φ^r, φ^d are equal to the greatest post fixed points of these functions, respectively. This allows us to compute the greatest solutions to the given inequalities (equations) contained in an arbitrary hesitant fuzzy set g of Q^X.

Recall that the image $Im(g)$ of a hesitant fuzzy set g on X is the set of all membership degrees that appear in g, i.e., $Im(g) = \{g(x) \mid x \in X\}$. Similarly, the image of a hesitant fuzzy relation R_i is the set of all membership degrees assigned by R_i to the ordered pairs in $X \times X$, i.e., $Im(R_i) = \{R_i(x, y) \mid x, y \in X\}$. The following theorem can be stated.

Theorem 4.10. *Assume that X and I are finite sets and let $\{R_i\}_{i\in I}$ be a collection of hesitant fuzzy relations on X. Let g be a hesitant fuzzy set on X. Construct a*

sequence $\{g_k\}_{k\in\mathbb{N}}$ *of hesitant fuzzy sets on* X *inductively as follows:*

$$g_1 = g, \quad g_{k+1} = g_k \cap \varphi^l(g_k), \quad \text{for each } k \in \mathbb{N}.$$

If the subalgebra of $\mathcal{Q}$*, generated by the set* $\bigcup_{i\in I}(Im(R_i)) \cup Im(g)$ *is finite, then the following assertions hold:*

(1) *The sequence* $\{g_k\}_{k\in\mathbb{N}}$ *is finite and decreasing, and there is the smallest number* $m \in \mathbb{N}$ *such that* $g_m = g_{m+1}$*,*

(2) g_m *is the greatest solution to the system* $w \circ R_i = w$ $(i \in I)$ *contained in* g*.*

Proof. (1) Let $\mathcal{D}$ denote the subalgebra $\bigcup_{i\in I}(Im(R_i)) \cup Im(g)$ and let D denote the carrier set of $\mathcal{D}$. By the hypothesis, D is a finite set, so the set D^X of all hesitant fuzzy sets on X with membership values in $\mathcal{D}$ is also finite. Since $\{g_k\}_{k\in\mathbb{N}} \subseteq D^X$, the sequence $\{g_k\}_{k\in\mathbb{N}}$ is also finite. Moreover, $\{g_k\}_{k\in\mathbb{N}}$ clearly forms a decreasing sequence, and for each $k \in \mathbb{N}$, we have $g_k \subseteq g$.

A finite decreasing sequence must stabilize; that is, starting from some index, all elements are equal. Consequently, there exist $m, n \in \mathbb{N}$ such that $g_{m+n} = g_m$, and then $g_m = g_{m+n} \subseteq g_{m+1} \subseteq g_m$. Hence, $g_{m+1} = g_m$, and m is the smallest natural number with this property. Therefore, we obtain:

$$g_m = g_{m+1} = g_m \cap \varphi^l(g_m),$$

and hence $g_m \subseteq \varphi^l(g_m)$. By Theorem 4.9, g_m is a solution to the system $w \circ R_i = w$ $(i \in I)$.

(2) Consider an arbitrary solution h to this system, satisfying $h \subseteq g$. Then we have $h \subseteq \varphi^l(h)$ and $h \subseteq g = g_1$. If $h \subseteq g_k$, for some $k \in \mathbb{N}$, then $\varphi^l(h) \subseteq \varphi^l(g_k)$. Hence, the following holds:

$$h \subseteq g_k \cap \varphi^l(h) \subseteq g_k \cap \varphi^l(g_k) = g_{k+1}.$$

Therefore, since $h \subseteq g_k$ for each $k \in \mathbb{N}$, then $h \subseteq g_m$. Consequently, g_m is the greatest solution to the system $w \circ R_i = w$ $(i \in I)$ contained in g. □

Remark 4.2. The Kleene fixed point theorem asserts that the greatest fixed point of the given function φ^l can be computed as the intersection of the descending Kleene chain $\{g_k\}_{k\in\mathbb{N}}$, which is defined in Theorem 4.10. Observe that the previous theorem gives sufficient conditions for a given descending sequence to be stabilized and the greatest solution to the system $w \circ R_i = w$ $(i \in I)$ contained in g can be computed

Algorithm 2 Computing the greatest solution to the system of equations $w \circ R_i = w$ $(i \in I)$ contained in g

Input: A family $\{R_i\}_{i \in I}$ of hesitant fuzzy relations on a finite set X (with finite index set I) and a hesitant fuzzy set g on X.
Output: The greatest solution to the system of equations $w \circ R_i = w$ $(i \in I)$ contained in g.

```
function COMPUTE φ^l(g, {R_i}_{i∈I})
    Initialize all values of HFS f to S
    for each i ∈ I do
        h ← (g ∘ R_i) ∩ (g ⧸ R_i)
        f ← f ∩ h
    end for
    return f
end function
g_current ← g
do
    g_previous ← g_current
    g_current ← g_previous ∩ COMPUTE φ^l(g_previous, {R_i}_{i∈I})
while g_current ≠ g_previous
return g_current
```

in this way. Therefore, the result can alternatively be interpreted within this fixed point framework.

Consider a system represented by the single equation $w \circ R = w$. In order to compute the greatest solution to this equation contained in a hesitant fuzzy set g on X, we first determine the greatest solution g' contained in g to the inequality $w \circ R \subseteq w$ and then compute the greatest solution to the inequality $w \subseteq w \circ R$ contained in g' (i.e., the greatest post-fixed point of the function $h \mapsto h \circ R$ contained in g'). The obtained hesitant fuzzy set is also the greatest solution to the equation $w \circ R = w$ contained in g.

Theorem 4.11. *Let R be a hesitant fuzzy relation on a finite universal set X, let $F = R^f$, and let g be a hesitant fuzzy set on X. Construct a sequence $\{h_k\}_{k \in \mathbb{N}}$ of hesitant fuzzy sets on X inductively as follows:*

$$h_1 = g \not\circ F, \quad h_{k+1} = h_k \circ R, \quad \text{for each } k \in \mathbb{N}. \tag{4.15}$$

If the subsemiring of $\mathcal{Q}^*$ generated by the set $\mathrm{Im}(R) \cup \mathrm{Im}(g)$ is finite, then the following holds:

(1) *The sequence $\{h_k\}_{k \in \mathbb{N}}$ is finite and decreasing, and there is the smallest number $m \in \mathbb{N}$ such that $h_m = h_{m+1}$,*

(2) *h_m is the greatest solution to equation $w \circ R = w$ contained in g.*

Proof. (1) According to Proposition 4.1, $h_1 = g \not\phi F$ is the greatest solution to the inequality $w \circ R \subseteq w$ contained in g. Hence, $h_1 \circ R \subseteq h_1$ and, therefore, $h_2 = h_1 \circ R \subseteq h_1$. Next, assume that $h_{k+1} \subseteq h_k$ for some $k \in \mathbb{N}$. Then:

$$h_{k+2} = h_{k+1} \circ R \subseteq h_k \circ R = h_{k+1}.$$

By induction, $h_{k+1} \subseteq h_k$ holds for any $k \in \mathbb{N}$. Therefore, $\{h_k\}_{k\in\mathbb{N}}$ is a decreasing sequence, and $h_k \subseteq g \not\phi F \subseteq g$, for any $k \in \mathbb{N}$. Similarly to the proof of Theorem 4.10, $\{h_k\}_{k\in\mathbb{N}}$ is a finite sequence, so there exists the smallest $m \in \mathbb{N}$ such that $h_m = h_{m+1}$.

(2) Since $h_m = h_{m+1} = h_m \circ R$, then h_m is a solution to the equation $w \circ R = w$. Assume h is an arbitrary solution to this equation contained in g. Then h is also a solution to the inequality $w \circ R \subseteq w$. According to Proposition 4.1, $g \not\phi F$ is the greatest solution to the inequality $w \circ R \subseteq w$. Therefore, we have $h \subseteq g \not\phi F$ and $(g \not\phi F) \circ R \subseteq (g \not\phi F)$. Consequently,

$$h = h \circ R \subseteq (g \not\phi F) \circ R \subseteq g \not\phi F = h_1.$$

If $h \subseteq h_k$, for some $k \in \mathbb{N}$, then

$$h = h \circ R \subseteq h_k \circ R = h_{k+1}.$$

Therefore, by induction, we obtain $h \subseteq h_k$, for every $k \in \mathbb{N}$, and consequently, $h \subseteq h_m$. Hence, h_m is the greatest solution to the equation $w \circ R = w$ contained in g. □

Remark 4.3. The iterative construction in Theorem 4.11 can also be viewed as the iteration of an isotone operator on a complete lattice of hesitant fuzzy sets, and therefore Knaster-Tarski fixed point theorem can be applied. The sequence $\{h_k\}_{k\in\mathbb{N}}$ is decsending (Kleene's chain) and if the conditions of the previous theorem are fulfilled, then the sequence stabilizes, i.e. there exists $m \in \mathbb{N}$ such that $h_m = h_{m+1}$. Hence, h_m is the greatest fixed point of the given operator $\varphi(h_k) = h_k \circ R$, i.e., the greatest solution to the equation $w \circ R = w$ contained in g.

Note that the previous algorithm provides an alternative approach when the system consists of a single equation. The algorithm requires only two variables, the input set g and a second variable to store each newly computed value. However, for the sake of readability and more intuitive notation, we have introduced an additional variable.

Algorithm 3 Computing the greatest solution to the equation $w \circ R = w$ contained in g

Input: A hesitant fuzzy relation R on a finite set X and a hesitant fuzzy set g on X.

Output: The greatest solution to the equation $w \circ R = w$ contained in g.

1: Compute the reflexive-transitive closure: $F \leftarrow R^f$
2: Compute the left residual: $L \leftarrow g \not\phi F$
3: Set $h_{current} \leftarrow L$
4: **do**
5: $\quad h_{previous} \leftarrow h_{current}$
6: $\quad h_{current} \leftarrow h_{previous} \circ R$
7: **while** $h_{current} \neq h_{previous}$
8: **return** $h_{current}$

4.3 Algebraic characterizations of the solutions sets

This section is devoted to analyzing families of hesitant fuzzy sets that form the solution sets to hesitant fuzzy relation equations, specifically when the relation is a hesitant fuzzy quasi-order. We first establish several properties of closure systems on families of hesitant fuzzy sets, which provide the foundation for the subsequent analysis.

Let $\mathcal{C}$ be a closure system in $\mathcal{Q}(X)$. The principal elements $\mathcal{C}_x$ and $\mathcal{C}_y$ of $\mathcal{C}$ satisfy:

- the *inclusion property* if

$$\mathcal{C}_x(y) = I(\mathcal{C}_x, \mathcal{C}_y) = \bigcap_{a \in X} \mathcal{C}_x(a) \to \mathcal{C}_y(a), \tag{4.16}$$

- the *reverse inclusion property* if

$$\mathcal{C}_x(y) = I(\mathcal{C}_y, \mathcal{C}_x) = \bigcap_{a \in X} \mathcal{C}_y(a) \to \mathcal{C}_x(a), \tag{4.17}$$

- the *equality property* if

$$\mathcal{C}_x(y) = E(\mathcal{C}_y, \mathcal{C}_x) = \bigcap_{a \in X} \mathcal{C}_x(a) \leftrightarrow \mathcal{C}_y(a), \tag{4.18}$$

for all $x, y \in X$.

Proposition 4.2. *A family $\mathcal{P}$ of normalized hesitant fuzzy sets on X is the collection of all aftersets of some hesitant fuzzy quasi-order on X if and only if $\mathcal{P}$ is the*

principal part of some closure system in $\mathcal{Q}(X)$ whose principal elements satisfy the reverse inclusion property.

Proof. Let $\mathcal{P}$ be the collection of all aftersets of F. Define a collection $\mathcal{C} \subseteq Q^X$ in the following way:

$$\mathcal{C} = \langle w \circ F \subseteq w \rangle = \{g \in \mathcal{Q}(X) \mid F \subseteq g \backslash g\}.$$

Recall that for any $x \in X$, the afterset xF is defined by $xF(y) = F(x, y)$ for all $y \in X$. Consequently,

$$xF(y) \circ F(x, y) \subseteq xF(y)$$

for all $x, y \in X$, which implies that $xF \in \mathcal{C}$ for all $x \in X$. By Theorem 4.5, $\mathcal{C}$ is a closure system in $\mathcal{Q}(X)$.

Now, fix an arbitrary $a \in X$ and define the set $K = \{g \in \mathcal{C} \mid 1_\otimes \subseteq g(a)\}$. By Lemma 2.2, we obtain:

$$\begin{aligned}\mathcal{C}_a(x) &= \bigcap_{g \in K} g(x) \supseteq \bigcap_{g \in K} (g(a) \to g(x)) \supseteq \bigcap_{g \in \mathcal{C}} (g(a) \to g(x)) \\ &= \bigcap_{g \in \mathcal{C}} (g \backslash g)(a, x) \supseteq F(a, x) = aF(x),\end{aligned}$$

for any $x \in X$. Hence, $aF \subseteq \mathcal{C}_a$. On the other hand, if $aF \in K$ it follows that:

$$\mathcal{C}_a = \bigcap_{g \in K} g \subseteq aF.$$

Therefore, $\mathcal{C}_a = aF$ for all $a \in X$, which implies that $\mathcal{P}$ represents the principal part of $\mathcal{C}$. According to Theorem 3.6, the following holds:

$$\begin{aligned}\bigcap_{x \in X} \mathcal{C}_b(x) \to \mathcal{C}_a(x) = \bigcap_{x \in X} bF(x) \to aF(x) = \bigcap_{x \in X} Fx(b) \to Fx(a) \\ = F(a, b) = aF(b) = \mathcal{C}_a(b),\end{aligned}$$

for all $a, b \in X$. Hence, principal elements of $\mathcal{C}$ satisfy the reverse inclusion property.

For the opposite direction, assume that $\mathcal{P}$ is the principal part of some closure system $\mathcal{C}$ in $\mathcal{Q}(X)$ whose principal elements satisfy the reverse inclusion property. Define a hesitant fuzzy relation F on X in the following way:

$$F(a, b) = \mathcal{C}_a(b) = I(\mathcal{C}_b, \mathcal{C}_a) = \bigcap_{x \in X} \mathcal{C}_b(x) \to \mathcal{C}_a(x),$$

for all $a, b \in X$. By definition of the principal element $\mathcal{C}_x$ of the closure system $\mathcal{C}$ generated by $x \in X$ (cf. (2.24)), we have $1_\otimes \subseteq \mathcal{C}_x(x)$ for every $x \in X$. Consequently, as $F(a, a) = \mathcal{C}_a(a)$, the relation F is reflexive.

Consider arbitrary $a, b, c \in X$. Using property (1.39), we obtain:

$$F(a, b) \otimes F(b, c) \subseteq (\mathcal{C}_b(x) \to \mathcal{C}_a(x)) \otimes (\mathcal{C}_c(x) \to \mathcal{C}_b(x)) \subseteq \mathcal{C}_c(x) \to \mathcal{C}_a(x), \tag{4.19}$$

for each $x \in X$. Consequently:

$$F(a, b) \otimes F(b, c) \subseteq \bigcap_{x \in X} \mathcal{C}_c(x) \to \mathcal{C}_a(x) = F(a, c).$$

Hence, F is a hesitant fuzzy quasi-order. According to (4.19), it holds $(aF)(b) = F(a, b) = \mathcal{C}_a(b)$ for all $a, b \in X$, and it follows that $aF = \mathcal{C}_a$ for all $a \in X$. Therefore, $\mathcal{P}$ consists of all F-aftersets. □

The corresponding properties of the aforementioned family $\mathcal{P}$, expressed in terms of foresets (or equivalence classes) and inclusion (or equality), are presented below and can be proved similarly to Proposition 4.2.

Proposition 4.3. *A family $\mathcal{P}$ of normalized hesitant fuzzy sets on X is the collection of all foresets of some hesitant fuzzy quasi-order on X if and only if $\mathcal{P}$ is the principal part of some closure system in $\mathcal{Q}(X)$ whose principal elements satisfy the inclusion property.*

Proposition 4.4. *A family $\mathcal{P}$ of normalized hesitant fuzzy sets on X is the collection of all equivalence classes of some hesitant fuzzy quasi-order on X if and only if $\mathcal{P}$ is the principal part of some closure system in $\mathcal{Q}(X)$ whose principal elements satisfy the equality property.*

As previously mentioned, the lattice of all hesitant fuzzy sets on X, equipped with the scalar multiplication is denoted by $\mathcal{Q}_\otimes(X)$, called $\mathcal{Q}$-lattice of hesitant fuzzy sets on X. Its complete sublattices equipped also with the scalar multiplication are said to be complete $\mathcal{Q}$-sublattices of $\mathcal{Q}_\otimes(X)$.

Theorem 4.12. *Let $\mathcal{C}$ be a collection of hesitant fuzzy sets on X. Then there exists a hesitant fuzzy quasi-order F on X such that $\mathcal{C} = \langle w \circ F = w \rangle$ if and only if the following hold:*

(1) *$\mathcal{C}$ is a complete $\mathcal{Q}$-sublattice of $\mathcal{Q}_\otimes(X)$;*

(2) *every element of $\mathcal{C}$ can be expressed as a linear combination of principal elements of $\mathcal{C}$;*

(3) *principal elements of* $\mathcal{C}$ *satisfy the reverse inclusion property.*

Proof. Let F be a hesitant fuzzy quasi-order on X and let $\mathcal{C} = \langle w \circ F = w \rangle$. By properties (2.36), (2.37), and using the reflexivity of F, for any family $\{g_i\}_{i\in I} \subseteq \mathcal{C}$ we have:

$$\Big(\bigcup_{i\in I} g_i\Big) \circ F = \bigcup_{i\in I}(g_i \circ F) = \bigcup_{i\in I} g_i, \quad \Big(\bigcap_{i\in I} g_i\Big) \circ F \subseteq \bigcap_{i\in I}(g_i \circ F) = \bigcap_{i\in I} g_i \subseteq \Big(\bigcap_{i\in I} g_i\Big) \circ F.$$

Consequently, $\mathcal{C}$ is closed under arbitrary joins and meets. Let now $\lambda \in Q$ be an arbitrary scalar and $g \in \mathcal{C}$. For any $x \in X$, using the properties of scalar multiplication, we obtain:

$$\begin{aligned}\Big((\lambda \otimes g) \circ F\Big)(x) &= \bigcup_{y\in X}\Big((\lambda \otimes g)(y) \otimes F(y,x)\Big) = \lambda \otimes \bigcup_{y\in X}\Big(g(y) \otimes F(y,x)\Big) \\ &= \lambda \otimes (g \circ F)(x) = \lambda \otimes g(x).\end{aligned}$$

Thus, $\mathcal{C}$ is closed under scalar multiplication. Consequently, $\mathcal{C}$ forms a complete $\mathcal{Q}$-sublattice of $\mathcal{Q}_{\otimes}(X)$.

Next, fix an arbitrary $x \in X$. Since F is a hesitant fuzzy-quasi order, the following holds:

$$\begin{aligned}(xF \circ F)(y) &= \bigcup_{z\in X}\Big(xF(z) \otimes F(z,y)\Big) = \bigcup_{z\in X}\Big(F(x,z) \otimes F(z,y)\Big) \\ &= (F \circ F)(x,y) = F(x,y) = xF(y),\end{aligned}$$

for all $y \in X$. Hence, $xF \in \mathcal{C}$ for all $x \in X$. Since F is reflexive, it follows that $1_{\otimes} \subseteq xF(x)$ for all $x \in X$. Therefore, $xF \in \{g \in \mathcal{C} \mid 1_{\otimes} \subseteq g(x)\}$, which implies that $\{g \in \mathcal{C} \mid 1_{\otimes} \subseteq g(x)\}$ is non-empty set. Consider now an arbitrary $g \in \mathcal{C}$ such that $1_{\otimes} \subseteq g(x)$ for some $x \in X$. Then for any $y \in X$ we have:

$$g(y) = (g \circ F)(y) = \bigcup_{z\in X}\Big(g(z) \otimes F(z,y)\Big) \supseteq g(x) \otimes F(x,y) \supseteq F(x,y) = xF(y),$$

and it follows that $xF \subseteq g$. Consequently, $xF \subseteq g$ for all $g \in \mathcal{C}$, and, hence, xF is the least element of the set $\{g \in \mathcal{C} \mid 1_{\otimes} \subseteq g(x)\}$. Therefore, $xF = \bigcap\{g \in \mathcal{C} \mid 1_{\otimes} \subseteq g(x)\} = \mathcal{C}_x$ for all $x \in X$. According to (3.20), for arbitrary $x, y \in X$ we obtain:

$$\begin{aligned}\bigcap_{a\in X} \mathcal{C}_y(a) \to \mathcal{C}_x(a) &= \bigcap_{a\in X} yF(a) \to xF(a) = \bigcap_{a\in X} Fa(y) \to Fa(x) \\ &= F(x,y) = \mathcal{C}_x(y).\end{aligned}$$

Therefore, principal elements of $\mathcal{C}$ satisfy the reverse inclusion property.

Consider now an arbitrary $g \in \mathcal{C}$. The following holds:

$$g(x) = (g \circ F)(x) = \bigcup_{y \in X} \Big(g(y) \otimes F(y,x) \Big) = \bigcup_{y \in X} \Big(g(y) \otimes \mathcal{C}_y(x) \Big)$$
$$= \Big(\bigcup_{y \in X} g(y) \otimes \mathcal{C}_y \Big)(x)$$

for all $x \in X$. Hence:

$$g = \bigcup_{y \in X} g(y) \otimes \mathcal{C}_y,$$

and, consequently, every member of $\mathcal{C}$ can be represented as a linear combination of its principal elements. Therefore, we proved that each of the properties (1), (2), and (3) holds.

Conversely, assume that $\mathcal{C}$ satisfies conditions (1), (2), and (3). Define a hesitant fuzzy relation F on X in the following way:

$$F(x,y) = \mathcal{C}_x(y) = \bigcap_{a \in X} \mathcal{C}_y(a) \to \mathcal{C}_x(a), \tag{4.20}$$

for all $x, y \in X$. It is easy to see that F is reflexive. Consider arbitrary $x, y, z \in X$. According to (4.20) the following holds:

$$F(x,y) \otimes F(y,z) = \Big(\bigcap_{a \in X} \mathcal{C}_y(a) \to \mathcal{C}_x(a) \Big) \otimes \Big(\bigcap_{a \in X} \mathcal{C}_z(a) \to \mathcal{C}_y(a) \Big)$$
$$\subseteq \Big(\mathcal{C}_y(a) \to \mathcal{C}_x(a) \Big) \otimes \Big(\mathcal{C}_z(a) \to \mathcal{C}_y(a) \Big).$$

Next, by property (1.39) we have:

$$\Big(\mathcal{C}_y(a) \to \mathcal{C}_x(a) \Big) \otimes \Big(\mathcal{C}_z(a) \to \mathcal{C}_y(a) \Big) \subseteq \mathcal{C}_z(a) \to \mathcal{C}_x(a).$$

Hence, for all $a \in X$ it holds $F(x,y) \otimes F(y,z) \subseteq \mathcal{C}_z(a) \to \mathcal{C}_x(a)$ and it follows:

$$F(x,y) \otimes F(y,z) \subseteq \bigcap_{a \in X} \mathcal{C}_z(a) \to \mathcal{C}_x(a) = F(x,z).$$

Therefore, F is transitive, and, consequently, F is a hesitant fuzzy quasi-order. By the definition of F, for an arbitrary $x \in X$, we have $\mathcal{C}_x = xF$.

Consider now an arbitrary $g \in \langle w \circ F = w \rangle$. Then for each $x \in X$ the following

holds:

$$g(x) = \bigcup_{y\in X}\Big(g(y)\otimes F(y,x)\Big) = \bigcup_{y\in X}\Big(g(y)\otimes \mathcal{C}_y(x)\Big) = \Big(\bigcup_{y\in X} g(y)\otimes \mathcal{C}_y\Big)(x).$$

Consequently, we obtain:

$$g = \bigcup_{y\in X} g(y)\otimes \mathcal{C}_y.$$

Since $\mathcal{C}_y \in \mathcal{C}$ for all $y \in X$, and $\mathcal{C}$ is closed under scalar products and arbitrary joins, then it follows that $g \in \mathcal{C}$. Therefore, $\langle w \circ F = w\rangle \subseteq \mathcal{C}$.

On the other hand, let $g \in \mathcal{C}$. According to assertion 2., there are scalars $\lambda_z \in Q$, $z \in X$, such that

$$g = \bigcup_{z\in X} \lambda_z \otimes \mathcal{C}_z,$$

and, for any $x \in X$, the following holds:

$$\begin{aligned}(g\circ F)(x) &= \bigcup_{y\in X}\Big(g(y)\otimes F(y,x)\Big) = \bigcup_{y\in X}\Big(\bigcup_{z\in X}\lambda_z\otimes \mathcal{C}_z(y)\Big)\otimes F(y,x)\\ &= \bigcup_{z\in X}\left(\lambda_z\otimes\Big(\bigcup_{y\in X}F(z,y)\otimes F(y,x)\Big)\right) = \bigcup_{z\in X}\left(\lambda_z\otimes\Big(F\circ F(z,x)\Big)\right)\\ &= \bigcup_{z\in X}\Big(\lambda_z\otimes \mathcal{C}_z(x)\Big) = g(x).\end{aligned}$$

Thus, $g \in \langle w \circ F = w\rangle$, so $\mathcal{C} \subseteq \langle w \circ F = w\rangle$. Accordingly, it has been proved that $\mathcal{C} = \langle w \circ F = w\rangle$. □

Recall that the set $\mathcal{F}(X)$ of all hesitant fuzzy quasi-orders on X is a closure system in the lattice $\mathcal{R}(X)$ of all hesitant fuzzy relations on X. Thus, $\mathcal{F}(X)$ is a complete lattice under inclusion.

The above results enables the construction of an order isomorphism between the lattice $\langle \mathcal{F}(X), \subseteq\rangle$ and a partially ordered set of particular complete $\mathcal{Q}$-sublattice of $\mathcal{Q}_\otimes(X)$, implying that the latter inherits the lattice structure. We explore this relationship below.

Theorem 4.13. *The mapping $F \mapsto \langle w \circ F = w\rangle$ is a dual order isomorphism of the lattice $\mathcal{F}(X)$ of all hesitant fuzzy quasi-orders on X onto the partially ordered set of all complete $\mathcal{Q}$-sublattices of $\mathcal{Q}_\otimes(X)$ satisfying conditions (2) and (3) of Theorem 4.12.*

Proof. According to the results of Theorem 4.12, we have to prove that for any two hesitant fuzzy quasi-orders F_1 and F_2 on X the following holds:

$$F_1 \subseteq F_2 \quad \Leftrightarrow \quad \langle w \circ F_2 = w \rangle \subseteq \langle w \circ F_1 = w \rangle. \tag{4.21}$$

Notice that the equivalence (4.21) ensures that φ is injection.

Let F_1 and F_2 be hesitant fuzzy quasi-oders on X. If $F_1 \subseteq F_2$ and $g \in \langle w \circ F_2 = w \rangle$, then we have:

$$g \subseteq g \circ F_1 \subseteq g \circ F_2 = g,$$

and, consequently, $g \circ F_1 = g$, i.e. $g \in \langle w \circ F_1 = w \rangle$. Conversely, assume that $\langle w \circ F_2 = w \rangle \subseteq \langle w \circ F_1 = w \rangle$. In that case, for each $x \in X$ we have that $xF_2 \in \langle w \circ F_2 = w \rangle \subseteq \langle w \circ F_1 = w \rangle$, i.e. $xF_2 \circ F_1 = xF_2$. Therefore, for arbitrary $x, y \in X$ we have

$$\begin{aligned} F_2(x,y) = xF_2(y) = (xF_2 \circ F_1)(y) &= \bigcup_{z \in X} xF_2(z) \otimes F_1(z,y) \\ &\supseteq xF_2(x) \otimes F_1(x,y) \supseteq F_1(x,y), \end{aligned}$$

and, hence, $F_1 \subseteq F_2$. From the above, we immediately obtain that (4.21) is satisfied. □

In the sequel, we present the second characterization theorem and outline the corresponding order isomorphism analogously to Theorems 4.12 and 4.13. The proofs of the following theorems are analogous to those already presented and are therefore omitted.

Theorem 4.14. *Let $\mathcal{G}$ be a collection of hesitant fuzzy sets on X. Then there exists a hesitant fuzzy quasi-order F on X such that $\mathcal{G} = \langle F \circ w = w \rangle$ if and only if the following holds:*

(1)' *$\mathcal{G}$ is a complete $\mathcal{Q}$-sublattice of $\mathcal{Q}_{\otimes}(X)$;*

(2)' *every element of $\mathcal{G}$ can be expressed as a linear combination of principal elements of $\mathcal{G}$;*

(3)' *principal elements of $\mathcal{G}$ satisfy the reverse inclusion property.*

Theorem 4.15. *The mapping $F \mapsto \langle F \circ w = w \rangle$ is a dual order isomorphism of the lattice $\mathcal{F}(X)$ of all hesitant fuzzy quasi-orders on X onto the partially ordered set of all complete $\mathcal{Q}$-sublattices of $\mathcal{Q}_{\otimes}(X)$ satisfying conditions (2)' and (3)' of Theorem 4.14.*

4.4 Hesitant fuzzy relation inequalities and equations with two unknown hesitant fuzzy sets

In this section, we study solution sets of systems of hesitant fuzzy relation inequalities and equations with two unknown hesitant fuzzy sets.

Recall that, unless otherwise specified, $\langle S, * \rangle$ denotes a semigroup, $Q = \mathcal{P}(S)$ its power set and $\mathcal{Q} = \{\mathcal{P}(S), \cup, \cap, \otimes, \emptyset, S\}$ the power set quantale of S. The power set quantale of hesitant fuzzy sets on a non-empty set X is denoted by

$$Q(X) = \langle Q^X, \cup, \cap, \otimes, h^X_\emptyset, h^X_S \rangle.$$

Proposition 4.5. *Let X and Y be non-empty sets, let $u \in Q^X$ and $w \in Q^Y$ be unknown hesitant fuzzy sets, and let $\{R_i\}_{i \in I} \subseteq Q^{X \times Y}$ be a given family of hesitant fuzzy relations. Define*

$$\mathscr{C}^l = \langle u \circ R_i \subseteq w \mid i \in I \rangle, \quad \mathscr{C}^r = \langle R_i \circ w \subseteq u \mid i \in I \rangle.$$

Then

(a) *$(h, g) \in \mathscr{C}^l$ implies $(h', g') \in \mathscr{C}^l$, for all $h' \in Q^X$, $g' \in Q^Y$ such that $h' \subseteq h$ and $g' \supseteq g$,*

(b) *$(h, g) \in \mathscr{C}^r$ implies $(h', g') \in \mathscr{C}^r$, for all $h' \in Q^X$ and $g' \in Q^Y$ such that $h' \supseteq h$, $g' \subseteq g$.*

Proof. (a) Let $(h, g) \in \mathscr{C}^l$. By the definition of $\mathscr{C}^l$, we have

$$(h \circ R_i)(y) \subseteq g(y) \quad \text{for all } y \in Y \text{ and } i \in I.$$

Fix $y \in Y$ and $i \in I$. Since

$$(h \circ R_i)(y) = \bigcup_{x \in X} h(x) \otimes R_i(x, y),$$

it follows that

$$\bigcup_{x \in X} h(x) \otimes R_i(x, y) \subseteq g(y).$$

Let $h' \in Q^X$ and $g' \in Q^Y$ be such that $h' \subseteq h$ and $g' \supseteq g$. Then, using the isotonicity

of $\otimes$, we obtain

$$(h' \circ R_i)(y) = \bigcup_{x \in X} h'(x) \otimes R_i(x,y) \subseteq \bigcup_{x \in X} h(x) \otimes R_i(x,y) \subseteq g(y) \subseteq g'(y).$$

Therefore, $h' \circ R_i \subseteq g'$ for all $i \in I$, and hence $(h', g') \in \mathscr{C}^l$.

(b) The statement can be proved analogously. □

Proposition 4.6. *Let X and Y be non-empty sets, let $u \in Q^X$ and $w \in Q^Y$ be unknown hesitant fuzzy sets, and let $\{R_i\}_{i \in I} \subseteq Q^{X \times Y}$ be a given family of hesitant fuzzy relations. Define*

$$\mathscr{C}^l = \langle u \circ R_i \subseteq w \mid i \in I \rangle, \quad \mathscr{C}^r = \langle R_i \circ w \subseteq u \mid i \in I \rangle,$$
$$\mathscr{C}^d = \langle u \circ R_i \subseteq w, R_i \circ w \subseteq u \mid i \in I \rangle.$$

Then the following statements hold:

(a) *$\mathscr{C}^l$, $\mathscr{C}^r$ and $\mathscr{C}^d$ contain the least and the greatest element of the lattice $Q^X \times Q^Y$.*

(b) *$\mathscr{C}^l$, $\mathscr{C}^r$ and $\mathscr{C}^d$ are closed under arbitrary joins and meets.*

(c) *$\mathscr{C}^l$ is closed under left scalar multiplication and right scalar residuation.*

(d) *$\mathscr{C}^r$ is closed under right scalar multiplication and left scalar residuation.*

Proof. (a) Since Q^X and Q^Y are complete lattices under pointwise inclusion, their direct product $Q^X \times Q^Y$, equipped with the componentwise order, is also a complete lattice. Its least and greatest elements are $(h^X_\emptyset, h^Y_\emptyset)$, (h^X_S, h^Y_S), respectively. For every $i \in I$,

$$h^X_\emptyset \circ R_i = h^Y_\emptyset, \quad R_i \circ h^Y_\emptyset = h^X_\emptyset,$$

hence, $(h^X_\emptyset, h^Y_\emptyset)$ satisfies all inequalities occurring in $\mathscr{C}^l$, $\mathscr{C}^r$ and $\mathscr{C}^d$. Moreover,

$$h^X_S \circ R_i \subseteq h^Y_S, \quad R_i \circ h^Y_S \subseteq h^X_S,$$

since h^X_S and h^Y_S are the greatest elements of Q^X and Q^Y, respectively. Therefore, (h^X_S, h^Y_S) also satisfies all inequalities, and consequently, $\mathscr{C}^l$, $\mathscr{C}^r$ and $\mathscr{C}^d$ contain both the least and the greatest element of $Q^X \times Q^Y$.

(b) We first prove that $\mathscr{C}^l$ is closed under arbitrary joins. Let $\{(h_j, g_j)\}_{j \in J}$ be a non-empty family of elements of $\mathscr{C}^l$. Then, for every $j \in J$ and every $i \in I$, we have

$$h_j \circ R_i \subseteq g_j.$$

Since composition distributes over arbitrary joins (2.36), we obtain, for every $i \in I$,

$$\left(\bigcup_{j\in J} h_j\right) \circ R_i = \bigcup_{j\in J} (h_j \circ R_i) \subseteq \bigcup_{j\in J} g_j.$$

Hence, $\left(\bigcup_{j\in J} h_j, \bigcup_{j\in J} g_j\right) \in \mathscr{C}^l$, so $\mathscr{C}^l$ is closed under arbitrary joins.

To prove that $\mathscr{C}^l$ is closed under arbitrary meets, we use property (2.37). Since $h_j \circ R_i \subseteq g_j$ for every $j \in J$, we obtain

$$\left(\bigcap_{j\in J} h_j\right) \circ R_i \subseteq \bigcap_{j\in J} g_j,$$

for every $i \in I$. Consequently, $\left(\bigcap_{j\in J} h_j, \bigcap_{j\in J} g_j\right) \in \mathscr{C}^l$, so $\mathscr{C}^l$ is closed under arbitrary meets.

The same arguments, using the corresponding inequalities for right composition, establish that $\mathscr{C}^r$ is closed under arbitrary joins and meets. Applying both arguments shows that $\mathscr{C}^d$ is also closed under arbitrary joins and meets.

(c) Let $\lambda \in Q$ be an arbitrary scalar and let $(h, g) \in \mathscr{C}^l$. Then $h \circ R_i \subseteq g$ holds for every $i \in I$. Fix an arbitrary $i \in I$ and $y \in Y$. Using the definition of scalar multiplication together with the isotonicity of $\otimes$, we obtain

$$\begin{aligned}((\lambda \otimes h) \circ R_i)(y) &= \bigcup_{x\in X} (\lambda \otimes h)(x) \otimes R_i(x, y) = \bigcup_{x\in X} (\lambda \otimes h(x) \otimes R_i(x, y)) \\ &= \lambda \otimes \bigcup_{x\in X} h(x) \otimes R_i(x, y) = \lambda \otimes (h \circ R_i)(y) \subseteq \lambda \otimes g(y) \\ &= (\lambda \otimes g)(y).\end{aligned}$$

Hence, $(\lambda \otimes h) \circ R_i \subseteq \lambda \otimes g$ for all $i \in I$ and consequently $(\lambda \otimes h, \lambda \otimes g) \in \mathscr{C}^l$.

We similarly prove the assertion for scalar residuation. Since $\otimes$ is commutative, the left and right scalar residuals coincide. Hence, we use the following definition of the right scalar residual

$$(\lambda \backslash_s h)(x) = \lambda \to h(x).$$

Using properties (1.38) and (1.49) and noting that the right residual is isotone in

its second argument, we obtain

$$\begin{aligned}\Big((\lambda\backslash_s h)\circ R_i\Big)(y) &= \bigcup_{x\in X}\Big((\lambda\backslash_s h)(x)\otimes R_i(x,y)\Big) = \bigcup_{x\in X}\Big((\lambda\to h(x))\otimes R_i(x,y)\Big)\\ &\subseteq \bigcup_{x\in X}\Big(\lambda\to(h(x)\otimes R_i(x,y))\Big) \subseteq \lambda\to\bigcup_{x\in X}\Big(h(x)\otimes R_i(x,y)\Big)\\ &= \lambda\to\Big(h\circ R_i\Big)(y) \subseteq \lambda\to g(y) = (\lambda\backslash_s g)(y).\end{aligned}$$

Hence, $(\lambda\backslash_s h)\circ R_i\subseteq\lambda\backslash_s g$ for all $i\in I$ and consequently $(\lambda\backslash_s h,\lambda\backslash_s g)\in\mathscr{C}^l$.

(d) The proof is analogous to that of part (c). □

To transform systems of hesitant fuzzy relation inequalities and equations with two unknown sets into equivalent single inequalities, we define the following functions on $Q^X\times Q^Y$:

$$\begin{aligned}&\phi^l((h,g)) = (g\not/ R, g), \quad \phi^r((h,g)) = (h, R\not\backslash h), \quad \phi^d((h,g)) = (g\not/ R, R\not\backslash h), \\ &\psi^l((h,g)) = (h, h\circ R), \quad \psi^r((h,g)) = (R\circ g, g), \quad \psi^d((h,g)) = (R\circ g, h\circ R),\end{aligned} \tag{4.22}$$

for all hesitant fuzzy sets $h\in Q^X$ and $g\in Q^Y$.

Theorem 4.16. *Let X and Y be non-empty sets, and let $u\in Q^X$ and $w\in Q^Y$ be unknown hesitant fuzzy sets. Let $\{R_i\}_{i\in I}\subseteq Q^{X\times Y}$ be a given family of hesitant fuzzy relations and define $R=\bigcup_{i\in I}R_i$. Then:*

$$\begin{aligned}\langle u\circ R_i\subseteq w \mid i\in I\rangle &= \langle u\subseteq w\not/ R_i \mid i\in I\rangle = \langle u\circ R\subseteq w\rangle \\ &= \langle u\subseteq w\not/ R\rangle = \langle\zeta\subseteq\phi^l(\zeta)\rangle = \langle\psi^l(\zeta)\subseteq\zeta\rangle,\end{aligned} \tag{4.23}$$

$$\begin{aligned}\langle R_i\circ w\subseteq u \mid i\in I\rangle &= \langle w\subseteq R_i\not\backslash u \mid i\in I\rangle = \langle R\circ w\subseteq u\rangle \\ &= \langle w\subseteq R\not\backslash u\rangle = \langle\zeta\subseteq\phi^r(\zeta)\rangle = \langle\psi^r(\zeta)\subseteq\zeta\rangle,\end{aligned} \tag{4.24}$$

$$\begin{aligned}\langle u\circ R_i\subseteq w,\, R_i\circ w\subseteq u \mid i\in I\rangle &= \langle u\subseteq w\not/ R_i,\, w\subseteq R_i\not\backslash u \mid i\in I\rangle \\ &= \langle u\circ R\subseteq w,\, R\circ w\subseteq u\rangle = \langle u\subseteq w\not/ R,\, w\subseteq R\not\backslash u\rangle \\ &= \langle\zeta\subseteq\phi^d(\zeta)\rangle = \langle\psi^d(\zeta)\subseteq\zeta\rangle,\end{aligned} \tag{4.25}$$

where ζ is an unknown taking values in $Q^X\times Q^Y$ and ϕ^l, ϕ^r, ϕ^d, ψ^l, ψ^r and ψ^d are functions defined in (4.22). Moreover, all functions ϕ^l, ϕ^r, ϕ^d, ψ^l, ψ^r and ψ^d are isotone.

Proof. We first establish the equalities in (4.23). The proof of (4.24) is analogous, and (4.25) is an immediate consequence of the first two cases.

The following equalities are immediate consequences of the residuation proper-

ties (2.45):

$$\langle u \circ R_i \subseteq w \mid i \in I \rangle = \langle u \subseteq w \not\circ R_i \mid i \in I \rangle, \quad \langle u \circ R \subseteq w \rangle = \langle u \subseteq w \not\circ R \rangle.$$

Consider the systems $\langle u \circ R_i \subseteq w \mid i \in I \rangle$ and $\langle u \circ R \subseteq w \rangle$, where $R = \bigcup_{i \in I} R_i$. Let $(h, g) \in \langle u \circ R_i \subseteq w \mid i \in I \rangle$. Then $h \circ R_i \subseteq g$ for every $i \in I$. Consequently,

$$h \circ R = h \circ \bigcup_{i \in I} R_i = \bigcup_{i \in I} (h \circ R_i) \subseteq g,$$

and thus $(h, g) \in \langle u \circ R \subseteq w \rangle$. Therefore, we obtain

$$\langle u \circ R_i \subseteq w \mid i \in I \rangle \subseteq \langle u \circ R \subseteq w \rangle.$$

Conversely, let $(h, g) \in \langle u \circ R \subseteq w \rangle$. Since $R_i \subseteq R$ for every $i \in I$, isotonicity of composition implies that

$$h \circ R_i \subseteq h \circ R \subseteq g.$$

Hence, $(h, g) \in \langle u \circ R_i \subseteq w \mid i \in I \rangle$, and consequently,

$$\langle u \circ R \subseteq w \rangle \subseteq \langle u \circ R_i \subseteq w \mid i \in I \rangle.$$

Therefore, $\langle u \circ R \subseteq w \rangle = \langle u \circ R_i \subseteq w \mid i \in I \rangle$.

We next consider the systems $\langle u \subseteq w \not\circ R \rangle$ and $\langle \zeta \subseteq \phi^l(\zeta) \rangle$. Let $\xi = (h, g) \in \langle u \subseteq w \not\circ R \rangle$. Then, we have

$$h \subseteq g \not\circ R \quad \Leftrightarrow \quad (h, g) \subseteq (g \not\circ R, g) \quad \Leftrightarrow \quad \xi \subseteq \phi^l(\xi). \tag{4.26}$$

Consequently, $(h, g) \in \langle \zeta \subseteq \phi^l(\zeta) \rangle$, and hence, $\langle u \subseteq w \not\circ R \rangle \subseteq \langle \zeta \subseteq \phi^l(\zeta) \rangle$.

Conversely, if $\xi = (h, g) \in \langle \zeta \subseteq \phi^l(\zeta) \rangle$, we obtain

$$\xi \subseteq \phi^l(\xi) \quad \Leftrightarrow \quad (h, g) \subseteq \phi^l\Big((h, g)\Big) \quad \Leftrightarrow \quad (h, g) \subseteq (g \not\circ R, g) \quad \Leftrightarrow \quad h \subseteq g \not\circ R.$$

Hence, $(h, g) \in \langle u \subseteq w \not\circ R \rangle$, and therefore $\langle \xi \subseteq \phi^l(\xi) \rangle \subseteq \langle u \subseteq w \not\circ R \rangle$. Combining both inclusions, we conclude $\langle \xi \subseteq \phi^l(\xi) \rangle = \langle u \subseteq w \not\circ R \rangle$.

It remains to consider the systems $\langle u \circ R \subseteq w \rangle$ and $\langle \psi^l(\zeta) \subseteq \zeta \rangle$. Let $\xi = (h, g) \in \langle u \circ R \subseteq w \rangle$. Then:

$$h \circ R \subseteq g \quad \Rightarrow \quad (h, g) \subseteq (h, h \circ R) \quad \Rightarrow \quad \xi \subseteq \psi^l(\xi),$$

and it follows that $\langle u \circ R \subseteq w \rangle \subseteq \langle \zeta \subseteq \psi^l(\zeta) \rangle$.

Conversely, let $\xi = (h, g) \in \langle \psi^l(\zeta) \subseteq \zeta \rangle$. Then,

$$\psi^l(\xi) \subseteq \xi \quad \Rightarrow \quad (h, h \circ R) \subseteq (h, g) \quad \Rightarrow \quad h \circ R \subseteq g,$$

and thus, $\langle \psi^l(\zeta) \subseteq \zeta \rangle \subseteq \langle u \circ R \subseteq w \rangle$. Hence, $\langle u \circ R \subseteq w \rangle = \langle \psi^l(\zeta) \subseteq \zeta \rangle$.

To prove that φ^l is isotone, let $(h, g), (h', g') \in Q^X \times Q^Y$ be such that $(h, g) \subseteq (h', g')$. Then $h \subseteq h'$ and $g \subseteq g'$. By the isotonicity of the left $\circ$-residual in its first argument and the isotonicity of the composition, we obtain

$$g \not\phi R \subseteq g' \not\phi R \qquad h \circ R \subseteq h' \circ R.$$

Consequently,

$$\varphi^l((h, g)) = (g \not\phi R, h \circ R) \subseteq (g' \not\phi R, h' \circ R) = \varphi^l((h', g')).$$

Therefore, φ^l is isotone. The isotonicity of the remaining functions can be proved similarly. □

Theorem 4.17. *Let X and Y be non-empty sets, let $\{R_i\}_{i \in I} \subseteq Q^{X \times Y}$ be a family of hesitant fuzzy relations, and let $R = \bigcup_{i \in I} R_i$. Furthermore, let $h_0 \in Q^X$ and $g_0 \in Q^Y$ be given hesitant fuzzy sets and let $u \in Q^X$ and $w \in Q^Y$ be unknown hesitant fuzzy sets. Then:*

(a) $(h_0 \cap (g_0 \not\phi R), g_0)$ *is the greatest solution to the system* $\langle u \circ R_i \subseteq w \mid i \in I \rangle$ *contained in* (h_0, g_0)*;*

(b) $(h_0, g_0 \cup (h_0 \circ R))$ *is the least solution to the system* $\langle u \circ R_i \subseteq w \mid i \in I \rangle$ *containing* (h_0, g_0)*;*

(c) $(h_0, g_0 \cap (R \backslash h_0))$ *is the greatest solution to the system* $\langle R_i \circ w \subseteq u \mid i \in I \rangle$ *contained in* (h_0, g_0)*;*

(d) $(h_0 \cup (R \circ g_0), g_0)$ *is the least solution to the system* $\langle R_i \circ w \subseteq u \mid i \in I \rangle$ *containing* (h_0, g_0).

Proof. (a) First, we show that $(h_0 \cap (g_0 / R), g_0)$ is a solution to the system $\langle u \circ R_i \subseteq w \mid i \in I \rangle$. By properties (2.37) and (2.50) we obtain:

$$\begin{aligned} \Big(h_0 \cap (g_0 \not\phi R)\Big) \circ R_i \subseteq \Big(h_0 \cap (g_0 \not\phi R)\Big) \circ R &\subseteq \Big(h_0 \circ R\Big) \cap \Big((g_0 \not\phi R) \circ R\Big) \\ &\subseteq \Big(h_0 \circ R\Big) \cap g_0 \subseteq g_0. \end{aligned}$$

Hence,

$$(h_0 \cap (g_0 \not\phi R), g_0) \in \langle u \circ R_i \subseteq w \mid i \in I \rangle.$$

Moreover, since $h_0 \cap (g_0 \not\phi R) \subseteq h_0$, it follows that

$$(h_0 \cap (g_0 \not\phi R), g_0) \subseteq (h_0, g_0).$$

To prove that $(h_0 \cap (g_0 \not\phi R), g_0)$ is the greatest solution to the system $\langle u \circ R_i \subseteq w \mid i \in I \rangle$ contained in (h_0, g_0), suppose, to the contrary, that there exists a solution (h_1, g_1) satisfying

$$(h_0 \cap (g_0 \not\phi R), g_0) \subsetneq (h_1, g_1) \subseteq (h_0, g_0).$$

As (h_1, g_1) is a solution and $g_1 \subseteq g_0$, we have $h_1 \circ R_i \subseteq g_1 \subseteq g_0$ for all $i \in I$. By the residuation property, this implies that $h_1 \subseteq g_0 \not\phi R_i$ for all $i \in I$. Hence, by Theorem 4.16, $h_1 \subseteq g_0 \not\phi R$. Since $h_1 \subseteq h_0$, it follows that

$$h_1 \subseteq h_0 \cap (g_0 \not\phi R).$$

Together with $g_1 \subseteq g_0$, this yields

$$(h_1, g_1) \subseteq (h_0 \cap (g_0 \not\phi R), g_0),$$

contradicting the assumption that $(h_0 \cap (g_0 \not\phi R), g_0) \subsetneq (h_1, g_1)$.

The proofs of (b), (c), and (d) are analogous. □

Theorem 4.18. *Let X and Y be finite non-empty sets, let $u \in Q^X$ and $w \in Q^Y$ be unknown hesitant fuzzy sets, and let $\{R_i\}_{i \in I} \subseteq Q^{X \times Y}$ be a family of hesitant fuzzy relations, where I is a finite index set. Further, let $(h_0, g_0) \in Q^X \times Q^Y$.*

Define inductively a sequence $\{h_k, g_k\}_{k \in \mathbb{N}} \subseteq Q^X \times Q^Y$ in one of the following two ways:

$$(h_1, g_1) = (h_0, g_0), \quad (h_{k+1}, g_{k+1}) = \Big(h_k \cap (g_k \not\phi R_i), g_k \cap (R_i \not\backslash h_k)\Big), \quad \forall k \in \mathbb{N}, \tag{4.27}$$

$$(h_1, g_1) = (h_0, g_0), \quad (h_{k+1}, g_{k+1}) = \Big(h_k \cup (R_i \circ g_k), g_k \cup (h_k \circ R_i)\Big), \quad \forall k \in \mathbb{N}. \tag{4.28}$$

If the subalgebra of $\mathcal{Q}$ generated by the set $\bigcup_{i \in I}(\operatorname{Im}(R_i)) \cup \operatorname{Im}(h_0) \cup \operatorname{Im}(g_0)$ is finite, then the following is true:

(a) *the sequence $\{(h_k, g_k)\}_{k \in \mathbb{N}}$ is finite and decreasing, and there is the smallest*

number $m \in \mathbb{N}$ *such that* $(h_m, g_m) = (h_{m+1}, g_{m+1})$,

(b) *for the sequence defined in* (4.27), (h_m, g_m) *is the greatest solution to system* $\langle u \circ R_i \subseteq w,\ R_i \circ w \subseteq u \mid i \in I\rangle$ *contained in* (h_0, g_0),

(c) *for the sequence defined in* (4.28), (h_m, g_m) *is the least solution to system* $\langle u \circ R_i \subseteq w,\ R_i \circ w \subseteq u \mid i \in I\rangle$ *containing* (h_0, g_0).

Proof. (a) First, observe that the sequence $\{(h_k, g_k)\}_{k\in\mathbb{N}}$ is decreasing and $(h_k, g_k) \subseteq (h_0, g_0)$ holds for all $k \in \mathbb{N}$. Next, let D denote the carrier set of the finite subalgebra of $\mathcal{Q}$. By the hypothesis, D is finite, and therefore D^X and D^Y are finite as well. Consequently, $D^X \times D^Y$ is finite. Since $\{(h_k, g_k)\}_{k\in\mathbb{N}}$ is a decreasing sequence in the finite poset $D^X \times D^Y$, it must eventually stabilize. Hence, there exists a least $m \in \mathbb{N}$ such that $(h_k, g_k) = (h_m, g_m)$, for all $k \geqslant m$. In particular, $(h_{m+1}, g_{m+1}) = (h_m, g_m)$.

(b) By Theorem 4.16, $\langle u \circ R_i \subseteq w,\ R_i \circ w \subseteq u \mid i \in I\rangle = \langle \zeta \subseteq \phi^d(\zeta)\rangle$. Therefore, by the definition of the sequence and part (a), we obtain

$$\begin{aligned}(h_m, g_m) = (h_{m+1}, g_{m+1}) &= \Big(h_m \cap (g_m \not\phi R_i), g_m \cap (R_i \not\backslash h_m)\Big)\\ &= (h_m, g_m) \cap \phi^d(h_m, g_m).\end{aligned}$$

Hence, $(h_m, g_m) \subseteq \phi^d((h_m, g_m))$, and consequently (h_m, g_m) is a solution to the system.

Let (h, g) be a solution to the system contained in (h_0, g_0). Then,

$$(h, g) \subseteq \phi^d((h, g)) \quad \text{and} \quad (h, g) \subseteq (h_0, g_0) = (h_1, g_1).$$

Assume now, as the induction hypothesis, that $(h, g) \subseteq (h_k, g_k)$ for some $k \in \mathbb{N}$. Since (h, g) is a solution to the system, then $(h, g) \subseteq \phi^d(h, g)$. By isotonicity of ϕ^d, we obtain $\phi^d(h, g) \subseteq \phi^d(h_k, g_k)$, and consequently,

$$(h, g) \subseteq (h_k, g_k) \cap \phi^d(h_k, g_k) = (h_{k+1}, g_{k+1}).$$

Hence, by mathematical induction, $(h, g) \subseteq (h_k, g_k)$ holds for all $k \in \mathbb{N}$. In particular, $(h, g) \subseteq (h_m, g_m)$. Consequently, (h_m, g_m) is the greatest solution to the system $\langle u \circ R_i \subseteq w,\ R_i \circ w \subseteq u \mid i \in I\rangle$ contained in (h_0, g_0).

The proof of part (c) is analogous. □

Proposition 4.7. *Let* X *and* Y *be non-empty sets, let* $u \in Q^X$ *and* $w \in Q^Y$ *be unknown hesitant fuzzy sets, and let* $\{R_i\}_{i\in I} \subseteq Q^{X\times Y}$ *be a given family of hesitant fuzzy relations. Define the systems* $\mathcal{D}^l = \langle u \circ R_i = w \mid i \in I\rangle$, $\mathcal{D}^r = \langle R_i \circ w = u \mid i \in I\rangle$ *and* $\mathcal{D}^d = \langle u \circ R_i = w,\ R_i \circ w = u \mid i \in I\rangle$. *Then:*

(a) $\mathcal{D}^l$, $\mathcal{D}^r$ *and* $\mathcal{D}^d$ *contain the least element of the lattice* $Q^X \times Q^Y$;

(b) $\mathcal{D}^l$, $\mathcal{D}^r$ *and* $\mathcal{D}^d$ *are closed under arbitrary joins;*

(c) $\mathcal{D}^l$ *is closed under left scalar multiplication;*

(d) $\mathcal{D}^r$ *is closed under right scalar multiplication.*

Proof. (a) Let $\emptyset_X$ and $\emptyset_Y$ denote the least elements of the lattices Q^X and Q^Y, respectively. For any $i \in I$ and any $y \in Y$,

$$(\emptyset_X \circ R_i)(y) = \bigcup_{x \in X} \emptyset_X(x) \otimes R_i(x, y) = \emptyset_Y(y).$$

Therefore, $(\emptyset_X, \emptyset_Y) \in \mathcal{D}^l$. The argument for $\mathcal{D}^r$ is completely analogous, and the claim for $\mathcal{D}^d$ follows immediately from the previous two cases.

(b) Let $\{(h_j, g_j)\}_{j \in J}$ be an arbitrary family of elements of $\mathcal{D}^l$. Since $(h_j, g_j) \in \mathcal{D}^l$ for every $j \in J$, we have

$$h_j \circ R_i = g_j,$$

for all $i \in I$ and $j \in J$. By the property (2.36), it follows that

$$\left(\bigcup_{j \in J} h_j \right) \circ R_i = \bigcup_{j \in J} (h_j \circ R_i) = \bigcup_{j \in J} g_j,$$

for every $i \in I$. Hence,

$$\left(\bigcup_{j \in J} h_j, \bigcup_{j \in J} g_j \right) \in \mathcal{D}^l.$$

Therefore, $\mathcal{D}^l$ is closed under arbitrary joins.

The same argument, using the corresponding distributivity property for right composition, shows that $\mathcal{D}^r$ is closed under arbitrary joins. Finally, since $\mathcal{D}^d = \mathcal{D}^l \cap \mathcal{D}^r$, the system $\mathcal{D}^d$ is also closed under arbitrary joins.

(c) Let $(h, g) \in \mathcal{D}^l$ and let $\lambda \in Q$ be an arbitrary scalar. Then $h \circ R_i = g$ holds for all $i \in I$. For every $y \in Y$, we obtain:

$$\begin{aligned} \Big((\lambda \otimes h) \circ R_i\Big)(y) &= \bigcup_{x \in X} \Big((\lambda \otimes h)(x) \otimes R_i(x, y)\Big) = \bigcup_{x \in X} \Big(\lambda \otimes h(x) \otimes R_i(x, y)\Big) \\ &= \lambda \otimes \bigcup_{x \in X} \Big(h(x) \otimes R_i(x, y)\Big) = \lambda \otimes \Big(h \circ R_i\Big)(y) = \lambda \otimes g(y). \end{aligned}$$

Hence, $(\lambda \otimes h, \lambda \otimes g) \in \mathcal{D}^l$.

(d) Let $(h, g) \in \mathcal{D}^r$ and let $\lambda \in Q$ be an arbitrary scalar. Then $R_i \circ g = h$ holds for all $i \in I$. For every $x \in X$ we obtain:

$$\begin{aligned}\Big(R_i \circ (g \otimes \lambda)\Big)(x) &= \bigcup_{y \in Y} \Big(R_i(x, y) \otimes (g \otimes \lambda)(y)\Big) = \bigcup_{y \in Y} \Big(R_i(x, y) \otimes g(y) \otimes \lambda\Big) \\ &= \bigcup_{y \in Y} \Big(R_i(x, y) \otimes g(y)\Big) \otimes \lambda = \Big(R_i \circ g\Big)(x) \otimes \lambda = h(x) \otimes \lambda.\end{aligned}$$

Therefore, $(h \otimes \lambda, g \otimes \lambda) \in \mathcal{D}^r$. □

Theorem 4.19. *Let X and Y be non-empty sets, let $u \in Q^X$ and $w \in Q^Y$ be unknown hesitant fuzzy sets, and let $\{R_i\}_{i \in I} \subseteq Q^{X \times Y}$ be a given family of hesitant fuzzy relations, where I is a non-empty index set. Then:*

$$\begin{aligned}&\langle u \circ R_i = w \mid i \in I\rangle = \langle \zeta \subseteq \nu^l(\zeta)\rangle, \quad \langle R_i \circ w = u \mid i \in I\rangle = \langle \zeta \subseteq \nu^r(\zeta)\rangle, \\ &\langle u \circ R_i = w,\ R_i \circ w = u \mid i \in I\rangle = \langle \zeta \subseteq \nu^d(\zeta)\rangle, \end{aligned} \tag{4.29}$$

where $\zeta \in Q^X \times Q^Y$ is an unknown, ν^l, ν^r and ν^d are functions from $Q^X \times Q^Y$ into itself, defined for any $(h, g) \in Q^X \times Q^Y$ as follows:

$$\begin{aligned}&\nu^l((h, g)) = \left(\bigcap_{i \in I}(g \not\phi R_i), \bigcap_{i \in I}(h \circ R_i)\right), \quad \nu^r((h, g)) = \left(\bigcap_{i \in I}(R_i \circ g), \bigcap_{i \in I}(R_i \mathbin{\backslash\!\!\!\phi} h)\right), \\ &\nu^d((h, g)) = \left(\bigcap_{i \in I}(g \not\phi R_i \cap R_i \circ g), \bigcap_{i \in I}(h \circ R_i \cap R_i \mathbin{\backslash\!\!\!\phi} h)\right) = (\nu^l \cap \nu^r)((h, g)).\end{aligned}$$

In addition, functions ν^l, ν^r and ν^d are isotone.

Proof. Let $(h, g) \in \langle u \circ R_i = w \mid i \in I\rangle$. Then $h \circ R_i = g$ holds for all $i \in I$, which implies

$$\bigcap_{i \in I}(h \circ R_i) = g.$$

By the residuation property (2.45), this is equivalent to

$$h \subseteq g \not\phi R_i, \quad \text{for all } i \in I.$$

Consequently,

$$h \subseteq \bigcap_{i \in I}(g \not\phi R_i),$$

and therefore $(h, g) \subseteq \nu^l(h, g)$.

Conversely, let $(h, g) \in \langle \zeta \subseteq \nu^l(\zeta) \rangle$. Then, by definition of ν^l, we have:

$$h \subseteq \bigcap_{i \in I} (g \not\phi R_i), \quad g \subseteq \bigcap_{i \in I} (h \circ R_i).$$

Consequently,

$$h \subseteq g \not\phi R_i, \quad g \subseteq h \circ R_i, \tag{4.30}$$

for all $i \in I$. By the residuation property (2.45), $h \subseteq g \not\phi R_i$ is equivalent to $h \circ R_i \subseteq g$ for all $i \in I$. Together with the second inequality in (4.30), this yields $h \circ R_i = g$ for all $i \in I$. Hence, $(h, g) \in \langle h \circ R_i = g \mid i \in I \rangle$.

The second equality in (4.29) can be shown in a similar way, and the third equality follows directly from the first two.

In the sequel, we show that ν^l is isotone. The isotonicity of ν^r follows analogously, and the isotonicity of ν^d then follows from these two results. Let $(h, g), (h', g') \in Q^X \times Q^Y$ such that $(h, g) \subseteq (h', g')$. Then, since the composition is isotone and the left residual is isotone in its first argument, we obtain $h \circ R_i \subseteq h' \circ R_i$ and $g \not\phi R_i \subseteq g' \not\phi R_i$ for all $i \in I$. Hence, the following componentwise inequality holds for each $i \in I$:

$$(g \not\phi R_i, h \circ R_i) \subseteq (g' \not\phi R_i, h' \circ R_i).$$

Taking the meet over all $i \in I$, we obtain:

$$\begin{aligned} \nu^l((h, g)) &= \left(\bigcap_{i \in I} (g \not\phi R_i), \bigcap_{i \in I} (h \circ R_i) \right) \subseteq \left(\bigcap_{i \in I} (g' \not\phi R_i), \bigcap_{i \in I} (h' \circ R_i) \right) \\ &= \nu^l((h', g')). \end{aligned}$$

Hence, ν^l is isotone. □

Theorem 4.20. *Let X and Y be non-empty finite sets, let $u \in Q^X$ and $w \in Q^Y$ be unknown hesitant fuzzy sets, and let $\{R_i\}_{i \in I} \subseteq Q^{X \times Y}$ be a given family of hesitant fuzzy relations, where I is a non-empty finite index set. Let $(h_0, g_0) \in Q^X \times Q^Y$.*

Define inductively a sequence $\{(h_k, g_k)\}_{k \in \mathbb{N}} \subseteq Q^X \times Q^Y$ in one of the following

three ways:

$$(h_1, g_1) = (h_0, g_0), \tag{4.31}$$
$$(h_{k+1}, g_{k+1}) = \left(h_k \cap \bigcap_{i\in I}(g_k \not\triangleright R_i), g_k \cap \bigcap_{i\in I}(h_k \circ R_i)\right),$$
$$(h_1, g_1) = (h_0, g_0), \tag{4.32}$$
$$(h_{k+1}, g_{k+1}) = \left(h_k \cap \bigcap_{i\in I}(R_i \circ g_k), g_k \cap \bigcap_{i\in I}(R_i \not\triangleleft h_k)\right),$$
$$(h_1, g_1) = (h_0, g_0), \tag{4.33}$$
$$(h_{k+1}, g_{k+1}) = \left(h_k \cap \bigcap_{i\in I}(g_k \not\triangleright R_i \cap R_i \circ g_k), g_k \cap \bigcap_{i\in I}(h_k \circ R_i \cap R_i \not\triangleleft h_k)\right),$$

for any $k \in \mathbb{N}$. If the subalgebra of $\mathcal{Q}$ generated by the set $\bigcup_{i\in I}(\operatorname{Im}(R_i)) \cup \operatorname{Im}(h_0) \cup \operatorname{Im}(g_0)$ is finite, then the following statements hold:

(a) *the sequence $\{(h_k, g_k)\}_{k\in\mathbb{N}}$ is finite and decreasing, and there is the smallest number $m \in \mathbb{N}$ such that $(h_m, g_m) = (h_{m+1}, g_{m+1})$,*

(b) *for the sequence defined in (4.31), (h_m, g_m) is the greatest solution to system $\langle u \circ R_i = w \mid i \in I\rangle$ contained in (h_0, g_0),*

(c) *for the sequence defined in (4.32), (h_m, g_m) is the greatest solution to system $\langle R_i \circ w = u \mid i \in I\rangle$ contained in (h_0, g_0),*

(d) *for the sequence defined in (4.33), (h_m, g_m) is the greatest solution to system $\langle u \circ R_i = w, R_i \circ w = u \mid i \in I\rangle$ contained in (h_0, g_0).*

Proof. (a) We first establish the result for the sequence (4.31). The argument for (4.32) is analogous, while (4.33) follows directly from these two cases.

First, notice that the sequence $\{(h_k, g_k)\}_{k\in\mathbb{N}}$ is decreasing and satisfies $(h_k, g_k) \subseteq (h_0, g_0)$ for all $k \in \mathbb{N}$. Next, let $\mathcal{D}$ denote the subalgebra of $\mathcal{Q}$ generated by the set $\bigcup_{i\in I} \operatorname{Im}(\{R_i\}) \cup \operatorname{Im}(h_0) \cup \operatorname{Im}(g_0)$, and let D denote its carrier set. By the hypothesis, D is a finite set, and X and Y are also finite sets, and hence D^X and D^Y are also finite sets. Therefore, $\{(h_k, g_k)\}_{k\in\mathbb{N}} \subseteq D^X \times D^Y$ is a decreasing sequence taking values in a finite set and it must stabilize after finitely many steps. Hence, there exists a smallest $m \in \mathbb{N}$ such that

$$(h_m, g_m) = (h_{m+1}, g_{m+1}).$$

(b) According to Theorem 4.19, $\langle u \circ R_i = w \mid i \in I\rangle = \langle \zeta \subseteq \nu^l(\zeta)\rangle$. From part

(a) it follows that:

$$(h_m, g_m) = (h_{m+1}, g_{m+1}) = \left(h_m \cap \bigcap_{i \in I} (g_m \not\circ R_i), g_m \cap \bigcap_{i \in I} (h_m \circ R_i) \right)$$
$$= (h_m, g_m) \cap \nu^l((h_m, g_m)) \subseteq \nu^l((h_m, g_m)).$$

Hence, (h_m, g_m) is a solution to the system $\langle u \circ R_i = w \mid i \in I \rangle$.

We now prove that (h_m, g_m) is the greatest solution. Let (h, g) be any solution to the system $\langle u \circ R_i = w \mid i \in I \rangle$ such that in $(h, g) \subseteq (h_0, g_0)$. Then, we have

$$(h, g) \subseteq \nu^l((h, g)) \quad \text{and} \quad (h, g) \subseteq (h_0, g_0) = (h_1, g_1).$$

We prove by mathematical induction that $(h, g) \subseteq (h_k, g_k)$ for all $k \in \mathbb{N}$. By assumption, $(h, g) \subseteq (h_1, g_1)$. Assume, as the induction step, that

$$(h, g) \subseteq (h_k, g_k)$$

for some $k \in \mathbb{N}$. Since ν^l is isotone and $(h, g) \subseteq \nu^l((h, g))$, we obtain

$$(h, g) \subseteq \nu^l((h, g)) \subseteq \nu^l((h_k, g_k)).$$

Consequently,

$$(h, g) \subseteq (h_k, g_k) \cap \nu^l((h_k, g_k)) = (h_{k+1}, g_{k+1}).$$

By mathematical induction, $(h, g) \subseteq (h_k, g_k)$ for all $k \in \mathbb{N}$. In particular, $(h, g) \subseteq (h_m, g_m)$. Since (h, g) was an arbitrary solution to the system contained in (h_0, g_0), it follows that (h_m, g_m) is the greatest solution to the system $\langle u \circ R_i = w \mid i \in I \rangle$ contained in (h_0, g_0).

Assertion (c) can be proved analogously, and (d) follows directly from (b) and (c). □

Theorem 4.21. *Let $u \in Q^X$ and $w \in Q^Y$ be unknown hesitant fuzzy sets, and let $\{R_i\}_{i \in I} \subseteq Q^{X \times Y}$ be a given family of hesitant fuzzy relations, where X and Y are non-empty sets. Then:*

$$\begin{aligned} &\langle w \not\circ R_i = u \mid i \in I \rangle = \langle \mu^l(\zeta) \subseteq \zeta \rangle, \qquad \langle R_i \not\backslash u = w \mid i \in I \rangle = \langle \mu^r(\zeta) \subseteq \zeta \rangle, \\ &\langle w \not\circ R_i = u,\ R_i \not\backslash u = w \mid i \in I \rangle = \langle \mu^d(\zeta) \subseteq \zeta \rangle, \end{aligned} \tag{4.34}$$

where $\zeta \in Q^X \times Q^Y$ is an unknown and μ^l, μ^r and μ^d are functions from $Q^X \times Q^Y$

into itself defined for any $(h, g) \in Q^X \times Q^Y$ *as follows:*

$$\mu^l((h,g)) = \Big(\bigcup_{i\in I}(g \not\phi R_i), \bigcup_{i\in I}(h \circ R_i)\Big), \quad \mu^r((h,g)) = \Big(\bigcup_{i\in I}(R_i \circ g), \bigcup_{i\in I}(R_i \backslash\!\!\!\phi\, h)\Big),$$
$$\mu^d((h,g)) = \Big(\bigcup_{i\in I}(g \not\phi R_i \cup R_i \circ g), \bigcup_{i\in I}(h \circ R_i \cup R_i \backslash\!\!\!\phi\, h)\Big) = (\mu^l \cup \mu^r)((h,g)).$$

In addition, functions μ^l, μ^r *and* μ^d *are isotone.*

Proof. We prove the first equality in (4.34). The second is obtained analogously, and the third follows from the preceding two.

The following chain of equalities completes the proof:

$$\begin{aligned}
\langle w \not\phi R_i = u \mid i \in I\rangle &= \langle w \not\phi R_i \subseteq u, u \subseteq w \not\phi R_i \mid i \in I\rangle \\
&= \langle w \not\phi R_i \subseteq u, u \circ R_i \subseteq w \mid i \in I\rangle \\
&= \langle \bigcup_{i\in I}(w \not\phi R_i) \subseteq u, \bigcup_{i\in I}(u \circ R_i) \subseteq w\rangle \\
&= \langle \Big(\bigcup_{i\in I}(w \not\phi R_i), \bigcup_{i\in I}(u \circ R_i)\Big) \subseteq (u,w)\rangle \\
&= \langle \mu^l((u,w)) \subseteq (u,w)\rangle.
\end{aligned}$$

In the sequel we show that μ^l is isotone. Let $(h,g),(h',g') \in Q^X \times Q^Y$ satisfy $(h,g) \subseteq (h',g')$. Since left $\circ$-residual is isotone in its first argument and composition $\circ$ is isotone in both arguments, we obtain

$$\mu^l((h,g)) = \Big(\bigcup_{i\in I}(g \not\phi R_i), \bigcup_{i\in I}(h \circ R_i)\Big) \subseteq \Big(\bigcup_{i\in I}(g' \not\phi R_i), \bigcup_{i\in I}(h' \circ R_i)\Big) = \mu^l((h',g')).$$

The isotonicity of μ^r is proved analogously, and the isotonicity of μ^d follows from the first two. □

Theorem 4.22. *Let* X *and* Y *be non-empty finite sets, let* $u \in Q^X$ *and* $w \in Q^Y$ *be unknown hesitant fuzzy sets, and let* $\{R_i\}_{i\in I} \subseteq Q^{X\times Y}$ *be a given family of hesitant fuzzy relations, where* I *is a non-empty finite index set. Let* $(h_0, g_0) \in Q^X \times Q^Y$.

Define inductively a sequence $\{(h_k, g_k)\}_{k\in\mathbb{N}} \subseteq Q^X \times Q^Y$ *in one of the following*

three ways:

$$(h_1, g_1) = (h_0, g_0), \tag{4.35}$$
$$(h_{k+1}, g_{k+1}) = \Big(h_k \cup \bigcup_{i\in I}(g_k \not\phi R_i), g_k \cup \bigcup_{i\in I}(h_k \circ R_i)\Big),$$

$$(h_1, g_1) = (h_0, g_0), \tag{4.36}$$
$$(h_{k+1}, g_{k+1}) = \Big(h_k \cup \bigcup_{i\in I}(R_i \circ g_k), g_k \cup \bigcup_{i\in I}(R_i \setminus h_k)\Big),$$

$$(h_1, g_1) = (h_0, g_0), \tag{4.37}$$
$$(h_{k+1}, g_{k+1}) = \Big(h_k \cup \bigcup_{i\in I}(g_k \not\phi R_i \cup R_i \circ g_k), g_k \cup \bigcup_{i\in I}(h_k \circ R_i \cup R_i \setminus h_k)\Big),$$

for any $k \in \mathbb{N}$. *If the subalgebra of* $\mathcal{Q}$ *generated by the set* $\bigcup_{i\in I}(\operatorname{Im}(R_i)) \cup \operatorname{Im}(h_0) \cup \operatorname{Im}(g_0)$ *is finite, then the following statements hold:*

(a) *the sequence* $\{(h_k, g_k)\}_{k\in\mathbb{N}}$ *is finite and nondecreasing, and there is the smallest number* $m \in \mathbb{N}$ *such that* $(h_m, g_m) = (h_{m+1}, g_{m+1})$;

(b) *for the sequence defined in* (4.35), (h_m, g_m) *is the least solution to the system* $\langle w \not\phi R_i = u \mid i \in I\rangle$ *containing* (h_0, g_0);

(c) *for the sequence defined in* (4.36), (h_m, g_m) *is the least solution to the system* $\langle R_i \setminus u = w \mid i \in I\rangle$ *containing* (h_0, g_0);

(d) *for the sequence defined in* (4.37), (h_m, g_m) *is the least solution to the system* $\langle w \not\phi R_i = u,\ R_i \setminus u = w \mid i \in I\rangle$ *containing* (h_0, g_0).

Proof. (a) Consider the sequence (4.35). Let $\mathcal{D}$ be a subalgebra of $\mathcal{Q}$ generated by the set $\bigcup_{i\in I}(\operatorname{Im}(R_i)) \cup \operatorname{Im}(h_0) \cup \operatorname{Im}(g_0)$. Since $\mathcal{D}$ is finite, its carrier set D is also finite, and therefore D^X and D^Y are finite as well. By the definition of the sequence, we have:

$$(h_{k+1}, g_{k+1}) = \Big(h_k \cup \bigcup_{i\in I}(g_k \not\phi R_i), g_k \cup \bigcup_{i\in I}(h_k \circ R_i)\Big) = (h_k, g_k) \cup \mu^l((h_k, g_k)).$$

Consequently, $(h_k, g_k) \subseteq (h_{k+1}, g_{k+1})$, for every $k \in \mathbb{N}$. Since the sequence is nondecreasing and takes values in a finite poset, it must eventually stabilize. Hence, there is a smallest $m \in \mathbb{N}$ such that

$$(h_m, g_m) = (h_{m+1}, g_{m+1}).$$

(b) According to Theorem 4.21, we have $\langle w \not\phi R_i = u \mid i \in I\rangle = \langle \mu^l(\zeta) \subseteq \zeta\rangle$. We

first observe that:

$$(h_m, g_m) = (h_{m+1}, g_{m+1}) = \Big(h_m \cup \bigcup_{i\in I}(g_m \not\phi\, R_i), g_m \cup \bigcup_{i\in I}(h_m \circ R_i)\Big)$$
$$= (h_m, g_m) \cup \mu^l((h_m, g_m)).$$

This immediately yields

$$\mu^l((h_m, g_m)) \subseteq (h_m, g_m),$$

and, consequently $(h_m, g_m) \in \langle w \not\phi\, R_i = u \mid i \in I\rangle$.

Next, let (h, g) be any solution to the system such that $(h_0, g_0) \subseteq (h, g)$. Consequently, $(h_1, g_1) \subseteq (h, g)$. Assume, by the induction hypothesis, that $(h_k, g_k) \subseteq (h, g)$ for some $k \in \mathbb{N}$. Then we obtain:

$$(h_{k+1}, g_{k+1}) = \Big(h_k \cup \bigcup_{i\in I}(g_k \not\phi\, R_i), g_k \cup \bigcup_{i\in I}(h_k \circ R_i)\Big) = (h_k, g_k) \cup \mu^l((h_k, g_k))$$
$$\subseteq (h, g) \cup \mu^l(h, g).$$

Since (h, g) is a solution, we have $\mu^l((h, g)) \subseteq (h, g)$, and therefore

$$(h, g) \cup \mu^l((h, g)) = (h, g).$$

Hence, $(h_{k+1}, g_{k+1}) \subseteq (h, g)$. By induction, it follows that $(h_k, g_k) \subseteq (h, g)$ for all $k \in \mathbb{N}$. In particular, $(h_m, g_m) \subseteq (h, g)$. Consequently, (h_m, g_m) is the least solution to the system that contains (h_0, g_0). □

Chapter 5

Weakly linear systems of hesitant fuzzy relation inequalities and equations

The study of systems of fuzzy relational equations and inequalities was initiated by Sanchez in the context of medical diagnosis [105, 107]. Since then, such systems have found applications in a wide range of areas, including fuzzy control, knowledge representation, decision making, pattern recognition, image processing, and the reduction of fuzzy automata; see, e.g., [28, 30, 38–40, 89, 90]. Linear systems of fuzzy relational equations and inequalities have been extensively investigated over various algebraic structures, including complete residuated lattices (cf. eg. [91, 94, 95]).

A particular class of systems of fuzzy relation inequalities and equations, called weakly linear systems, was introduced in [60]. Since then, both homogeneous and heterogeneous weakly linear systems have been investigated, with applications to fuzzy automata and social network analysis [57, 61].

Motivated by these developments, we extend the study of weakly linear systems to the hesitant fuzzy setting, considering systems of hesitant fuzzy relation inequalities and equations.

Let $I = I_1 \cup I_2 \cup I_3 \cup I_4$ be a non-empty index set. A *weakly linear system of hesitant fuzzy relation inequalities*, called *hesitant weakly linear system* (HWLS), is

defined as a family of hesitant fuzzy inequalities of the following form:

$$A_i \circ U \subseteq U \circ B_i \quad (i \in I_1), \tag{$hs^{1.1}$}$$
$$U \circ B_i \subseteq A_i \circ U \quad (i \in I_2), \tag{$hs^{1.2}$}$$
$$U^{-1} \circ A_i \subseteq B_i \circ U^{-1} \quad (i \in I_3), \tag{$hs^{1.3}$}$$
$$B_i \circ U^{-1} \subseteq U^{-1} \circ A_i \quad (i \in I_4), \tag{$hs^{1.4}$}$$

where $\{A_i\}_{i\in I} \subseteq Q^{X\times X}$ and $\{B_i\}_{i\in I} \subseteq Q^{Y\times Y}$ are given families of hesitant fuzzy relations and U is an unknown hesitant fuzzy relation between X and Y. We distinguish two types of HWLSs. A system is called *homotypical* if $X = Y$ and $A_i = B_i$ for all $i \in I$; otherwise, it is called *heterotypical*. The system is said to be *finite* if the index set I is finite. It is straightforward to verify that the zero hesitant fuzzy relation on X is a solution to any HWLS. This solution is called the trivial solution.

The chapter is organized as follows. Section 5.1 deals with heterotypical weakly linear systems, while Section 5.2 focuses on homotypical weakly linear systems. Section 5.3 is devoted to the study of the greatest solution to a homotypical weakly linear system of hesitant fuzzy relation inequalities and equations.

5.1 Heterotipical weakly linear systems

This section is devoted to the heterotypical HWLSs. For simplicity of notation, in what follows, the heterotypical HWLSs corresponding to different values of k will be denoted by $HS^{1.k}(X, Y; A, B, V)$.

The following two theorems hold for both heterotypical and homotypical weakly linear systems. We present the proofs only for the heterotypical case, since the proofs for the homotypical case are analogous.

The next theorem establishes the existence of the greatest solution for every such system.

Theorem 5.1. *The set of all solutions to the system ($hs^{1.1}$)-($hs^{1.4}$) forms a complete lattice, which is a complete join-subsemilattice of the lattice $\mathcal{R}(X, Y)$, and therefore admits a greatest solution.*

Proof. Let $\{U_j\}_{j\in J} \subseteq \mathcal{R}(X, Y)$ be the family of all solutions to the system ($hs^{1.1}$)-($hs^{1.4}$) and define $U = \bigcup_{j\in J} U_j$. We show that U is also a solution. Consider

inequality ($hs^{1.1}$). For all $i \in I_1$ and $j \in J$, we have:

$$A_i \circ U_j \subseteq U_j \circ B_i.$$

Taking joins over all $j \in J$, we obtain

$$\bigcup_{j \in J} (A_i \circ U_j) \subseteq \bigcup_{j \in J} (U_j \circ B_i),$$

for all $i \in I_1$. Using the distributivity of composition over arbitrary joins given in (2.42), it follows that

$$A_i \circ \left(\bigcup_{j \in J} U_j \right) \subseteq \left(\bigcup_{j \in J} U_j \right) \circ B_i,$$

for all $i \in I_1$. Hence, U satisfies inequality ($hs^{1.1}$). The remaining inequalities are verified analogously. Therefore, U is a solution to the system ($hs^{1.1}$)-($hs^{1.4}$).

This proves that the set of all solutions is closed under arbitrary joins, and thus forms a complete join-subsemilattice of $\mathcal{R}(X, Y)$. Since, the zero hesitant fuzzy relation is also a solution to the system, the set is non-empty, and therefore it forms a complete lattice.

Moreover, by construction, U is the join of all solutions, and therefore $U_j \subseteq U$ for all $j \in J$. Consequently, U is the greatest solution to the system. □

Recall that two systems of equations or inequalities are said to be equivalent if they have the same set of solutions. The following theorem establishes the equivalence of the considered systems.

Theorem 5.2. *Any WLS can be equivalently represented by a system consisting only of the inequalities* ($hs^{1.1}$) *and* ($hs^{1.2}$).

Proof. Assume that the index set I_3 is non-empty, and let R denote a hesitant fuzzy relation satisfying the system given by ($hs^{1.3}$). By applying properties (2.38) and (2.40), for all $i \in I_3$, we obtain:

$$\begin{aligned} R^{-1} \circ A_i \subseteq B_i \circ R^{-1} \quad &\Leftrightarrow \quad (R^{-1} \circ A_i)^{-1} \subseteq (B_i \circ R^{-1})^{-1} \\ &\Leftrightarrow \quad A_i \circ R^{-1} \subseteq R^{-1} \circ B_i. \end{aligned}$$

Consequently, the system ($hs^{1.3}$) is equivalent to the corresponding system ($hs^{1.1}$).

Similarly, assuming that the index set I_4 is non-empty, it can be shown that the system given by ($hs^{1.4}$) is equivalent to the corresponding system ($hs^{1.2}$). □

As a consequence of Theorem 5.2, it is sufficient to consider only WLS consisting of the hesitant fuzzy inequalities ($hs^{1.1}$) and ($hs^{1.2}$). To express the last system as a single inequality, we introduce the mapping $\phi : \mathcal{R}(X,Y) \mapsto \mathcal{R}(X,Y)$ defined by:

$$\phi(R) = \left(\bigcap_{i \in I_1} A_i \backslash (R \circ B_i) \right) \cap \left(\bigcap_{i \in I_2} (A_i \circ R) / B_i \right), \tag{5.1}$$

for any $R \in Q^{X \times Y}$. If either I_1 or I_2 is empty, the corresponding meet in (5.1) is omitted, since the infimum of the empty set coincides with the greatest element, which in this case is the corresponding full hesitant fuzzy relation.

Theorem 5.3. *Let $\phi : \mathcal{R}(X,Y) \mapsto \mathcal{R}(X,Y)$ be the mapping defined by (5.1). The following statements hold:*

(1) *The system ($hs^{1.1}$)–($hs^{1.2}$) is equivalent to the inequality $U \subseteq \phi(U)$.*

(2) *The mapping ϕ is isotone.*

Proof. (1) By applying the residuation property (2.64) to inequalities ($hs^{1.1}$) and ($hs^{1.2}$), we obtain:

$$U \subseteq \left(\bigcap_{i \in I_1} A_i \backslash (U \circ B_i) \right) \quad \text{and} \quad U \subseteq \left(\bigcap_{i \in I_2} (A_i \circ U) / B_i \right).$$

Consequently,

$$U \subseteq \left(\bigcap_{i \in I_1} A_i \backslash (U \circ B_i) \right) \cap \left(\bigcap_{i \in I_2} (A_i \circ U) / B_i \right),$$

which implies that U satisfies the inequality $U \subseteq \phi(U)$. Therefore, every solution to the system ($hs^{1.1}$)–($hs^{1.2}$) is also a solution to $U \subseteq \phi(U)$.

(2) This is a direct consequence of properties (2.41), (2.67). □

In what follows, we study systems obtained by combining inequalities from the system ($hs^{1.1}$)-($hs^{1.4}$), over the same index set I. More precisely, we consider the following systems of hesitant fuzzy inequalities:

$$A_i \circ U \subseteq U \circ B_i, \quad B_i \circ U^{-1} \subseteq U^{-1} \circ A_i \quad (i \in I), \quad U \subseteq V, \tag{$hs^{1.5}$}$$

$$U \circ B_i \subseteq A_i \circ U, \quad U^{-1} \circ A_i \subseteq B_i \circ U^{-1} \quad (i \in I), \quad U \subseteq V, \tag{$hs^{1.6}$}$$

where $\{A_i\}_{i\in I} \subseteq Q^{X\times X}$ and $\{B_i\}_{i\in I} \subseteq Q^{Y\times Y}$ are given families of hesitant fuzzy relations and U is an unknown hesitant fuzzy relation between X and Y. For brevity, we denote the system ($hs^{1.5}$) by $HS^{1.5}(X, Y; A, B, V)$, while the system ($hs^{1.6}$) is denoted by $HS^{1.6}(X, Y; A, B, V)$.

Proposition 5.1. *Let X, Y and Z be non-empty sets. Let R be a solution to $HS^{1.k}(X, Y; A, B, V)$, let T be a solution to $HS^{1.k}(Y, Z; B, C, W)$, where $k \in \{5, 6\}$. Then $R \circ T$ is a solution to $HS^{1.k}(X, Z; A, C, V \circ W)$.*

Proof. We prove the assertion for the system ($hs^{1.6}$). The proof for ($hs^{1.5}$) is similar and is therefore omitted.

It follows from the assumptions that

$$R \circ B_i \subseteq A_i \circ R, \quad R^{-1} \circ A_i \subseteq B_i \circ R^{-1} \quad (i \in I), \quad R \subseteq V, \tag{5.2}$$

$$S \circ C_i \subseteq B_i \circ S, \quad S^{-1} \circ B_i \subseteq C_i \circ R^{-1} \quad (i \in I), \quad S \subseteq W. \tag{5.3}$$

Using associativity and monotonicity of the composition together with (5.2) and (5.2), we obtain

$$(R \circ S) \circ C_i = R \circ (S \circ C_i) \subseteq R \circ (B_i \circ S) \subseteq A_i \circ (R \circ S).$$

Moreover, monotonicity of the composition implies $R \circ S \subseteq V \circ W$. □

Theorem 5.4. *Each of the HWLSs ($hs^{1.5}$) and ($hs^{1.6}$) admits a greatest solution, possibly empty. Moreover, if V is a partial hesitant fuzzy function, then these greatest solutions are partial hesitant fuzzy functions as well.*

Proof. The existence of greatest solutions to the HWLSs ($hs^{1.5}$) and ($hs^{1.6}$) is proved analogously to Theorem 5.1.

Next, let V be a partial hesitant fuzzy function (see Definition 3.6) and let R be the greatest solution to the system ($hs^{1.5}$). Then $R \circ R^{-1} \circ R \subseteq V \circ V^{-1} \circ V \subseteq V$. Furthermore, by associativity and monotonicity of the composition, we have

$$\begin{aligned} A_i \circ R \circ R^{-1} \circ R &\subseteq R \circ B_i \circ R^{-1} \circ R \subseteq R \circ R^{-1} \circ A_i \circ R \\ &\subseteq R \circ R^{-1} \circ R \circ B_i, \end{aligned}$$

for all $i \in I$. Consequently, $R \circ R^{-1} \circ R$ is a solution to the HWLS ($hs^{1.5}$).

The proof for ($hs^{1.6}$) follows in a similar way and is therefore omitted. □

In order to express the HWLSs ($hs^{1.5}$) and ($hs^{1.6}$) in the form of a single inequality, we introduce the functions $\phi^{1.k} : \mathcal{R}(X, Y) \to \mathcal{R}(X, Y)$, $k \in \{5, 6\}$, as

follows:

$$\phi^{1.5}(R) = \bigcap_{i\in I}[(A_i \backslash (R \circ B_i)) \cap (B_i \backslash (R^{-1} \circ A_i))^{-1}],$$
$$\phi^{1.6}(R) = \bigcap_{i\in I}[((A_i \circ R) \not\backslash B_i) \cap ((B_i \circ R^{-1}) \not\backslash A_i)^{-1}],$$

for all $R \in Q^{X\times Y}$.

Theorem 5.5. *The HWLSs ($hs^{1.5}$) and ($hs^{1.6}$) are equivalent to the systems $U \subseteq \phi^{1.5}(U)$ and $U \subseteq \phi^{1.6}(U)$, respectively.*

Proof. We prove that ($hs^{1.5}$) is equivalent to $U \subseteq \phi^{1.5}(U)$. The other equivalence can be proved analogously.

Fix $i \in I$. For an arbitrary hesitant fuzzy relation $R \in Q^{X\times Y}$, we have

$$\begin{aligned} R \circ B_i \subseteq A_i \circ R \quad &\Leftrightarrow \quad R(x,y) \otimes B_i(y,y') \subseteq (A_i \circ R)(x,y'), \\ R^{-1} \circ A_i \subseteq B_i \circ R^{-1} \quad &\Leftrightarrow \quad R^{-1}(y,x) \otimes A_i(x,x') \subseteq (B_i \circ R^{-1})(y,x'), \end{aligned}$$

for all $x, x' \in X$ and $y, y' \in Y$. By the residuation property, it follows that

$$R(x,y) \subseteq \bigcap_{y'\in Y} B_i(y,y') \to (A_i \circ R)(x,y') = ((A_i \circ R) \not\backslash B_i)(x,y),$$

for every $x \in X, y \in Y$. Analogously,

$$R^{-1}(y,x) \subseteq \bigcap_{x'\in X} A_i(x,x') \to (B_i \circ R^{-1})(y,x') = ((B_i \circ R^{-1}) \not\backslash A_i)(y,x),$$

which is equivalent to

$$R(x,y) \subseteq ((B_i \circ R^{-1}) \not\backslash A_i)^{-1}(x,y),$$

for every $x \in X$ and $y \in Y$. Since $i \in I$ was arbitrary, we obtain

$$R(x,y) \subseteq \bigcap_{i\in I}[((A_i \circ R) \not\backslash B_i) \cap ((B_i \circ R^{-1}) \not\backslash A_i)^{-1}](x,y).$$

Consequently, $R(x,y) \subseteq \phi^{1.5}(R)$. □

5.2 Homotypical weakly linear systems

Let X be a non-empty set and let $\{A_i\}_{i\in I} \subseteq Q^{X\times X}$ be a given family of hesitant fuzzy relations on X. In this section, we investigate the following homotypical weakly

linear systems, where $U \in Q^{X\times X}$ denotes an unknown hesitant fuzzy relation on X:

$$U \circ A_i \subseteq A_i \circ U \quad (i \in I), \tag{$hs^{2,1}$}$$
$$A_i \circ U \subseteq U \circ A_i \quad (i \in I), \tag{$hs^{2,2}$}$$
$$U \circ A_i = A_i \circ U \quad (i \in I), \tag{$hs^{2,3}$}$$
$$U^{-1} \circ A_i \subseteq A_i \circ U^{-1} \quad (i \in I), \tag{$hs^{2,4}$}$$
$$A_i \circ U^{-1} \subseteq U^{-1} \circ A_i \quad (i \in I), \tag{$hs^{2,5}$}$$
$$U^{-1} \circ A_i = A_i \circ U^{-1} \quad (i \in I). \tag{$hs^{2,6}$}$$

We say that a hesitant fuzzy relation U on X is *right regular*, *left regular*, or *regular* with respect to the family $\{A_i\}_{i\in I}$ if it satisfies system ($hs^{2,1}$), ($hs^{2,2}$), or ($hs^{2,3}$), respectively. Similarly, systems ($hs^{2,4}$)-($hs^{2,6}$) address, respectively, left regularity, right regularity, and regularity of the inverse hesitant fuzzy relation U^{-1} with respect to the family $\{A_i\}_{i\in I}$. Recall that the operations $\oslash$ and $\oslash$, used throughout this section, were introduced in Subsection 2.2.3; see (2.76), (2.77).

Let $\mathcal{F}(X)$ denote the lattice of all hesitant fuzzy quasi-orders on X. We define the mappings ϕ^f_{rr}, ϕ^f_{lr} and ϕ^f_r of $\mathcal{F}(X)$ into itself as follows:

$$\begin{aligned} &\phi^f_{rr}(F) = \bigcap_{i\in I} A_i \oslash F, \quad \phi^f_{lr}(F) = \bigcap_{i\in I} F \oslash A_i, \\ &\phi^f_r(F) = \bigcap_{i\in I} (A_i \oslash F) \cap (F \oslash A_i) = \phi^f_{rr}(F) \cap \phi^f_{lr}(F), \end{aligned} \tag{5.4}$$

for every $F \in \mathcal{F}(X)$. Recall that, by Theorem 2.2, the relations $A_i \oslash F$ and $F \oslash A_i$ are hesitant fuzzy quasi-orders for all $i \in I$. Consequently, ϕ^f_{rr}, ϕ^f_{lr} and ϕ^f_r are hesitant fuzzy quasi-order on X.

The following theorem describes the right regular hesitant fuzzy quasi-orders which are solutions to the system ($hs^{2,1}$).

Theorem 5.6. *Let F be a hesitant fuzzy quasi-order on X and let U denote an unknown hesitant fuzzy relation. The following conditions are equivalent:*

(1) *F is a solution to the system ($hs^{2,1}$)*

(2) *F is a solution to the system:*

$$U \circ A_i \circ U = A_i \circ U, \quad i \in I. \tag{5.5}$$

(3) *F is a solution to the system:*

$$U \subseteq \phi_{rr}^{f}(U). \tag{5.6}$$

Proof. (1) $\Leftrightarrow$ (2) Let F be a solution to ($hs^{2,1}$), i.e., $F \circ A_i \subseteq A_i \circ F$ for all $i \in I$. By property (2.41) and transitivity of F, we obtain:

$$F \circ A_i \circ F \subseteq A_i \circ F \circ F \subseteq A_i \circ F,$$

for all $i \in I$. On the other hand, by the reflexivity of F, it follows that:

$$A_i \circ F \subseteq F \circ A_i \circ F,$$

for all $i \in I$. Hence, F is a solution to the equation (5.5).

Conversely, suppose that F is a solution to (5.5). By the reflexivity of F, for every $i \in I$, we obtain:

$$F \circ A_i \subseteq F \circ A_i \circ F = F \circ A_i.$$

Together with the reverse inclusion established above, this yields the equality (5.5).

(2) $\Leftrightarrow$ (3) Assume that F satisfies (5.5). Then F is a solution to the system

$$U \circ (A_i \circ F) = A_i \circ F,$$

for all $i \in I$, where U is an unknown hesitant fuzzy relation. By Theorem 2.2, the greatest solution to this system is $A_i \oslash F = (A_i \circ F) \oint (A_i \circ F)$. Since F itself is a solution, it follows that $F \subseteq A_i \oslash F$ for all $i \in I$. Consequently,

$$F \subseteq \bigcap_{i \in I} A_i \oslash F = \phi_{rr}^{f}(F).$$

Conversely, assume that F is a solution to the system (5.6). Then, for every $i \in I$ the following holds:

$$F \subseteq A_i \oslash F = (A_i \circ F) \oint (A_i \circ F).$$

By the residuation property, it follows that:

$$F \circ A_i \circ F \subseteq A_i \circ F.$$

On the other hand, the reflexivity of F implies:

$$A_i \circ F \subseteq F \circ A_i \circ F,$$

for all $i \in I$. Consequently, F satisfies (5.5). □

The following theorem provides a characterization of left regular hesitant fuzzy quasi-orders determined by system ($hs^{1.2}$).

Theorem 5.7. *Let F be a hesitant fuzzy quasi-order on X and let U denote an unknown hesitant fuzzy relation. The following conditions are equivalent:*

(1) *F is a solution to the system ($hs^{2,2}$)*

(2) *F is a solution to the systems*

$$U \circ A_i \circ U = U \circ A_i, \quad i \in I. \tag{5.7}$$

(3) *F is a solution to the system:*

$$U \subseteq \phi^f_{lr}(U). \tag{5.8}$$

Proof. The proof is similar to that of Theorem 5.6 and is therefore omitted. □

As a direct consequence of Theorems 5.6 and 5.7, we state the following theorem, which characterizes regular hesitant fuzzy quasi-orders with respect to $\{A_i\}_{i\in I}$.

Theorem 5.8. *Let F be a hesitant fuzzy quasi-order on X and let U denote an unknown hesitant fuzzy relation. The following conditions are equivalent:*

(1) *F is a solution to the system ($hs^{2,3}$).*

(2) *F is a solution the systems ($hs^{2,1}$) and ($hs^{2,2}$).*

(3) *F is a solution to the system:*

$$U \subseteq \phi^f_{r}(U). \tag{5.9}$$

Proof. The proof is a direct consequence of Theorems 5.6 and 5.7 and is therefore omitted. □

We next consider the following systems of hesitant fuzzy relation inequalities and equations. As before, U denotes an unknown hesitant fuzzy relation on X.

$$\begin{aligned}
&U \circ A_i \subseteq A_i \circ U, && U^{-1} \circ A_i \subseteq A_i \circ U^{-1} && (i \in I), && (hs^{2,7})\\
&A_i \circ U \subseteq U \circ A_i, && A_i \circ U^{-1} \subseteq U^{-1} \circ A_i && (i \in I), && (hs^{2,8})\\
&U \circ A_i = A_i \circ U, && U^{-1} \circ A_i^{-1} = A_i \circ U^{-1} && (i \in I). && (hs^{2,9})
\end{aligned}$$

It follows that R is a solution to $(hs^{2,7})$, $(hs^{2,8})$ and $(hs^{2,9})$ if and only if both R and R^{-1} are solutions to $(hs^{2,1})$, $(hs^{2,2})$ and $(hs^{2,3})$, respectively.

Theorem 5.9. *The systems* $(hs^{2,7})$-$(hs^{2,9})$ *have the greatest solutions.*

Proof. Consider the system $(hs^{2,7})$. Clearly, the zero hesitant fuzzy relation is a solution, so the solution set is non-empty. Let $\{R_j\}_{j\in J}$ denote the family of all hesitant fuzzy relations on X, that solve $(hs^{2,7})$ and define $R = \bigcup_{j\in J} R_j$. Then R is itself a solution to $(hs^{2,7})$ and, moreover, it is the greatest solution. Similarly, the result holds for the other systems. □

In the following, we characterize the solutions to the systems $(hs^{2,7})$-$(hs^{2,9})$ that are hesitant fuzzy equivalences, specifically the right regular, left regular, and regular fuzzy equivalences with respect to the family $\{A_i\}_{i\in I}$. Consider the lattice $\mathcal{E}(X)$ of hesitant fuzzy equivalences on X and define the functions ϕ^e_{rr}, ϕ^e_{lr} and ϕ^e_r of $\mathcal{E}(X)$ into itself as follows:

$$\begin{aligned}
&\phi^e_{rr}(E) = \bigcap_{i\in I} A_i \hat{\oslash} E, \quad \phi^e_{lr}(E) = \bigcap_{i\in I} E \hat{\oslash} A_i,\\
&\phi^e_r(E) = \bigcap_{i\in I} (A_i \hat{\oslash} E) \cap (E \hat{\oslash} A_i),
\end{aligned} \tag{5.10}$$

for all $E \in \mathcal{E}(X)$. Since $A_i \hat{\oslash} E$ and $E \hat{\oslash} A_i$ are hesitant fuzzy equivalences, it follows that $\phi^e_{rr}(E)$, $\phi^e_{lr}(E)$ and $\phi^e_r(E)$ are also hesitant fuzzy equivalences. In the sequel, we present three theorems providing characterizations of the right regular, left regular and regular hesitant fuzzy equivalences with respect to $\{A_i\}_{i\in I}$.

Theorem 5.10. *Let E be a hesitant fuzzy equivalence on X. The following statements are equivalent:*

(1) *E is a solution to the system $(hs^{2,7})$.*

(2) *E is a solution to the system $(hs^{2,1})$.*

(3) *E is a solution to the system (5.5).*

(4) *E is a solution to the system*

$$U \subseteq \phi^e_{rr}(U),$$

where U is an unknown hesitant fuzzy relation on X.

Proof. This is an immediate consequence of Theorem 5.6 and the symmetry of E. □

Theorem 5.11. *Let E be a hesitant fuzzy equivalence on X. The following statements are equivalent:*

(1) *E is a solution to the system ($hs^{2,8}$).*

(2) *E is a solution to the system ($hs^{2,2}$).*

(3) *E is a solution to the system (5.7).*

(4) *E is a solution to the system*

$$U \subseteq \phi^e_{lr}(U),$$

where U is an unknown hesitant fuzzy relation on X.

Proof. The proof is similar to that of Theorem 5.10 and is therefore omitted. □

Theorem 5.12. *Let E be a hesitant fuzzy equivalence on X. The following statements are equivalent:*

(1) *E is a solution to the system ($hs^{2,9}$).*

(2) *E is a solution to the system ($hs^{2,3}$).*

(3) *E is a solution to the system ($hs^{2,1}$) and ($hs^{2,2}$).*

(4) *E is a solution to the system (5.5) and (5.7).*

(5) *E is a solution to the system*

$$U \subseteq \phi^e_{r}(U),$$

where U is an unknown hesitant fuzzy relation on X.

Proof. This is an immediate consequence of Theorems 5.10 and 5.11. □

5.3 The greatest solution to a homotypical weakly linear system

In this section, we present effective algorithms for computing the greatest solution to the homotypical weakly linear systems introduced in the preceding sections, contained in a given hesitant fuzzy relation. The proposed algorithms are based on a fixed-point approach, allowing the greatest solution to be obtained through an iterative procedure. More precisely, the set of all hesitant fuzzy relations on X, ordered by set inclusion, forms a complete lattice, allowing us to define operators that are monotone self-maps on this lattice. Consequently, the solutions to these systems can be characterized as fixed points (or pre-fixed points) of the corresponding operators.

The same approach applies to computing the greatest solutions to heterotypical hesitant weakly linear systems. However, in this dissertation, we restrict our consideration to the homotypical case.

Let X be a non-empty set and let $\phi : \mathcal{R}(X) \to \mathcal{R}(X)$ be an isotone function. Following the terminology of [101], we say that hesitant fuzzy relation R on X is called a *fixed point* of ϕ if $R = \phi(R)$, a *pre-fixed point* of ϕ if $\phi(R) \subseteq R$ and a *post-fixed point* of ϕ if $R \subseteq \phi(R)$. The Knaster-Tarski fixed point theorem, formulated for isotone operators on complete lattices, states that the sets of fixed, pre-fixed, and post-fixed points each form complete lattices, with the least fixed point coinciding with the least pre-fixed point and the greatest fixed point coinciding with the greatest post-fixed point.

In order to compute the greatest solution to the WLS contained in a given hesitant fuzzy relation, we apply the Kleene fixed point theorem. Specifically, the theorem guarantees that the greatest fixed point of a meet-continuous operator ϕ can be obtained as the meet of the descending Kleene sequence, defined by

$$\begin{aligned} &R_1 = I, \\ &R_{k+1} = \phi(R_k), \quad k \in \mathbb{N}, \end{aligned} \tag{5.11}$$

where $\mathbf{I}$ denotes the full hesitant fuzzy relation on X. This sequence is descending because ϕ is isotone and $\phi(\mathbf{I}) \subseteq \mathbf{I}$. If ϕ is only isotone (not necessarily meet-continuous), we have:

$$\hat{R} \subseteq \bigcap_{k \in \mathbb{N}} R_k, \tag{5.12}$$

where $\hat{R}$ denotes the greatest fixed point of ϕ. If both the underlying structure of

membership values and X are finite, then the lattice $\mathcal{R}(X)$ is also finite and the sequence $\{R_k\}_{k\in\mathbb{N}}$ is necessarily finite. Hence, we have a finite descending sequence, that stabilizes after finite number of steps. Its least element coincides with $\hat{R}$ and, consequently, the equality in (5.12) holds.

Building on the results of fixed-point theory, we now turn to statements outlining algorithms for finding the greatest solutions to the homotypical weakly linear systems considered here.

Definition 5.1. A hesitant fuzzy relation $R \in Q^{X\times X}$ is called *image-finite* if, for each $x \in X$, the set $\mathrm{Im}(R) = \{y \in X \mid R(x, y) \neq \emptyset\}$ is finite. The sequence $\{R_k\}_{k\in\mathbb{N}}$ is said to be *image-finite* if the set $\bigcup_{k\in\mathbb{N}} \mathrm{Im}(R_k)$ is finite.

Definition 5.2. The function $\phi : \mathcal{R}(X) \mapsto \mathcal{R}(X)$ is called *image-localized* if there exists a finite subset $Q_1 \subseteq Q$ such that, for every $R \in \mathcal{R}(X)$,

$$\mathrm{Im}(\phi(R)) \subseteq \langle Q_1 \cup \mathrm{Im}(R)\rangle,$$

where $\langle Q_1 \cup \mathrm{Im}(R)\rangle$ denotes the subalgebra of $\mathcal{Q}$ generated by $Q_1 \cup \mathrm{Im}(R)$.

In the following, we present a procedure for computing the greatest post-fixed point of ϕ contained in a given hesitant fuzzy relation $W \in \mathcal{R}(X)$, where X is a finite set, that is, the greatest solution to the system:

$$U \subseteq \phi(U), \quad U \subseteq W. \tag{5.13}$$

For this purpose, we define the following sequence:

$$\begin{aligned} &R_1 = W, \\ &R_{k+1} = R_k \cap \phi(R_k), \quad k \in \mathbb{N}. \end{aligned} \tag{5.14}$$

Let $\hat{R}$ be the greatest post-fixed point of ϕ contained in W. It is straightforward to verify that:

$$\hat{R} \subseteq \bigcap_{k\in\mathbb{N}} R_k. \tag{5.15}$$

We next analyze conditions under which equality in (5.14) holds.

Lemma 5.1. *Let X be a finite set and let $\{R_k\}_{k\in\mathbb{N}}$ be the sequence of hesitant fuzzy relations on X defined by (5.14). If the sequence $\{R_k\}_{k\in\mathbb{N}}$ is image-finite, then it is finite.*

Proof. Let $\mathcal{D}$ denote the subalgebra generated by $\bigcup_{k\in\mathbb{N}} \mathrm{Im}(R_k)$ and let D denote its carrier set. By assumption, $\{R_k\}_{k\in\mathbb{N}}$ is image-finite, that is, D is a finite set. Consequently, the set $D^{X\times X}$ of all hesitant fuzzy relations on X with membership values in $\mathcal{D}$ is also finite. Since $\{R_k\}_{k\in\mathbb{N}} \subseteq D^{X\times X}$, it can take only finitely many distinct values.

Moreover, the sequence $\{R_k\}_{k\in\mathbb{N}}$ is descending by definition, and hence it stabilizeS after finitely many steps. Therefore, the sequence $\{R_k\}_{k\in\mathbb{N}}$ is finite. □

Theorem 5.13. *Assume that the function ϕ is image-localized, let W be a hesitant fuzzy relation on X, and let $\{R_k\}_{k\in\mathbb{N}}$ be the sequence of hesitant fuzzy relations on X defined by (5.14). Then:*

$$\bigcup_{k\in\mathbb{N}} \mathrm{Im}(R_k) \subseteq \langle Q_1 \cup \mathrm{Im}(W)\rangle. \tag{5.16}$$

In particular, if $\langle Q_1 \cup \phi(\mathrm{Im}(W))\rangle$ is a finite subalgebra of $\mathcal{Q}$, then the sequence $\{R_k\}_{k\in\mathbb{N}}$ is finite.

Proof. We prove the claim by induction on k. For $k = 1$, we have $R_1 = W$ by definition, and therefore the following holds:

$$\mathrm{Im}(R_k) \subseteq \langle Q_1 \cup \mathrm{Im}(W)\rangle. \tag{5.17}$$

Assume that (5.17) holds for some $k \in \mathbb{N}$. Then $X \cup \mathrm{Im}(R_k) \subseteq \langle X \cup \mathrm{Im}(W)\rangle$ holds, which implies $\langle Q_1 \cup \mathrm{Im}(R_k)\rangle \subseteq \langle Q_1 \cup \mathrm{Im}(W)\rangle$. In addition, since ϕ is image-localized, we have

$$\mathrm{Im}(\phi(R_k)) \subseteq \langle Q_1 \cup \mathrm{Im}(R_k)\rangle \subseteq \langle Q_1 \cup \mathrm{Im}(W)\rangle.$$

Since, $\langle Q_1 \cup \mathrm{Im}(W)\rangle$ is a subalgebra of $\mathcal{Q}$, it is closed under finite meets. Moreover, since $\mathrm{Im}(R_k) \subseteq \langle Q_1 \cup \mathrm{Im}(W)\rangle$, $\mathrm{Im}(R_k) \subseteq \langle Q_1 \cup \mathrm{Im}(W)\rangle$ and $R_{k+1} = R_k \cap \phi(R_k)$, we obtain that $\phi(R_{k+1}) \subseteq \langle Q_1 \cup \mathrm{Im}(W)\rangle$. Thus, by induction, statement (5.17) holds for every $k \in \mathbb{N}$, which immediately implies (5.16).

Hence, if $\langle Q_1 \cup \mathrm{Im}(W)\rangle$ is finite, the sequence $\{R_k\}_{k\in\mathbb{N}}$ is finite as a direct consequence of (5.16) and Lemma 5.1. □

In what follows, we analyze the greatest post-fixed points of the functions ϕ^f_{rr}, ϕ^f_{lr}, and ϕ^f_r introduced in (5.4), and ϕ^e_{rr}, ϕ^e_{lr}, and ϕ^e_r introduced in (5.10), assuming that the membership values belong to a commutative power set quantale. For this purpose, we assume that X and I are finite sets, and hence $\{A_i\}_{i\in I}$ is a finite family of hesitant fuzzy relations on X.

Theorem 5.14. *The functions ϕ^f_{rr}, ϕ^f_{lr}, ϕ^f_r, ϕ^e_{rr}, ϕ^e_{lr}, and ϕ^e_r are isotone and image-localized.*

Proof. We first consider ϕ^f_{rr}. To prove isotonicity of ϕ^f_{rr}, let $F, G \in \mathcal{F}(X)$ be such that $F \subseteq G$. By Lemma 2.8, we obtain $F \circ G = G$. Let $i \in I$ and $x, y \in X$ be arbitrary elements. Then, by (1.37) for all z, t we obtain:

$$(A_i \circ F)(y,z) \to (A_i \circ F)(x,z) \subseteq (A_i \circ F)(y,z) \otimes G(z,t) \to (A_i \circ F)(x,z) \otimes G(z,t).$$

Next, by definition (2.75) of the operation $\oslash$ and by property (1.48), we have:

$$\begin{aligned}
(A_i \oslash F)(x,y) &\subseteq \bigcap_{z \in X} (A_i \circ F)(y,z) \to (A_i \circ F)(x,z) \\
&\subseteq \bigcap_{z \in X} \Big((A_i \circ F)(y,z) \otimes G(z,t) \to (A_i \circ F)(x,z) \otimes G(z,t) \Big) \\
&\subseteq \Big(\bigcup_{z \in X} (A_i \circ F)(y,z) \otimes G(z,t) \Big) \to \Big(\bigcup_{z \in X} (A_i \circ F)(x,z) \otimes G(z,t) \Big) \\
&= (A_i \circ F \circ G)(y,t) \to (A_i \circ F \circ G)(x,t) \\
&= (A_i \circ G)(y,t) \to (A_i \circ G)(x,t).
\end{aligned}$$

Since $t \in X$ is arbitrary, it follows that:

$$(A_i \oslash F)(x,y) \subseteq \bigcap_{t \in X} (A_i \circ G)(y,t) \to (A_i \circ G)(x,t) = (A_i \oslash G)(x,y)$$

for all $x, y \in X$ and, hence, $A_i \oslash F \subseteq A_i \oslash G$. Consequently:

$$\phi^f_{rr}(F) = \bigcap_{i \in I} A_i \oslash F \subseteq \bigcap_{i \in I} A_i \oslash G,$$

that is, ϕ^f_{rr} is isotone.

For any $F \in \mathcal{F}(X)$, we have that $\mathrm{Im}(\phi^f_{rr}(F)) \subseteq \langle Q_1 \cup \mathrm{Im}(F) \rangle$, where $Q_1 = \bigcup_{i \in I} A_i$. Since X and I are finite sets by assumption, then ϕ^f_{rr} is image-localized.

The proofs for the remaining functions are analogous. □

In the following we formulate two theorems for determining the greatest solution to the system $U \subseteq \phi(U)$ contained in a given hesitant fuzzy relation W on X. In the first result W is assumed to be a hesitant fuzzy quasi-order on X and $\phi \in \{\phi^f_{rr}, \phi^f_{lr}, \phi^f_r\}$, while in the second theorem W is assumed to be hesitant fuzzy equivalence on X and $\phi \in \{\phi^e_{rr}, \phi^e_{lr}, \phi^e_r\}$. The proofs follow directly from Theorems 5.13 and 5.14.

Theorem 5.15. *Let ϕ be one of the functions ϕ^f_{rr}, ϕ^f_{lr}, or ϕ^f_r, and let W be a given hesitant fuzzy quasi-order on X. Let $\{R_k\}_{k\in\mathbb{N}}$ be the sequence of hesitant fuzzy relations on X defined by (5.14). If*

$$\langle \bigcup_{i\in I} \mathrm{Im}(A_i) \cup \mathrm{Im}(W)\rangle$$

is a finite subalgebra of $\mathcal{Q}$, the sequence $\{R_k\}_{k\in\mathbb{N}}$ stabilizes after finitely many steps.

Moreover, if k is the smallest natural number such that $R_k = R_{k+1}$, then R_k is the greatest solution to the system

$$U \subseteq \phi(U), \qquad U \subseteq W.$$

Theorem 5.16. *Let ϕ be one of the functions ϕ^e_{rr}, ϕ^e_{lr}, or ϕ^e_r, and let W be a given hesitant fuzzy quasi-order on X. Let $\{R_k\}_{k\in\mathbb{N}}$ be the sequence of hesitant fuzzy relations on X defined by (5.14). If*

$$\langle \bigcup_{i\in I} \mathrm{Im}(A_i) \cup \mathrm{Im}(W)\rangle$$

is a finite subalgebra of $\mathcal{Q}$, the sequence $\{R_k\}_{k\in\mathbb{N}}$ stabilizes after finitely many steps.

Moreover, if k is the smallest natural number such that $R_k = R_{k+1}$, then R_k is the greatest solution to the system

$$U \subseteq \phi(U), \qquad U \subseteq W.$$

Example 5.1. Let $X = \{x_1, x_2\}$ be a universal set and let $\mathcal{Q}$ be the Gödel power set quantale, defined in Example 1.4. Let A_1 and A_2 be hesitant fuzzy relations on X over $\mathcal{Q}$, given by,

$$A_1 = \begin{bmatrix} \{1\} & \{0.2\} \\ \emptyset & \{0.1\} \end{bmatrix}, \quad A_2 = \begin{bmatrix} \{1\} & \emptyset \\ \emptyset & \emptyset \end{bmatrix}.$$

Using the described method, we compute the greatest solution to the system ($hs^{2,1}$), i.e., the greatest right regular hesitant fuzzy quasi-order F^{rr} on X. We start from the full hesitant fuzzy relation I on X over $\mathcal{Q}$, and construct the sequence defined

by (5.14). We obtain,

$$R_1^{rr} = I,$$
$$R_2^{rr} = I \cap \phi_{rr}^f(I) = \begin{bmatrix} [0,1] & [0,1] \\ [0,1] & [0,1] \end{bmatrix} \cap \begin{bmatrix} [0,1] & [0,1] \\ \emptyset & [0,1] \end{bmatrix} = \begin{bmatrix} [0,1] & [0,1] \\ \emptyset & [0,1] \end{bmatrix}.$$

Since $R_3^{rr} = R_2^{rr}$, the sequence stabilizes at R_2^{rr}. Hence, $F^{rr} = R_2^{rr}$.

Similarly, to determine the greatest left regular hesitant fuzzy quasi-order F^{lr} on X, we construct the sequence defined by (5.14) with ϕ_{lr}^f, i.e.,

$$R_1^{lr} = I,$$
$$R_2^{lr} = I \cap \phi_{lr}^f(I) = \begin{bmatrix} [0,1] & [0,1] \\ [0,1] & [0,1] \end{bmatrix} \cap \begin{bmatrix} [0,1] & \emptyset \\ [0,0.2] & [0,1] \end{bmatrix} = \begin{bmatrix} [0,1] & \emptyset \\ [0,0.2] & [0,1] \end{bmatrix}.$$

In this case $R_3^{lr} = R_2^{lr}$, which shows that the sequence has stabilized at R_2^{lr} and $F^{lr} = R_2^{lr}$. Applying the same procedure with ϕ_r^f, we obtain the greatest regular hesitant fuzzy quasi-order F^r on X,

$$R_1^r = I,$$
$$R_2^r = I \cap \phi_r^f(I) = \begin{bmatrix} [0,1] & [0,1] \\ [0,1] & [0,1] \end{bmatrix} \cap \begin{bmatrix} [0,1] & \emptyset \\ \emptyset & [0,1] \end{bmatrix} = \begin{bmatrix} [0,1] & \emptyset \\ \emptyset & [0,1] \end{bmatrix}.$$

For the resulting sequence, $R_3^r = R_2^r$, so the sequence stabilizes at R_2^r. Consequently, $F^r = R_2^r$.

The same procedure can be applied to the operators ϕ_{rr}^e, ϕ_{lr}^e, and ϕ_r^e to compute the greatest right regular, left regular, and regular hesitant fuzzy equivalences, respectively. In all three cases, the corresponding sequences stabilize after the second iteration, and we obtain

$$E^{rr} = E^{lr} = E^r = \begin{bmatrix} [0,1] & \emptyset \\ \emptyset & [0,1] \end{bmatrix}.$$

Theorems 5.15 and 5.16 complete the study of the greatest solutions to the considered weakly linear systems in the hesitant fuzzy setting. Together with the preceding results, they demonstrate that the approach developed in this chapter provides effective methods for computing the greatest solutions to weakly linear

systems of hesitant fuzzy relation inequalities and equations under suitable finiteness conditions.

One possible direction for further research suggested by the results of this dissertation is to investigate their connections with simulation and bisimulation notions in the hesitant fuzzy setting. Bisimulations are widely used in the study of state-based systems, including fuzzy automata (cf. [26, 27]), fuzzy transition systems (cf. [131]), fuzzy Kripke models (cf. [42, 113, 114]), and weighted networks with weights in a quantale (cf. [111]), where they provide a natural framework for comparing system behavior.

Several equations and inequalities studied in this dissertation are analogous to those occurring in the definitions of simulations and bisimulations for fuzzy automata and fuzzy Kripke models. For instance, inequality ($hs^{1.3}$) is analogous to the second condition in the definition of simulation for fuzzy automata in [26] and for fuzzy Kripke models in [114]. This observation suggests that the results developed in this dissertation may therefore provide useful tools for investigating systems of hesitant fuzzy relation equations and inequalities arising in the study of simulation- and bisimulation-like notions on suitable hesitant fuzzy structures. This approach could potentially lead to new notions of bisimulation in the hesitant fuzzy setting, conceptually different from the approximate bisimulations previously studied for fuzzy automata [110] and fuzzy Kripke models [115, 116].

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Her academic and research activities are focused on various topics in mathematics and computer science, particularly mathematical logic, fuzzy and hesitant fuzzy relations, relation inequalities and equations, information technology, and mathematics education. Her research work has resulted in several scientific publications and research papers.

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Наслов дисертације:

Неодлучне фази релацијске неједначине и једначине над кванталом подскупова полугрупе

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Scientific field, **SF**:	**mathematics**
Scientific discipline, **SD**:	**Algebra and Mathematical Logic**
Subject/Key words, **S/KW**:	**Hesitant fuzzy sets, hesitant fuzzy relations, hesitant fuzzy relation inequalities, hesitant fuzzy relation equations**
UDC	**519.713:510.64+512.13/.53(043.)**
Holding data, **HD**:	**library**
Note, **N**:	

Abstract, **AB**:		The main objective of the dissertation is to develop a new algebraic framework for hesitant fuzzy sets based on the power set quantale of a semigroup. The research employs algebraic and order-theoretic methods, as well as constructive and algorithmic approaches, together with abstraction and generalization. Within this framework, hesitant fuzzy equivalence relations and quasi-orders are characterized, including the corresponding equivalence classes and partitions. Hesitant fuzzy relational inequalities and equations involving one or two unknown hesitant fuzzy sets are then studied, and algorithms for determining their greatest solutions are developed. The study is further extended to weakly linear systems, including both homotypical systems, with relations on a single set, and heterotypical systems, with relations between two, possibly different, sets. Effective procedures for determining greatest solutions of homotypical weakly linear systems are also established. The obtained results provide a general theoretical setting for extending classical and fuzzy relational methods to the context of hesitant fuzzy set theory, while preserving the residuation property.
Accepted by the Scientific Board on, **ASB**:		14 September 2026
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Defended Board, **DB**:	President:	
	Member:	
	Member:	
	Member:	
	Member:	
	Member, Mentor:	

Прилог 4/1

ПРИРОДНО - МАТЕМАТИЧКИ ФАКУЛТЕТ
НИШ

КЉУЧНА ДОКУМЕНТАЦИЈСКА ИНФОРМАЦИЈА

Редни број, **РБР**:	
Идентификациони број, **ИБР**:	
Тип документације, **ТД**:	монографска
Тип записа, **ТЗ**:	текстуални / графички
Врста рада, **ВР**:	докторска дисертација
Аутор, **АУ**:	Марија Р. Тасић
Ментор, **МН**:	Мирослав Д. Ћирић
Наслов рада, **НР**:	Неодлучне фази релацијске неједначине и једначине над кванталом подскупова полугрупе
Језик публикације, **ЈП**:	енглески
Језик извода, **ЈИ**:	енглески и српски
Земља публиковања, **ЗП**:	Србија
Уже географско подручје, **УГП**:	Србија
Година, **ГО**:	2026.
Издавач, **ИЗ**:	ауторски репринт
Место и адреса, **МА**:	Ниш, Вишеградска 33.
Физички опис рада, **ФО**: (поглавља/страна/ цитата/табела/слика/графика/прилога)	5 поглавља; 181 страна; 142 референце
Научна област, **НО**:	математика
Научна дисциплина, **НД**:	алгебра и математичка логика
Предметна одредница/Кључне речи, **ПО**:	Неодлучни фази скупови, неодлучне фази релације, неодлучне фази релацијске неједначине, неодлучне фази релацијске једначине
УДК	519.713:510.64+512.13/.53(043.)
Чува се, **ЧУ**:	библиотека
Важна напомена, **ВН**:	

Извод, **ИЗ**:	Основни циљ докторске дисертације јесте развој новог алгебарског оквира за неодлучне фази скупове заснованог на кванталу подскупова полугрупе. У истраживању се користе алгебарске и теоријско-уређајне методе, као и конструктивни и алгоритамски приступи уз примену апстракције и генерализације. У оквиру овог приступа карактеришу се неодлучне фази релације еквиваленције и квази-уређења, као и одговарајуће класе еквиваленције и партиције. Затим, проучавају се неодлучне фази релацијске неједначине и једначине које укључују један или два непозната неодлучна фази скупа и развијају се алгоритми за одређивање њихових највећих решења. Истраживање се даље проширује на слабо линеарне системе, укључујући хомогене системе, са релацијама на истом скупу, и хетерогене системе, са релацијама између два различита скупа. На крају, развијају се ефикасне процедуре за одређивање највећих решења хомогених слабо линеарних система. Добијени резултати пружају општи теоријски оквир за проширивање класичних и фази-релацијских метода на контекст теорије неодлучних фази скупова, уз очување својства резидуације.
Датум прихватања теме, **ДП**:	14.09.2026.
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Чланови комисије, **КО**: Председник:	
Члан:	
Члан:	
Члан:	
Члан:	
Члан, ментор:	