UNIVERSITY OF NIŠ
FACULTY OF SCIENCE AND MATHEMATICS
DEPARTMENT OF MATHEMATICS

Jelena S. Milošević

Asymptotic analysis of the solutions of nonlinear differential equations and Karamata's regularly varying functions

PhD thesis

Niš, 2015

UNIVERZITET U NIŠU
PRIRODNO-MATEMATIČKI FAKULTET
DEPARTMAN ZA MATEMATIKU

Jelena S. Milošević

Asimptotska analiza rešenja nelinearnih diferencijalnih jednačina i Karamatine pravilno promenljive funkcije

Doktorska disertacija

Niš, 2015

To the loving memory of my mother,
Milanka Živković

Supervisor	**Jelena Manojlović**
	Full Professor Faculty of Sciences and Mathematics University of Niš, Serbia

Members of Commitee

Прилог 4/1

	ПРИРОДНО - МАТЕМАТИЧКИ ФАКУЛТЕТ **НИШ**
	КЉУЧНА ДОКУМЕНТАЦИЈСКА ИНФОРМАЦИЈА

Редни број, **РБР**:	
Идентификациони број, **ИБР**:	
Тип документације, **ТД**:	монографска
Тип записа, **ТЗ**:	текстуални
Врста рада, **ВР**:	докторска дисертација
Аутор, **АУ**:	Јелена С. Милошевић
Ментор, **МН**:	Јелена В. Манојловић
Наслов рада, **НР**:	Асимптотска анализа решења нелинеарних диференцијалних једначина и Караматине правилно променљиве функције
Језик публикације, **ЈП**:	енглески
Језик извода, **ЈИ**:	енглески
Земља публиковања, **ЗП**:	Србија
Уже географско подручје, **УГП**:	Србија
Година, **ГО**:	2015.
Издавач, **ИЗ**:	ауторски репринт
Место и адреса, **МА**:	Ниш, Вишеградска 33.
Физички опис рада, **ФО**: (поглавља/страна/ цитата/табела/слика/графика/прилога)	168 стр.
Научна област, **НО**:	математике науке (математичка анализа)
Научна дисциплина, **НД**:	диференцијалне једначине
Предметна одредница/Кључне речи, **ПО**:	правилно променљиве функције, асимптотско понашање решења, позитивна решења, нелинеарне диференцијалне једначине
УДК	517.957 (043.3)
Чува се, **ЧУ**:	библиотека
Важна напомена, **ВН**:	
Извод, **ИЗ**:	У овој дисертацији аутор даје темељне и комплетне информације о постојању правилно променљивих решења нелинеарних диференцијалних једначина другог и четвртог реда, као и о асимиптотском понашању тих решења у бесконачности.
Датум прихватања теме, **ДП**:	24.11.2014.
Датум одбране, **ДО**:	
Чланови комисије, **КО**: Председник:	
Члан:	
Члан, ментор:	

Прилог 4/2

ПРИРОДНО - МАТЕМАТИЧКИ ФАКУЛТЕТ
НИШ

KEY WORDS DOCUMENTATION

Accession number, **ANO**:	
Identification number, **INO**:	
Document type, **DT**:	monograph
Type of record, **TR**:	textual
Contents code, **CC**:	doctoral dissertation
Author, **AU**:	Jelena S. Milošević
Mentor, **MN**:	Jelena V. Manojlović
Title, **TI**:	Asymptotic analysis of the solutions of nonlinear differential equations and Karamata's regularly varying functions
Language of text, **LT**:	English
Language of abstract, **LA**:	English
Country of publication, **CP**:	Serbia
Locality of publication, **LP**:	Serbia
Publication year, **PY**:	2015
Publisher, **PB**:	author's reprint
Publication place, **PP**:	Niš, Višegradska 33.
Physical description, **PD**: (chapters/pages/ref./tables/pictures/graphs/appendixes)	168 p.
Scientific field, **SF**:	mathematics
Scientific discipline, **SD**:	differential equations
Subject/Key words, **S/KW**:	regularly varying functions, asymptotic behavior of solutions, positive solutions, nonlinear differential equations
UC	517.957 (043.3)
Holding data, **HD**:	library
Note, **N**:	
Abstract, **AB**:	This dissertation is concerned with thorough and complete information is acquired about the existence of regularly varying solutions of second and fourth order nonlinear differential equations and their accurate asymptotic behavior at infinity.
Accepted by the Scientific Board on, **ASB**:	24.11.2014.
Defended on, **DE**:	
Defended Board, **DB**: President:	
Member:	
Member, Mentor:	

Contents

Summary

Differential equations are mainly used to describe the change of quantities or behavior of certain systems in applications. When linear differential equations are used, there exist several methods, such as Laplace transform method, that can be used to solve the equation analytically. If the equation is nonlinear it is, in general, not possible to write the solution using formulas. In that case, the numerical approximation approach is the only way to find the solution. However, in most applications in biology, chemistry and physics, one is not interested in the analytic form of the solution, but is more interested in so-called qualitative properties of the solution, such as periodicity, stability, oscillation, asymptotic behavior of nonoscillatory solutions and so on. If these questions can be answered without solving the differential equation, especially when analytical solutions are unavailable, we can still get a very good understanding of the system, which is in fact the main objective of qualitative analysis of differential equations. The foundations of the qualitative theory of differential equations were laid at the end of the 19th century by H. Poincaré and A.M. Lyapunov. Anyway, intensive development of this discipline began only in the last forty years. During that time, the new methods were developed and many important results were obtained.

The equation of the form $x''(t) + q(t)|x(t)|^{\lambda}\mathrm{sgn}x(t) = 0, \lambda \neq 1$ is probably the most studied nonlinear second order differential equation. It is also known as the Emden-Fowler, or Thomas-Fermi equation, depending on the sign of the coefficient $q(t)$. The equation of this form has attracted the attention of R. Emden at the end of nineteenth century in the early theories of the dynamics of gases in astrophysics, while E. Fermi and L.H.Tomas used it in their works on the study of the distribution of electrons in heavy atoms, during the thirties of the last century. The classical Thomas-Fermi atomic model is described by the following nonlinear singular boundary value problem

$$x'' = \frac{1}{\sqrt{t}}x^{3/2}, \qquad x(0) = 1,\ x(\infty) = 0,$$

(see Thomas [69] and Fermi [12]). The equation of this type also appears in the study of fluid mechanics, relativistic mechanics, nuclear physics, as well as in the study of chemical reactions of systems.

The equations

$$\left(p(t)|x'(t)|^{\alpha-1}\, x'(t)\right)' + q(t)|x(t)|^{\beta-1}\, x(t) = 0, \tag{E$_1$}$$

$$(p(t)\varphi(|x'(t)|)\,\operatorname{sgn} x'(t))' + q(t)\psi(x(t)) = 0, \tag{E$_2$}$$

$$\left(p(t)|x''(t)|^{\alpha-1}\, x''(t)\right)'' + q(t)|x(t)|^{\beta-1}\, x(t) = 0 \tag{E}$$

are considered to be a natural generalization of Emden-Fowler equation.

The properties of solutions of (E_1), such as existence, uniqueness, continuability and oscillatory and nonoscillatory properties of solutions have been investigated in detail (see monographs [9], [28], [57], [58] and [3–8, 24, 25, 29, 59–62, 68, 70–72]). Unlike the equation (E_1), the more general equation (E_2) has been far less investigated under certain assumptions on nonlinear functions φ, ψ. Oscillation criteria, as well as the classification and existence of the nonoscillatory solutions have been treated in [11, 41, 42]. Studying oscillation and asymptotic behavior of nonoscillatory solutions for the fourth order nonlinear equation (E) was initiated by Wu [73] and Kamo and Usami [23] in 2002. and afterwards developed in [26, 38, 43, 46, 63, 64, 74].

A study of the asymptotic behavior of solutions of nonlinear differential equations is accomplished by introducing an appropriate classification of solutions. More precisely all continuable solutions are divided in several disjoint subsets, whereby it is desirable to fully characterize these subsets by means of necessary and sufficient integral conditions which involve only coefficients of equation. For some subsets this problem is solvable with relative ease, but there are always some "difficult" solutions for which only either necessary or sufficient conditions have been already established. But even if it is possible to establish necessary and sufficient conditions for the existence of these solutions, determining their precise asymptotic behavior is a notoriously difficult problem, under assumption that coefficients are continuous functions.

The recent development of asymptotic analysis of differential equations by the means of regular variation (initiated by the monograph of Marić [47]), suggested to investigate the problem in the framework of regularly varying functions (also known as Karamata functions). Since the precise asymptotic behavior of solutions is still an open problem for a wide class of nonlinear high-order differential equations, its study in the framework of regular variation became the subject of the research in this dissertation, whereby equations (E_1), (E_2) and (E) are under consideration. It is shown that assuming that coefficients are regularly varying, not only necessary and sufficient conditions for the existence of all possible types of regularly varying solutions can be established, but also that the precise information can be acquired about the asymptotic behavior at infinity of these solutions. All the results of this dissertation generalize, extend or improve analogous ones that exist in the literature.

The notion of regular variation was introduced by a Serbian mathematician Jovan Karamata (1902-1967) in 1930 (see [27]). The theory of regular variation which is basically a chapter of mathematical analysis, found it's application in many different mathematical fields such as analytic number theory, complex analysis, probability theory, game theory and differential equations. So, further development of regular variation theory was carried out by the so-called Karamata's school (Avakumović, Aljančić, Bašajski, Bojanić, Tomić, Marić, Adamović, Arandjelović), as well as Bingham, Goldie, Teugels, Seneta, Geluk, de Haan and many others. Even today, Karamata is one of the most frequently cited Serbian mathematicians.

The first paper connecting regular variation and the differential equations is the one of V.G. Avakumović [1] in 1947. However, his paper did not attract much attention – regularly varying functions were totally distant from the theory of differential equations at that time, until about thirty years later, when Marić and Tomić further extended and developed the study of asymptotic of solutions of differential equations via regular variation [48–52]. After the monograph of Marić [47] appeared, numerous papers in that spirit have been published, dealing also with some more general differential equations of the second order, the ones of higher orders and some systems, functional differential equations, difference and dynamic ones and also some partial differential ones. Many recent interesting contributions, devoted in particular to the study of Emden-Fowler type equations

$$x''(t) \pm q(t)x(t)^{\gamma} = 0 \quad \text{and} \quad x''(t) \pm q(t)\phi(x(t)) = 0 \tag{A}$$

in the framework of regular variation are due to Jaroš, Kusano, Manojlović, Marić, Tanigawa (see for instance [30, 31, 35, 40, 45] and the references therein). Further, asymptotic behavior of positive solutions of the fourth order nonlinear differential equations

$$x^{(4)}(t) \pm q(t)x(t)^{\gamma} = 0 \quad \text{and} \quad \left(|x''(t)|^{\alpha-1} x''(t)\right)'' \pm q(t)|x(t)|^{\beta-1} x(t) = 0 \tag{B}$$

in the framework of regular variation has been investigated in [32,33,39]. Other important works related to systems and high-order differential equations were carried out by Jaroš, Kusano, Manojlović, Matucci and Řehák [16–20, 34, 53, 65].

That the class of classical Karamata functions is well suited for the study of the linear differential equation

$$x''(t) + q(t)x(t) = 0$$

has been shown by Marić, Tomić [51] and Howard, Marić [14]. However, the class of classical Karamata functions is not sufficient to properly describe the asymptotic behavior of positive solutions of the self-adjoint differential equation

$$(p(t)x'(t))' + q(t)x(t) = 0.$$

For this reason, Jaroš and Kusano introduced in [15] the class of generalized Karamata functions. Considering equations (E_1) and (E) as generalizations of equations (A) and (B), it is natural to expect that the class of the generalized regularly varying functions (or generalized Karamata functions) is an appropriate framework for asymptotic analysis, in order to make the dependence of solutions on the coefficient $p(t)$ clear. Therefore, the study of asymptotic behavior of solutions of equations (E_1) and (E) in the framework of generalized Karamata functions has been imposed as one of the main tasks of this dissertation.

The dissertation consists of four chapters. In the first chapter, some basic definitions and theorems are introduced, as well as the overview of regular variation theory.

In the second chapter Emden-Fowler second order differential equation (E_1) under two different integral conditions is studied. Papers [11, 41, 42] deal with a classification of nonoscillatory solutions, based on suitable integral criteria. Positive solutions are classified according to asymptotic behavior at infinity as dominant, intermediate and subdominant solutions and the necessary and sufficient conditions for the existence of dominant and subdominant solutions were obtained. As regards the existence of intermediate solutions for (E_1), although sufficient conditions can be obtained with relative ease, the problem of establishing necessary and sufficient conditions turns out to be extremely difficult to solve and thus, has been an open problem for a long time. Nevertheless, the problem has recently been solved by Kamo, Usami [25] and Naito [60]. The asymptotic behavior of dominant and subdominant solutions is obvious because they are asymptotic to a positive constant or to a positive constant multiplied by an appropriate function, while this is not the case of the so-called intermediate solutions. Therefore, the precise asymptotic formulas for all possible types of intermediate solutions, under the assumption that coefficients are generalized regularly varying functions, are given in this chapter. Because of the presence of general $p(t) \not\equiv 1$ in the differential operator of equation (E_1) and motivated by papers [15, 22] on second order linear and half-linear differential equations, we decided to choose the class of generalized regularly varying functions as the basic framework for our asymptotic analysis. Such a choice proves to be appropriate in the sense that complete analysis can be conducted for all possible generalized regularly varying solutions of equation (E_1) if $p(t)$ and $q(t)$ are assumed to be generalized regularly varying functions. *The results in Section 2.2 are original results published in [36].* The results in Section 2.4 are achieved in [21]. As a direct consequence of results from these two sections, in Section 2.5, under the assumption that coefficients are regularly varying functions in the sense of Karamata, overall structure of regularly varying solutions of (E_1) is established. Finally, Section 2.6 contains some illustrative examples.

Regularly varying functions can be understood as a (nontrivial) extension of functions asymptotically equivalent to power ones. Therefore, when considering the second order quasilinear differential equation (E_2) in the framework of regular variation it is natural to assume that nonlinearities φ and ψ are itself regular varying function. Thus, the third chapter is devoted to the study of positive solutions of (E_2) under two different integral conditions. If the coefficients are regularly varying functions, the asymptotic behavior of intermediate regularly varying solutions is determined and necessary and sufficient conditions for existence are established. *All results presented in this chapter are original, and are published in [54] and [56].*

Finally, the fourth chapter deals with the fourth order quasilinear differential equation (E). Under the certain integral conditions the existence of minimal and maximal solutions was considered in [73]. An additional condition gives two more solutions (see [64]), which are called, along with the previous two, the primitive solutions. The existence of these solutions was also established in [64]. However, neither the existence nor asymptotic behavior of intermediate solutions have not yet been studied in the existent literature and they are presented in this chapter. In Section 4.1 the detailed classification is done, and two more types of intermediate solutions are obtained. In Sections 4.2 and 4.4 the existence of intermediate solutions is determined, under the assumption that coefficients are positive continuous functions. Assuming that coefficients are generalized regularly varying functions, the precise asymptotic formulas of intermediate generalized regularly varying solutions are determined in Sections 4.3 and 4.5. *All of the results in Chapter 4 are original and are published in [37] and [55].*

At the end, I would like to express my sincere gratitude to my mentor Professor Jelena Manojlović for a great commitment during our joint research and writing of PhD thesis. I also want to thank my family for all their love and encouragement.

Rezime

Diferencijalne jednačine se u praksi najčešće upotrebljavaju za modeliranje ponašanja pojedinih sistema. U slučaju da se za to koriste linearne diferencijalne jednačine, one mogu biti analitički rešene primenom nekog od postojećih metoda, kao što je, na primer, metod Laplasovih transformacija. Ako je jednačina nelinearna, u opštem slučaju nije moguće dobiti analitički oblik njenog rešenja. Tada jedino numerički pristup omogućava pronalaženje rešenja. Ipak, u najvećem broju primena u biologiji, hemiji i fizici, analitičko rešenje nije od primarnog interesa. Najčešće su mnogo bitnija tzv. kvalitativna svojstva rešenja, kao što su periodičnost, stabilnost, oscilatornost, asimptotsko ponašanje neoscilatornih rešenja, itd. Ukoliko se ova svojstva mogu odrediti bez rešavanja jednačine, posebno kada sama rešenja i nije moguće analitički odrediti, onda se modelirani sistemi mogu veoma dobro razumeti i opisati. Stoga je ispitivanje ovakvih svojstava jedan od osnovnih zadataka kvalitativne analize diferencijalnih jednačina. Osnove ove teorije postavili su, još u XIX veku, H. Poincaré i A. M. Lyapunov. I pored toga, intenzivni razvoj ove matematičke discipline je započeo tek u poslednjih četrdesetak godina. U tom vremenskom periodu su razvijeni novi metodi i dobijeni mnogi važni rezultati.

Jednačina oblika $x''(t) + q(t)|x(t)|^{\lambda}\mathrm{sgn}x(t) = 0, \lambda \neq 1$ je, verovatno, najproučavanija nelinearna diferencijalna jednačina drugog reda. Poznata je i kao jednačina Emden–Fowler–a, odnosno Thomas–Fermi-a, u zavisnosti od znaka koeficijenta $q(t)$. Jednačina ovog oblika je privukla pažnju R. Emdena krajem devetanestog veka tokom rane faze razvoja teorije dinamike gasova u astrofizici, dok su je E. Fermi i L.H. Tomas koristili u svojim proučavanjima distribucije elektrona u teškim atomima tokom tridesetih godina prošlog veka. Klasičan Thomas–Fermi model atoma je opisan sledećom nelinearnom singularnom jednačinom

$$x'' = \frac{1}{\sqrt{t}}x^{3/2}, \qquad x(0) = 1,\ x(\infty) = 0,$$

(videti Thomas [69] i Fermi [12]). Jednačina ovog tipa se takodje pojavljuje u proučavanju mehanike fluida, relativističkoj mehanici, nuklearnoj fizici, kao i u proučavanju različitih hemijskih reakcija sistema.

Jednačine

$$\text{(E}_1\text{)} \qquad \big(p(t)|x'(t)|^{\alpha-1}\, x'(t)\big)' + q(t)|x(t)|^{\beta-1}\, x(t) = 0,$$

$$\text{(E}_2\text{)} \qquad (p(t)\varphi(|x'(t)|)\,\operatorname{sgn} x'(t))' + q(t)\psi(x(t)) = 0,$$

$$\text{(E)} \qquad \big(p(t)|x''(t)|^{\alpha-1}\, x''(t)\big)'' + q(t)|x(t)|^{\beta-1}\, x(t) = 0$$

se smatraju prirodnim generalizacijama jednačine Emden-Fowler.

Neka od svojstava rešenja jednačine (E_1), kao što su egzistencija, jedinstvenost, neprekidnost, oscilatornost i svojstva neoscilatornih rešenja, su detaljno proučena (videti monografije [9], [28], [57], [58], kao i [3–8, 24, 25, 29, 59–62, 68, 70–72]). Za razliku od jednačine (E_1), opštija jednačina (E_2) je mnogo manje izučavana pod specifičnim pretpostavkama za nelinearne funkcije φ i ψ. Kriterijumi oscilatornosti, kao i klasifikacija i postojanje neoscilatornih rešenja su izučavani u [11, 41, 42]. Proučavanje oscilatornosti i asimptotskog ponašanja neoscilatornih rešenja nelinearne diferencijalne jednačine četvrtog reda su inicirali Wu [73] i Kamo i Usami [23] 2002. godine, a kasnije je razvijeno u [26, 38, 43, 46, 63, 64, 74].

Osnovni zadatak u proučavanju asimptotskog ponašanja rešenja nelinearnih diferencijalnih jednačina je klasifikacija tih rešenja. Preciznije, sva produživa rešenja se dele u disjunktne skupove, koje je poželjno okarakterisati potrebnim i dovoljnim integralnim uslovima koji zavise od koeficijenata jednačine. Za neke od ovih skupova rešenje problema je relativno jednostavno, ali postoje i "teški" slučajevi za koje su poznati ili samo potrebni ili samo dovoljni uslovi. Medjutim, čak i kada je moguće odrediti potrebne i dovoljne uslove za postojanje ovih rešenja, odredjivanje njihovog asimptotskog ponašanja je ekstremno težak problem, pod pretpostavkom da su koeficijenti neprekidne funkcije.

Nedavni razvoj asimptotske analize diferencijalnih jednačina korišćenjem pravilno promenljivih funkcija (iniciran monografijom Marića, videti [47]), sugeriše proučavanje problema u okviru pravilno promenljivih funkcija (poznatih i kao Karamatine funkcije). Kako je odredjivanje asismptotskog ponašanja rešenja još uvek otvoren problem za široku klasu nelinearnih diferencijalnih jednačina višeg reda, proučavanje ovog problema u klasi pravilno promenljivih funkcija je postalo predmet izučavanja ove disertacije. Posmatrajući jednačine (E_1), (E_2) i(E) pokazujemo da se, pod pretpostavkom da su koeficijenti pravilno promenljive funkcije, mogu odrediti ne samo potrebni i dovoljni uslovi za postojanje pravilno promenljivih rešenja, već se može precizno utvrditi i njihovo asimptotsko ponašanje u beskonačnosti. Rezultati dati u ovoj disertaciji generalizuju, proširuju i poboljšavaju odgovarajuće, do sada poznate, rezultate.

Pojam pravilno promenljive funkcije je 1930. godine uveo srpski matematičar Jovan Karamata (1902–1967) (videti [27]). Ova teorija, koja je u suštini deo matematičke analize, je našla primenu u mnogim oblastima matematike, kao što su teorija brojeva, kompleksna analiza, teorija verovatnoće, teorija igara i teorija diferencijalnih jednačina. Dalji razvoj teorije pravilno promenljivih funkcija nastavili su pripadnici tzv. Karamatine škole (Avakumović, Aljančić, Bašajski, Bojanić, Tomić, Marić, Adamović, Arandjelović), kao i Bingham, Goldie, Teugels, Seneta, Geluk, de Haan i mnogi drugi. Čak i danas, Karamata je jedan od najcitiranijih srpskih matematičara.

Prvi rad koji povezuje pravilno promenljive funkcije i diferencijalne jednačine je delo V.G. Avakumovića [1] iz 1947. godine. Taj rad, medjutim, nije privukao previše pažnje – u to vreme teorija pravilno promenljivih funkcija nije primenjivana u teoriji diferencijalnih jednačina – sve do nekih trideset godina kasnije, kada su Marić i Tomić u svojim radovima [48–52] nastavili i dalje razvili istraživanje diferencijalnih jednačina koristeći pravilno promenljive funkcije. Posle pojavljivanja Marićeve monografije [47], objavljen je veliki broj radova u kojima je korišćen sličan pristup i koji su se bavili opštijim diferencijalnim jednačinama drugog reda, jednačinama višeg reda i sistemima, funkcionalnim diferencijalnim jednačinama, diferencnim i parcijalnim diferencijalnim jednačinama. Za neke od skorijih interesantnih rezultata, posvećenih jednačinama Emden–Fowler tipa

$$\text{(A)} \qquad x''(t) \pm q(t)x(t)^\gamma = 0 \quad \text{and} \quad x''(t) \pm q(t)\phi(x(t)) = 0$$

korišćenjem pravilno promenljivih funkcija, treba pogledati radove autora Jaroš, Kusano, Manojlović, Marić, Tanigawa (videti [30, 31, 35, 40, 45] i reference u njima). Dalje, asimptotsko ponašanje pozitivnih rešenja nelinearnih diferencijalnih jednačina četvrtog reda

$$\text{(B)} \qquad x^{(4)}(t) \pm q(t)x(t)^\gamma = 0 \quad \text{and} \quad \left(|x''(t)|^{\alpha-1}x''(t)\right)'' \pm q(t)|x(t)|^{\beta-1}x(t) = 0$$

korišćenjem pravilno promenljivih funkcija je ispitivano u [32, 33, 39]. Jaroš, Kusano, Manojlović, Matucci and Řehák su došli do rezultata vezanih za sisteme diferencijalnih jednačina i za diferencijalne jednačine višeg reda u [16–20, 34, 53, 65].

Marić i Tomić u [51], kao i Howard i Marić u [14], su pokazali da je klasa klasičnih Karamatinih funkcija pogodna za proučavanje linearne diferencijalne jednačine

$$x''(t) + q(t)x(t) = 0.$$

Ipak, ova klasa funkcija nije pogodna za odgovarajuće opisivanje asimptotskog ponašanja pozitivnih rešenja samoadjungovane diferencijalne jednačine

$$(p(t)x'(t))' + q(t)x(t) = 0.$$

Zbog toga su Jaroš i Kusano u [15] uveli klasu generalisanih Karamatinih funkcija. Ako se jednačine (E_1) i (E) posmatraju kao generalizacije jednačina (A) i (B), prirodno je očekivati da će klasa generalisanih pravilno promenljivih funkcija (generalsane Karamatine funkcije) biti pogodna za asimptotsku analizu, u cilju odredjivanja zavisnosti prirode rešenja od koeficijenta $p(t)$. Zato je to izučavanje postalo jedan od glavnih zadataka koji treba rešiti u ovoj disertaciji.

Disertacija se sastoji od četiri poglavlja. U prvom poglavlju date su osnovne definicije i teoreme koje se koriste kao alati u istraživanju, kao i pregled teorije pravilno promenljivih funkcija.

U drugom poglavlju je istraživana Emden–Fowler diferencijalna jednačina drugog reda (E_1) pod dva različita integralna uslova. Radovi [11, 41, 42] se bave klasifikacijom neoscilatornih rešenja, pod odgovarajućim integralnim kriterijumom. Pozitivna rešenja su klasifikovana u odnosu na asimptotsko ponašanje u beskonačnosti kao domnantna, uklještena i subdominantna rešenja i dati su potrebni i dovoljni uslovi za postojanje dominantnih i subdominantnih rešenja. Što se tiče postojanja uklještenih rešenja za (E_1), iako se dovoljni uslovi mogu dobiti relativno jednostavno, problem utvrdjivanja potrebnih i dovoljnih uslova je ekstremno težak i dugo je predstavljao otvoren problem. Problem su nedavno rešili Kamo, Usami [25] i Naito [60]. Asimptotsko ponašanje dominantnih i subdominantnih rešenja je očigledno jer se ona asimptotsko ponašaju kao konstantna funkcija ili kao konstantna funkcija pomnožena odgovarajućom funkcijom. Ovo, medjutim, nije slučaj sa uklještenim rešenjima. Zato u ovom poglavlju dajemo asimptotske formule za sve moguće tipove uklještenih rešenja, pod pretpostavkom da su koeficijenti generalisane pravilno promenljive funkcije. Zbog pojave opšte funkcije $p(t) \not\equiv 1$ u diferencijalnom operatoru jednačine (E_1), a motivisano radovima [15, 22] koji su vezani za linearne i polulinearne diferencijalne jednačine drugog reda, odlučili smo da za ovu analizu koristimo klasu generalisanih pravilno promenljivih funkcija. Ispostavlja se da je ovakav izbor odgovarajući u smislu da je moguće izvesti kompletnu analizu svih mogućih generalisanih pravilno promenljivih rešenja jednačine (E_1), ako pretpostavimo da su $p(t)$ i $q(t)$ generalisane pravilno promenljive funkcije. *Rezultati iz Sekcije 2.2 su originalni rezultati objavljeni u [36].* Rezultati iz Sekcije 2.4 su objavljeni u [21]. Kao direktna posledica rezultata iz ove dve sekcije, u Sekciji 2.5 je, pod pretpostavkom da su koeficijenti pravilno promenljive funkcije u Karamatinom smislu, opisana je kompletna struktura pravilno promenljivih rešenja jednačine (E_1). Konačno, Sekcija 2.6 sadrži neke ilustrativne primere.

Pravilno promenljive funkcije se mogu shvatiti kao (netrivijalno) uopštenje funkcija koje su asimptotski ekvivalentne stepenim funkcijama. Zbog toga, kada govorimo o kvazilinearnog diferencijalnoj jednačini drugog reda (E_2) u okviru teorije pravilno promenljivih funkcija, prirodno je pretpostaviti da su funkcije φ i ψ i same pravilno promenljive funkcije. Treće poglavlje je posvećeno izučavanju pozitivnih rešenja jednačine (E_2) pod dva integralna uslova. Ako su koeficijenti pravilno promenljive funkcije, precizno je utvrdjeno asimptotsko ponašanje uklještenih pravilno promenljivih rešenja, kao i potrebni i dovoljni uslovi za njihovo postojanje. *Svi rezultati u ovom poglavlju su originalni i objavljeni u [54] i [56].*

Na kraju, četvrto poglavlje se bavi kvazilinearnom diferencijalnom jednačinom četvrtog reda (E) . Pod odredjenim integralnim uslovima, postojanje minimalnih i maksimalnih rešenja je posmatrano [73]. Dodatni uslov daje još dva tipa rešenja (videti [64]), koja su nazvana, zajedno sa prethodna dva, primitivna rešenja. Postojanje ovih rešenja je takodje utvrdjeno u [64]. Medjutim, ni postojanje ni asimptotsko ponašanje uklještenih rešenja još uvek nisu razmatrani. U Sekciji 4.1 je data detaljna klasifikacija i dobijena su još dva tipa uklještenih rešenja. U Sekcijama 4.2 i 4.4 je utvrdjeno postojanje uklještenih rešenja, pod pretpostavkom da su koeficijenti pozitivne neprekidne funkcije. Pod pretpostavkom da su koeficijenti generalisane pravilno promenljive funkcije, u sekcijama 4.3 i 4.5 su odredjene asimptotske formule generalisanih pravilno promenljivih rešenja ove jednačine. *Svi rezultati u Poglavlju 4 su originalni i objavljeni su u [37] i [55].*

Chapter 1

Introduction

1.1 Basic concepts and theorems

First, we define some relations that are used later in the dissertation.

Definition 1.1.1 *For positive functions $f(t)$ and $g(t)$ we define the* asymptotic equivalence relation $\sim$ *as*

$$f(t) \sim g(t), \quad t \to \infty \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{g(t)}{f(t)} = 1.$$

Definition 1.1.2 *For positive functions $f(t)$ and $g(t)$ we define the* asymptotic similarity relation $\simeq$ *as*

$$f(t) \simeq g(t), \quad t \to \infty \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{g(t)}{f(t)} = const > 0.$$

Definition 1.1.3 *For positive functions $f(t)$ and $g(t)$ we define the* dominance relation $\prec$ *as*

$$f(t) \prec g(t), \quad t \to \infty \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{g(t)}{f(t)} = \infty.$$

Since we are interested in asymptotic behavior of nonoscillatory solutions of second and fourth order nonlinear differential equations, we give the definition of these solutions as well as of the positive and negative solutions.

Definition 1.1.4 *A solution $x(t)$ of* (E_1) *is said to be* nonoscillatory *if there exists $t_0 \in \mathbb{R}$ so that $x(t) \neq 0$ when $t \geq t_0$. Otherwise the solution is* oscillatory.

Definition 1.1.5 *A solution $x(t)$ of* (E_1) *is* positive *(*negative*) if $x(t) > 0$ ($x(t) < 0$) for a sufficiently large t.*

Note that the solution is nonoscillatory if it is either positive or negative. If $x(t)$ is a solution of differential equations which are the subject of our research, then $-x(t)$ is also a solution. Therefore, we can, without loss of generality, restrict our study of nonoscillatory solutions to positive solutions.

Next, we give some basic definitions.

Definition 1.1.6 *Let X be a normed vector space. A subset $E \subseteq X$ is said to be* convex *if for any $x, y \in E$ and $t \in [0,1]$ we have $tx + (1-t)y \in E$.*

Definition 1.1.7 *Let X be a Banach space. A set $E \subseteq X$ is said to be* compact *if every sequence in E has a subsequence that converges to a limit that is also in E. Set E is* relatively compact *(or* precompact*) if its closure is compact.*

Definition 1.1.8 *Let X and Y be two metric spaces, and $\mathcal{F}$ a family of functions from X to Y. The family $\mathcal{F}$ is* equicontinuous at a point $x_0 \in X$ *if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(f(x_0), f(x)) < \varepsilon$ for all $f \in \mathcal{F}$ and all x such that $d(x_0, x) < \delta$. The family is* equicontinuous *on X if it is equicontinuous at each point of X.*

The family $\mathcal{F}$ is uniformly equicontinuous *if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(f(x_1), f(x_2)) < \varepsilon$ for all $f \in \mathcal{F}$ and all $x_1, x_2 \in \mathcal{F}$ such that $d(x_1, x_2) < \delta$.*

Definition 1.1.9 *The family $\mathcal{F}$ of functions from $\mathcal{C}([a,b], \mathbb{R})$ is* uniformly bounded on $[a,b]$ *if there exists a positive real number K so that $|f(t)| \leq K$ for all $t \in [a,b]$ and all $f \in \mathcal{F}$.*

In the theory of differential equations we usually use fixed point technique to determine the existence of the solutions. In our case we use the Schauder - Tychonoff fixed point theorem.

Theorem 1.1.1 (Schauder - Tychonoff fixed point theorem) *Let E be closed, convex, nonempty subset of a locally convex topological vector space X. Let T be continuous mapping from E to itself, such that TE is relatively compact. Then T has a fixed point.*

In the process of proving that operator T from the previous theorem is continuous, one of the steps requires the usage of the Lebesgue's Dominated Convergence Theorem.

Theorem 1.1.2 (Lebesgue's Dominated Convergence Theorem) *Let (f_n) be a sequence of real-valued measurable functions on a measurable set E, such that $\lim_{t\to\infty} f_n(x) = f(x)$, almost everywhere on E and for every $n \in \mathbb{N}$. Also, let $g(x)$ be an integrable on E, such that $|f_n(x)| \leq g(x)$ almost everywhere on E. Then*

$$\lim_{n\to\infty} \int_E f_n(x)dx = \int_E f(x)dx.$$

On the other hand, in order to use Schauder - Tychonoff fixed point theorem, we need to prove that the image of operator is relatively compact. For that the Arzela-Ascoli Theorem turns out to be a useful tool.

Theorem 1.1.3 (THE ARZELA-ASCOLI THEOREM) *The set E of continuous functions from $\mathcal{C}([a,b],\mathbb{R})$ is relatively compact if and only if it is uniformly bounded and equicontinuous on $[a,b]$.*

It should be noticed that the previous theorem cannot be used directly for problems that are defined on infinite interval $[a,\infty)$ (and this is the case in majority of problems solved in this dissertation). To overcome this, we use the result from [44] that enables us to use the theorem in our research.

After the construction of intermediate solutions with the help of the Schauder-Tychonoff fixed point theorem, to finish the proof of the "if" part of our main results we prove the regularity of those solutions using the generalized L'Hospital rule (see [13]):

Lemma 1.1.1 *Let $f,g \in C^1[T,\infty)$. Let*

$$\lim_{t\to\infty} g(t) = \infty \quad \textit{and} \quad g'(t) > 0 \quad \textit{for all large } t. \tag{1.1.1}$$

Then

$$\liminf_{t\to\infty} \frac{f'(t)}{g'(t)} \le \liminf_{t\to\infty} \frac{f(t)}{g(t)} \le \limsup_{t\to\infty} \frac{f(t)}{g(t)} \le \limsup_{t\to\infty} \frac{f'(t)}{g'(t)}.$$

If we replace (1.1.1) *with condition*

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = 0 \qquad g'(t) < 0 \quad \textit{for all large } t,$$

then the same conclusion holds.

1.2 Theory of regularly varying functions

In this section we recall the definition and some basic properties of regularly varying functions introduced by J. Karamata in [27].

Definition 1.2.1 *A measurable function $f : [a,\infty) \to (0,\infty)$, $a > 0$ is regularly varying at infinity of index $\rho \in \mathbb{R}$ (in the sense of Karamata) if*

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \textit{for all } \lambda > 0. \tag{1.2.1}$$

Definition 1.2.2 *A measurable function* $f : (0, a) \to (0, \infty),\ a > 0$ *is regularly varying at zero of index* $\rho \in \mathbb{R}$ *if*

$$\lim_{t \to 0+} \frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \quad \textit{for all}\ \lambda > 0. \tag{1.2.2}$$

The set of regularly varying functions of index ρ at infinity (zero) is denoted by $\mathrm{RV}(\rho)$ ($\mathcal{RV}(\rho)$). If, in particular, $\rho = 0$, then the function f is called *slowly varying at infinity (zero)* and is denoted by SV ($\mathcal{SV}$). It is clear that if the function $f(t)$ is regularly varying at zero of index ρ then the function $f(1/t)$ is regularly varying at infinity of index $-\rho$. When we say only regularly or slowly varying function, we mean that function is regularly or slowly varying at infinity.

It follows from Definition 1.2.1 that any function $f(t) \in \mathrm{RV}(\rho)$ is written as

$$f(t) = t^{\rho}\, g(t), \quad g(t) \in \mathrm{SV}. \tag{1.2.3}$$

If, in particular, the function $g(t)$ converges to a positive constant as $t \to \infty$, it is called a *trivial slowly varying function*, denoted by $g(t) \in \mathrm{tr} - \mathrm{SV}$, and the function $f(t)$ is called a *trivial regularly varying of index* ρ, denoted by $f(t) \in \mathrm{tr} - \mathrm{RV}(\rho)$. Otherwise, the function $g(t)$ is called a *nontrivial slowly varying*, denoted by $g(t) \in \mathrm{ntr} - \mathrm{SV}$, and the function $f(t)$ is called a *nontrivial regularly varying of index* ρ, denoted by $f(t) \in \mathrm{ntr} - \mathrm{RV}(\rho)$. Similar terminology is used for the set $\mathcal{RV}$.

Example 1.2.1 From (1.2.3) we see that the class of slowly varying functions is of fundamental importance in the theory of regular variation. Trivial examples of slowly varying functions are (positive, measurable) functions tending to positive constant as $t \to \infty$, or in particular positive constants. The simplest non-trivial example is $\log t$ or $\log_n t$ or

$$\prod_{k=1}^{N} (\log_n t)^{\alpha_k},\ \alpha_k \in \mathbb{R},\ k \in \{1, \ldots, N\},$$

where $\log_n t$ denotes the n-th iteration of the logarithm. Non-logarithmic examples are given by

$$\exp\left\{ \prod_{k=1}^{N} (\log_n t)^{\beta_k} \right\},\ \beta_k \in (0,1),\ k \in \{1, \ldots, N\}.$$

The function

$$L(t) = \exp\left\{ (\log t)^{\theta} \cos(\log t)^{\theta} \right\}, \quad \theta \in \left(0, \frac{1}{2}\right),$$

is an example of slowly varying functions which are oscillating in the sense that

$$\liminf_{t \to \infty} L(t) = 0, \quad \limsup_{t \to \infty} L(t) = \infty.$$

For a comprehensive treatise on regular variation the reader is referred to N.H. Bingham et al. [2]. See also E. Seneta [67].

Next three theorems are the most important in the theory of regular variation. Uniform Convergence Theorem was given by Karamata in 1930 in the continuous case, and by Korevaar in 1949 in the measurable case.

Theorem 1.2.1 (Uniform Convergence Theorem) *The relation* (1.2.1) *in the definition 1.2.1 holds uniformly on each compact* $\lambda-$*set in* $(0, \infty)$.

The next theorem answers the question about the functions that can satisfy (1.2.1).

Theorem 1.2.2 (Representation Theorem) $f(t) \in \mathrm{RV}(\rho)$ *if and only if* $f(t)$ *is represented in the form*

$$f(t) = c(t) \exp\left(\int_{t_0}^{t} \frac{\delta(s)}{s}\, ds\right), \quad t \geq t_0,$$

for some $t_0 > 0$ *and for some measurable functions* $c(t)$ *and* $\delta(t)$ *such that*

$$\lim_{t\to\infty} c(t) = c_0 \in (0, \infty) \quad \text{and} \quad \lim_{t\to\infty} \delta(t) = \rho.$$

The famous Karamata's Integration Theorem gives information about the asymptotic behavior of the integral of regularly varying functions, and it is of prime importance to our research.

Proposition 1.2.1 (Karamata's Integration Theorem) *Let* $L(t) \in \mathrm{SV}$. *Then,*

(i) *If* $\alpha > -1$,
$$\int_a^t s^{\alpha} L(s)\, ds \sim \frac{t^{\alpha+1} L(t)}{\alpha+1}, \quad t \to \infty;$$

(ii) *If* $\alpha < -1$,
$$\int_t^{\infty} s^{\alpha}\, L(s)\, ds \sim -\frac{t^{\alpha+1} L(t)}{\alpha+1}, \quad t \to \infty;$$

(iii) *If* $\alpha = -1$, *the integral* $\int_a^{\infty} s^{-1} L(s)\, ds$ *may or may not be convergent. The integral* $m_1(t) = \int_a^t s^{-1}\, L(s)\, ds$ *is a new slowly varying function and* $L(t)/m_1(t) \to 0$, $t \to \infty$. *In the case* $\int_a^{\infty} s^{-1} L(s)\, ds < \infty$, *again* $m_2(t) = \int_t^{\infty} s^{-1}\, L(s)\, ds \in \mathrm{SV}$ *and* $L(t)/m_2(t) \to 0$, $t \to \infty$.

The following results concern the basic operations with functions that preserves regular variation.

Proposition 1.2.2 *Let $g_1(t) \in \mathrm{RV}(\sigma_1),\ g_2(t) \in \mathrm{RV}(\sigma_2),\ g_3(t) \in \mathcal{RV}(\sigma_3)$. Then,*

(i) $(g_1(t))^\alpha \in \mathrm{RV}(\alpha\sigma_1)$ *for any* $\alpha \in \mathbb{R}$*;*

(ii) $g_1(t) + g_2(t) \in \mathrm{RV}(\sigma),\ \sigma = \max(\sigma_1, \sigma_2)$*;*

(iii) $g_1(t)g_2(t) \in \mathrm{RV}(\sigma_1 + \sigma_2)$*;*

(iv) $g_1(g_2(t)) \in \mathrm{RV}(\sigma_1\sigma_2)$*, if* $g_2(t) \to \infty$*, as* $t \to \infty$*;* $g_3(g_2(t)) \in \mathrm{RV}(\sigma_3\sigma_2)$*, if* $g_2(t) \to 0$*, as* $t \to \infty$*.*

A slowly varying function $L(t)$ may or may not be bounded, but as $t \to \infty$ it can neither grow to infinity too fast, nor decay to zero too fast, as we see from following proposition.

Proposition 1.2.3 *For any $\varepsilon > 0$ and $L(t) \in \mathrm{SV}$ one has*

$$t^\varepsilon L(t) \to \infty, \qquad t^{-\varepsilon} L(t) \to 0, \quad t \to \infty.$$

Proposition 1.2.4 *If $f(t) \sim t^\alpha l(t)$ as $t \to \infty$ with $l(t) \in \mathrm{SV}$, then $f(t)$ is a regularly varying function of index α i.e. $f(t) = t^\alpha l^*(t),\ l^*(t) \in \mathrm{SV}$, where in general $l^*(t) \neq l(t)$, but $l^*(t) \sim l(t)$ as $t \to \infty$.*

In some cases (for instance, the measuring of scales of growth or asymptotic behavior) slowly varying functions are of interest only to within asymptotic equivalence. Since regularly varying functions are not monotone functions in general, the next result shows that any regularly varying function with non-zero index is asymptotic to a monotone function.

Proposition 1.2.5 *A positive measurable function $l(t)$ belongs to* SV *if and only if for every $\alpha > 0$ there exist a non-decreasing function Ψ and a non-increasing function ψ with*

$$t^\alpha l(t) \sim \Psi(t) \quad \textit{and} \quad t^{-\alpha} l(t) \sim \psi(t), \quad t \to \infty.$$

Since regularly varying functions have no inverse function in general, the next result give the existence of an asymptotic inverse of regularly varying functions of positive index.

Proposition 1.2.6 *For the function $f(t) \in \mathrm{RV}(\alpha), \alpha > 0$, there exists $g(t) \in \mathrm{RV}(1/\alpha)$ such that*

$$f(g(t)) \sim g(f(t)) \sim t \quad \textit{as} \quad t \to \infty.$$

Here g is an asymptotic inverse of f (and it is determined uniquely to within asymptotic equivalence).

Note, that the same result holds for $t \to 0$ i.e. when $f(t) \in \mathcal{RV}(\alpha), \alpha > 0$.

Proposition 1.2.7 *For the function $f(t) \in \mathcal{RV}(\alpha), \alpha > 0$, there exists $g(t) \in \mathcal{RV}(1/\alpha)$ such that*

$$f(g(t)) \sim g(f(t)) \sim t \quad as \quad t \to 0.$$

Proof. Since $f(t) \in \mathcal{RV}(\alpha)$, we have $f(1/t) \in \mathrm{RV}(-\alpha)$ and $1/f(1/t) \in \mathrm{RV}(\alpha)$. We can apply the Proposition 1.2.6 to the function $\tilde{f}(t) = 1/f(1/t)$. Then, there exists $\tilde{g} \in \mathrm{RV}(1/\alpha)$ such that

$$\tilde{f}(\tilde{g}(t)) \sim \tilde{g}(\tilde{f}(t)) \sim t \quad \text{as} \quad t \to \infty.$$

Then, it is easy to show that the function $g(t) = 1/\tilde{g}(1/t) \in \mathcal{RV}(1/\alpha)$ is an asymptotic inverse of f.□

In Chapter 2 and 4 we treat the asymptotic behavior of positive solutions of differential equations under consideration in the framework of generalized regularly varying functions. These functions were introduced in [15] by Jaroš and Kusano.

Definition 1.2.3 *Let $R(t) : [0, \infty) \to (0, \infty)$ be continuously differential function such that*

$$R'(t) > 0, \quad t > 0, \quad \textit{and} \quad \lim_{t\to\infty} R(t) = \infty. \tag{1.2.4}$$

A measurable function $f : [0, \infty) \to (0, \infty)$ is said to be regularly varying of index $\rho \in \mathbb{R}$ with respect to $R(t)$ if $f \circ R^{-1}$ is defined for all large t and is regularly varying function of index ρ in the sense of Karamata, where R^{-1} denotes the inverse function of R.

The symbol $\mathrm{RV}_R(\rho)$ is used to denote the totality of regularly varying functions of index $\rho \in \mathbb{R}$ with respect to $R(t)$. The symbol SV_R is often used for $\mathrm{RV}_R(0)$. It is easy to see that if $f(t) \in \mathrm{RV}_R(\rho)$, then $f(t) = R(t)^\rho\, g(t)$, $g(t) \in \mathrm{SV}_R$. If

$$\lim_{t\to\infty} \frac{f(t)}{R(t)^\rho} = \lim_{t\to\infty} g(t) = \text{const} > 0$$

then $f(t)$ is said to be a *trivial regularly varying function of index ρ with respect to $R(t)$* and it is denoted by $f(t) \in tr - \mathrm{RV}_R(\rho)$. Otherwise, $f(t)$ is said to be a *nontrivial regularly varying function of index ρ with respect to $R(t)$* and it is denoted by $f(t) \in \mathrm{ntr} - \mathrm{RV}_R(\rho)$. Also, from Definition 1.2.3 it follows that $f \in \mathrm{RV}_R(\rho)$ if and only if it can be written in the form $f(t) = g(R(t))$, $g(t) \in \mathrm{RV}(\rho)$. It is clear that $\mathrm{RV}(\rho) = \mathrm{RV}_t(\rho)$.

Example 1.2.2 We emphasize that there exists a function which is regularly varying in generalized sense, but is not regularly varying in the sense of Karamata, so that, roughly speaking, the class of generalized Karamata functions is larger than that of classical Karamata functions. In fact, using the notation

$$\exp_0 t = t, \quad \exp_n t = \exp(\exp_{n-1} t),$$

$$\log_0 t = t, \quad \log(\log_{n-1} t), \quad n = 1, 2, \ldots,$$

we define the functions $\phi_n(t)$ and $f_n(t)$ for $n \in \mathbb{Z}$ by

$$\phi_n(t) = \exp_n t, \quad \phi_{-n}(t) = \log_n t, \quad n = 0, 1, 2, \ldots,$$

and

$$f_n(t) = 2 + \sin \phi_n(t), \quad n = 0, \pm 1, \pm 2, \ldots.$$

Since $\phi_n^{-1}(t) = \phi_{-n}(t)$ and $\phi_m \circ \phi_n(t) = \phi_{m+n}(t)$ for any $m, n \in \mathbb{Z}$, we have

$$f_n \circ \phi_m^{-1} = f_{n-m}(t)$$

for any $n, m \in \mathbb{Z}$, from which, by taking into account the facts that

$$f_n(t) \in \mathrm{SV} \quad \text{for} \quad n \leqq -2 \quad \text{and} \quad f_n(t) \notin \mathrm{SV} \quad \text{for} \quad n \geqq -1,$$

we conclude that

$$f_n(t) \notin \mathrm{SV} \quad \text{and} \quad f_n(t) \in \mathrm{SV}_{\phi_m} \quad \text{if} \quad n \geqq -1 \quad \text{and} \quad m \geqq n + 2.$$

Example 1.2.3 **(i)** Let $R \in \mathrm{RV}(m)$, $m > 0$. Then, $R^{-1} \in \mathrm{RV}(\frac{1}{m})$ and hence

$$f \in \mathrm{RV}(\rho) \quad \Longrightarrow \quad f \in \mathrm{RV}_R\Big(\frac{\rho}{m}\Big).$$

(ii) Let $R(t) = e^t$. Then, $R^{-1}(t) = \log t$.

(a) Consider $f(t) = \exp(t^\alpha)$, $\alpha > 0$:
- If $\alpha < 1$, then $f \in \mathrm{SV}_R$;
- If $\alpha = 1$, then $f \in \mathrm{RV}_R(1)$;
- If $\alpha > 1$, then f is rapidly varying, so that $f \notin \mathrm{RV}_R = \cup_{\rho \in \mathbb{R}} \mathrm{RV}_R(\rho)$.

(b) If $f \in \mathrm{RV}(\rho)$, then $f \in \mathrm{SV}_R$

(iii) Let $R(t) = \log t$. Since $R^{-1}(t) = e^t$, we see that

(a) if $f(t) = (\log t)^\beta$, then $f \in \mathrm{RV}_R(\beta)$;
(b) if $f(t) = (\log \log t)^\gamma$, then $f \in \mathrm{SV}_R$.

The similar properties of regularly varying functions given in Proposition 1.2.2 are true in the case of generalized regularly varying functions.

Proposition 1.2.8 *Let* $g_i(t) \in \mathrm{RV}_R(\sigma_i)$, $i = 1, 2$. *Then,*

(i) $(g_1(t))^\alpha \in \mathrm{RV}_R(\alpha\sigma_1)$, *for any* $\alpha \in \mathbb{R}$*;*

(ii) $g_1(t) + g_2(t) \in \mathrm{RV}_R(\sigma)$, $\sigma = \max(\sigma_1, \sigma_2)$*;*

(iii) $g_1(t)g_2(t) \in \mathrm{RV}_R(\sigma_1 + \sigma_2)$*;*

(iv) $g_1(g_2(t)) \in \mathrm{RV}_R(\sigma_1\sigma_2)$ *if* $g_2(t) \to \infty$ *as* $t \to \infty$.

Proposition 1.2.9 *If* $l(t) \in \mathrm{SV}_R$, *then for any* $\varepsilon > 0$,

$$\lim_{t\to\infty} R(t)^\varepsilon l(t) = \infty, \qquad \lim_{t\to\infty} R(t)^{-\varepsilon} l(t) = 0.$$

Also it is possible to generalized the Karamata's Integration Theorem.

Proposition 1.2.10 (Generalized Karamata's Integration Theorem) *Let* R *be a positive function which is continuously differential on* $[0, \infty)$ *and satisfies* (1.2.4). *Then, for any* $f(t) \in \mathrm{SV}_R$*:*

(i) *If* $\alpha > -1$, *then*

$$\int_a^t R'(s)R(s)^\alpha f(s)\, ds \sim \frac{R(t)^{\alpha+1} f(t)}{\alpha + 1}, \quad t \to \infty;$$

(ii) *If* $\alpha < -1$, *then* $\int_a^\infty R'(t)\; R(t)^\alpha\; f(t)\; dt < \infty$, *and*

$$\int_t^\infty R'(s)\; R(s)^\alpha\; f(s)\; ds \sim -\frac{R(t)^{\alpha+1} f(t)}{\alpha + 1}, \quad t \to \infty;$$

(iii) *If* $\alpha = -1$, *then*

$$\int_a^t R'(s)\, R(s)^{-1}\, f(s)\; ds \in \mathrm{SV}_R \quad \text{and} \quad \int_t^\infty R'(s)\, R(s)^{-1}\, f(s)\; ds \in \mathrm{SV}_R.$$

Next result is proved in [10] and we use it very often in our proofs. It help us in dealing with the asymptotic relations.

Proposition 1.2.11 *Let* $F : [a, \infty) \to (0, \infty)$ *be a measurable function and* x_1, x_2 *positive functions defined on* $[a, \infty)$ *such that* $x_i(t) \to \infty$, $t \to \infty$, $i = 1, 2$. *Then:*

$$F \in \mathrm{RV}(\rho),\ \ \rho \neq 0 \quad \text{iff} \quad x_1(t) \simeq x_2(t),\ t \to \infty \Longrightarrow F(x_1(t)) \simeq F(x_2(t)),\ t \to \infty.$$

Chapter 2

Asymptotic behavior of positive solutions of Emden-Fowler second order differential equation

In this chapter we study Emden-Fowler second order differential equation

$$(E_1) \qquad \big(p(t)|x'(t)|^{\alpha-1}\, x'(t)\big)' + q(t)|x(t)|^{\beta-1}\, x(t) = 0, \quad t \geq a > 0, \quad \alpha > \beta > 0$$

under two different integral conditions:

$$(C_1) \qquad \int_a^\infty \frac{dt}{p(t)^{\frac{1}{\alpha}}} < \infty,$$

$$(C_2) \qquad \int_a^\infty \frac{dt}{p(t)^{\frac{1}{\alpha}}} = \infty.$$

In both cases, the study of nonoscillatory solutions of the equation (E_1) consists of three basic tasks:

Task 1. Determine the three types of positive solutions of (E_1) according to their behavior at infinity. Under both conditions (C_1) and (C_2) three types of solutions are obtained, and they are usually referred as dominant, intermediate and subdominant solutions.

Task 2. Establish the necessary and sufficient conditions for the existence of such solutions.

Task 3. Determine the precise asymptotic formulas for the intermediate solutions of (E_1) only, because the asymptotic behavior at infinity of both dominant and subdominant solutions is obvious.

Assuming that the coefficients of (E_1) are positive, continuous functions, the first two tasks have been already completely resolved (see [11, 41, 60]). To accomplish the most difficult Task 3, we consider the equation (E_1) in the framework of regular variation, assuming that coefficients are generalized regularly varying functions.

The results related to the asymptotic behavior of the solution of the equation (E_1) under the condition (C_1), presented in Section 2.2, represent original results, given in the paper [36], and the results related to the equation (E_1) under the condition (C_2), which are presented in Section 2.4, are given in the paper [21].

2.1 Classification and existence of positive solutions of (E_1) under the condition (C_1)

In this section, we assume that $p, q : [a, \infty) \to (0, \infty)$ are continuous functions and that (C_1) holds. The condition (C_1) enables us to define the decreasing function $\pi(t)$ as

$$\pi(t) = \int_t^\infty \frac{ds}{p(s)^{\frac{1}{\alpha}}}\ , \quad t \geq a.$$

Definition 2.1.1 *A solution of* (E_1) *is a function* $x(t) : [T, \infty) \to \mathbb{R}$, $T \geq a$, *which is continuously differentiable together with* $p(t)|x'(t)|^{\alpha-1}x'(t)$ *on* $[T, \infty)$ *and satisfies the equation* (E_1) *at every point of* $[T, \infty)$.

Since we are interested in the existence and asymptotic behavior at infinity of positive solutions of (E_1), we begin by classifying the set of all possible positive solutions of (E_1) according to their asymptotic behavior at infinity.

Let $x(t)$ be a positive solution of the equation (E_1) on $[t_0, \infty)$. It is easy to verify that any nonoscillatory solution of (E_1) is eventually monotone, since (C_1) holds. Thus $p(t)|x'(t)|^{\alpha-1}x'(t)$ is either positive or negative, and since it is decreasing, the following three cases are possible:

$$\begin{aligned}
&\text{(a)} \qquad \lim_{t\to\infty} p(t)x'(t)^\alpha = \text{const} \geq 0,\\
&\text{(b)} \qquad \lim_{t\to\infty} p(t)(-x'(t))^\alpha = \text{const} > 0,\\
&\text{(c)} \qquad \lim_{t\to\infty} p(t)(-x'(t))^\alpha = \infty.
\end{aligned}$$

Let case (a) occur. Then, $0 < p(t)x'(t)^\alpha \leq C^\alpha$ or $0 < x'(t) \leq Cp(t)^{-1/\alpha}$ on $[t_0, \infty)$ for some constant $C > 0$. Integration of the last inequality on $[t_0, t]$ shows that $x(t) \leq x(t_0) + C\int_{t_0}^t p(s)^{-1/\alpha}\, ds \leq x(t_0) + C\pi(t_0)$, and so $x(t)$ increases to a finite constant $c_0 > 0$ as $t \to \infty$.

Let case (b) occur. Since $x(t)$ is positive and decreasing, it follows that $x(t)$ tends to a nonnegative constant c_0 as $t \to \infty$. If $c_0 > 0$, then $x(t) \sim c_0$, $t \to \infty$. On the other hand, if $c_0 = 0$, we have $p(t)(-x'(t))^\alpha \sim c_1$, $t \to \infty$, from which it follows

$$-x'(t) \sim \left(\frac{c_1}{p(t)}\right)^{\frac{1}{\alpha}}, \ t \to \infty.$$

Integrating the above relation on $[t, \infty)$ we get $x(t) \sim c_1^{\frac{1}{\alpha}} \pi(t)$ as $t \to \infty$.

Let case (c) occur. Integration (E_1) on $[t_0, t]$, using the fact that $x(t)$ tends to a nonnegative constant c_0 as $t \to \infty$, gives

$$p(t)(-x'(t))^\alpha = c_1 + \int_{t_0}^{t} q(s)\, x(s)^\beta \, ds, \quad t \geq t_0, \quad (c_1 = p(t_0)(-x'(t_0))^\alpha \geq 0),$$

which implies that $\int_{t_0}^{\infty} q(s)\, x(s)^\beta \, ds = \infty$. Integrating the above from t to ∞, we find that

$$x(t) = c_0 + \int_t^\infty \left(\frac{1}{p(s)}\left(c_1 + \int_{t_0}^{s} q(r) x(r)^\beta \, dr\right)\right)^{\frac{1}{\alpha}} ds, \quad t \geq t_0.$$

If $c_0 > 0$, then $x(t) \sim c_0$, $t \to \infty$, and if $c_0 = 0$, using the L'Hospital's rule, we easily see that $\lim\limits_{t\to\infty} \dfrac{x(t)}{\pi(t)} = \infty$.

The above observation leads to the following conventional classification of positive solutions of (E_1) according to their asymptotic behavior at infinity:

$$\lim_{t\to\infty} x(t) = \text{const} > 0; \tag{2.1.1}$$

$$\lim_{t\to\infty} x(t) = 0, \lim_{t\to\infty} \frac{x(t)}{\pi(t)} = \infty; \tag{2.1.2}$$

$$\lim_{t\to\infty} \frac{x(t)}{\pi(t)} = \text{const} > 0. \tag{2.1.3}$$

Positive solutions of type (2.1.1), (2.1.2), and (2.1.3) are usually called, respectively, *dominant*, *intermediate* and *subdominant* solutions, although solutions of type (2.1.2) are referred by some authors as *slowly decaying*. Indeed, if $x(t)$, $y(t)$, $z(t)$ are positive solutions of (E_1), respectively, of type (2.1.3), (2.1.2), (2.1.1), we have

$$x(t) < y(t) < z(t) \quad \text{for large } t\,.$$

It should be noticed that the existence of each of the above types of solutions for the equation (E_1) with continuous coefficients $p(t)$, $q(t)$ can be completely characterized by the convergence (or divergence) of integrals

$$Z_\alpha = \int_a^\infty \left(\frac{1}{p(t)} \int_a^t q(s)\, ds\right)^{\frac{1}{\alpha}} dt, \qquad W_\beta = \int_a^\infty q(t)\, \pi(t)^\beta \, dt.$$

In fact, the sharp conditions for the existence of positive solutions of (E_1) as well as dominant and subdominant positive solutions have long been known (see [41]). As regards the existence of intermediate solutions for (E_1), although sufficient conditions can be obtained with relative ease (see [41]), the problem of establishing necessary and sufficient conditions turns out to be extremely difficult to solve and thus, has been an open problem for a long time. Nevertheless, the problem has recently been solved by Kamo, Usami [25, Theorem 1.2].

Theorem 2.1.1 *Let $p(t), q(t) \in C[a, \infty)$ and (C_1) holds.*

(a) Equation (E_1) has a positive solution if and only if $Z_\alpha < \infty$;

(b) Equation (E_1) has a positive solution of type (2.1.1) if and only of $Z_\alpha < \infty$;

(c) Equation (E_1) has a positive solution of type (2.1.2) if and only of $Z_\alpha < \infty$ and $W_\beta = \infty$;

(d) Equation (E_1) has a positive solution of type (2.1.3) if and only of $W_\beta < \infty$.

Once the existence of solutions of (E) has been established, the next task is to acquire as detailed information as possible about the qualitative properties of its solutions. Of particular importance is to investigate the possibility of deriving the precise asymptotic formula of intermediate positive solutions of (E). There seems to be only a few of such information in the existing literature. Recently, Kamo and Usami in [25, Theorem 1.4] determined the asymptotic forms of intermediate solutions of (E) assuming that $p(t), q(t)$ behave like power functions, and afterwards Naito [60, Theorems 4.3, 4.4] generalized their results.(see Remark 2.2.1)

2.2 Asymptotic behavior of intermediate solutions of (E_1) under the condition (C_1)

Our goal in this section is to show that the class of generalized regularly varying functions with respect to $1/\pi(t)$ is a well suited framework for the asymptotic analysis of intermediate solutions of (E_1) under the condition (C_1) in the sense that thorough information can be acquired about the existence and asymptotic behavior of $\mathrm{RV}_{1/\pi}$ – solutions of (E_1) provided the coefficients $p(t)$ and $q(t)$ are $\mathrm{RV}_{1/\pi}$ – functions.

We assume that $p(t)$ and $q(t)$ are generalized regularly varying functions of indices η and σ with respect to $1/\pi(t)$ and are expressed in the form

$$(2.2.1)\quad p(t) = \pi(t)^{-\eta}\, l_p(t),\ l_p(t) \in \mathrm{SV}_{1/\pi} \text{ and } q(t) = \pi(t)^{-\sigma}\, l_q(t),\ l_q(t) \in \mathrm{SV}_{1/\pi},$$

and search for the intermediate solutions $x(t) \in \mathrm{RV}_{1/\pi}(\rho)$ of (E_1), which are represented as

$$x(t) = \pi(t)^{-\rho}\, l_x(t), \;\; l_x(t) \;\in \mathrm{SV}_{1/\pi}. \tag{2.2.2}$$

We denote $P(t) = \pi(t)^{-1}$ and rewrite (2.2.1) in the form

$$q(t)p(t)^{1/\alpha} = P(t)^{\mu+2}\, l(t), \quad l(t) \in \mathrm{SV}_{1/\pi}, \tag{2.2.3}$$

where $\mu = \sigma + \frac{\eta}{\alpha} - 2$, $l(t) = l_p(t)^{\frac{1}{\alpha}}\, l_q(t)$. Moreover, since $p(t)^{-1/\alpha} = P'(t)\, P(t)^{-2}$, from (2.2.3) we have

$$q(t) = P'(t)\, P(t)^{\mu}\, l(t), \quad l(t) \in \mathrm{SV}_{1/\pi}\,. \tag{2.2.4}$$

Let us interpret the necessary and sufficient condition for the existence of intermediate solutions of (E_1) in the language of regular variation. Since

$$\int_a^{\infty} q(t)\,\pi(t)^{\beta}\,dt = \int_a^{\infty} P'(t)\,P(t)^{\mu-\beta}\,l(t)\,dt,$$

it is easy to see that

$$\begin{aligned} W_\beta = \infty \iff\; & (i) \quad \mu - \beta > -1, \quad \text{or} \\ & (ii) \quad \mu - \beta = -1 \quad \text{and} \quad \int_a^{\infty} P'(t)\,P(t)^{-1}\,l(t)\,dt = \infty, \end{aligned}$$

or equivalently

$$\begin{aligned} W_\beta = \infty \iff\; & (i) \quad \sigma > \beta - \frac{\eta}{\alpha} + 1, \quad \text{or} \\ & (ii) \quad \sigma = \beta - \frac{\eta}{\alpha} + 1 \quad \text{and} \quad \int_a^{\infty} P'(t)\,P(t)^{-1}\,l(t)\,dt = \infty. \end{aligned}$$

Moreover, assuming that $\sigma > \beta - \frac{\eta}{\alpha} + 1$ i.e. $\mu > \beta - 1 > -1$, application of Generalized Karamata's integration theorem gives

$$\int_a^t q(s)ds = \int_a^t P'(s)P(s)^{\mu}l(s)\,ds \sim \frac{1}{\mu+1}P(t)^{\mu+1}\,l(t), \quad t \to \infty,$$

from which it follows that

$$\int_t^{\infty}\left(\frac{1}{p(s)}\int_a^s q(r)dr\right)^{\frac{1}{\alpha}} ds \sim \frac{1}{(\mu+1)^{\frac{1}{\alpha}}}\int_t^{\infty} P'(s)P(s)^{\frac{\mu+1}{\alpha}-2}\,l(s)^{\frac{1}{\alpha}}\,ds, \quad t \to \infty.$$

For condition $Z_\alpha < \infty$ to hold it is necessary that

$$(i)\quad \frac{\mu+1}{\alpha} - 2 < -1, \quad \text{or}$$
$$(ii)\quad \frac{\mu+1}{\alpha} - 2 = -1 \quad \text{and} \quad \int_a^\infty P'(t)P(t)^{-1}\, l(t)^{\frac{1}{\alpha}}\, dt < \infty,$$

or equivalently

$$(i)\quad \sigma < \alpha - \frac{\eta}{\alpha} + 1, \quad \text{or}$$
$$(ii)\quad \sigma = \alpha - \frac{\eta}{\alpha} + 1 \quad \text{and} \quad \int_a^\infty P'(t)P(t)^{-1}\, l(t)^{\frac{1}{\alpha}}\, dt < \infty.$$

The above observation, with the statement (c) of Theorem 2.1.1, suggests to carry out the study of intermediate solutions of (E_1) by distinguishing the three cases:

$$\sigma = \alpha - \frac{\eta}{\alpha} + 1 \quad \text{and} \quad Z_\alpha < \infty, \tag{2.2.5}$$
$$\beta - \frac{\eta}{\alpha} + 1 < \sigma < \alpha - \frac{\eta}{\alpha} + 1, \tag{2.2.6}$$
$$\sigma = \beta - \frac{\eta}{\alpha} + 1 \quad \text{and} \quad W_\beta = \infty. \tag{2.2.7}$$

Suppose that (E_1) has an intermediate solution $x(t)$ on $[t_0, \infty)$. Since $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} p(t)|x'(t)|^{\alpha-1}x'(t) = \infty$, integrating ($E_1$) first from t_0 to t and then on $[t, \infty)$, we have

$$x(t) = \int_t^\infty \left(\frac{1}{p(s)}\left(p(t_0)(-x'(t_0))^\alpha + \int_{t_0}^s q(r)x(r)^\beta\, dr\right)\right)^{\frac{1}{\alpha}} ds, \quad t \geq t_0. \tag{2.2.8}$$

It follows therefore that $x(t)$ satisfies the integral asymptotic relation

$$x(t) \sim \int_t^\infty \left(\frac{1}{p(s)}\int_{t_0}^s q(r)x(r)^\beta\, dr\right)^{\frac{1}{\alpha}} ds, \quad t \to \infty, \tag{2.2.9}$$

which is regarded as an "approximation" of (2.2.8) at infinity. A common way of determining the desired intermediate solution of (E_1) would be by solving the integral equation (2.2.8) with the help of fixed point technique. For that purpose Schauder-Tychonoff fixed point theorem should be applied to the integral operator

$$\mathcal{F}x(t) = \int_t^\infty \left(\frac{1}{p(s)}\int_{t_0}^s q(r)x(r)^\beta\, dr\right)^{\frac{1}{\alpha}} ds, \quad t \geq t_0, \tag{2.2.10}$$

acting on some closed convex subset $\mathcal{X}$ of $C[t_0,\infty)$, which should be chosen in such a way that $\mathcal{F}$ is a continuous self-map on $\mathcal{X}$ and sent it into a relatively compact subset of $C[t_0,\infty)$. However, to establish the existence of intermediate solutions with precise asymptotic behavior, the set with required properties for application Schauder-Tychonoff fixed point theorem will be found by the means of generalized regularly varying functions satisfying the integral asymptotic relation (2.2.9). In fact, to show the existence of solution $x(t)$ such that $x(t) \sim X(t)$, $t \to \infty$, we first construct such solutions as a fixed point of the integral operator $\mathcal{F}$ defined as (2.2.10) on the set

$$\mathcal{X} = \{x(t) \in C[t_0,\infty) : mX(t) \leq x(t) \leq MX(t),\ t \geq t_0\}.$$

With the help of generalized L'Hospital rule prove that such solutions must be generalized regularly varying satisfying desired asymptotic formula. Note that generalized regularly varying function $X(t)$ will be determined in terms of generalized regularly varying coefficients $p(t)$ and $q(t)$ and parameters α, β. For that purpose, we prove the next three Lemmas.

Lemma 2.2.1 *Suppose that* (2.2.5) *holds. The function*

$$Y_1(t) = \left(\frac{\alpha-\beta}{\alpha}\int_t^\infty \left(\frac{1}{p(s)}\int_a^s q(r)\,dr\right)^{\frac{1}{\alpha}} ds\right)^{\frac{\alpha}{\alpha-\beta}} \tag{2.2.11}$$

satisfies the asymptotic relation (2.2.9).

Proof. Let (2.2.5) hold. Then, $\mu = \alpha - 1$. Using (2.2.4) and Generalized Karamata's integration theorem (Proposition 1.2.10-(i)) we have

$$\left(\int_a^t q(s)\,ds\right)^{\frac{1}{\alpha}} = \left(\int_a^t P'(s)\,P(s)^{\alpha-1}\,l(s)\,ds\right)^{\frac{1}{\alpha}} \sim \frac{P(t)\,l(t)^{\frac{1}{\alpha}}}{\alpha^{\frac{1}{\alpha}}}, \quad t \to \infty,$$

implying that

$$\begin{aligned}\int_t^\infty \left(\frac{1}{p(s)}\int_a^s q(r)\,dr\right)^{\frac{1}{\alpha}} ds &\sim \int_t^\infty \frac{P(s)\,l(s)^{\frac{1}{\alpha}}}{\alpha^{\frac{1}{\alpha}}\,p(s)^{\frac{1}{\alpha}}}\,ds \\ &\sim \frac{1}{\alpha^{\frac{1}{\alpha}}}\int_t^\infty P'(s)P(s)^{-1}l(s)^{\frac{1}{\alpha}}\,ds, \quad t \to \infty.\end{aligned} \tag{2.2.12}$$

Due to the Proposition 1.2.10-(iii), this shows that $Y_1(t) \in \mathrm{ntr} - \mathrm{SV}_P$. Another application of Generalized Karamata's integration theorem gives

$$\begin{aligned}\left(\frac{1}{p(t)}\int_a^t q(s)Y_1(s)^\beta\,ds\right)^{\frac{1}{\alpha}} &= \left(\frac{1}{p(t)}\int_a^t P'(s)\,P(s)^{\alpha-1}\,l(s)\,Y_1(s)^\beta\,ds\right)^{\frac{1}{\alpha}} \\ &\sim \frac{P(t)\,l(t)^{\frac{1}{\alpha}}\,Y_1(t)^{\frac{\beta}{\alpha}}}{\alpha^{\frac{1}{\alpha}}\,p(t)^{\frac{1}{\alpha}}}, \quad t \to \infty.\end{aligned} \tag{2.2.13}$$

Integrating (2.2.13) on $[t,\infty)$, and using both (2.2.11) and (2.2.12), we obtain

$$
\begin{aligned}
&\int_t^\infty \left(\frac{1}{p(s)}\int_a^s q(r)\,Y_1(r)^\beta\,dr\right)^{\frac{1}{\alpha}} ds\\
&\sim \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \int_t^\infty \frac{P(s)\,l(s)^{\frac{1}{\alpha}}}{\alpha^{\frac{1}{\alpha}}\,p(s)^{\frac{1}{\alpha}}}\left(\int_s^\infty \frac{P(r)\,l(r)^{\frac{1}{\alpha}}}{\alpha^{\frac{1}{\alpha}}\,p(r)^{\frac{1}{\alpha}}}dr\right)^{\frac{\beta}{\alpha-\beta}} ds\\
&= \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} \int_t^\infty \left(\frac{P(s)\,l(s)^{\frac{1}{\alpha}}}{\alpha^{\frac{1}{\alpha}}\,p(s)^{\frac{1}{\alpha}}}\right)^{\frac{\alpha}{\alpha-\beta}} ds \sim Y_1(t), \quad t\to\infty.
\end{aligned}
\tag{2.2.14}
$$

This completes the proof of Lemma 2.2.1.□

Lemma 2.2.2 *Suppose that* (2.2.6) *holds and let* ρ *be defined as*

$$
\rho = \frac{\sigma-\alpha-1+\frac{\eta}{\alpha}}{\alpha-\beta}. \tag{2.2.15}
$$

The function

$$
Y_2(t) = \left(\frac{\pi(t)^{\alpha+1}\,p(t)^{\frac{1}{\alpha}}\,q(t)}{\alpha(-\rho)^\alpha\,(\rho+1)}\right)^{\frac{1}{\alpha-\beta}} \tag{2.2.16}
$$

satisfies the asymptotic relation (2.2.9).

Proof. We denote $\lambda = \alpha\,(-\rho)^\alpha\,(\rho+1)$. Using (2.2.3), $Y_2(t)\in \mathrm{RV}_{1/\pi}(\rho)$ is exprresed in the form

$$
Y_2(t) = \lambda^{-\frac{1}{\alpha-\beta}}\,P(t)^\rho\,l(t)^{\frac{1}{\alpha-\beta}},
$$

so that by application of Generalized Karamata's integration theorem we have

$$
\begin{aligned}
&\left(\frac{1}{p(t)}\int_a^t q(s)\,Y_2(s)^\beta\,ds\right)^{\frac{1}{\alpha}}\\
&\sim \lambda^{-\frac{\beta}{\alpha(\alpha-\beta)}}\left(\frac{1}{p(t)}\int_a^t P'(s)\,P(s)^{\mu+\rho\beta}\,l(s)^{\frac{\alpha}{\alpha-\beta}}\,ds\right)^{\frac{1}{\alpha}}\\
&\sim \lambda^{-\frac{\beta}{\alpha(\alpha-\beta)}}\,\frac{P'(t)}{(\mu+\rho\,\beta+1)^{\frac{1}{\alpha}}}\,P(t)^{\frac{\mu+\rho\,\beta+1}{\alpha}-2}\,l(t)^{\frac{1}{\alpha-\beta}}, \quad t\to\infty.
\end{aligned}
\tag{2.2.17}
$$

Since $\mu+\beta\,\rho+1 = \alpha(\rho+1)$, we integrate (2.2.17) on $[t,\infty)$ and use Generalized

Karamata's integration theorem once more to get

$$\int_t^\infty \left(\frac{1}{p(s)} \int_a^s q(r)\, Y_2(r)^\beta\, dr\right)^{\frac{1}{\alpha}} ds$$

$$\sim \lambda^{-\frac{\beta}{\alpha(\alpha-\beta)}} \frac{1}{(\alpha(\rho+1))^{\frac{1}{\alpha}}} \int_t^\infty P'(s)\, P(s)^{\rho-1}\, l(s)^{\frac{1}{\alpha-\beta}}\, ds$$

$$\sim \lambda^{-\frac{\beta}{\alpha(\alpha-\beta)}} \frac{1}{(\alpha\,(-\rho)^\alpha\,(\rho+1))^{\frac{1}{\alpha}}} P(t)^\rho\, l(t)^{\frac{1}{\alpha-\beta}} = Y_2(t), \quad t\to\infty.$$

This completes the proof of Lemma 2.2.2. □

Lemma 2.2.3 *Suppose that* (2.2.7) *holds. The function*

$$Y_3(t) = \pi(t) \left(\frac{\alpha-\beta}{\alpha} \int_a^t q(s)\, \pi(s)^\beta\, ds\right)^{\frac{1}{\alpha-\beta}} \tag{2.2.18}$$

satisfies the asymptotic relation (2.2.9).

Proof. Let (2.2.7) holds. Then, $\mu = \beta - 1$, and by using (2.2.4) we have $q(t)\,\pi(t)^\beta = P'(t)\,P(t)^{-1}\,l(t)$, $l(t) \in \mathrm{SV}_P$, implying, due to Proposition (1.2.10)-(iii), that

$$\int_a^t q(s)\,\pi(s)^\beta\, ds \in \mathrm{SV}_P. \tag{2.2.19}$$

Using (2.2.18) and (2.2.19) we conclude that $Y_3(t) \in \mathrm{ntr}-\mathrm{RV}_P(-1)$. On the other hand, by a simple calculation, we obtain

$$\begin{aligned}\int_a^t q(s)\,Y_3(s)^\beta\, ds &= \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \int_a^t q(s)\pi(s)^\beta \left(\int_a^s q(r)\,\pi(r)^\beta\, dr\right)^{\frac{\beta}{\alpha-\beta}} ds \\ &= \left(\frac{\alpha-\beta}{\alpha} \int_a^t q(s)\,\pi(s)^\beta\, ds\right)^{\frac{\alpha}{\alpha-\beta}}. \end{aligned} \tag{2.2.20}$$

From (2.2.19) and (2.2.20) we get

$$\int_a^t q(s) Y_3(s)^\beta\, ds \in \mathrm{SV}_P. \tag{2.2.21}$$

Multiplying (2.2.21) with $p(t)^{-\frac{1}{\alpha}}$ and integrating on $[t,\infty)$, by application of Proposition 1.2.10 as $t\to\infty$ we have

$$\int_t^\infty \left(\frac{1}{p(s)} \int_a^s q(r) Y_3(r)^\beta dr\right)^{\frac{1}{\alpha}} ds \;\sim\; \int_t^\infty P'(s) P(s)^{-2} \left(\int_a^s q(r) Y_3(r)^\beta dr\right)^{\frac{1}{\alpha}} ds$$

$$\sim P(t)^{-1} \left(\int_a^t q(s) Y_3(s)^\beta ds\right)^{\frac{1}{\alpha}} \;\sim\; P(t)^{-1} \left(\frac{\alpha-\beta}{\alpha} \int_a^t q(s)\pi(s)^\beta ds\right)^{\frac{1}{\alpha-\beta}} = Y_3(t),$$

where we use (2.2.20) in the last step. This completes the proof of Lemma 2.2.3. □

Since $c_1\pi(t) \le x(t) \le c_2$, for some positive constants c_1 and c_2 and all large t, the regularity index ρ of $x(t)$ must satisfy $-1 \le \rho \le 0$, while the slowly varying part of $x(t)$ satisfies

$$l_x(t) = \frac{x(t)}{\pi(t)} \to \infty \;\text{ as } t \to \infty \quad \text{or} \quad l_x(t) = x(t) \to 0 \;\text{ as } t \to \infty,$$

according as $\rho = -1$ or $\rho = 0$. Therefore, the class of intermediate regularly varying solutions with respect to $1/\pi(t)$ is divided into three types of subclasses

$$\text{(2.2.22)} \qquad \mathrm{ntr}-\mathrm{RV}_{1/\pi}(-1), \quad \text{or} \quad \mathrm{RV}(\rho) \;\text{ with }\; \rho \in (-1,0), \quad \text{or} \quad \mathrm{ntr}-\mathrm{SV}_{1/\pi}.$$

We will show that if (E_1) has intermediate regularly varying solutions with respect to $1/\pi(t)$ then all of them are members of only one of the subclasses in (2.2.22) and have one and the same asymptotic behavior at infinity.

Theorem 2.2.1 *Let $p(t) \in \mathrm{RV}_{1/\pi}(\eta)$, $q(t) \in \mathrm{RV}_{1/\pi}(\sigma)$ and (C_1) hold. Equation (E_1) has intermediate solutions $x(t) \in \mathrm{ntr}-\mathrm{SV}_{1/\pi}$ if and only if (2.2.5) holds. The asymptotic behavior of any such solution $x(t)$ is governed by the unique formula $x(t) \sim Y_1(t)$, $t \to \infty$, where $Y_1(t)$ is given by (2.2.11).*

Theorem 2.2.2 *Let $p(t) \in \mathrm{RV}_{1/\pi}(\eta)$, $q(t) \in \mathrm{RV}_{1/\pi}(\sigma)$ and (C_1) hold. Equation (E_1) has intermediate solutions $x(t) \in \mathrm{RV}_{1/\pi}(\rho)$ with $\rho \in (-1,0)$ if and only if (2.2.6) holds, in which case ρ is given by (2.2.15) and the asymptotic behavior of any such solution $x(t)$ is governed by the unique formula $x(t) \sim Y_2(t)$, $t \to \infty$, where $Y_2(t)$ is given by (2.2.16).*

Theorem 2.2.3 *Let $p(t) \in \mathrm{RV}_{1/\pi}(\eta)$, $q(t) \in \mathrm{RV}_{1/\pi}(\sigma)$ and (C_1) hold. Equation (E_1) has intermediate solutions $x(t) \in \mathrm{ntr}-\mathrm{RV}_{1/\pi}(-1)$ if and only if (2.2.7) holds. The asymptotic behavior of any such solution $x(t)$ is governed by the unique formula $x(t) \sim Y_3(t)$, $t \to \infty$, where $Y_3(t)$ is given by (2.2.18).*

Proof of the "only if" part of Theorems 2.2.1, 2.2.2, 2.2.3: Suppose that equation (E_1) has an intermediate solution $x(t) \in \mathrm{RV}_{1/\pi}(\rho)$, $\rho \in [-1,0]$ defined on $[t_0,\infty)$. Integrating equation (E_1) from t_0 to t using (2.2.2) and (2.2.4) we have

$$\text{(2.2.23)} \qquad p(t)\,(-x'(t))^{\alpha} \sim \int_{t_0}^{t} q(s)x(s)^{\beta}\,ds = \int_{t_0}^{t} P'(s)\,P(s)^{\mu+\rho\beta}\,l(s)\,l_x(s)^{\beta}\,ds,$$

as $t \to \infty$. Since $\lim\limits_{t\to\infty} p(t)(-x'(t))^{\alpha} = \infty$, the divergence of the last integral in (2.2.23) implies that it must be $\mu + \rho\beta \ge -1$. We distinguish two cases:

$$(a) \quad \mu + \rho\beta = -1, \quad (b) \quad \mu + \rho\beta > -1.$$

Assume that (a) holds. Then, by Proposition 1.2.10-(iii)

$$\left(\int_{t_0}^{t} q(s)x(s)^{\beta}\,ds\right)^{\frac{1}{\alpha}} = \left(\int_{t_0}^{t} P'(s)\,P(s)^{-1}\,l(s)l_x(s)^{\beta}\,ds\right)^{\frac{1}{\alpha}} \in \mathrm{SV}_P. \tag{2.2.24}$$

Thus, multiplying (2.2.24) with $p(t)^{-\frac{1}{\alpha}} = P'(t)\,P(t)^{-2}$ and integrating from t to ∞, by the Generalized Karamata's integration theorem we have

$$\begin{aligned} x(t) &\sim \int_t^{\infty} P'(s)\,P(s)^{-2}\left(\int_{t_0}^{s} P'(r)\,P(r)^{-1}\,l(r)\,l_x(r)^{\beta}\,dr\right)^{\frac{1}{\alpha}} ds \\ &\sim P(t)^{-1}\left(\int_{t_0}^{t} P'(s)\,P(s)^{-1}\,l(s)\,l_x(s)^{\beta}\,ds\right)^{\frac{1}{\alpha}} \in \mathrm{RV}_{1/\pi}(-1), \quad t\to\infty. \end{aligned} \tag{2.2.25}$$

Assume that (b) holds. Then, by an application of Proposition 1.2.10-(i), (2.2.23) implies

$$\int_{t_0}^{t} q(s)\,x(s)^{\beta}\,ds \sim \frac{1}{\mu+\rho\beta+1}\;P(t)^{\mu+\rho\,\beta+1}\,l(t)\,l_x(t)^{\beta}, \quad t\to\infty, \tag{2.2.26}$$

so that

$$-x'(t) \sim \left(\frac{1}{p(t)}\int_{t_0}^{t} q(s)x(s)^{\beta}\,ds\right)^{\frac{1}{\alpha}} \sim \frac{P'(t)\,P(t)^{\frac{\mu+\rho\,\beta+1}{\alpha}-2}\,l(t)^{\frac{1}{\alpha}}\,l_x(t)^{\frac{\beta}{\alpha}}}{(\mu+\rho\beta+1)^{\frac{1}{\alpha}}}, \tag{2.2.27}$$

as $t\to\infty$. Since $\lim_{t\to\infty} x(t) = 0$, the last function in (2.2.27) is integrable on $[t_0,\infty)$, so it must be $\frac{\mu+\rho\beta+1}{\alpha} \le 1$, and we must distinguish the two possibilities:

$$(b.1)\quad \frac{\mu+\rho\beta+1}{\alpha} < 1, \quad (b.2)\quad \frac{\mu+\rho\beta+1}{\alpha} = 1.$$

If $(b.1)$ holds, integration of (2.2.27) from t to ∞, by an application of Generalized Karamata's integral theorem implies

$$x(t) \sim \frac{P(t)^{\frac{\mu+\rho\beta+1}{\alpha}-1}\,l(t)^{\frac{1}{\alpha}}\,l_x(t)^{\frac{\beta}{\alpha}}}{-\left(\frac{\mu+\rho\beta+1}{\alpha}-1\right)\,(\mu+\rho\beta+1)^{\frac{1}{\alpha}}} \in \mathrm{RV}_{1/\pi}\left(\frac{\mu+\rho\beta+1-\alpha}{\alpha}\right), \tag{2.2.28}$$

as $t\to\infty$. On the other hand, if $(b.2)$ holds, integration of (2.2.27) from t to ∞ gives

$$x(t) \sim \alpha^{-\frac{1}{\alpha}}\int_t^{\infty} P'(s)\,P(s)^{-1}\,l(s)^{\frac{1}{\alpha}}\,l_x(s)^{\frac{\beta}{\alpha}}\,ds \in \mathrm{SV}_{1/\pi}, \quad t\to\infty. \tag{2.2.29}$$

Suppose that equation (E_1) has an intermediate solution $x(t)$ that belong to $\mathrm{ntr} - \mathrm{RV}_{1/\pi}(-1)$ on $[t_0, \infty)$. From the above observation this is possible only when the case (a) holds. In that case $\rho = -1$, $\mu = \beta - 1$ i.e. $\sigma = \beta - \frac{\eta}{\alpha} + 1$ and $x(t)$ must satisfy the asymptotic behavior (2.2.25). Since $x(t) = P(t)^{-1}\, l_x(t)$, $l_x(t) \in \mathrm{SV}_P$, from (2.2.25) we have

$$l_x(t) \sim \left(\int_{t_0}^{t} P'(s)\, P(s)^{-1}\, l(s)\, l_x(s)^{\beta}\, ds \right)^{\frac{1}{\alpha}} = \nu(t)^{\frac{1}{\alpha}}, \quad t \to \infty. \tag{2.2.30}$$

Now, we get the differential asymptotic relation for $\nu(t)$:

$$\nu(t)^{-\frac{\beta}{\alpha}}\, \nu'(t) \sim q(t)\ P(t)^{-\beta},\ t \to \infty. \tag{2.2.31}$$

Integrating (2.2.31) on $[t_0, t]$ we have

$$\nu(t) \sim \left(\frac{\alpha - \beta}{\alpha} \int_{t_0}^{t} q(s)\, \pi(s)^{\beta}\, ds \right)^{\frac{\alpha}{\alpha-\beta}},\ t \to \infty. \tag{2.2.32}$$

From (2.2.30), since $\lim_{t\to\infty} l_x(t) = \infty$, we have $\lim_{t\to\infty} \nu(t) = \infty$ which implies $W_\beta = \infty$. Combining (2.2.32) with (2.2.25) gives us $x(t) \sim Y_3(t)$, $t \to \infty$, where $Y_3(t)$ is given by (2.2.18). This completes the "only if" part of the proof of Theorem 2.2.3.

Next, we assume that equation (E_1) has an intermediate solution $x(t) \in \mathrm{RV}_{1/\pi}(\rho)$ with $\rho \in (-1, 0)$ on $[t_0, \infty)$. For such $x(t)$ only case $(b.1)$ is possible and $x(t)$ must satisfy the asymptotic relation (2.2.28), which shows that

$$\rho = \frac{\mu + \rho\,\beta + 1 - \alpha}{\alpha},$$

implying that the regularity of $x(t)$ is given by (2.2.15). Thus, hypothesis $\rho \in (-1, 0)$ determines the range of σ as $\beta - \frac{\eta}{\alpha} + 1 < \sigma < \alpha - \frac{\eta}{\alpha} + 1$. Using (2.2.3) we rewrite (2.2.28) in the form

$$x(t) \sim \frac{\left(P(t)^{-\alpha-1}\, p(t)^{\frac{1}{\alpha}}\, q(t)\right)^{\frac{1}{\alpha}}\, x(t)^{\frac{\beta}{\alpha}}}{(-\rho)\ (\alpha(\rho+1))^{1/\alpha}}, \quad t \to \infty,$$

which leads us to the asymptotic formula $x(t) \sim Y_2(t)$, $t \to \infty$, where $Y_2(t)$ is given by (2.2.16). This completes the "only if" part of the proof of Theorem 2.2.2.

Finally, if we assume that equation (E_1) has an intermediate solution $x(t) \in \mathrm{SV}_{1/\pi}$, the case $(b.2)$ is the only possibility for $x(t)$, which means that $\rho = 0$, $\mu = \alpha - 1$ i.e. $\sigma = \alpha - \frac{\eta}{\alpha} + 1$ and $x(t)$ satisfies (2.2.29). Letting

$$\xi(t) = \frac{1}{\alpha^{\frac{1}{\alpha}}} \int_{t}^{\infty} P'(s)\, P(s)^{-1}\, l(s)^{\frac{1}{\alpha}}\, l_x(s)^{\frac{\beta}{\alpha}}\, ds$$

we transform (2.2.29) into the differential asymptotic relation:

$$\xi(t)^{-\frac{\beta}{\alpha}}\xi'(t) \sim -\frac{1}{\alpha^{\frac{1}{\alpha}}}P(t)\,p(t)^{-\frac{1}{\alpha}}\,l(t)^{\frac{1}{\alpha}},\ t \to \infty. \tag{2.2.33}$$

Since

$$\left(\frac{1}{p(t)}\int_a^t q(s)\,ds\right)^{\frac{1}{\alpha}} \sim \frac{1}{\alpha^{\frac{1}{\alpha}}}P(t)\,p(t)^{-\frac{1}{\alpha}}\,l(t)^{\frac{1}{\alpha}},\ t \to \infty,$$

integration of (2.2.33) from t to ∞ combined with the fact that $\xi(t) \sim x(t) \to 0$, $t \to \infty$, shows that $Z_\alpha < \infty$ and that the asymptotic expression for $x(t)$ is

$$x(t) \sim \left(\frac{\alpha-\beta}{\alpha}\int_t^\infty \left(\frac{1}{p(s)}\int_a^s q(r)\,dr\right)^{\frac{1}{\alpha}} ds\right)^{\frac{\alpha}{\alpha-\beta}} = Y_1(t), \quad t \to \infty.$$

This completes the "only if" part of the proof of Theorem 2.2.1. □

Proof of the "if" part of Theorems 2.2.1, 2.2.2, 2.2.3: Suppose that (2.2.5) or (2.2.6) or (2.2.7) holds. From Lemmas 2.2.1, 2.2.2 and 2.2.3 it is known that $Y_i(t)$, $i = 1,2,3$, defined by (2.2.11), (2.2.16) and (2.2.18) satisfy the asymptotic relation (2.2.9). We perform the simultaneous proof for $Y_i(t)$, $i = 1,2,3$ so the subscripts $i = 1,2,3$ will be deleted in the rest of the proof. By (2.2.9) there exists $T_0 > a$ such that

$$\frac{Y(t)}{2} \le \int_t^\infty \left(\frac{1}{p(s)}\int_{T_0}^s q(r)\,Y(r)^\beta\,dr\right)^{\frac{1}{\alpha}} ds \le 2Y(t), \ \ t \ge T_0. \tag{2.2.34}$$

Let such a T_0 be fixed. Choose positive constants $m \in (0,1)$ and $M > 1$ such that

$$m^{1-\frac{\beta}{\alpha}} \le \frac{1}{2} \ \text{ and } \ M^{1-\frac{\beta}{\alpha}} \ge 2. \tag{2.2.35}$$

Let us define the set

$$\mathcal{X} := \{x(t) \in C[T_0,\infty) : mY(t) \le x(t) \le MY(t),\ t \ge T_0\}. \tag{2.2.36}$$

It is clear that $\mathcal{X}$ is a closed, convex subset of the locally convex space $C[T_0,\infty)$ equipped with the topology of uniform convergence on compact subintervals of $[T_0,\infty)$. We define the integral operator

$$\mathcal{F}x(t) = \int_t^\infty \left(\frac{1}{p(s)}\int_{T_0}^s q(r)x(r)^\beta\,dr\right)^{\frac{1}{\alpha}} ds, \ \ t \ge T_0, \tag{2.2.37}$$

and let it act on set $\mathcal{X}$ defined above. We show that $\mathcal{F}$ is a continuous self-map on $\mathcal{X}$ such that $\mathcal{F}(\mathcal{X})$ is relatively compact in $C[T_0,\infty)$.

(i) $\mathcal{F}(\mathcal{X}) \subset \mathcal{X}$. Let $x(t) \in \mathcal{X}$. Using (2.2.34), (2.2.35) and (2.2.36) we get

$$\mathcal{F}x(t) \le M^{\frac{\beta}{\alpha}} \int_t^{\infty} \left(\frac{1}{p(s)} \int_{T_0}^{s} q(r) Y(r)^{\beta}\, dr \right)^{\frac{1}{\alpha}} ds \le 2M^{\frac{\beta}{\alpha}}\, Y(t) \le M\, Y(t), \quad t \ge T_0,$$

and

$$\mathcal{F}x(t) \ge m^{\frac{\beta}{\alpha}} \int_t^{\infty} \left(\frac{1}{p(s)} \int_{T_0}^{s} q(r) Y(r)^{\beta}\, dr \right)^{\frac{1}{\alpha}} ds \ge m^{\frac{\beta}{\alpha}}\, \frac{Y(t)}{2} \ge m\, Y(t), \quad t \ge T_0.$$

This shows that $\mathcal{F}x(t) \in \mathcal{X}$, that is, $\mathcal{F}$ maps $\mathcal{X}$ into itself.

(ii)$\mathcal{F}$ *is continuous on* $\mathcal{X}$. Let $\{x_n(t)\}$ be a sequence in $\mathcal{X}$ converging to $x(t) \in \mathcal{X}$ uniformly on compact subintervals of $[T_0, \infty)$. Let $T_1 > T_0$ be arbitrary fixed. Then, by (2.2.37) we have

$$|\mathcal{F}x_n(t) - \mathcal{F}x(t)| \le \int_t^{\infty} \frac{1}{p(s)^{\frac{1}{\alpha}}}\, F_n(s)\, ds, \quad t \in [T_0, T_1], \tag{2.2.38}$$

where

$$F_n(t) = \left| \left(\int_{T_0}^{t} q(s)\, x_n(s)^{\beta}\, ds \right)^{\frac{1}{\alpha}} - \left(\int_{T_0}^{t} q(s)\, x(s)^{\beta}\, ds \right)^{\frac{1}{\alpha}} \right|.$$

By the Lebesgue dominated convergence theorem we have $\lim_{n\to\infty} F_n(t) = 0$ for each $t \in [T_0, T_1]$. In addition, using this fact and

$$|F_n(t)| \le 2M^{\frac{\beta}{\alpha}} \left(\int_{T_0}^{T_1} q(s)\, Y(s)^{\beta}\, ds \right)^{\frac{1}{\alpha}}, \qquad t \in [T_0, T_1],$$

an application of the Lebesgue dominated convergence theorem gives

$$\lim_{n\to\infty} \int_t^{\infty} \frac{1}{p(s)^{\frac{1}{\alpha}}}\, F_n(s)\, ds = 0\,.$$

Therefore, $|\mathcal{F}x_n(t) - \mathcal{F}x(t)| \to 0$, $n \to \infty$ uniformly on $[T_0, T_1] \subset [T_0, \infty)$, which proves the continuity of $\mathcal{F}$ on $\mathcal{X}$.

(iii) $\mathcal{F}(\mathcal{X})$ *is relatively compact.* The inclusion $\mathcal{F}(\mathcal{X}) \subset \mathcal{X}$ ensures that $\mathcal{F}(\mathcal{X})$ is locally uniformly bounded on $[T_0, \infty)$. Differentiation of (2.2.37) gives

$$-\frac{M^{\frac{\beta}{\alpha}}}{p(t)^{\frac{1}{\alpha}}} \left(\int_{T_0}^{t} q(s)\, Y(s)^{\beta}\, ds \right)^{\frac{1}{\alpha}} \le (\mathcal{F}x)'(t) \le 0, \;\; t \in [T_0, \infty), \;\; x(t) \in \mathcal{X},$$

which implies that $\mathcal{F}(\mathcal{X})$ is locally equicontinuous on $[T_0, \infty)$. Therefore, by the Arzela-Ascoli theorem, we conclude that $\mathcal{F}(\mathcal{X})$ is a relatively compact subset of $C[T_0, \infty)$.

Thus, all the conditions of the Schauder-Tychonoff fixed point theorem are fulfilled and so there exists a fixed point $x(t) \in \mathcal{X}$ of $\mathcal{F}$, which satisfies integral equation

$$x(t) = \int_t^\infty \left(\frac{1}{p(s)} \int_{T_0}^s q(r)x(r)^\beta \, dr \right)^{\frac{1}{\alpha}} ds, \quad t \geq T_0. \tag{2.2.39}$$

Differentiating the above twice shows that $x(t)$ is a solution of (E$_1$) on $[T_0, \infty)$. It is clear from (2.2.36) that $x(t)$ is an intermediate solution of (E$_1$).

Finally, we show that intermediate solutions constructed above are indeed a regularly varying function with respect to $1/\pi(t)$. Denote

$$J(t) = \int_t^\infty \left(\frac{1}{p(s)} \int_{T_0}^s q(r)Y(r)^\beta \, dr \right)^{\frac{1}{\alpha}} ds.$$

Due to (2.2.36) we get

$$0 < \liminf_{t\to\infty} \frac{x(t)}{Y(t)} \leq \limsup_{t\to\infty} \frac{x(t)}{Y(t)} < \infty.$$

Applying Lemma 1.1.1 and using that $Y(t) \sim J(t)$, $t \to \infty$, we obtain

$$\begin{aligned}
L = \limsup_{t\to\infty} \frac{x(t)}{J(t)} \quad &\leq \quad \limsup_{t\to\infty} \frac{x'(t)}{J'(t)} = \limsup_{t\to\infty} \frac{\left(\displaystyle\int_{T_0}^t q(s)x(s)^\beta \, ds \right)^{\frac{1}{\alpha}}}{\left(\displaystyle\int_{T_0}^t q(s)Y(s)^\beta \, ds \right)^{\frac{1}{\alpha}}} \\
&= \quad \left(\limsup_{t\to\infty} \frac{\int_{T_0}^t q(s)x(s)^\beta \, ds}{\int_{T_0}^t q(s)Y(s)^\beta \, ds} \right)^{\frac{1}{\alpha}} \leq \left(\limsup_{t\to\infty} \frac{q(t)x(t)^\beta}{q(t)Y(t)^\beta} \right)^{\frac{1}{\alpha}} \\
&= \quad \left(\limsup_{t\to\infty} \frac{x(t)}{Y(t)} \right)^{\frac{\beta}{\alpha}} = \left(\limsup_{t\to\infty} \frac{x(t)}{J(t)} \right)^{\frac{\beta}{\alpha}} = L^{\frac{\beta}{\alpha}}.
\end{aligned}$$

Since $0 < \frac{\beta}{\alpha} < 1$ and $0 < L < \infty$, the above fact implies $0 < L \leq 1$. In the same manner we can prove that $l = \liminf_{t\to\infty} x(t)/J(t)$ satisfies $1 \leq l < \infty$. Then, in view of the trivial inequality $l \leq L$, we obtain $l = L = 1$. This means $x(t) \sim J(t)$, $t \to \infty$, which in view of $J(t) \sim Y(t)$, $t \to \infty$, shows that $x(t) \sim Y(t)$, $t \to \infty$. Therefore, $x(t)$ is a regularly varying function with respect to $1/\pi(t)$ whose regularity index ρ is -1 or $(\sigma - \alpha - 1 + \frac{\eta}{\alpha})/(\alpha - \beta)$ or 0 according as the regularity index σ of the coefficient $q(t)$ is respectively, $\sigma = \beta - \frac{\eta}{\alpha} + 1$ or $\sigma \in (\beta - \frac{\eta}{\alpha} + 1, \alpha - \frac{\eta}{\alpha} + 1)$ or $\sigma = \alpha - \frac{\eta}{\alpha} + 1$. Thus, the if part of Theorems 2.2.1, 2.2.2 and 2.2.3 has been proved. □

Our main results (Theorems 2.2.1, 2.2.2, 2.2.3) combined with Theorem 2.1.1 enable us to describe in full details the simple and clear structure of $\mathrm{RV}_{1/\pi}-$solutions of equation (E_1) with $\mathrm{RV}_{1/\pi}-$coefficients under the condition (C_1). We denote by $\mathcal{R}_{1/\pi}$ the class of all regularly varying solutions with respect to $1/\pi(t)$ of equation (E_1) under the condition (C_1) and introduce the following symbols for subclasses of $\mathcal{R}_{1/\pi}$:

$$\mathcal{R}_{1/\pi}(\rho) = \mathcal{R}_{1/\pi} \cap \mathrm{RV}_{1/\pi}(\rho),$$

$$\mathrm{tr} - \mathcal{R}_{1/\pi}(\rho) = \mathcal{R}_{1/\pi} \cap \mathrm{tr} - \mathrm{RV}_{1/\pi}(\rho), \qquad \mathrm{ntr} - \mathcal{R}_{1/\pi}(\rho) = \mathcal{R}_{1/\pi} \cap \mathrm{ntr} - \mathrm{RV}_{1/\pi}(\rho).$$

Corollary 2.2.1 *Let* $p \in \mathrm{RV}_{1/\pi}(\eta)$, $q \in \mathrm{RV}_{1/\pi}(\sigma)$ *and* (C_1) *holds.*

(i) *If* $\sigma < \beta - \frac{\eta}{\alpha} + 1$, *then* $\mathcal{R}_{1/\pi} = \mathrm{tr} - \mathcal{R}_{1/\pi}(-1) \cup \mathrm{tr} - \mathcal{R}_{1/\pi}(0)$;

(ii) *If* $\sigma = \beta - \frac{\eta}{\alpha} + 1$ *and* $W_\beta < \infty$, *then* $\mathcal{R}_{1/\pi} = \mathrm{tr} - \mathcal{R}_{1/\pi}(-1) \cup \mathrm{tr} - \mathcal{R}_{1/\pi}(0)$;

(iii) *If* $\sigma = \beta - \frac{\eta}{\alpha} + 1$ *and* $W_\beta = \infty$, *then* $\mathcal{R}_{1/\pi} = \mathrm{ntr} - \mathcal{R}_{1/\pi}(-1) \cup \mathrm{tr} - \mathcal{R}_{1/\pi}(0)$;

(iv) *If* $\beta - \frac{\eta}{\alpha} + 1 < \sigma < \alpha - \frac{\eta}{\alpha} + 1$, *then* $\mathcal{R}_{1/\pi} = \mathcal{R}_{1/\pi}\left(\frac{\sigma-\alpha-1+\frac{\eta}{\alpha}}{\alpha-\beta}\right) \cup \mathrm{tr} - \mathcal{R}_{1/\pi}(0)$;

(v) *If* $\sigma = \alpha - \frac{\eta}{\alpha} + 1$ *and* $Z_\alpha < \infty$, *then* $\mathcal{R}_{1/\pi} = \mathrm{ntr} - \mathcal{R}_{1/\pi}(0) \cup \mathrm{tr} - \mathcal{R}_{1/\pi}(0)$;

(vi) *If* $\sigma = \alpha - \frac{\eta}{\alpha} + 1$ *and* $Z_\alpha = \infty$, *then* $\mathcal{R}_{1/\pi} = \emptyset$;

(vii) *If* $\sigma > \alpha - \frac{\eta}{\alpha} + 1$, *then* $\mathcal{R}_{1/\pi} = \emptyset$.

Remark 2.2.1 As mentioned in the Section 2.1, in some recent papers ([25] and [60]) the asymptotic forms of intermediate solutions of (E_1) have been obtained. So, we end this paper by comparing our main results with earlier ones, wanting to point out that our results are an improvement over existing results in several directions. Naito [60, Theorems 4.3,4.4] determined asymptotic forms of intermediate solutions of (E) assuming that

$$q(t) \sim \frac{\kappa}{p(t)^{1/\alpha}} \pi(t)^{-\mu} \omega_0(\pi(t)), \quad t \to \infty, \tag{2.2.40}$$

where κ is a positive constant, $\omega_0(t)$ is a positive continuously differentiable function on an interval $(0, \tau_0]$, $0 < \tau_0 < 1$ and either

$$\beta + 1 < \mu < \alpha + 1 \quad \text{and} \quad \lim_{s \to 0+} \frac{s\, \omega_0'(s)}{\omega_0(s)} = 0\,, \tag{2.2.41}$$

or

$$\mu = \beta + 1 \quad \text{and} \quad \lim_{s \to 0+} \frac{s\, |\log s|\, \omega_0'(s)}{\omega_0(s)} = 0\,. \tag{2.2.42}$$

It has been proved that equation (E$_1$) has a slowly decaying solution and moreover, every slowly decaying solution $x(t)$ of (E$_1$) satisfies

$$(2.2.43)\qquad x(t) \sim \left(\frac{\kappa}{\alpha(1-\nu)\nu^{\alpha}}\right)^{\frac{1}{\alpha-\beta}} \pi(t)^{\nu}\omega_0(\pi(t))^{\frac{1}{\alpha-\beta}}, \qquad t \to \infty\,,$$

where $\nu = (\alpha - \mu + 1)/(\alpha - \beta)$, or

$$(2.2.44)\qquad x(t) \sim \left(\frac{\kappa(\alpha-\beta)}{\alpha}\right)^{\frac{1}{\alpha-\beta}} \pi(t)\big(|\log \pi(t)|\,\omega_0(\pi(t))\big)^{\frac{1}{\alpha-\beta}}, \qquad t \to \infty$$

according to whether (2.2.41) or (2.2.42) holds, respectively. The assumption on $w_0(t)$ in (2.2.41) and (2.2.42) shows that $w_0(t)$ is slowly varying at zero, so that $w_0(1/t)$ is slowly varying at infinity. This implies that the condition (2.2.40) means that $p(t)^{1/\alpha}q(t)$ is in fact regularly varying at infinity with respect to $1/\pi(t)$ of index μ and that an intermediate positive solution satisfying (2.2.43) or (2.2.44) is regularly varying at infinity with respect to $1/\pi(t)$ of index $-\nu$ or -1, respectively. Thus results presented in [60] is essentially concerned with the existence of generalized regularly varying solutions of a particular equation of the form (E$_1$) with generalized regularly varying coefficients $p(t)$ and $q(t)$. It should be notice that this results are covered by our results because asymptotic formulas (2.2.43) and (2.2.44) follow from Theorem 2.2.2 and Theorem 2.2.3, respectively, applied to the special case under consideration. However, the use of theory of regular variation allows us to reduce the assumption on function $w_0(t)$ in (2.2.40) from continuous differentiability to only continuity. Finally, we emphasize that our main results provide sharp criteria for equation (E$_1$) to possess three possible types of intermediate regularly varying solutions with respect to $1/\pi(t)$, listed in (2.2.22), while in [60] only sufficient conditions for the existence of intermediate solutions are given, and neither the existence nor the asymptotic behavior for $x(t) \in \mathrm{SV}_{1/\pi}$ has been investigated.

2.3 Classification and existence of positive solutions of (E$_1$) under the condition (C$_2$)

We assume that $p, q : [a, \infty) \to (0, \infty)$ are continuous functions and that (C$_2$) holds. The condition (C$_2$) enables us to define the increasing function $\Pi(t)$ as

$$(2.3.1)\qquad \Pi(t) = \int_a^t \frac{ds}{p(s)^{\frac{1}{\alpha}}}\,, \quad t \geq a.$$

It is easily seen (Elbert and Kusano [11]) that if $x(t)$ is an eventually positive solution of (E$_1$), then there are positive constants c_1 and c_2 such that

$$(2.3.2)\qquad c_1 \leq x(t) \leq c_2\Pi(t), \quad \text{for all large} \quad t.$$

More precisely, the asymptotic behavior of any positive solution $x(t)$ of (E_1) falls into one of the following three types:

$$\lim_{t\to\infty}\frac{x(t)}{\Pi(t)} = \text{const} > 0. \tag{2.3.3}$$

$$\lim_{t\to\infty} x(t) = \infty \quad \text{and} \quad \lim_{t\to\infty}\frac{x(t)}{\Pi(t)} = 0; \tag{2.3.4}$$

$$\lim_{t\to\infty} x(t) = \text{const} > 0; \tag{2.3.5}$$

Solutions of type (2.3.3), (2.3.4), (2.3.5) are often called, respectively, *dominant*, *intermediate* and *subdominant* solutions. It should be noticed (see [11], [41] and [60]) that oscillation of all solutions, as well as the existence of the positive solutions of each of the above types for the equation (E_1) with continuous coefficients $p(t)$, $q(t)$ can be completely characterized by the convergence (or divergence) of integrals:

$$I_\beta = \int_a^\infty q(t)\,\Pi(t)^\beta dt, \qquad J_\alpha = \int_a^\infty \left(\frac{1}{p(t)}\int_t^\infty q(s)ds\right)^{\frac{1}{\alpha}} dt.$$

Theorem 2.3.1 *Let $p(t), q(t) \in C[a,\infty)$ and* (C_2) *holds.*

(a) All solutions of (E_1) *are oscillatory if and only if $I_\beta = \infty$;*

(b) Equation (E_1) *has a positive solution of type* (2.3.3) *if and only if $I_\beta < \infty$;*

(c) Equation (E_1) *has a positive solution of type* (2.3.4) *if and only if $J_\alpha = \infty$ and $I_\beta < \infty$;*

(d) Equation (E_1) *has a positive solution of type* (2.3.5) *if and only if $J_\alpha < \infty$;*

2.4 Asymptotic behavior of intermediate solutions of (E_1) under the condition (C_2)

We assume that (C_2) holds and that the functions $p(t)$ and $q(t)$ are generalized regularly varying functions of indices η and σ with respect to $\Pi(t)$, which is defined with (2.3.1), and search for the intermediate solutions $x(t) \in \mathrm{RV}_\Pi(\rho)$ of (E_1). Since (2.3.2) holds, the regularity index ρ of $x(t)$ satisfies $\rho \in [0,1]$, while for the slowly varying part $l_x(t)$ of $x(t)$ it is true that either $l_x(t) \to \infty$ or $l_x(t) \to 0$ as $t \to \infty$ according as $\rho = 0$ or $\rho = 1$. Therefore, it is natural to divide the totality of intermediate RV_Π-solutions of (E_1) into the following three disjoint subclasses:

$$\text{ntr-RV}_\Pi(0), \qquad \mathrm{RV}_\Pi(\rho) \;\text{ with }\; \rho \in (0,1), \qquad \text{ntr-RV}_\Pi(1)\,.$$

Our main results formulated below characterize completely the membership of each of the three subclasses of solutions and show that all members of each subclass enjoy one and the same asymptotic behavior as $t \to \infty$.

Theorem 2.4.1 *Let $p(t) \in \mathrm{RV}_\Pi(\eta)$, $q(t) \in \mathrm{RV}_\Pi(\sigma)$ and* (C$_2$) *hold. Equation* (E$_1$) *has intermediate solutions $x(t) \in \mathrm{ntr}-\mathrm{RV}_\Pi(1)$ if and only if*

$$\sigma = -\frac{\eta}{\alpha} - \beta - 1 \quad \text{and} \quad I_\beta < \infty, \tag{2.4.1}$$

in which case the asymptotic behavior of any such solution $x(t)$ is governed by the unique formula

$$x(t) \sim \Pi(t)\left(\frac{\alpha-\beta}{\alpha}\int_t^\infty \Pi(s)^\beta q(s)ds\right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty. \tag{2.4.2}$$

Theorem 2.4.2 *Let $p(t) \in \mathrm{RV}_\Pi(\eta)$, $q(t) \in \mathrm{RV}_\Pi(\sigma)$ and* (C$_2$) *hold. Equation* (E$_1$) *has intermediate solutions $x(t) \in \mathrm{RV}_\Pi(\rho)$, $\rho \in (0,1)$, if and only if*

$$-\frac{\eta}{\alpha} - \alpha - 1 < \sigma < -\frac{\eta}{\alpha} - \beta - 1 \tag{2.4.3}$$

in which case ρ is defined by

$$\rho = \frac{\frac{\eta}{\alpha} + \sigma + \alpha + 1}{\alpha - \beta}, \tag{2.4.4}$$

and the asymptotic behavior of any such solution $x(t)$ is governed by the unique formula

$$x(t) \sim \left(\frac{\Pi(t)^{\alpha+1} p(t)^{\frac{1}{\alpha}} q(t)}{\alpha(1-\rho)\rho^\alpha}\right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty. \tag{2.4.5}$$

Theorem 2.4.3 *Let $p(t) \in \mathrm{RV}_\Pi(\eta)$, $q(t) \in \mathrm{RV}_\Pi(\sigma)$ and* (C$_2$) *hold. Equation* (E$_1$) *has intermediate solutions $x(t) \in \mathrm{ntr}-\mathrm{SV}_\Pi$ if and only if*

$$\sigma = -\frac{\eta}{\alpha} - \alpha - 1 \quad \text{and} \quad J_\alpha = \infty \tag{2.4.6}$$

in which case the asymptotic behavior of any such solution $x(t)$ is governed by the unique formula

$$x(t) \sim \left(\frac{\alpha-\beta}{\alpha}\int_a^t \left(\frac{1}{p(s)}\int_s^\infty q(r)dr\right)^{\frac{1}{\alpha}} ds\right)^{\frac{\alpha}{\alpha-\beta}}, \quad t \to \infty. \tag{2.4.7}$$

Main results (Theorems 2.4.1, 2.4.2, 2.4.3) combined with Theorem 2.3.1 enable us to describe in full details the simple and clear structure of RV_Π-solutions of equation (E_1) under the condition (C_2) with RV_Π-coefficients. We denote by $\mathcal{R}_\Pi$ the class of all regularly varying solutions with respect to $\Pi(t)$ of equation (E_1) under the condition (C_2) and introduce the following symbols for some important subclasses of $\mathcal{R}_\Pi$:

$$\mathcal{R}_\Pi(\rho) = \mathcal{R}_\Pi \cap \mathrm{RV}_\Pi(\rho), \qquad \mathrm{tr} - \mathcal{R}_\Pi(\rho) = \mathcal{R}_\Pi \ \cap \ \text{tr-}\mathrm{RV}_\Pi(\rho),$$

$$\mathrm{ntr} - \mathcal{R}_\Pi(\rho) = \mathcal{R}_\Pi \ \cap \ \text{ntr-}\mathrm{RV}_\Pi(\rho).$$

Corollary 2.4.1 *Let* $p \in \mathrm{RV}_\Pi(\eta)$, $q \in \mathrm{RV}_\Pi(\sigma)$ *and* (C_2) *holds .*

(i) *If* $\sigma < -\frac{\eta}{\alpha} - \alpha - 1$, *then* $\mathcal{R}_\Pi = tr - \mathcal{R}_\Pi(1) \cup tr - \mathcal{R}_\Pi(0)$.

(ii) *If* $\sigma = -\frac{\eta}{\alpha} - \alpha - 1$ *and* $J_\alpha < \infty$, *then* $\mathcal{R}_\Pi = tr - \mathcal{R}_\Pi(1) \cup tr - \mathcal{R}_\Pi(0)$.

(iii) *If* $\sigma = -\frac{\eta}{\alpha} - \alpha - 1$ *and* $J_\alpha = \infty$, *then* $\mathcal{R}_\Pi = tr - \mathcal{R}_\Pi(1) \cup ntr - \mathcal{R}_\Pi(0)$.

(iv) *If* $-\frac{\eta}{\alpha} - \alpha - 1 < \sigma < -\frac{\eta}{\alpha} - \beta - 1$, *then* $\mathcal{R}_\Pi = tr - \mathcal{R}_\Pi(1) \cup \mathcal{R}_\Pi\left(\frac{\frac{\eta}{\alpha}+\sigma+\alpha+1}{\alpha-\beta}\right)$.

(v) *If* $\sigma = -\frac{\eta}{\alpha} - \beta - 1$ *and* $I_\beta < \infty$, *then* $\mathcal{R}_\Pi = tr - \mathcal{R}_\Pi(1) \cup ntr - \mathcal{R}_\Pi(1)$.

(vi) *If* $\sigma = -\frac{\eta}{\alpha} - \beta - 1$ *and* $I_\beta = \infty$, *then* $\mathcal{R}_\Pi = \emptyset$.

(vii) *If* $\sigma > -\frac{\eta}{\alpha} - \beta - 1$, *then* $\mathcal{R}_\Pi = \emptyset$.

2.5 Asymptotic behavior of intermediate regularly varying solutions of (E_1) in the sense of Karamata

As mentioned before, the class of classical Karamata functions is the subset of the class of generalized Karamata functions, so in case the coefficients $p(t)$ and $q(t)$ of equation (E_1) are *regularly varying* the detailed information can be acquired about the existence and asymptotic behavior of *regularly varying solutions* $x(t)$ of (E_1).

We suppose that $p(t) \in \mathrm{RV}(\eta)$, $q(t) \in \mathrm{RV}(\sigma)$ and search for solutions $x(t)$ of (E_1) belonging to the class $\mathrm{RV}(\rho)$. We require first that $p(t)$ satisfies condition (C_1), which implies $\eta \geq \alpha$. Our attention is focused on the case where $\eta > \alpha$, since not all functions $p(t)$ with $\eta = \alpha$ satisfy (C_1). As is easily seen, if $\eta > \alpha$, then

$$\pi(t) \in \mathrm{RV}\left(\frac{\alpha-\eta}{\alpha}\right), \ P(t) = \frac{1}{\pi(t)} \in \mathrm{RV}\left(\frac{\eta-\alpha}{\alpha}\right) \text{ and } P^{-1}(t) \in \mathrm{RV}\left(\frac{\alpha}{\eta-\alpha}\right),$$

where P^{-1} denotes inverse of P. It follows that $p(t)$, $q(t)$ and $x(t)$ can be considered as generalized regularly varying functions with respect to $1/\pi(t)$. More precisely,

$$p(t) \in \mathrm{RV}_{1/\pi}\left(\frac{\alpha\,\eta}{\eta-\alpha}\right), \quad q(t) \in \mathrm{RV}_{1/\pi}\left(\frac{\alpha\,\sigma}{\eta-\alpha}\right), \quad x(t) \in \mathrm{RV}_{1/\pi}\left(\frac{\alpha\,\rho}{\eta-\alpha}\right).$$

The above observation makes it possible to apply our main results to the present situation, giving rise to new results on the asymptotic analysis of equation (E_1) under the condition (C_1) which are formulated in terms of generalized Karamata functions. Translating the obtained results into the language of classical Karamata functions then provides the accurate information about all possible regularly varying solutions for the equation (E_1) under the condition (C_1) with regularly varying coefficients $p(t)$ and $q(t)$.

Corollary 2.5.1 *Let* $p(t) \in \mathrm{RV}(\eta)$, $q(t) \in \mathrm{RV}(\sigma)$ *and* (C_1) *holds. Equation* (E_1) *has intermediate nontrivial slowly varying solutions* $x(t)$ *if and only if*

$$\sigma = -1-\alpha+\eta \ \ \textit{and} \ \ Z_\alpha < \infty.$$

Any such solution $x(t)$ *enjoys one and the same asymptotic behavior* $x(t) \sim Y_1(t)$, $t \to \infty$, *where* $Y_1(t)$ *is given by* (2.2.11).

Corollary 2.5.2 *Let* $p(t) \in \mathrm{RV}(\eta)$, $q(t) \in \mathrm{RV}(\sigma)$ *and* (C_1) *holds. Equation* (E_1) *has intermediate solutions* $x(t) \in \mathrm{RV}(\rho)$ *with* $\rho \in (1-\frac{\eta}{\alpha}, 0)$ *if and only if*

$$-1-\beta+\frac{\beta}{\alpha}\eta < \sigma < -1-\alpha+\eta,$$

in which case ρ *is given by*

$$\rho = \frac{1+\alpha+\sigma-\eta}{\alpha-\beta}$$

and the asymptotic behavior of any such solution $x(t)$ *is governed by the unique formula*

$$x(t) \sim \left(\frac{t^{1+\alpha}\,p(t)^{-1}q(t)}{(-\rho)^\alpha(\alpha(\rho-1)+\eta)}\right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty.$$

Corollary 2.5.3 *Let* $p(t) \in \mathrm{RV}(\eta)$, $q(t) \in \mathrm{RV}(\sigma)$ *and* (C_1) *holds. Equation* (E_1) *has intermediate solutions* $x(t) \in \mathrm{ntr}-\mathrm{RV}(1-\frac{\eta}{\alpha})$ *if and only if*

$$\sigma = -1-\beta+\frac{\beta}{\alpha}\eta \ \ \textit{and} \ \ W_\beta = \infty.$$

Any such solution $x(t)$ *enjoys one and the same asymptotic behavior* $x(t) \sim Y_3(t)$, $t \to \infty$, *where* $Y_3(t)$ *is given by* (2.2.18).

Now, we suppose that $p(t) \in \mathrm{RV}(\eta)$ and $q(t) \in \mathrm{RV}(\sigma)$ and (C_2) holds. Note that condition (C_2) is satisfied if $\eta \leq \alpha$. In what follows, we assume that $\eta < \alpha$, excluding the case $\eta = \alpha$ because of computational difficulty. Then, it is easy to see that

$$\Pi(t) \in \mathrm{RV}\Big(\frac{\alpha-\eta}{\alpha}\Big) \quad \text{and} \quad \Pi^{-1}(t) \in \mathrm{RV}\Big(\frac{\alpha}{\alpha-\eta}\Big),$$

so that

$$p(t) \in \mathrm{RV}_{\Pi}\Big(\frac{\alpha\eta}{\alpha-\eta}\Big), \quad q(t) \in \mathrm{RV}_{\Pi}\Big(\frac{\alpha\sigma}{\alpha-\eta}\Big) \quad \text{and} \quad x(t) \in \mathrm{RV}_{\Pi}\Big(\frac{\alpha\rho}{\alpha-\eta}\Big).$$

Having the above observation in mind, we easily see that our theory of generalized regularly varying solutions can be applied to the present situation, giving birth to the following results as corollaries to Theorems 2.4.1-2.4.3.

Corollary 2.5.4 *Let $p(t) \in \mathrm{RV}(\eta)$ and $q(t) \in \mathrm{RV}(\sigma)$ and (C_2) holds. Equation (E_1) has intermediate solutions $x(t) \in \mathrm{ntr}-\mathrm{RV}\big(1-\frac{\eta}{\alpha}\big)$ if and only if*

$$\sigma = \frac{\beta}{\alpha}\eta - \beta - 1 \quad \textit{and} \quad I_\beta < \infty,$$

in which case any such solution $x(t)$ enjoys one and the same asymptotic behavior (2.4.2).

Corollary 2.5.5 *Let $p(t) \in \mathrm{RV}(\eta)$ and $q(t) \in \mathrm{RV}(\sigma)$ and (C_2) holds. Equation (E_1) has intermediate solutions $x(t) \in \mathrm{RV}(\rho)$, $\rho \in \big(0, 1-\frac{\eta}{\alpha}\big)$, if and only if*

$$\eta - \alpha - 1 < \sigma < \frac{\beta}{\alpha}\eta - \beta - 1,$$

in which case ρ is given by

$$\rho = \frac{-\eta + \sigma + \alpha + 1}{\alpha - \beta},$$

and any such solution $x(t)$ enjoys one and the same asymptotic behavior

$$x(t) \sim \left(\frac{t^{\alpha+1}p(t)^{-1}q(t)}{\big((1-\rho)\alpha-\eta\big)\rho^\alpha}\right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty.$$

Corollary 2.5.6 *Let $p(t) \in \mathrm{RV}(\eta)$ and $q(t) \in \mathrm{RV}(\sigma)$ and (C_2) holds. Equation (E_1) has intermediate solutions $x(t) \in \mathrm{ntr}-\mathrm{SV}$ if and only if*

$$\sigma = \eta - \alpha - 1 \quad \textit{and} \quad J_\alpha = \infty,$$

in which case any such solution $x(t)$ enjoys one and the same asymptotic behavior (2.4.7).

2.6 Examples

Now, we present four examples that illustrate results presented in previous sections. First example illustrates Theorems 2.2.1-2.2.3.

Example 2.6.1 Consider the equation

$$\big(e^{\alpha t}\,|x'|^{\alpha-1}\,x'\big)' + q(t)|x|^{\beta-1}\,x = 0\,,\ \ t>2, \quad \alpha>\beta>0. \tag{2.6.1}$$

Here $p(t)=e^{\alpha t}$ satisfies (C_1) and $P(t)=1/\pi(t)=e^t$, i.e. $p(t)\in \mathrm{RV}_{e^t}(\alpha)$, so that $\eta=\alpha$.

(i) Suppose that

$$q(t)\sim \frac{\alpha}{2^\alpha}\,t^{-\frac{\alpha}{2}}\,e^{\alpha\,t+(\beta-\alpha)\,\sqrt{t}}\,r(t),\quad t\to\infty, \tag{2.6.2}$$

where $r(t)$ is continuous function on $(2,\infty)$, such that $\lim\limits_{t\to\infty} r(t)=1$. Then, $q(t)\in \mathrm{RV}_{e^t}(\alpha)$, i.e. $\eta=\sigma=\alpha$, so that $\sigma=\alpha-\frac{\eta}{\alpha}+1$ and we see that

$$\int_t^\infty \left(\frac{1}{p(s)}\int_2^s q(r)\,dr\right)^{\frac{1}{\alpha}} ds \sim \frac{\alpha^{\frac{1}{\alpha}}}{2}\int_t^\infty e^{-s}\left(\int_2^s r^{-\frac{\alpha}{2}}\,e^{\alpha\,r+(\beta-\alpha)\,\sqrt{r}}\,dr\right)^{\frac{1}{\alpha}} ds$$

$$\sim \frac{1}{2}\int_t^\infty \frac{e^{\frac{\beta-\alpha}{\alpha}\,\sqrt{s}}\,ds}{\sqrt{s}} \sim \frac{\alpha}{\alpha-\beta}\,e^{\frac{\beta-\alpha}{\alpha}\,\sqrt{t}}\to 0\,,\quad t\to\infty\,,$$

implying $Z_\alpha<\infty$. Therefore, by Theorem 2.2.1 there exist nontrivial SV-solutions with respect to e^t of (2.6.1) and any such solution $x(t)$ has asymptotic behavior

$$x(t)\ \sim\ \left(\frac{\alpha-\beta}{2\,\alpha^{1-\frac{1}{\alpha}}}\int_t^\infty e^{-s}\ \left(\int_2^s r^{-\frac{\alpha}{2}}\,e^{\alpha\,r+(\beta-\alpha)\sqrt{r}}\,dr\right)^{\frac{1}{\alpha}} ds\right)^{\frac{\alpha}{\alpha-\beta}}$$

$$\sim\ \left(\frac{\alpha-\beta}{\alpha}\,\frac{\alpha}{\alpha-\beta}\,e^{\frac{\beta-\alpha}{\alpha}\,\sqrt{t}}\right)^{\frac{\alpha}{\alpha-\beta}} \sim e^{-\sqrt{t}},\quad t\to\infty.$$

If in (2.6.2) instead of " $\sim$ " one has " $=$ " and in particular $r(t)=1-\frac{1}{2\sqrt{t}}-\frac{1}{2t}$, then (2.6.1) possesses an exact solution $x(t)=e^{-\sqrt{t}}$.

(ii) Suppose that

$$q(t)\sim \frac{\alpha}{2^{\alpha+1}}\,t^{\alpha-\beta}\,e^{\frac{\alpha+\beta}{2}\,t}\,r(t),\quad t\to\infty, \tag{2.6.3}$$

where $r(t)$ is continuous function on $(2,\infty)$ such that $\lim\limits_{t\to\infty} r(t)=1$. It is clear that $q(t)$ is regularly varying with respect to e^t of index

$$\sigma=\frac{\alpha+\beta}{2}\in\Big(\beta-\frac{\eta}{\alpha}+1\ ,\ \alpha-\frac{\eta}{\alpha}+1\Big)=(\beta,\alpha)$$

and that

$$\rho = \frac{\sigma - \alpha + 1 + \frac{\eta}{\alpha}}{\alpha - \beta} = -\frac{1}{2}.$$

By Theorem 2.2.2 there exist regularly varying solutions of index ρ with respect to e^t of (2.6.1) and any such solution $x(t)$ has asymptotic behavior

$$x(t) \sim \left(\frac{2^{\alpha+1}}{\alpha} \, \pi(t)^{1+\alpha} \, p(t)^{\frac{1}{\alpha}} \, q(t) \right)^{\frac{1}{\alpha-\beta}} \sim t \, e^{-\frac{t}{2}}, \quad t \to \infty.$$

Observe that if in (2.6.3) instead " $\sim$ " one has " $=$ " and $r(t) = \left(1 - \frac{2}{t}\right)^{\alpha-1}$, then $x(t) = t\, e^{-\frac{t}{2}}$ is an exact solution of (2.6.1).

(iii) Suppose that

$$q(t) \sim \alpha \, t^{\alpha-\beta-1} \, e^{\beta\, t} \, r(t), \quad t \to \infty, \tag{2.6.4}$$

where $r(t)$ is continuous function on $(2, \infty)$ such that $\lim\limits_{t\to\infty} r(t) = 1$. Here, $q(t) \in \mathrm{RV}_{e^t}(\beta)$. Therefore, $\sigma = \beta - \frac{\eta}{\alpha} + 1$ and

$$\int_2^t q(s)\, \pi(s)^\beta \, ds \sim \alpha \int_2^t s^{\alpha-\beta-1} \, ds \sim \frac{\alpha}{\alpha - \beta} \, t^{\alpha-\beta} \to \infty\,, \quad t \to \infty,$$

implying $W_\beta = \infty$. By Theorem 2.2.3 there exist solutions $x(t) \in \mathrm{ntr} - \mathrm{RV}_{e^t}(-1)$ of (2.6.1) and any such solution $x(t)$ has asymptotic behavior

$$x(t) \sim e^{-t} \left(\frac{\alpha - \beta}{\alpha} \frac{\alpha}{\alpha - \beta} \, t^{\alpha-\beta} \right)^{\frac{1}{\alpha-\beta}} \sim t\, e^{-t}, \quad t \to \infty.$$

If in (2.6.4) instead of " $\sim$ " one has " $=$ " and in particular $r(t) = \left(1 - \frac{1}{t}\right)^{\alpha-1}$, then (2.6.1) has an exact solution $x(t) = t\, e^{-t}$.

In the following example we consider equation with regularly varying function in the sense of Karamata, applying Corollaries 2.5.1-2.5.3.

Example 2.6.2 Consider the equation

$$\left(t^{2\alpha} \left(\frac{2\sqrt{\log t}}{\log t - 1} \right)^{\alpha} |x'|^{\alpha-1} \, x' \right)' + q(t)|x|^{\beta-1}\, x = 0\,, \;\; t > e, \quad \alpha > \beta > 0. \tag{2.6.5}$$

Here $p(t) = t^{2\alpha} \left(\frac{2\sqrt{\log t}}{\log t - 1} \right)^{\alpha} \in \mathrm{RV}(2\alpha)$, $\pi(t) \sim \frac{1}{2t}\sqrt{\log t} \in \mathrm{RV}(-1)$, $t \to \infty$ and $p(t)$ satisfies the condition (C_1).

(i) Suppose that

$$q(t) \sim \alpha t^{\alpha-1}(\log t)^{\frac{\beta}{2}-2\alpha}\, r(t), \quad t \to \infty \tag{2.6.6}$$

where $r(t)$ is a positive continuous function on (e,∞) such that $\lim_{t\to\infty} r(t) = 1$. The regularity index of $q(t)$ is $\sigma = \alpha - 1$, and thus $\sigma = -1-\alpha+\eta$ and

$$\int_t^\infty \left(\frac{1}{p(s)}\int_e^s q(r)\,dr\right)^{\frac{1}{\alpha}} ds \sim \frac{1}{2}\int_t^\infty \frac{\log s - 1}{s}\,(\log s)^{\frac{\beta-5\alpha}{2\alpha}}\,ds \sim \frac{\alpha}{\alpha-\beta}\,(\log t)^{\frac{\beta-\alpha}{2\alpha}} \to 0,$$

as $t\to\infty$, implying $Z_\alpha < \infty$. Therefore, by Corollary 2.5.1 there exist nontrivial SV-solutions of (2.6.5) and any such solution $x(t)$ has asymptotic behavior

$$x(t) \sim \left(\frac{\alpha-\beta}{\alpha}\,\frac{\alpha}{\alpha-\beta}\,(\log t)^{\frac{\beta-\alpha}{2\alpha}}\right)^{\frac{\alpha}{\alpha-\beta}} \sim (\log t)^{-\frac{1}{2}}, \quad t\to\infty.$$

If in (2.6.6) instead " $\sim$ " one has " $=$ " and in particular

$$r(t) = \left(1 - \frac{3}{\log t} - \frac{1}{\log^2 t}\right)\left(1-\frac{1}{\log t}\right)^{-\alpha-1},$$

then (2.6.5) has an exact nontrivial SV-solution $x(t) = (\log t)^{-\frac{1}{2}}$.

(ii) Suppose that

$$q(t) \sim \frac{\alpha}{2}\, t^{\frac{\alpha+\beta}{2}-1}\,(\log t)^{-\frac{\beta}{2}}, \quad t\to\infty. \tag{2.6.7}$$

It is clear that now $q(t)$ is regularly varying function of index

$$\sigma = \frac{\alpha+\beta}{2} - 1 \in \left(-1-\beta+\frac{\beta}{\alpha}\eta\ ,\ -1-\alpha+\eta\right) = (\beta-1, \alpha-1)$$

and that

$$\rho = \frac{1+\alpha+\sigma-\eta}{\alpha-\beta} = -\frac{1}{2}.$$

By Corollary 2.5.2 there exist nontrivial regularly varying solutions of index ρ of (2.6.5) and any such solution $x(t)$ has asymptotic behavior

$$x(t) \sim \left(\frac{2^{\alpha+1}}{\alpha}\,t^{1+\alpha}\,\frac{q(t)}{p(t)}\right)^{\frac{1}{\alpha-\beta}} \sim \sqrt{\frac{\log t}{t}}, \quad t\to\infty.$$

Observe that if in (2.6.7) instead " $\sim$ " one has " $=$ ", then $x(t) = \sqrt{\frac{\log t}{t}}$ is an exact solution of (2.6.5).

(iii) Suppose that

$$q(t) \sim \alpha\, 2^{\alpha-1}\, t^{\beta-1}\, (\log t)^{\frac{\alpha}{2}-\beta-1}, \quad t \to \infty. \tag{2.6.8}$$

Here, $q(t) \in \mathrm{RV}\,(\beta-1)$, so that $\sigma = -1-\beta+\frac{\beta}{\alpha}\eta$ and

$$\int_e^t q(s)\,\pi(s)^\beta\, ds \sim \alpha\; 2^{\alpha-\beta-1}\int_e^t \frac{(\log s)^{\frac{\alpha-\beta}{2}-1}}{s}\, ds \sim \frac{\alpha}{\alpha-\beta} 2^{\alpha-\beta}(\log t)^{\frac{\alpha-\beta}{2}} \to \infty,$$

as $t \to \infty$, implying $W_\beta = \infty$. By Corollary 2.5.3 there exist nontrivial solutions $x(t) \in \mathrm{RV}(1-\frac{\eta}{\alpha}) = \mathrm{RV}(-1)$ of (2.6.5) and any such solution $x(t)$ has asymptotic behavior

$$x(t) \sim \frac{\sqrt{\log t}}{2t}\left(\frac{\alpha-\beta}{\alpha}\,\frac{\alpha}{\alpha-\beta}\; 2^{\alpha-\beta}\; (\log t)^{\frac{\alpha-\beta}{2}}\right)^{\frac{1}{\alpha-\beta}} \sim \frac{\log t}{t}, \quad t \to \infty.$$

Observe that if in (2.6.8) instead " $\sim$ " one has " $=$ ", then $x(t) = \frac{\log t}{t}$ is an exact solution of (2.6.5).

In the following two examples we illustrate results of Theorems 2.4.1-2.4.3 and its Corollaries 2.5.4-2.5.6.

Example 2.6.3 Consider the equation

$$(e^{-\alpha t}|x'|^{\alpha-1}x')' + q(t)|x|^{\beta-1}x = 0, \quad \alpha > \beta > 0. \tag{2.6.9}$$

Here $p(t) = e^{-\alpha t}$ satisfies (C_2). Since $\Pi(t) \sim e^t$, $t \to \infty$, $p \in \mathrm{RV}_\Pi(-\alpha)$, i.e., $\eta = -\alpha$.

(i) Suppose that

$$q(t) \sim \frac{\alpha(\alpha-\beta)}{2t\sqrt{\log t}}\, e^{-\beta t-(\alpha-\beta)^2\sqrt{\log t}}, \quad t \to \infty.$$

It is clear that $q \in \mathrm{RV}_\Pi(-\beta)$, i.e., $\sigma = -\beta$, and so we see that $\sigma = -\frac{\eta}{\alpha} - \beta - 1$ and that

$$\begin{aligned}\int_t^\infty \Pi(s)^\beta q(s) ds &\sim \alpha(\alpha-\beta)\int_t^\infty \frac{e^{-(\alpha-\beta)^2\sqrt{\log s}}}{2s\sqrt{\log s}} ds \\ &= \frac{\alpha}{\alpha-\beta} e^{-(\alpha-\beta)^2\sqrt{\log t}} \longrightarrow 0, \quad t \to \infty,\end{aligned}$$

which implies that $I_\beta < \infty$. Therefore, from Theorem 2.4.1 it follows that equation (2.6.9) has intermediate solutions in $\mathrm{RV}_\Pi(1)$ all of which enjoy the unique asymptotic behavior

$$x(t) \sim e^{t-(\alpha-\beta)\sqrt{\log t}}, \quad t \to \infty.$$

(ii) Suppose that

$$q(t) \sim e^{-\frac{2\alpha+\beta}{3}t + t^\theta \cos\, t^\theta}, \quad t \to \infty,$$

where $\theta \in \left(0, \frac{1}{2}\right)$. Here $q \in \mathrm{RV}_\Pi(\sigma)$ with $\sigma = -\frac{2\alpha+\beta}{3}$. Since σ satisfies

$$-\frac{\eta}{\alpha} - \alpha - 1 = -\alpha < \sigma < -\beta = -\frac{\eta}{\alpha} - \beta - 1,$$

by Theorem 2.4.2 the equation (2.6.9) has intermediate solutions in $\mathrm{RV}_\Pi(\rho)$ with ρ given by (2.4.4), i.e.,

$$\rho = \frac{\frac{\eta}{\alpha} + \sigma + \alpha + 1}{\alpha - \beta} = \frac{1}{3},$$

and moreover all such solutions $x \in \mathrm{RV}_\Pi(1/3)$ enjoy the unique asymptotic behavior

$$x(t) \sim \left(\frac{3^{\alpha+1}}{2\alpha}\right)^{\frac{1}{\alpha-\beta}} e^{\frac{t}{3} + \frac{t^\theta \cos\, t^\theta}{\alpha-\beta}}, \quad t \to \infty.$$

(iii) Suppose that

$$q(t) \sim \frac{\alpha}{2^\alpha t^{\frac{\alpha}{2}}}\, e^{-\alpha t + \alpha\sqrt{t}}, \quad t \to \infty.$$

It is clear that $q \in \mathrm{RV}_\Pi(-\alpha)$, i.e., $\sigma = -\alpha = -\frac{\eta}{\alpha} - \alpha - 1$. Then, we have

$$\int_t^\infty q(s)ds \sim \frac{e^{-\alpha t + \alpha\sqrt{t}}}{2^\alpha t^{\frac{\alpha}{2}}}$$

and

$$\int_a^t \left(\frac{1}{p(s)} \int_s^\infty q(r)dr\right)^{\frac{1}{\alpha}} ds \sim \int_a^t \frac{e^{\sqrt{s}}}{2\sqrt{s}}\, ds \sim e^{\sqrt{t}},$$

as $t \to \infty$. Consequently, $J_\alpha = \infty$ and thus Theorem 2.4.3 ensures that (2.6.9) has intermediate solutions in SV_Π all of which obey the asymptotic formula

$$x(t) \sim \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} e^{\frac{\alpha}{\alpha-\beta}\sqrt{t}}, \quad t \to \infty.$$

Example 2.6.4 Consider the equation

$$\left(t^{\frac{\alpha}{2}} (\log t)^\alpha |x'|^{\alpha-1} x'\right)' + q(t)|x|^{\beta-1}x = 0, \quad \alpha > \beta > 0. \tag{2.6.10}$$

Here

$$p(t) = t^{\frac{\alpha}{2}} (\log t)^\alpha \in \mathrm{RV}\left(\frac{\alpha}{2}\right), \quad \Pi(t) \sim \frac{2\sqrt{t}}{\log t} \in \mathrm{RV}\left(\frac{1}{2}\right), \quad t \to \infty.$$

so that $\eta = \dfrac{\alpha}{2}$ and $p(t)$ satisfies (C_2).

(i) Suppose that

$$q(t) \sim \frac{1}{2^\beta t^{1+\frac{\beta}{2}}(\log t)^{1-\beta}(\log\log t)^{2-\frac{\beta}{\alpha}}}, \quad t \to \infty.$$

The regularity index of $q(t)$ is $\sigma = -1 - \frac{\beta}{2} = \frac{\beta}{\alpha}\eta - \beta - 1$, and

$$\begin{aligned}\int_t^\infty \Pi(s)^\beta q(s)ds \;&\sim\; \int_t^\infty \frac{ds}{s\log s(\log\log s)^{2-\frac{\beta}{\alpha}}} \\ &= \frac{\alpha}{\alpha-\beta}(\log\log t)^{\frac{\beta-\alpha}{\alpha}} \to 0, \quad t \to \infty,\end{aligned}$$

implying that $I_\beta < \infty$. Hence, from Corollary 2.5.4 it follows that (2.6.10) has intermediate solutions in $\mathrm{RV}\big(\frac{1}{2}\big)$ all of which obey the unique asymptotic formula

$$x(t) \sim \frac{2\sqrt{t}}{\log t\big(\log\log t\big)^{\frac{1}{\alpha}}}, \quad t \to \infty.$$

(ii) Suppose that

$$q(t) \sim t^{-\frac{\alpha+\beta+4}{4}} e^{(\log t)^\theta \cos\,(\log t)^\theta}, \quad t \to \infty,$$

where $\theta \in \big(0, \frac{1}{2}\big)$. The regularity index of $q(t)$ is $\sigma = -\frac{\alpha+\beta+4}{4}$ which satisfies

$$\eta - \alpha - 1 = -\frac{\alpha}{2} - 1 < \sigma < -\frac{\beta}{2} - 1 = \frac{\beta}{\alpha}\eta - \beta - 1.$$

Therefore, applying Corollary 2.5.5 we conclude that equation (2.6.10) has intermediate solutions in $\mathrm{RV}\big(\frac{1}{4}\big)$ all of which obey the unique asymptotic formula

$$x(t) \sim \left(\frac{4^{\alpha+1}}{\alpha}\right)^{\frac{1}{\alpha-\beta}} t^{\frac{1}{4}} \left[\frac{e^{(\log t)^\theta \cos\,(\log t)^\theta}}{(\log t)^\alpha}\right]^{\frac{1}{\alpha-\beta}}, \quad t \to \infty.$$

(iii) Suppose that

$$q(t) \sim \frac{\alpha}{2} t^{-\frac{\alpha}{2}-1}\left(\frac{\log t}{\log\log t}\right)^{\alpha-\beta}, \quad t \to \infty.$$

A simple computation shows that

$$\left(\frac{1}{p(t)}\int_t^\infty q(s)ds\right)^{\frac{1}{\alpha}} \sim \frac{1}{t\log t}\left(\frac{\log t}{\log\log t}\right)^{1-\frac{\beta}{\alpha}}, \quad t \to \infty,$$

implying that

$$\int_{\exp(e)}^{t} \left(\frac{1}{p(s)} \int_{s}^{\infty} q(r)dr \right)^{\frac{1}{\alpha}} ds \sim \int_{e}^{\log t} \frac{u^{-\frac{\beta}{\alpha}}}{(\log u)^{1-\frac{\beta}{\alpha}}} \sim \frac{\alpha}{\alpha-\beta} \left(\frac{\log t}{\log\log t} \right)^{1-\frac{\beta}{\alpha}}, \quad t \to \infty.$$

Consequently, by Corollary 2.5.6 equation (2.6.10) has slowly varying intermediate solutions $x(t)$ whose asymptotic behavior is governed by the unique formula

$$x(t) \sim \frac{\log t}{\log\log t}, \quad t \to \infty.$$

Chapter 3

Asymptotic behavior of positive solutions of quasilinear second order differential equation

The aim of this chapter is to establish the existence and asymptotic behavior of positive solutions at infinity of quasilinear second order equation

$$(\mathrm{E}_2) \qquad (p(t)\varphi(|x'(t)|)\,\mathrm{sgn}\, x'(t))' + q(t)\psi(x(t)) = 0, \quad t \geq a > 0,$$

under two different conditions

$$(\mathrm{C}_1) \qquad \int_a^\infty \varphi^{-1}(p(t)^{-1})\, dt < \infty,$$

$$(\mathrm{C}_2) \qquad \int_a^\infty \varphi^{-1}(p(t)^{-1})\, dt = \infty,$$

assuming that the coefficients $p(t)$ and $q(t)$ are regularly varying in the sense of Karamata. Therefore, the results presented in this chapter are generalization of results given in Section 2.5. Unlike the equation (E_1), in the case of the equation (E_2) with positive continuous coefficients, the necessary condition for the existence of intermediate solutions is still an open problem. If the coefficients of the equation, as well as the functions φ and ψ, are regularly varying functions it turns out that it is possible not only to determine the necessary and sufficient conditions for the existence of intermediate regularly varying solutions of this equation, but also the precise information about the asymptotic behavior at infinity of these solutions can be acquired.

The whole chapter is based on the original results contained in [54] and [56].

3.1 Classification and existence of positive decreasing solutions of (E_2) under the condition (C_1)

In this section we classify the set of positive decreasing solutions of (E_2) according to their asymptotic behavior at infinity under the assumptions that the functions $p, q : [a,\infty) \to (0,\infty)$ and $\varphi, \psi : (0,\infty) \to (0,\infty)$ are continuous, φ is increasing and that (C_1) holds. The condition (C_1) enables us to define the decreasing function $\pi(t)$ as

$$\pi(t) = \int_t^\infty \varphi^{-1}(p(s)^{-1})\, ds\ , \quad t \geq a.$$

Definition 3.1.1 *By a solution of* (E_2) *we mean a function* $x(t) : [T,\infty) \to \mathbb{R}$, $T \geq a$, *which is continuously differentiable together with* $p(t)\varphi(|x'(t)|)$ *on* $[T,\infty)$ *and satisfies the equation (E) at every point of* $[T,\infty)$.

It is easily seen (see [42]) that if $x(t)$ is a positive decreasing solution of (E_2), then there are positive constants c_1 and c_2, such that for all large t

$$c_1\pi(t) \leq x(t) \leq c_2. \tag{3.1.1}$$

More precisely, the asymptotic behavior of any positive decreasing solution $x(t)$ of (E_2) falls into one of the following three types:

$$\lim_{t\to\infty} \frac{x(t)}{\pi(t)} = \text{const} > 0; \tag{3.1.2}$$

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} p(t)\varphi(-x'(t)) = \infty; \tag{3.1.3}$$

$$\lim_{t\to\infty} x(t) = \text{const} > 0. \tag{3.1.4}$$

Solutions of type (3.1.2), (3.1.3), (3.1.4) are often called, respectively, *subdominant, intermediate* and *dominant* solutions.

It is known (see [42]) that the existence of positive solutions of subdominant and dominant type for the equation (E_2) with continuous coefficients $p(t), q(t), \varphi(s)$ and $\psi(s)$ can be completely characterized by the convergence or divergence of integrals

$$W = \int_a^\infty q(t)\psi(\pi(t))dt, \quad Z = \int_a^\infty \varphi^{-1}\left(p(t)^{-1}\int_a^t q(s)ds\right) dt.$$

Theorem 3.1.1 *Let* $p(t), q(t) \in C[a,\infty)$, $\varphi(s), \psi(s) \in C[0,\infty)$ *and* (C_1) *holds.*

(a) *Equation (E) has a positive solution of type* (3.1.2) *if and only if* $W < \infty$.

(b) *Equation (E) has a positive solution of type* (3.1.4) *if and only if* $Z < \infty$.

(c) *Equation (E) has a positive solution of type* (3.1.3) *if* $W = \infty$ *and* $Z < \infty$.

Note that only the sufficient condition for the existence of intermediate solutions of (E_2) is given in Theorem 3.1.1(c).

3.2 Asymptotic behavior of intermediate solutions of (E_2) under the condition (C_1)

This section is devoted to the study of the existence and asymptotic behavior of intermediate regularly varying solutions of the equation (E_2) with assumptions that (C_1) holds, φ is increasing and

$$\begin{aligned} &\varphi(s) \in \mathcal{RV}(\alpha), \quad \alpha > 0; \quad \psi(s) \in \mathcal{RV}(\beta), \quad \alpha > \beta > 0; \\ &p(t) \in \mathrm{RV}(\eta), \quad \eta > \alpha; \qquad q(t) \in \mathrm{RV}(\sigma), \quad \sigma \in \mathbb{R}. \end{aligned} \tag{3.2.1}$$

Using (1.2.3), we can express $\varphi(s)$, $\psi(s)$, $p(t)$ and $q(t)$ as

$$\varphi(s) = s^{\alpha} L_1(s), \quad L_1(s) \in \mathcal{SV}; \quad \psi(s) = s^{\beta} L_2(s), \quad L_2(s) \in \mathcal{SV}; \tag{3.2.2}$$

$$p(t) = t^{\eta} l_p(t), \quad l_p(t) \in \mathrm{SV}; \qquad q(t) = t^{\sigma} l_q(t), \quad l_q(t) \in \mathrm{SV}. \tag{3.2.3}$$

Since $\varphi(s)$ is an increasing function, then $\varphi(s)$ has the inverse function, denoted by $\varphi^{-1}(s)$ and from (3.2.2) we conclude that

$$\varphi^{-1}(s) \in \mathcal{RV}(1/\alpha) \quad \Rightarrow \quad \varphi^{-1}(s) = s^{1/\alpha} L(s), \quad L(s) \in \mathcal{SV}. \tag{3.2.4}$$

We also need two additional requirements for the slowly varying parts of φ and ψ:

$$L(tu(t)) \sim L(t), \quad t \to 0, \quad \forall u(t) \in \mathcal{SV} \cap C^1(\mathbb{R}); \tag{3.2.5}$$

$$L_2(tu(t)) \sim L_2(t), \quad t \to 0, \quad \forall u(t) \in \mathcal{SV} \cap C^1(\mathbb{R}). \tag{3.2.6}$$

It is easy to check that this is satisfied by e.g.

$$L_0(t) = \prod_{k=1}^{N} (\log_k t)^{\alpha_k}, \ \alpha_k \in \mathbb{R}, \quad \text{but not by} \quad L_0(t) = \exp \prod_{k=1}^{N} (\log_k t)^{\beta_k}, \ \beta_k \in (0,1),$$

where $\log_k t = \log \log_{k-1} t, \ k = 1, 2, \ldots$.

Remark 3.2.1 The condition (3.2.5) implies an useful property of the function φ^{-1}. For $u(t) \in \mathrm{SV} \cap C^1(\mathbb{R})$ and $\lambda \in \mathbb{R}^-$, applying Proposition 1.2.2-(iv), we have $u(t^{\frac{1}{\lambda}}) \in \mathcal{SV} \cap C^1(\mathbb{R})$. Using the substitution $t^\lambda = s$ ($s \to 0$ as $t \to \infty$) and (3.2.5) we obtain

$$L(t^\lambda u(t)) = L(s\, u(s^{\frac{1}{\lambda}})) \sim L(s) = L(t^\lambda),\ \ t \to \infty,\ \ \forall \lambda \in \mathbb{R}^-,\ \ \forall u(t) \in \mathrm{SV} \cap C^1(\mathbb{R}),$$

from which it follows that

$$\text{(3.2.7)}\quad \varphi^{-1}(t^\lambda u(t)) \sim \varphi^{-1}(t^\lambda) u(t)^{\frac{1}{\alpha}}, \quad t \to \infty, \quad \forall \lambda \in \mathbb{R}^-, \quad \forall u(t) \in \mathrm{SV} \cap C^1(\mathbb{R}).$$

Similarly, the condition (3.2.6) implies an useful property of the function ψ:

$$\text{(3.2.8)}\qquad \psi(t^\lambda u(t)) \sim \psi(t^\lambda) u(t)^\beta, \quad t \to \infty, \quad \forall \lambda \in \mathbb{R}^-, \quad \forall u(t) \in \mathrm{SV} \cap C^1(\mathbb{R}).$$

We seek such solutions $x(t)$ of (E_2) that can be expressed in the form

$$\text{(3.2.9)}\qquad x(t) = t^\rho l_x(t), \quad l_x(t) \in \mathrm{SV}.$$

First, we express the function $\pi(t)$ in the framework of regular variation. Using (3.2.3), (3.2.7) and (3.2.4) we have as $t \to \infty$

$$\pi(t) = \int_t^\infty \varphi^{-1}(s^{-\eta} l_p(s)^{-1}) ds \sim \int_t^\infty \varphi^{-1}(s^{-\eta}) l_p(s)^{-\frac{1}{\alpha}} ds \sim \int_t^\infty s^{-\frac{\eta}{\alpha}} L(s^{-\eta}) l_p(s)^{-\frac{1}{\alpha}} ds.$$

Applying Karamata's integration theorem (Proposition 1.2.1) to the last integral in the above relation we obtain

$$\text{(3.2.10)}\qquad \pi(t) \sim \frac{\alpha}{\eta - \alpha}\, t^{1-\frac{\eta}{\alpha}}\, L(t^{-\eta})\, l_p(t)^{-\frac{1}{\alpha}}, \quad t \to \infty.$$

Clearly, $\pi(t) \in \mathrm{RV}(1 - \frac{\eta}{\alpha})$.

Our main tool in establishing necessary and sufficient condition for the existence and precise asymptotic forms of intermediate positive solutions of (E_2) will be Schauder-Tychonoff fixed point theorem combined with theory of regular variation. To that end, the closed convex subset $\mathcal{X}$ of $C[t_0, \infty)$, which should be chosen in such a way that $\mathcal{F}$ is a continuous self-map on $\mathcal{X}$ and send it into a relatively compact subset of $C[t_0, \infty)$, will be now found by means of regularly varying functions satisfying the integral asymptotic relation

$$\text{(3.2.11)}\qquad x(t) \sim \int_t^\infty \varphi^{-1}\left(p(s)^{-1} \int_{t_0}^s q(r)\psi(x(r))\, dr\right) ds, \quad t \to \infty.$$

Thus, the proof of the "if" part of our main results is performed in three steps:

(i) the analysis of the integral asymptotic relation (3.2.11),

(ii) the construction of intermediate solutions by means of the fixed point technique, and

(iii) the verification of the regularity of those solutions with the help of the generalized L'Hospital rule (see [13]).

To simplify the "if" part of proof of our main results we now take the frst step and prove the next three Lemmas verifying that regularly varying functions $X_i(t),\ i = 1, 2, 3$ defined, respectively by

$$X_1(t) = \pi(t)\left(\frac{\alpha-\beta}{\alpha}\int_a^t q(s)\,\psi(\pi(s))\,ds\right)^{\frac{1}{\alpha-\beta}}, \tag{3.2.12}$$

$$X_2(t) = \Psi^{-1}\left(\frac{\alpha}{\alpha-\beta}\,\frac{t^{2-\rho+\frac{1}{\alpha}}}{(-\rho)[\alpha(\rho-1)+\eta]^{\frac{1}{\alpha}}}\varphi^{-1}(t^{\alpha(\rho-1)})\,p(t)^{-\frac{1}{\alpha}}\,q(t)^{\frac{1}{\alpha}}\right), \tag{3.2.13}$$

$$X_3(t) = \Psi^{-1}\left(\int_t^\infty \varphi^{-1}\left(p(s)^{-1}\int_a^s q(r)\,dr\right)ds\right) \tag{3.2.14}$$

satisfy the integral asymptotic relation (3.2.11)

Lemma 3.2.1 *Suppose that*

$$\sigma = \frac{\beta}{\alpha}\eta - \beta - 1 \quad \textit{and} \quad \int_a^\infty q(t)\,\psi(\pi(t))\,dt = \infty, \tag{3.2.15}$$

holds. The function $X_1(t) \in \mathrm{ntr}-\mathrm{RV}(1-\frac{\eta}{\alpha})$ *satisfies the asymptotic relation* (3.2.11).

Proof. Let (3.2.15) hold. Since $\sigma = \frac{\beta}{\alpha}\eta - \beta - 1$, using (3.2.10), (3.2.2) and (3.2.3), by Proposition 1.2.2 we obtain that $q(t)\psi(\pi(t)) \in \mathrm{RV}(-1)$ so that $\int_{t_0}^t q(s)\,\psi(\pi(s))\,ds \in \mathrm{SV}$ by Proposition 1.2.1-(iii). In view of (3.2.10) and (3.2.12), we conclude that $X_1(t) \in \mathrm{ntr}-\mathrm{RV}(1-\frac{\eta}{\alpha})$. Using (3.2.10), we get

$$\int_{t_0}^t q(s)\,\psi(\pi(s))\,ds \sim \int_{t_0}^t s^{\beta(\frac{\eta}{\alpha}-1)}\,q(s)\,\psi(s^{1-\frac{\eta}{\alpha}})\,\pi(s)^\beta\,ds,\ t\to\infty. \tag{3.2.16}$$

This, combined with (3.2.12), gives the following expression for $X_1(t)$:

$$X_1(t) \sim \pi(t)\left(\frac{\alpha-\beta}{\alpha}\int_{t_0}^t s^{\beta(\frac{\eta}{\alpha}-1)}\,q(s)\,\psi(s^{1-\frac{\eta}{\alpha}})\,\pi(s)^\beta\,ds\right)^{\frac{1}{\alpha-\beta}},\ t\to\infty. \tag{3.2.17}$$

Next, we integrate $q(t)\psi(X_1(t))$ on $[t_0,t]$. Since $X_1(t)=t^{1-\frac{\eta}{\alpha}}l_1(t),\ l_1(t)\in\mathrm{SV}$, due to (3.2.8), we obtain

$$\begin{aligned}(3.2.18)\qquad \int_{t_0}^{t} q(s)\psi(X_1(s))\,ds &\sim \int_{t_0}^{t} q(s)\,\psi(s^{1-\frac{\eta}{\alpha}})\,l_1(s)^{\beta}\,ds\\ &\sim \int_{t_0}^{t} s^{\beta(\frac{\eta}{\alpha}-1)}\,q(s)\,\psi(s^{1-\frac{\eta}{\alpha}})\,X_1(s)^{\beta}\,ds,\ t\to\infty.\end{aligned}$$

Changing (3.2.17) in the last integral in (3.2.18), by a simple calculation we have

$$\begin{aligned}(3.2.19)\ \int_{t_0}^{t} q(s)\psi(X_1(s))\,ds &\sim \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}}\\ &\times \int_{t_0}^{t} s^{\beta(\frac{\eta}{\alpha}-1)}\,q(s)\,\psi(s^{1-\frac{\eta}{\alpha}})\,\pi(s)^{\beta}\left(\int_{t_0}^{s} r^{\beta(\frac{\eta}{\alpha}-1)}\,q(r)\,\psi(r^{1-\frac{\eta}{\alpha}})\,\pi(r)^{\beta}\,dr\right)^{\frac{\beta}{\alpha-\beta}} ds\\ &= \left(\frac{\alpha-\beta}{\alpha}\int_{t_0}^{t} s^{\beta(\frac{\eta}{\alpha}-1)}\,q(s)\,\psi(s^{1-\frac{\eta}{\alpha}})\,\pi(s)^{\beta}\,ds\right)^{\frac{\alpha}{\alpha-\beta}}\\ &\sim \left(\frac{\alpha-\beta}{\alpha}\int_{t_0}^{t} q(s)\,\psi(\pi(s))\,ds\right)^{\frac{\alpha}{\alpha-\beta}},\ t\to\infty,\end{aligned}$$

where we use (3.2.16) in the last step. Since $\int_{t_0}^{t} q(s)\psi(X_1(s))\,ds\in\mathrm{SV}$, (3.2.3), (3.2.4) and (3.2.7) gives

$$\begin{aligned}\varphi^{-1}\left(p(t)^{-1}\int_{t_0}^{t} q(s)\,\psi(X_1(s))\,ds\right) &= \varphi^{-1}\left(t^{-\eta}\,l_p(t)^{-1}\int_{t_0}^{t} q(s)\,\psi(X_1(s))\,ds\right)\\ (3.2.20)\qquad &\sim \varphi^{-1}(t^{-\eta})\,l_p(t)^{-\frac{1}{\alpha}}\left(\int_{t_0}^{t} q(s)\,\psi(X_1(s))\,ds\right)^{\frac{1}{\alpha}}\\ &= t^{-\frac{\eta}{\alpha}}L(t^{-\eta})\,l_p(t)^{-\frac{1}{\alpha}}\left(\int_{t_0}^{t} q(s)\,\psi(X_1(s))\,ds\right)^{\frac{1}{\alpha}},\end{aligned}$$

as $t\to\infty$. Integrating (3.2.20) on $[t,\infty)$, we conclude via Proposition 1.2.1 that

$$\begin{aligned}&\int_{t}^{\infty}\varphi^{-1}\left(p(s)^{-1}\int_{t_0}^{s} q(r)\,\psi(X_1(r))\,dr\right)ds\\ &\sim \frac{\alpha}{\eta-\alpha}t^{1-\frac{\eta}{\alpha}}\,L(t^{-\eta})\,l_p(t)^{-\frac{1}{\alpha}}\left(\int_{t_0}^{t} q(s)\,\psi(X_1(s))\,ds\right)^{\frac{1}{\alpha}},\quad t\to\infty,\end{aligned}$$

which, combined with (3.2.10) and (3.2.19), shows that $X_1(t)$ satisfies the asymptotic relation (3.2.11). This completes the proof of Lemma 3.2.1. □

Lemma 3.2.2 *Suppose that*

$$\frac{\beta}{\alpha}\eta - \beta - 1 < \sigma < \eta - \alpha - 1 \tag{3.2.21}$$

holds and let ρ *be defined by*

$$\rho = \frac{\sigma + \alpha + 1 - \eta}{\alpha - \beta}. \tag{3.2.22}$$

The function $X_2(t) \in \mathrm{RV}(\rho)$ *given by* (3.2.13) *satisfies the asymptotic relation* (3.2.11).

Proof. Let (3.2.21) hold. Using (3.2.3), (3.2.4) and (3.2.31) by Proposition 1.2.2, we conclude that $X_2(t) \in \mathrm{RV}(\rho)$, with ρ given by (3.2.22). Thus, $X_2(t)$ is expressed as $X_2(t) = t^{\rho} l_2(t)$, $l_2(t) \in \mathrm{SV}$. Then, we get

$$\begin{aligned}\int_{t_0}^{t} q(s)\psi(X_2(s))ds &= \int_{t_0}^{t} q(s)\frac{\psi(X_2(s))}{X_2(s)^{\alpha}} X_2(s)^{\alpha}\, ds \\ &\sim (-\rho)^{\alpha}[\alpha(\rho-1)+\eta] \int_{t_0}^{t} q(s)s^{-\sigma-\alpha-1+\eta} L(s^{\alpha(\rho-1)})^{-\alpha} l_p(s) l_q(s)^{-1} X_2(s)^{\alpha} ds \\ &= (-\rho)^{\alpha}[\alpha(\rho-1)+\eta] \int_{t_0}^{t} s^{\alpha(\rho-1)+\eta-1}\, L(s^{\alpha(\rho-1)})^{-\alpha}\, l_p(s)\, l_2(s)^{\alpha}\, ds,\ t \to \infty.\end{aligned} \tag{3.2.23}$$

Applying Proposition 1.2.1 on the last integral in (3.2.23) and then multiplying the result with $p(t)^{-1}$ we obtain

$$p(t)^{-1} \int_{t_0}^{t} q(s)\psi(X_2(s))\, ds \sim (-\rho)^{\alpha} t^{\alpha(\rho-1)}\, L(t^{\alpha(\rho-1)})^{-\alpha}\, l_2(t)^{\alpha},\ t \to \infty,$$

from which, applying Proposition 1.2.11, it readily follows that

$$\begin{aligned}&\varphi^{-1}\left(p(t)^{-1} \int_{t_0}^{t} q(s)\psi(X_2(s))\, ds\right) \\ &\sim (-\rho)\, \varphi^{-1}(t^{\alpha(\rho-1)})\, L(t^{\alpha(\rho-1)})^{-1}\, l_2(t) = (-\rho)\, t^{\rho-1}\, l_2(t), \quad t \to \infty,\end{aligned}$$

where we use (3.2.4) and (3.2.7) in two last steps. Integration of the above relation on $[t, \infty)$ with application of Proposition 1.2.1 yields

$$\begin{aligned}&\int_{t}^{\infty} \varphi^{-1}\left(p(s)^{-1} \int_{t_0}^{s} q(r)\psi(X_2(r))\, dr\right) ds \\ &\sim (-\rho) \int_{t}^{\infty} s^{\rho-1}\, l_2(s)\, ds \sim t^{\rho}\, l_2(t) = X_2(t),\ t \to \infty.\end{aligned}$$

This completes the proof of Lemma 3.2.2. □

Lemma 3.2.3 *Suppose that*

$$\sigma = \eta - \alpha - 1 \quad \text{and} \quad \int_a^\infty \varphi^{-1}\left(p(t)^{-1}\int_a^t q(s)\,ds\right)dt < \infty, \tag{3.2.24}$$

holds. The function $X_3(t) \in \mathrm{ntr}-\mathrm{SV}$ *given by* (3.2.14) *satisfies the asymptotic relation* (3.2.11).

Proof. Let (3.2.24) hold. Using first (3.2.3) and Proposition 1.2.1 (which is possible since $\sigma > -1$) and then (3.2.7) and (3.2.4) we get

$$\begin{aligned}\varphi^{-1}\left(p(t)^{-1}\int_{t_0}^t q(s)ds\right) &= \varphi^{-1}\left(t^{-\eta}\,l_p(t)^{-1}\int_{t_0}^t s^\sigma l_q(s)\,ds\right)\\ &\sim \varphi^{-1}\left((\sigma+1)^{-1}t^{\sigma+1-\eta}l_p(t)^{-1}l_q(t)\right)\\ &\sim (\sigma+1)^{-\frac{1}{\alpha}}t^{\frac{\sigma+1-\eta}{\alpha}}L(t^{\sigma+1-\eta})l_p(t)^{-\frac{1}{\alpha}}l_q(t)^{\frac{1}{\alpha}},\ t\to\infty.\end{aligned} \tag{3.2.25}$$

Integration of (3.2.25) on $[t,\infty)$ and application of Proposition 1.2.1-(iii) since $\sigma = \eta - \alpha - 1$ gives

$$\begin{aligned}&\int_t^\infty \varphi^{-1}\left(p(s)^{-1}\int_{t_0}^s q(r)dr\right)ds\\ &\sim (\eta-\alpha)^{-\frac{1}{\alpha}}\int_t^\infty s^{-1}\,L(s^{-\alpha})\,l_p(s)^{-\frac{1}{\alpha}}\,l_q(s)^{\frac{1}{\alpha}}\,ds \in \mathrm{SV}, \quad t\to\infty.\end{aligned} \tag{3.2.26}$$

From (3.2.14) and (3.2.26), by Proposition 1.2.2-(iv), we find that $X_3(t) \in \mathrm{ntr}-\mathrm{SV}$ and $\psi(X_3(t)) \in \mathrm{ntr}-\mathrm{SV}$. Integrate $q(t)\,\psi(X_3(t))$ on $[t_0,t]$, applying Proposition 1.2.1 and using (3.2.3) we obtain

$$\begin{aligned}&\int_{t_0}^t q(s)\psi(X_3(s))ds = \int_{t_0}^t s^\sigma l_q(s)\psi(X_3(s))ds\\ &\sim \frac{t^{\sigma+1}}{\sigma+1}\,l_q(t)\psi(X_3(t)) = \frac{t^{\eta-\alpha}}{\eta-\alpha}l_q(t)\psi(X_3(t)), \quad t\to\infty,\end{aligned}$$

from which using Proposition 1.2.11, (3.2.7) and (3.2.4) follows that

$$\begin{aligned}\varphi^{-1}\left(p(t)^{-1}\int_{t_0}^t q(s)\psi(X_3(s))ds\right) &\sim \varphi^{-1}\left((\eta-\alpha)^{-1}\,t^{-\alpha}\,l_p(t)^{-1}\,l_q(t)\psi(X_3(t))\right)\\ &\sim (\eta-\alpha)^{-\frac{1}{\alpha}}\,t^{-1}L(t^{-\alpha})\,l_p(t)^{-\frac{1}{\alpha}}\,l_q(t)^{\frac{1}{\alpha}}\psi(X_3(t))^{\frac{1}{\alpha}}\\ &\sim \varphi^{-1}\left(p(t)^{-1}\int_{t_0}^t q(s)ds\right)\psi(X_3(t))^{\frac{1}{\alpha}},\ t\to\infty.\end{aligned} \tag{3.2.27}$$

On the other hand, we rewrite (3.2.14) as

$$\Psi(X_3(t)) = \int_t^\infty \varphi^{-1}\left(p(s)^{-1}\int_{t_0}^s q(r)dr\right)ds. \tag{3.2.28}$$

Since

$$\Psi(X_3(t)) = \int_0^{X_3(t)} \frac{dv}{\psi(v)^{\frac{1}{\alpha}}},$$

differentiation of (3.2.28) gives

$$X_3'(t) = -\varphi^{-1}\left(p(t)^{-1}\int_{t_0}^t q(s)ds\right)\psi(X_3(t))^{\frac{1}{\alpha}}. \tag{3.2.29}$$

Integrating (3.2.29) on $[t,\infty)$ and combine with (3.2.27) we have

$$X_3(t) \sim \int_t^\infty \varphi^{-1}\left(p(s)^{-1}\int_{t_0}^s q(r)\,\psi(X_3(r))\,dr\right)ds,\ t\to\infty.$$

This completes the proof of Lemma 3.2.3.□

To state our main results, we will need the function

$$\Psi(y) = \int_0^y \frac{dv}{\psi(v)^{\frac{1}{\alpha}}},\ y>0, \tag{3.2.30}$$

which is clearly increasing on $(0,\infty)$. From (3.2.2), (3.2.30) and Proposition 1.2.1 we get

$$\begin{aligned}\Psi(y) &= \int_0^y v^{-\frac{\beta}{\alpha}} L_2(v)^{-\frac{1}{\alpha}}\,dv \\ &\sim \frac{\alpha}{\alpha-\beta}\,y^{1-\frac{\beta}{\alpha}}\,L_2(y)^{-\frac{1}{\alpha}} = \frac{\alpha}{\alpha-\beta}\frac{y}{\psi(y)^{\frac{1}{\alpha}}},\quad y\to\infty,\end{aligned} \tag{3.2.31}$$

implying $\Psi(y)\in\mathcal{RV}(\frac{\alpha-\beta}{\alpha})$ and $\Psi^{-1}(y)\in\mathcal{RV}(\frac{\alpha}{\alpha-\beta})$ with $\frac{\alpha-\beta}{\alpha}>0$.

In view of (3.1.1), the regularity index ρ of $x(t)$ must satisfy $1-\frac{\eta}{\alpha}\le\rho\le 0$. Therefore, the class of intermediate regularly varying solutions of (E_2) is divided into three types of subclasses:

$$\mathrm{ntr}-\mathrm{RV}\left(1-\frac{\eta}{\alpha}\right),\quad \mathrm{RV}(\rho),\ \rho\in\left(1-\frac{\eta}{\alpha},0\right),\quad \mathrm{ntr}-\mathrm{SV}. \tag{3.2.32}$$

Our main results formulated below characterize completely the membership of each of the three subclasses of solutions (3.2.32) and show that all members of each subclass enjoy one and the same asymptotic behavior as $t\to\infty$.

Theorem 3.2.1 *Suppose that* (3.2.1), (3.2.7), (3.2.8) *and* (C_1) *hold. Equation* (E_2) *has intermediate solutions* $x(t) \in \mathrm{ntr} - \mathrm{RV}(1 - \frac{\eta}{\alpha})$ *if and only if* (3.2.15) *holds, in which case any such solution* $x(t)$ *has the asymptotic behavior* $x(t) \sim X_1(t)$, $t \to \infty$, *where* $X_1(t)$ *is given by* (3.2.12).

Theorem 3.2.2 *Suppose that* (3.2.1), (3.2.7), (3.2.8) *and* (C_1) *hold. Equation* (E_2) *has intermediate solutions* $x(t) \in \mathrm{RV}(\rho)$ *with* $\rho \in (1 - \frac{\eta}{\alpha}, 0)$ *if and only if* (3.2.21) *holds, in which case* ρ *is given by* (3.2.22) *and any such solution* $x(t)$ *has the asymptotic behavior* $x(t) \sim X_2(t)$, $t \to \infty$, *where* $x_2(t)$ *is given by* (3.2.13).

Theorem 3.2.3 *Suppose that* (3.2.1), (3.2.7), (3.2.8) *and* (C_1) *hold. Equation* (E_2) *has intermediate solutions* $x(t) \in \mathrm{ntr} - \mathrm{SV}$ *if and only if* (3.2.24) *holds, in which case any such solution* $x(t)$ *has the asymptotic behavior* $x(t) \sim X_3(t)$, $t \to \infty$, *where* $X_3(t)$ *is given by* (3.2.14).

Proof of the "only if" part of Theorems 3.2.1, 3.2.2, 3.2.3: Suppose that the equation (E_2) has an intermediate solution $x(t) \in \mathrm{RV}(\rho)$ with $\rho \in [1 - \frac{\eta}{\alpha}, 0]$ defined on $[t_0, \infty)$. Integration of equation (E_2) from t_0 to t using (3.2.2), (3.2.3) and (3.2.9) gives

$$p(t)\varphi(-x'(t)) \sim \int_{t_0}^{t} q(s)\,\psi(x(s))\,ds = \int_{t_0}^{t} s^{\sigma+\beta\rho}\,l_q(s) l_x(s)^\beta\, L_2(x(s))\,ds, \tag{3.2.33}$$

as $t \to \infty$, implying the divergence of the last integral in (3.2.33) i.e. implying that $\sigma + \beta\rho \geq -1$. We distinguish the two cases:

$$(a) \quad \sigma + \beta\rho = -1, \quad (b) \quad \sigma + \beta\rho > -1.$$

Assume that (a) holds. Multiplying (3.2.33) with $p(t)^{-1}$ we get

$$\varphi(-x'(t)) \sim p(t)^{-1}\,\xi(t), \quad t \to \infty, \quad \xi(t) = \int_{t_0}^{t} s^{-1} l_q(s)\, l_x(s)^\beta L_2(x(s))\,ds. \tag{3.2.34}$$

Clearly, $\xi(t) \in \mathrm{SV}$ and $\lim_{t\to\infty} \xi(t) = \infty$. From (3.2.34), using (3.2.3) and (3.2.7) we have

$$\begin{aligned} -x'(t) &\sim \varphi^{-1}\left(p(t)^{-1}\xi(t)\right) \\ &= \varphi^{-1}(t^{-\eta}\, l_p(t)^{-1}\xi(t)) \sim \varphi^{-1}(t^{-\eta})\, l_p(t)^{-\frac{1}{\alpha}}\, \xi(t)^{\frac{1}{\alpha}}, \quad t \to \infty. \end{aligned} \tag{3.2.35}$$

Integrating (3.2.35) from t to ∞, using (3.2.4) we find via Karamata's integration theorem that

$$x(t) \sim \frac{\alpha}{\eta - \alpha}\, t^{1-\frac{\eta}{\alpha}}\, L(t^{-\eta})\, l_p(t)^{-\frac{1}{\alpha}}\, \xi(t)^{\frac{1}{\alpha}} \in \mathrm{RV}\left(1 - \frac{\eta}{\alpha}\right), \quad t \to \infty. \tag{3.2.36}$$

Using (3.2.10) we rewrite (3.2.36) in the form

$$x(t) \sim \pi(t)\, \xi(t)^{\frac{1}{\alpha}}, \quad t \to \infty. \tag{3.2.37}$$

Assume that (b) holds. Applying Proposition 1.2.1 to the last integral in (3.2.33) we have

$$p(t)\varphi(-x'(t)) \sim \frac{t^{\sigma+\beta\rho+1}}{\sigma+\beta\rho+1} l_q(t)\, l_x(t)^{\beta} L_2(x(t)), \quad t \to \infty. \tag{3.2.38}$$

Multiplying (3.2.38) with $p(t)^{-1}$ and then using Proposition 1.2.11, (3.2.3), (3.2.7) and (3.2.4) we have

$$\begin{aligned} -x'(t) &\sim \varphi^{-1}\left(t^{\sigma+\beta\rho+1-\eta}(\sigma+\beta\rho+1)^{-1}\, l_p(t)^{-1}\, l_q(t)\, l_x(t)^{\beta} L_2(x(t))\right) \\ &\sim (\sigma+\beta\rho+1)^{-\frac{1}{\alpha}}\, t^{\frac{\sigma+\beta\rho+1-\eta}{\alpha}}\, L(t^{\sigma+\beta\rho+1-\eta})\, l_p(t)^{-\frac{1}{\alpha}}\, l_q(t)^{\frac{1}{\alpha}}\, l_x(t)^{\frac{\beta}{\alpha}} L_2(x(t))^{\frac{1}{\alpha}}, \end{aligned} \tag{3.2.39}$$

as $t \to \infty$. Integration of (3.2.39) on $[t, \infty)$ leads to

$$\begin{aligned} x(t) &\sim (\sigma+\beta\rho+1)^{-\frac{1}{\alpha}} \\ &\times \int_t^\infty s^{\frac{\sigma+\beta\rho+1-\eta}{\alpha}} L(s^{\sigma+\beta\rho+1-\eta}) l_p(s)^{-\frac{1}{\alpha}} l_q(s)^{\frac{1}{\alpha}} l_x(s)^{\frac{\beta}{\alpha}} L_2(x(s))^{\frac{1}{\alpha}} ds, \ t \to \infty. \end{aligned} \tag{3.2.40}$$

Since the above integral tends to zero as $t \to \infty$(note that $x(t) \to 0, t \to \infty$), we consider the following two cases separately:

$$(b.1) \quad \frac{\sigma+\beta\rho+1-\eta}{\alpha} < -1, \quad (b.2) \quad \frac{\sigma+\beta\rho+1-\eta}{\alpha} = -1.$$

Assume that $(b.1)$ holds. Applying Proposition 1.2.1 to the integral in (3.2.40), we get

$$\begin{aligned} x(t) \sim & -\frac{\alpha}{\sigma+\beta\rho+1-\eta+\alpha} (\sigma+\beta\rho+1)^{-\frac{1}{\alpha}}\, t^{\frac{\sigma+\beta\rho+1-\eta+\alpha}{\alpha}} \\ & \times L(t^{\sigma+\beta\rho+1-\eta})\, l_p(t)^{-\frac{1}{\alpha}}\, l_q(t)^{\frac{1}{\alpha}}\, l_x(t)^{\frac{\beta}{\alpha}} L_2(x(t))^{\frac{1}{\alpha}}, \ t \to \infty, \end{aligned}$$

so that $x(t) \in \mathrm{RV}\left(\frac{\sigma+\beta\rho+1-\eta+\alpha}{\alpha}\right)$.

Assume that $(b.2)$ holds. Then, (3.2.40) shows that $x(t) \in \mathrm{SV}$, that is $\rho = 0$, and hence $\sigma = \eta - \alpha - 1$. Since $\sigma + \beta\rho + 1 = \eta - \alpha$, (3.2.40) reduced to

$$x(t) \sim (\eta-\alpha)^{-\frac{1}{\alpha}} \int_t^\infty s^{-1} L(s^{-\alpha})\, l_p(s)^{-\frac{1}{\alpha}}\, l_q(s)^{\frac{1}{\alpha}} l_x(s)^{\frac{\beta}{\alpha}} L_2(x(s))^{\frac{1}{\alpha}}\, ds \in \mathrm{SV}, \tag{3.2.41}$$

as $t \to \infty$.

Let us now suppose that $x(t)$ is an intermediate solution of (E_2) belonging to $\mathrm{ntr}-\mathrm{RV}(1-\frac{\eta}{\alpha})$. Then, the case (a) is the only possibility for $x(t)$, which means that $\rho = 1-\frac{\eta}{\alpha}$, $\sigma = \frac{\beta}{\alpha}\eta - \beta - 1$ and (3.2.37) is satisfied by $x(t)$. Differentiation of $\xi(t)$, defined in (3.2.34), using (3.2.2), (3.2.3) and (3.2.9) leads to

$$\xi'(t) = t^{-1}\, l_q(t)\, l_x(t)^{\beta}\, L_2(x(t)) \sim q(t)\,\psi(x(t)),\ t\to\infty.$$

Noting that $x(t) \sim \pi(t)\xi(t)^{\frac{1}{\alpha}}$, $t\to\infty$ and using (3.2.8), one can transform the above relation into

$$\xi'(t) \sim q(t)\,\psi(\pi(t)\xi(t)^{\frac{1}{\alpha}}) \sim q(t)\psi(\pi(t))\xi(t)^{\frac{\beta}{\alpha}},\ t\to\infty.$$

So, we get the differential asymptotic relation for $\xi(t)$:

$$\xi(t)^{-\frac{\beta}{\alpha}}\,\xi'(t) \sim q(t)\,\psi(\pi(t)),\ t\to\infty. \tag{3.2.42}$$

Integration of (3.2.42) on $[t_0,t]$ yields

$$\xi(t) \sim \left(\frac{\alpha-\beta}{\alpha}\int_{t_0}^{t} q(s)\,\psi(\pi(s))\,ds\right)^{\frac{\alpha}{\alpha-\beta}},\ t\to\infty. \tag{3.2.43}$$

Since $\lim_{t\to\infty}\xi(t)=\infty$, from (3.2.43) we have $\int_{t_0}^{\infty} q(t)\,\psi(\pi(t))\,dt = \infty$. Thus, the condition (3.2.15) is satisfied. Combining (3.2.43) with (3.2.37) gives $x(t)\sim X_1(t)$, $t\to\infty$, where $X_1(t)$ is given by (3.2.12). This proves the "only if" part of Theorem 3.2.1.

Next, suppose that $x(t)$ is an intermediate solution of (E_2) belonging to $\mathrm{RV}(\rho)$, $\rho\in(1-\frac{\eta}{\alpha},0)$. This is possible only when $(b.1)$ holds, in which case $x(t)$ must satisfy the asymptotic relation (4.3.30). Therefore,

$$\rho = \frac{\sigma+\beta\rho+1-\eta+\alpha}{\alpha} \quad\Rightarrow\quad \rho = \frac{\sigma+\alpha+1-\eta}{\alpha-\beta},$$

which justifies (3.2.22). An elementary calculation shows that

$$1-\frac{\eta}{\alpha} < \rho < 0 \quad\Rightarrow\quad \frac{\beta}{\alpha}\eta-\beta-1<\sigma<\eta-\alpha-1,$$

which determines the range (3.2.21) of σ. Since $\sigma+\beta\rho+1-\eta+\alpha = \alpha\,\rho$ and $\sigma+\beta\rho+1 = \alpha(\rho-1)+\eta$, (4.3.30) reduced to

$$\begin{aligned} x(t) &\sim \frac{t^{\rho}}{(-\rho)(\alpha(\rho-1)+\eta)^{\frac{1}{\alpha}}}\, L(t^{\alpha(\rho-1)})\, l_p(t)^{-\frac{1}{\alpha}}\, l_q(t)^{\frac{1}{\alpha}}\, l_x(t)^{\frac{\beta}{\alpha}}\, L_2(x(t))^{\frac{1}{\alpha}} \\ &= \frac{t^{2-\rho+\frac{1}{\alpha}}}{(-\rho)(\alpha(\rho-1)+\eta)^{\frac{1}{\alpha}}}\,\varphi^{-1}(t^{\alpha(\rho-1)})\, p(t)^{-\frac{1}{\alpha}}\, q(t)^{\frac{1}{\alpha}}\,\psi(x(t))^{\frac{1}{\alpha}},\ t\to\infty, \end{aligned} \tag{3.2.44}$$

where we use (3.2.2), (3.2.3), (3.2.4) and (3.2.9) in the last step. From (3.2.44) using (3.2.31) we get

$$\Psi(x(t)) \sim \frac{\alpha}{\alpha-\beta}\,\frac{x(t)}{\psi(x(t))^{\frac{1}{\alpha}}} \sim \frac{\alpha}{\alpha-\beta}\,\frac{t^{2-\rho+\frac{1}{\alpha}}}{(-\rho)(\alpha(\rho-1)+\eta)^{\frac{1}{\alpha}}}\,\varphi^{-1}(t^{\alpha(\rho-1)})\,p(t)^{-\frac{1}{\alpha}}\,q(t)^{\frac{1}{\alpha}},$$

as $t \to \infty$. Thus, we conclude that $x(t)$ enjoys the asymptotic formula $x(t) \sim X_2(t)$, $t \to \infty$, where $X_2(t)$ is given by (3.2.13). This proves the "only if" part of the Theorem 3.2.2.

Finally, suppose that $x(t)$ is an intermediate solution of (E_2) belonging to $\mathrm{ntr}-\mathrm{SV}$. From the above observation this is possible only when the case $(b.2)$ holds, in which case $\rho = 0$, $\sigma = \eta - \alpha - 1$ and $x(t) = l_x(t)$ must satisfy the asymptotic behavior (3.2.41). Denote the right-hand side of (3.2.41) by $\mu(t)$. Then, $\mu(t) \to 0,\ t \to \infty$ and satisfies

$$\begin{aligned}\mu'(t) &= -(\eta-\alpha)^{-\frac{1}{\alpha}}\,t^{-1}\,L(t^{-\alpha})\,l_p(t)^{-\frac{1}{\alpha}}\,l_q(t)^{\frac{1}{\alpha}}l_x(t)^{\frac{\beta}{\alpha}}L_2(x(t))^{\frac{1}{\alpha}}\\ &= -(\eta-\alpha)^{-\frac{1}{\alpha}}\,t^{-1}\,L(t^{-\alpha})\,l_p(t)^{-\frac{1}{\alpha}}\,l_q(t)^{\frac{1}{\alpha}}\,\psi(x(t))^{\frac{1}{\alpha}},\ t\to\infty,\end{aligned}$$

where we use (3.2.2) in the last step. Since (3.2.41) is equivalent to $x(t) \sim \mu(t)$, $t \to \infty$, from the above using (3.2.25) we obtain

$$\frac{\mu'(t)}{\psi(\mu(t))^{\frac{1}{\alpha}}} \sim -\varphi^{-1}\left(p(t)^{-1}\int_{t_0}^{t} q(s)ds\right),\ t\to\infty.$$

An integration of the last relation over $[t,\infty)$ gives

$$\int_0^{\mu(t)} \frac{dv}{\psi(v)^{\frac{1}{\alpha}}} \sim \Psi(\mu(t)) \sim \int_t^{\infty} \varphi^{-1}\left(p(s)^{-1}\int_{t_0}^{s} q(r)dr\right)ds,\ t\to\infty,$$

or

$$x(t) \sim \mu(t) \sim \Psi^{-1}\left(\int_t^{\infty} \varphi^{-1}\left(p(s)^{-1}\int_{t_0}^{s} q(r)dr\right)ds\right),\ t\to\infty.$$

Since $\lim_{t\to\infty}\mu(t) = 0$, from the above relation we have convergence of integral $\int_a^{\infty}\varphi^{-1}\left(p(t)^{-1}\int_a^t q(s)ds\right)dt$, so the condition (3.2.24) is satisfied. Thus, it has been shown that $x(t) \sim X_3(t)$, $t \to \infty$, where $X_3(t)$ is given by (3.2.14). This completes the "only if" part of the proof of Theorem 3.2.3. □

Proof of the "if" part of Theorems 3.2.1, 3.2.2, 3.2.3: Suppose that (3.2.15), (3.2.21) or (3.2.24) holds. From Lemmas 3.2.1, 3.2.2 and 3.2.3 it is known that $X_i(t),\ i = 1,2,3$ defined by (3.2.12),(3.2.13) and (3.2.14) satisfy the asymptotic relation (3.2.11). We preform the simultaneous proof for $X_i(t),\ i = 1,2,3$ so the subscript $i = 1,2,3$ will be deleted in the rest of proof. By (3.2.11) there exists $T_0 > a$ such that

$$\frac{X(t)}{2} \le \int_t^\infty \varphi^{-1}\left(\frac{1}{p(s)}\int_{T_0}^s q(r)\,\psi(X(r))\,dr\right) ds \le 2X(t),\ \ t \ge T_0. \tag{3.2.45}$$

Applying Proposition 1.2.5 to the function $\psi(s) \in \mathcal{RV}(\beta),\ \beta > 0$ we see that there exists a constant $A > 1$ such that

$$\psi(s_1) \le A\psi(s_2) \quad \text{for each} \quad 0 \le s_1 \le s_2 < a. \tag{3.2.46}$$

Now we choose positive constants m and M such that

$$m^{1-\frac{\beta}{\alpha}} \le \frac{1}{4(2A)^{1/\alpha}}, \quad M^{1-\frac{\beta}{\alpha}} \ge 4(2A)^{1/\alpha}. \tag{3.2.47}$$

In addition, since $X(t) \to 0$ as $t \to \infty$, from (1.2.2), for $\lambda > 0$ we have

$$\frac{\lambda^\beta}{2}\psi(X(t)) \le \psi(\lambda X(t)) \le 2\lambda^\beta\psi(X(t)), \tag{3.2.48}$$

for all sufficiently large t. Also, since $Q(t) = p(t)^{-1}\int_{t_0}^t q(s)\psi(X(s))\,ds \to 0$ as $t \to \infty$, from (1.2.2), for $\lambda > 0$ we have

$$\frac{\lambda^{1/\alpha}}{2}\varphi^{-1}(Q(t)) \le \varphi^{-1}(\lambda Q(t)) \le 2\lambda^{1/\alpha}\varphi^{-1}(Q(t)), \tag{3.2.49}$$

for all sufficiently large t. Define the integral operator $\mathcal{F}$ by

$$\mathcal{F}x(t) = \int_t^\infty \varphi^{-1}\left(\frac{1}{p(s)}\int_{T_0}^s q(r)\psi(x(r))\,dr\right) ds, \quad t \ge T_0,$$

and let it act on the set

$$\mathcal{X} := \{x(t) \in C[T_0,\infty) : mX(t) \le x(t) \le MX(t),\ t \ge T_0\}. \tag{3.2.50}$$

It is clear that $\mathcal{X}$ is a closed convex subset of the locally convex space $C[T_0,\infty)$ equipped with the topology of uniform convergence on compact subintervals of $[T_0,\infty)$.

Let $x(t) \in \mathcal{X}$. Using first (3.2.46) and (3.2.50) , and then (3.2.48) we get

$$\begin{aligned}
\mathcal{F}x(t) &\le \int_t^\infty \varphi^{-1}\left(\frac{A}{p(s)}\int_{T_0}^s q(r)\psi(MX(r))\,dr\right) ds \\
&\le \int_t^\infty \varphi^{-1}\left(\frac{2AM^\beta}{p(s)}\int_{T_0}^s q(r)\psi(X(r))\,dr\right) ds, \quad t \ge T_0,
\end{aligned}$$

from which, using (3.2.49), (3.2.45) and (3.2.47), follows that

$$\begin{aligned}\mathcal{F}x(t) &\leq 2(2AM^\beta)^{1/\alpha}\int_t^\infty \varphi^{-1}\left(\frac{1}{p(s)}\int_{T_0}^s q(r)\psi(X(r))\,dr\right)ds\\ &\leq 4(2AM^\beta)^{1/\alpha}X(t) \leq MX(t), \quad t\geq T_0.\end{aligned}$$

On the other hand, using (3.2.50), (3.2.46) and (3.2.48) we obtain

$$\begin{aligned}\mathcal{F}x(t) &\geq \int_t^\infty \varphi^{-1}\left(\frac{1}{Ap(s)}\int_{T_0}^s q(r)\psi(mX(r))\,dr\right)ds\\ &\geq \int_t^\infty \varphi^{-1}\left(\frac{m^\beta}{2Ap(s)}\int_{T_0}^s q(r)\psi(X(r))\,dr\right)ds, \quad t\geq T.\end{aligned}$$

From the above using (3.2.49) and (3.2.47) we conclude

$$\begin{aligned}\mathcal{F}x(t) &\geq \frac{1}{2}\left(\frac{m^\beta}{2A}\right)^{\frac{1}{\alpha}}\int_t^\infty \varphi^{-1}\left(\frac{1}{p(s)}\int_{T_0}^s q(r)\psi(X(r))\,dr\right)ds\\ &\geq \frac{1}{4}\left(\frac{m^\beta}{2A}\right)^{\frac{1}{\alpha}}X(t)\geq mX(t), \quad t\geq T_0.\end{aligned}$$

This shows that $\mathcal{F}x(t)\in\mathcal{X}$, that is, $\mathcal{F}$ maps $\mathcal{X}$ into itself.

Furthermore it can be verified that $\mathcal{F}$ is a continuous map and that $\mathcal{F}(\mathcal{X})$ is relatively compact in $C[T_0,\infty)$.

Thus, all the hypotheses of the Schauder-Tychonoff fixed point theorem are fulfilled and so there exists a fixed point $x(t)\in\mathcal{X}$ of $\mathcal{F}$, which satisfies integral equation

$$x(t)=\int_t^\infty \varphi^{-1}\left(\frac{1}{p(s)}\int_{T_0}^s q(r)\psi(x(r))\,dr\right)ds, \quad t\geq T_0.$$

Differentiating the above twice shows that $x(t)$ is a solution of (E_2) on $[T_0,\infty)$. It is clear from (3.2.50) that $x(t)$ is an intermediate solution of (E_2).

Therefore, the existence of three types of intermediate solutions of (E_2) has been established. The proof of our main results will be completed with the verification that the intermediate solutions of (E_2) constructed above are actually regularly varying functions.

We defined the function

$$J(t)=\int_t^\infty \varphi^{-1}\left(\frac{1}{p(s)}\int_{T_0}^s q(r)\psi(X(r))\,dr\right)ds, \quad t\geq T_0,$$

and put

$$l=\liminf_{t\to\infty}\frac{x(t)}{J(t)}, \quad L=\limsup_{t\to\infty}\frac{x(t)}{J(t)}.$$

Since $x \in \mathcal{X}$, it is clear that

$$0 < \liminf_{t\to\infty} \frac{x(t)}{X(t)} \le \limsup_{t\to\infty} \frac{x(t)}{X(t)} < \infty.$$

By Lemmas 3.2.1, 3.2.2 and 3.2.3 we have

$$J(t) \sim X(t), \quad t \to \infty. \tag{3.2.51}$$

If we denote with

$$f(t) = \frac{1}{p(t)} \int_{T_0}^{t} q(s)\psi(x(s))ds \quad \text{and} \quad g(t) = \frac{1}{p(t)} \int_{T_0}^{t} q(s)\psi(X(s))ds,$$

using (3.2.4) and Lemma 1.1.1 we see that

$$\begin{aligned} L &\le \limsup_{t\to\infty} \frac{x'(t)}{J'(t)} = \limsup_{t\to\infty} \frac{\varphi^{-1}(f(t))}{\varphi^{-1}(g(t))} = \limsup_{t\to\infty} \frac{f(t)^{\frac{1}{\alpha}} L(f(t))}{g(t)^{\frac{1}{\alpha}} L(g(t))} \\ &\le \limsup_{t\to\infty} \left(\frac{f(t)}{g(t)}\right)^{\frac{1}{\alpha}} \limsup_{t\to\infty} \frac{L\left(\frac{f(t)}{g(t)} g(t)\right)}{L(g(t))}. \end{aligned} \tag{3.2.52}$$

Using (3.2.48) and (3.2.46) we obtain $m_1 = \dfrac{m^\beta}{2A} \le \dfrac{f(t)}{g(t)} \le 2AM^\beta = M_1$ implying by Uniform convergence theorem ([2],Theorem 1.2.1) that

$$\left| \frac{L\left(\frac{f(t)}{g(t)} g(t)\right)}{L(g(t))} - 1 \right| \le \sup_{\lambda \in [m_1, M_1]} \left| \frac{L(\lambda g(t))}{L(g(t))} - 1 \right| \longrightarrow 0, \quad t \to \infty. \tag{3.2.53}$$

In the view of (3.2.53), from (3.2.52) it follows

$$L \le \limsup_{t\to\infty} \left(\frac{f(t)}{g(t)}\right)^{\frac{1}{\alpha}} = \left(\limsup_{t\to\infty} \frac{\int_{T_0}^t q(s)\, \psi(x(s))\, ds}{\int_{T_0}^t q(s)\, \psi(X(s))\, ds} \right)^{\frac{1}{\alpha}}. \tag{3.2.54}$$

Similarly, using (3.2.2) and Lemma 1.1.1 we have

$$\begin{aligned} \limsup_{t\to\infty} \frac{\int_{T_0}^t q(s)\, \psi(x(s))\, ds}{\int_{T_0}^t q(s)\, \psi(X(s))\, ds} &\le \limsup_{t\to\infty} \frac{\psi(x(t))}{\psi(X(t))} \\ = \limsup_{t\to\infty} \frac{x(t)^\beta L_2(x(t))}{X(t)^\beta L_2(X(t))} &\le \limsup_{t\to\infty} \left(\frac{x(t)}{X(t)}\right)^\beta \limsup_{t\to\infty} \frac{L_2\left(\frac{x(t)}{X(t)} X(t)\right)}{L_2(X(t))}. \end{aligned} \tag{3.2.55}$$

Since $m \leq \dfrac{x(t)}{X(t)} \leq M,\ t \geq T_0$, using Uniform convergence theorem we conclude

(3.2.56) $$\left| \frac{L_2\left(\frac{x(t)}{X(t)}X(t)\right)}{L_2(X(t))} - 1 \right| \leq \sup_{\lambda\in[m,M]} \left| \frac{L_2\left(\lambda X(t)\right)}{L_2(X(t))} - 1 \right| \longrightarrow 0, \quad t \to \infty.$$

In the view of (3.2.56), from (3.2.51) and (3.2.55) it follows

(3.2.57)
$$\limsup_{t\to\infty} \frac{\int_{T_0}^t q(s)\,\psi(x(s))\,ds}{\int_{T_0}^t q(s)\,\psi(X(s))\,ds} \leq \left(\limsup_{t\to\infty} \frac{x(t)}{X(t)}\right)^{\beta} = \left(\limsup_{t\to\infty} \frac{x(t)}{J(t)}\right)^{\beta} = L^{\beta}.$$

From (3.2.54) and (3.2.57), it follows that $L \leq L^{\frac{\beta}{\alpha}}$, implying that $0 < L \leq 1$ because $\alpha > \beta$. If we argue similarly by taking the inferior limits instead of the superior limits, we are led to the inequality $l \geq l^{\frac{\beta}{\alpha}}$, which implies that $l \geq 1$. Thus we conclude that $l = L = 1$, i.e. $\lim_{t\to\infty} x(t)/J(t) = 1$. This combined with (3.2.51) shows that $x(t) \sim X(t)$, $t \to \infty$, which yields that $x(t)$ is a regularly varying function whose regularity index ρ is $1-\frac{\eta}{\alpha}$, $\frac{\sigma+\alpha+1-\eta}{\alpha-\beta}$, or 0 according as $\sigma = \frac{\beta}{\alpha}\eta-\beta-1$, $\frac{\beta}{\alpha}\eta-\beta-1 < \sigma < \eta-\alpha-1$, or $\sigma = \eta-\alpha-1$. Thus, the if part of Theorems 3.2.1, 3.2.2, 3.2.3 has been proved. □

3.3 Classification and existence of positive increasing solutions of (E_2) under the condition (C_2)

In this section, we assume that $p, q : [a,\infty) \to (0,\infty)$ and $\varphi, \psi : (0,\infty) \to (0,\infty)$ are continuous functions, φ is increasing and that (C_2) holds. We use the function $P(t)$ defined as

(3.3.1) $$P(t) = \int_a^t \varphi^{-1}(p(s)^{-1})\,ds, \quad t \geq a.$$

We begin by classification the set of increasing positive solutions of (E_2) according to their asymptotic behavior at infinity. It is easily seen (see [11]) that if $x(t)$ is an increasing positive solution of (E_2), then we have the following classification of increasing positive solutions of (E_2) into three types according to their asymptotic behavior at infinity:

$$\lim_{t\to\infty} x(t) = \text{const} > 0, \tag{3.3.2}$$

$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} p(t)\varphi(x'(t)) = 0, \tag{3.3.3}$$

$$\lim_{t\to\infty} \frac{x(t)}{P(t)} = \text{const} > 0. \tag{3.3.4}$$

Solutions of type (3.3.2), (3.3.3), (3.3.4) are often called, respectively, *subdominant, intermediate* and *dominant* solutions.

It is well known (see [11], [41]) that the existence of subdominant and dominant solutions for the equation (E_2) with continuous coefficients $p(t), q(t), \varphi(s)$ and $\psi(s)$ can be completely characterized by the convergence of the integrals

$$I = \int_a^\infty q(t)\psi(P(t))dt, \quad J = \int_a^\infty \varphi^{-1}\left(p(t)^{-1}\int_t^\infty q(s)ds\right)dt.$$

Theorem 3.3.1 *Let $p(t), q(t) \in C[a,\infty)$, $\varphi(s), \psi(s) \in C[0,\infty)$ and (C_2) hold.*

(a) Equation (E_2) has an increasing positive solution of type (3.3.2) if and only if $J < \infty$.

(b) Equation (E_2) has an increasing positive solution of type (3.3.4) if and only if $I < \infty$.

(c) Equation (E_2) has an increasing positive solution of type (3.3.3) if $J = \infty$ and $I < \infty$.

For the existence of intermediate solutions for (E_2), sufficient conditions can be obtained with relative ease. But the problem of establishing necessary and sufficient conditions turns out to be extremely difficult to solve and thus, has been an open problem for a long time.

3.4 Asymptotic behavior of intermediate solutions of (E_2) under the condition (C_2)

We assume that (C_2) hold, φ is increasing and

$$\begin{gathered}\varphi(s) \in \mathcal{RV}(\alpha), \quad \alpha > 0; \quad \psi(s) \in \mathrm{RV}(\beta), \quad \alpha > \beta > 0; \\ p(t) \in \mathrm{RV}(\eta), \quad \eta \in (0,\alpha); \qquad q(t) \in \mathrm{RV}(\sigma), \quad \sigma \in \mathbb{R}.\end{gathered} \tag{3.4.1}$$

Using the notation (1.2.3), we can express $\varphi(s)$, $\psi(s)$, $p(t)$ and $q(t)$ as

$$\varphi(s) = s^{\alpha} L_1(s), \quad L_1(s) \in \mathcal{SV}; \;\; \psi(s) = s^{\beta} L_2(s), \quad L_2(s) \in \mathrm{SV}; \tag{3.4.2}$$

$$p(t) = t^{\eta} l_p(t), \quad l_p(t) \in \mathrm{SV}; \qquad q(t) = t^{\sigma} l_q(t), \quad l_q(t) \in \mathrm{SV}. \tag{3.4.3}$$

Since $\varphi(s)$ is an increasing function, so $\varphi(s)$ has the inverse function, denoted by $\varphi^{-1}(s)$ and from (3.4.2) we conclude that

$$\varphi^{-1}(s) \in \mathcal{RV}(1/\alpha) \quad \Rightarrow \quad \varphi^{-1}(s) = s^{1/\alpha} L(s), \quad L(s) \in \mathcal{SV}. \tag{3.4.4}$$

We also need the additional requirements for the slowly varying parts of φ and ψ:

$$L(tu(t)) \sim L(t), \quad t \to 0, \quad \forall u(t) \in \mathcal{SV} \cap C^1(\mathbb{R}); \tag{3.4.5}$$

$$L_2(tu(t)) \sim L_2(t), \quad t \to \infty, \quad \forall u(t) \in \mathrm{SV} \cap C^1(\mathbb{R}). \tag{3.4.6}$$

Analogous to the Remark 3.2.1 we obtain

$$\varphi^{-1}(t^{\lambda} u(t)) \sim \varphi^{-1}(t^{\lambda}) u(t)^{\frac{1}{\alpha}}, \quad t \to \infty, \quad \forall \lambda \in \mathbb{R}^-, \quad \forall u(t) \in \mathrm{SV} \cap C^1(\mathbb{R}), \tag{3.4.7}$$

and

$$\psi(t^{\lambda} u(t)) \sim \psi(t^{\lambda}) u(t)^{\beta}, \quad t \to \infty, \quad \forall \lambda \in \mathbb{R}^+, \quad \forall u(t) \in \mathrm{SV} \cap C^1(\mathbb{R}). \tag{3.4.8}$$

We seek such solutions $x(t)$ of (E_2) that can be expressed in the form

$$x(t) = t^{\rho} l_x(t), \quad l_x(t) \in \mathrm{SV}. \tag{3.4.9}$$

Since $\eta > 0$, applying Proposition 1.2.3, we have $\lim_{t\to\infty} p(t) = \infty$. Then, applying Proposition 1.2.2-(iv), we get $\varphi^{-1}\left(p(t)^{-1}\right) \in \mathcal{RV}\left(-\frac{\eta}{\alpha}\right)$ so that the assumption $\eta < \alpha$ ensures that we may apply Karamata's integration theorem (Proposition 1.2.1) to the integral in (3.3.1). Using (3.4.3), (3.4.7), (3.4.4) and Proposition 1.2.1 we obtain

$$\begin{aligned} P(t) &= \int_a^t \varphi^{-1}(s^{-\eta}\, l_p(s)^{-1})\, ds \sim \int_a^t \varphi^{-1}(s^{-\eta})\, l_p(s)^{-\frac{1}{\alpha}}\, ds \\ &= \int_a^t s^{-\frac{\eta}{\alpha}} L(s^{-\eta}) l_p(s)^{-\frac{1}{\alpha}}\, ds \sim \frac{\alpha}{\alpha-\eta}\, t^{1-\frac{\eta}{\alpha}}\, L(t^{-\eta})\, l_p(t)^{-\frac{1}{\alpha}}, \quad t \to \infty, \end{aligned} \tag{3.4.10}$$

implying that $P(t) \in \mathrm{RV}(1 - \frac{\eta}{\alpha})$. Since $\eta < \alpha$ by Proposition 1.2.3 we have $\lim_{t\to\infty} P(t) = \infty$.

We emphasize that we exclude the case $\eta = \alpha$ because of computational difficulty and the fact that integral

$$\int_a^t \varphi^{-1}\left(p(s)^{-1}\right) ds = \int_a^t s^{-1} L(s^{-\eta}) l_p(s)^{-\frac{1}{\alpha}} ds$$

might be either convergent or divergent.

Let $x(t)$ be an intermediate solution of (E_2) defined on $[t_0, \infty)$. Integrating of equation (E_2) first on (t_0, ∞) and then on $[t_0, t]$ gives

$$x(t) = x(t_0) + \int_{t_0}^t \varphi^{-1}\left(p(s)^{-1} \int_s^\infty q(r)\psi(x(r))\, dr\right) ds, \quad t \geq t_0. \tag{3.4.11}$$

It follows therefore that $x(t)$ satisfies the integral asymptotic relation

$$x(t) \sim \int_b^t \varphi^{-1}\left(p(s)^{-1} \int_s^\infty q(r)\psi(x(r))\, dr\right) ds, \quad t \to \infty, \tag{3.4.12}$$

for any $b \geq a$, which is regarded as an "approximation" of (3.4.11) at infinity. A common way of determining the desired intermediate solution of (E_2) would be by solving the integral equation (3.4.11) with the help of fixed point technique. For this purpose Schauder-Tychonoff fixed point theorem should be applied to the integral operator

$$\mathcal{F}x(t) = x_0 + \int_{t_0}^t \varphi^{-1}\left(p(s)^{-1} \int_s^\infty q(r)\psi(x(r))\, dr\right) ds, \quad t \geq t_0, \quad x_0 \in \mathbb{R},$$

acting on some closed convex subsets $\mathcal{X}$ of $C[t_0, \infty)$, which should be chosen in such a way that $\mathcal{F}$ is a continuous self-map on $\mathcal{X}$ and send it into a relatively compact subset of $C[t_0, \infty)$. That such choices of $\mathcal{X}$ are feasible is guaranteed by the existence of three types of regularly varying functions that determine exactly the asymptotic behavior of all possible solutions of (3.4.12). We begin by proving three results verifying that regularly varying functions $X_i(t)$, $i = 1, 2, 3$ defined, respectively by

$$X_1(t) = \Psi^{-1}\left(\int_a^t \varphi^{-1}\left(p(s)^{-1} \int_s^\infty q(r)\, dr\right) ds\right), \tag{3.4.13}$$

$$X_2(t) = \Psi^{-1}\left(\frac{\alpha}{\alpha - \beta} \frac{t^{2-\rho+\frac{1}{\alpha}}}{\rho[\alpha(1-\rho) - \eta]^{\frac{1}{\alpha}}} \varphi^{-1}(t^{\alpha(\rho-1)})\, p(t)^{-\frac{1}{\alpha}}\, q(t)^{\frac{1}{\alpha}}\right), \tag{3.4.14}$$

$$X_3(t) = P(t)\left(\frac{\alpha - \beta}{\alpha} \int_t^\infty q(s)\psi(P(s))\, ds\right)^{\frac{1}{\alpha-\beta}} \tag{3.4.15}$$

satisfy the integral asymptotic relation (3.4.12).

Lemma 3.4.1 *Suppose that*

$$(3.4.16)\qquad \sigma = \eta - \alpha - 1 \quad \textit{and} \quad \int_a^\infty \varphi^{-1}\left(p(t)^{-1}\int_t^\infty q(s)\,ds\right)dt = \infty,$$

holds. The function $X_1(t) \in \mathrm{ntr}-\mathrm{SV}$ *given by* (3.4.13) *satisfies the asymptotic relation* (3.4.12).

Proof. Let (3.4.16) hold. Since $\eta < \alpha$, from (3.4.16) we have $\sigma < -1$, so we can apply Proposition 1.2.1 to the integral

$$\int_t^\infty q(s)ds = \int_t^\infty s^\sigma l_q(s)\,ds \sim (-(\sigma+1))^{-1} t^{\sigma+1} l_q(t), \quad t \to \infty.$$

Using the above relation, (3.4.3), (3.4.7) and (3.4.4) we get

$$(3.4.17)\qquad \varphi^{-1}\left(p(t)^{-1}\int_t^\infty q(s)ds\right) = \varphi^{-1}\left(t^{-\eta}\, l_p(t)^{-1}\int_t^\infty s^\sigma l_q(s)\,ds\right)$$

$$\sim (-(\sigma+1))^{-\frac{1}{\alpha}}\varphi^{-1}(t^{\sigma+1-\eta}) l_p(t)^{-\frac{1}{\alpha}} l_q(t)^{\frac{1}{\alpha}}$$
$$= (-(\sigma+1))^{-\frac{1}{\alpha}} t^{\frac{\sigma+1-\eta}{\alpha}} L(t^{\sigma+1-\eta}) l_p(t)^{-\frac{1}{\alpha}} l_q(t)^{\frac{1}{\alpha}},\ t \to \infty.$$

Since $\sigma = \eta - \alpha - 1$ we can rewrite (3.4.17) in the form

$$(3.4.18)\quad \varphi^{-1}\left(p(t)^{-1}\int_t^\infty q(s)ds\right) \sim (\alpha-\eta)^{-\frac{1}{\alpha}} t^{-1} L(t^{-\alpha}) l_p(t)^{-\frac{1}{\alpha}} l_q(t)^{\frac{1}{\alpha}}, \quad t \to \infty.$$

Application of Proposition 1.2.1-(iii) to (3.4.18) gives

$$(3.4.19)\qquad \int_{t_0}^t \varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)dr\right)ds \in \mathrm{SV}.$$

From (3.4.13) and (3.4.19), by Proposition 1.2.2-(iv), we find that $X_1(t) \in \mathrm{ntr}-\mathrm{SV}$ and $\psi(X_1(t)) \in \mathrm{ntr}-\mathrm{SV}$. We integrate $q(t)\,\psi(X_1(t))$ on $[t,\infty)$. Applying Proposition 1.2.1 (which is possible since $\sigma < -1$) and using (3.4.3) we obtain

$$\int_t^\infty q(s)\psi(X_1(s))ds = \int_t^\infty s^\sigma l_q(s)\psi(X_1(s))ds$$
$$\sim \frac{t^{\sigma+1}}{-(\sigma+1)}\, l_q(t)\psi(X_1(t)) = \frac{t^{\eta-\alpha}}{\alpha-\eta} l_q(t)\psi(X_1(t)), \quad t\to\infty,$$

from which it readily follows that

$$p(t)^{-1}\int_t^\infty q(s)\,\psi(X_1(s))\,ds \sim \frac{t^{-\alpha}}{\alpha-\eta}\, l_p(t)^{-1}\, l_q(t)\psi(X_1(t)),\ t\to\infty.$$

From the above relation, using Proposition 1.2.11, (3.4.7) and (3.4.4) we conclude

$$\begin{aligned}\varphi^{-1}\left(p(t)^{-1}\int_t^\infty q(s)\,\psi(X_1(s))\,ds\right) &\sim \varphi^{-1}\left((\alpha-\eta)^{-1}\,t^{-\alpha}\,l_p(t)^{-1}\,l_q(t)\psi(X_1(t))\right)\\ &\sim (\alpha-\eta)^{-\frac{1}{\alpha}}\varphi^{-1}(t^{-\alpha})\,l_p(t)^{-\frac{1}{\alpha}}\,l_q(t)^{\frac{1}{\alpha}}\psi(X_1(t))^{\frac{1}{\alpha}}\\ &= (\alpha-\eta)^{-\frac{1}{\alpha}}\,t^{-1}L(t^{-\alpha})\,l_p(t)^{-\frac{1}{\alpha}}\,l_q(t)^{\frac{1}{\alpha}}\psi(X_1(t))^{\frac{1}{\alpha}},\ t\to\infty.\end{aligned} \tag{3.4.20}$$

In view of (3.4.18), integrating (3.4.20) from t_0 to t, we get

$$\begin{aligned}&\int_{t_0}^t \varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)\,\psi(X_1(r))\,dr\right)ds\\ \sim\ &\int_{t_0}^t \varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)dr\right)\psi(X_1(s))^{\frac{1}{\alpha}}\,ds,\ t\to\infty.\end{aligned} \tag{3.4.21}$$

On the other hand, we rewrite (3.4.13) as

$$\Psi(X_1(t)) = \int_{t_0}^t \varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)dr\right)ds, \quad t\geq t_0. \tag{3.4.22}$$

Since

$$\Psi(X_1(t)) = \int_0^{X_1(t)} \frac{dv}{\psi(v)^{\frac{1}{\alpha}}},$$

differentiation of (3.4.22) gives

$$X_1'(t) = \varphi^{-1}\left(p(t)^{-1}\int_t^\infty q(s)ds\right)\,\psi(X_1(t))^{\frac{1}{\alpha}},\ t\geq t_0. \tag{3.4.23}$$

Integrating (3.4.23) on $[t_0,t]$ and combining with (3.4.21) we obtain

$$X_1(t)\sim\int_{t_0}^t X_1'(s)\,ds\sim\int_{t_0}^t\varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)\,\psi(X_1(r))\,dr\right)ds,\ t\to\infty.$$

This completes the proof of Lemma 3.4.1.□

Lemma 3.4.2 *Suppose that*

$$\eta-\alpha-1<\sigma<\frac{\beta}{\alpha}\eta-\beta-1 \tag{3.4.24}$$

holds and let ρ *be defined by* (3.2.22). *The function* $X_2(t)\in\mathrm{RV}(\rho)$ *given by* (3.4.14) *satisfies the asymptotic relation* (3.4.12).

Proof. Let (3.4.24) hold. Using (3.4.3) and (3.4.4) we rewrite (3.4.14) in the form

$$\Psi(X_2(t)) = \frac{\alpha}{\alpha-\beta}\,\frac{t^{\frac{\sigma+\alpha+1-\eta}{\alpha}}}{\rho[\alpha(1-\rho)-\eta]^{\frac{1}{\alpha}}} L(t^{\alpha(\rho-1)})\, l_p(t)^{-\frac{1}{\alpha}}\, l_q(t)^{\frac{1}{\alpha}},\ t \geq t_0, \tag{3.4.25}$$

from which using (3.4.35) follows

$$\frac{X_2(t)}{\psi(X_2(t))^{\frac{1}{\alpha}}} \sim \frac{t^{\frac{\sigma+\alpha+1-\eta}{\alpha}}}{\rho[\alpha(1-\rho)-\eta]^{\frac{1}{\alpha}}} L(t^{\alpha(\rho-1)})\, l_p(t)^{-\frac{1}{\alpha}}\, l_q(t)^{\frac{1}{\alpha}},\ t \to \infty. \tag{3.4.26}$$

Since $\frac{\sigma+\alpha+1-\eta}{\alpha} > 0$, by Proposition 1.2.3, we conclude that the function on the right-hand side of relation (3.4.25) tends to ∞ as $t \to \infty$. From (3.4.25) using the previous conclusion and $\Psi^{-1} \in \mathrm{RV}(\frac{\alpha}{\alpha-\beta})$ with application of Proposition 1.2.2-(iv), we obtain $X_2(t) \in \mathrm{RV}(\rho)$, with ρ given by (3.2.22). Thus, $X_2(t)$ is expressed as $X_2(t) = t^\rho l_2(t)$, $l_2(t) \in \mathrm{SV}$. Then, using (3.4.26) we get

$$\begin{aligned} \int_t^\infty q(s)\psi(X_2(s))\,ds &= \int_t^\infty q(s)\,\frac{\psi(X_2(s))}{X_2(s)^\alpha}\,X_2(s)^\alpha\,ds \\ &\sim \rho^\alpha[\alpha(1-\rho)-\eta]\int_t^\infty q(s)s^{-\sigma-\alpha-1+\eta}L(s^{\alpha(\rho-1)})^{-\alpha}l_p(s)l_q(s)^{-1}X_2(s)^\alpha\,ds \\ &= \rho^\alpha[\alpha(1-\rho)-\eta]\int_t^\infty s^{\alpha(\rho-1)+\eta-1}L(s^{\alpha(\rho-1)})^{-\alpha}l_p(s)l_2(s)^\alpha ds,\ t\to\infty. \end{aligned} \tag{3.4.27}$$

Since $\sigma+\beta+1 < \frac{\beta}{\alpha}\eta$, we have $\alpha(\rho-1)+\eta < 0$ implying that we can apply Proposition 1.2.1 on the last integral in (3.4.27) and then multiplying the result with $p(t)^{-1}$ we obtain

$$p(t)^{-1}\int_t^\infty q(s)\psi(X_2(s))\,ds \sim \rho^\alpha t^{\alpha(\rho-1)}\,L(t^{\alpha(\rho-1)})^{-\alpha}\,l_2(t)^\alpha,\ t\to\infty,$$

from which, applying Proposition 1.2.11, it readily follows as $t \to \infty$ that

$$\varphi^{-1}\left(p(t)^{-1}\int_t^\infty q(s)\psi(X_2(s))\,ds\right) \sim \rho\,\varphi^{-1}(t^{\alpha(\rho-1)})\,L(t^{\alpha(\rho-1)})^{-1}\,l_2(t) \sim \rho\,t^{\rho-1}\,l_2(t),$$

where we use (3.4.4) and (3.4.7) in the two last steps. Integration on the above relation from t_0 to t with application of Proposition 1.2.1(which is possible since $\rho > 0$) then yields

$$\begin{aligned} &\int_{t_0}^t \varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)\psi(X_2(r))\,dr\right)\,ds \\ &\sim \rho\int_{t_0}^t s^{\rho-1}\,l_2(s)\,ds \sim t^\rho\,l_2(t) = X_2(t),\ t\to\infty. \end{aligned}$$

This completes the proof of Lemma 3.4.2. □

Lemma 3.4.3 *Suppose that*

$$\sigma = \frac{\beta}{\alpha}\eta - \beta - 1 \quad \textit{and} \quad \int_a^\infty q(t)\,\psi(P(t))\,dt < \infty, \tag{3.4.28}$$

holds. The function $X_3(t) \in \mathrm{ntr} - \mathrm{RV}(1 - \frac{\eta}{\alpha})$ *given by* (3.4.15) *satisfies the asymptotic relation* (3.4.12).

Proof. Let (3.4.28) hold. Since $\sigma = \frac{\beta}{\alpha}\eta - \beta - 1$, using (3.4.2), (3.4.3) and (3.4.10), by Proposition 1.2.2 we get $q(t)\,\psi(P(t)) \in \mathrm{RV}(-1)$ so that $\int_t^\infty q(s)\,\psi(P(s))\,ds \in \mathrm{SV}$ by Proposition 1.2.1-(iii). In view of (3.4.10) and (3.4.15), we conclude that $X_3(t) \in \mathrm{ntr} - \mathrm{RV}(1 - \frac{\eta}{\alpha})$. Using (3.4.8) and (3.4.10) we have

$$\int_t^\infty q(s)\,\psi(P(s))\,ds \sim \int_t^\infty s^{\beta(\frac{\eta}{\alpha}-1)}\,q(s)\,\psi(s^{1-\frac{\eta}{\alpha}})\,P(s)^\beta\,ds,\ t \to \infty. \tag{3.4.29}$$

This, combined with (3.4.15), gives the following expression for $X_3(t)$:

$$X_3(t) \sim P(t)\left(\frac{\alpha-\beta}{\alpha}\int_t^\infty s^{\beta(\frac{\eta}{\alpha}-1)}\,q(s)\,\psi(s^{1-\frac{\eta}{\alpha}})\,P(s)^\beta\,ds\right)^{\frac{1}{\alpha-\beta}},\ t \to \infty. \tag{3.4.30}$$

Next, we integrate $q(t)\psi(X_3(t))$ on $[t,\infty)$. Since $X_3(t) = t^{1-\frac{\eta}{\alpha}} l_3(t),\ l_3(t) \in \mathrm{SV}$, due to (3.4.8), we obtain

$$\begin{aligned}\int_t^\infty q(s)\psi(X_3(s))\,ds &= \int_t^\infty q(s)\psi(s^{1-\frac{\eta}{\alpha}} l_3(s))\,ds \sim \int_t^\infty q(s)\,\psi(s^{1-\frac{\eta}{\alpha}})\,l_3(s)^\beta\,ds \\ &= \int_t^\infty s^{\beta(\frac{\eta}{\alpha}-1)}\,q(s)\,\psi(s^{1-\frac{\eta}{\alpha}})\,X_3(s)^\beta\,ds,\ t \to \infty.\end{aligned} \tag{3.4.31}$$

Changing (3.4.30) in the last integral in (3.4.31), by a simple calculation we have

$$\begin{aligned}\int_t^\infty q(s)\psi(X_3(s))\,ds &\sim \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \\ &\times \int_t^\infty s^{\beta(\frac{\eta}{\alpha}-1)} q(s)\psi(s^{1-\frac{\eta}{\alpha}})P(s)^\beta\left(\int_s^\infty r^{\beta(\frac{\eta}{\alpha}-1)}q(r)\psi(r^{1-\frac{\eta}{\alpha}})P(r)^\beta dr\right)^{\frac{\beta}{\alpha-\beta}} ds \\ &= \left(\frac{\alpha-\beta}{\alpha}\int_t^\infty s^{\beta(\frac{\eta}{\alpha}-1)}\,q(s)\,\psi(s^{1-\frac{\eta}{\alpha}})\,P(s)^\beta\,ds\right)^{\frac{\alpha}{\alpha-\beta}} \\ &\sim \left(\frac{\alpha-\beta}{\alpha}\int_t^\infty q(s)\,\psi(P(s))\,ds\right)^{\frac{\alpha}{\alpha-\beta}},\ t \to \infty,\end{aligned} \tag{3.4.32}$$

where we use (3.4.29) in the last step. Since $\int_t^\infty q(s)\psi(X_3(s))\,ds \in \mathrm{SV}$, (3.4.3), (3.4.4) and (3.4.7) give

$$\begin{aligned}
\varphi^{-1}\left(p(t)^{-1}\int_t^\infty q(s)\psi(X_3(s))ds\right) &= \varphi^{-1}\left(t^{-\eta}l_p(t)^{-1}\int_t^\infty q(s)\psi(X_3(s))ds\right)\\
&\sim \varphi^{-1}(t^{-\eta})l_p(t)^{-\frac{1}{\alpha}}\left(\int_t^\infty q(s)\psi(X_3(s))ds\right)^{\frac{1}{\alpha}} \qquad (3.4.33)\\
&= t^{-\frac{\eta}{\alpha}}L(t^{-\eta})\,l_p(t)^{-\frac{1}{\alpha}}\left(\int_t^\infty q(s)\,\psi(X_3(s))\,ds\right)^{\frac{1}{\alpha}},
\end{aligned}$$

as $t\to\infty$. Integrating (3.4.33) from t_0 to t, we conclude via Proposition 1.2.1 that

$$\begin{aligned}
&\int_{t_0}^t \varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)\,\psi(X_3(r))\,dr\right)ds\\
&\sim \frac{\alpha}{\alpha-\eta}t^{1-\frac{\eta}{\alpha}}\,L(t^{-\eta})\,l_p(t)^{-\frac{1}{\alpha}}\left(\int_t^\infty q(s)\,\psi(X_3(s))\,ds\right)^{\frac{1}{\alpha}}, \quad t\to\infty.
\end{aligned}$$

This, combined with (3.4.10) and (3.4.32), shows that $X_3(t)$ satisfies the asymptotic relation (3.4.12). This completes the proof of Lemma 3.4.3. □

Since there are positive constants c_1 and c_2 such that $c_1 \le x(t) \le c_2P(t)$, for all large t, the regularity index ρ of $x(t)$ must satisfy $0\le\rho\le 1-\frac{\eta}{\alpha}$. Therefore, the class of intermediate regularly varying solutions of (E_2) is divided into three types of subclasses:

$$\mathrm{ntr}-\mathrm{SV},\quad \mathrm{RV}(\rho),\ \rho\in\left(0,1-\frac{\eta}{\alpha}\right),\quad \mathrm{ntr}-\mathrm{RV}\left(1-\frac{\eta}{\alpha}\right).$$

To state our main results, we will need the function

$$\Psi(y)=\int_0^y \frac{dv}{\psi(v)^{\frac{1}{\alpha}}},\ y>0, \qquad (3.4.34)$$

which is clearly increasing on $(0,\infty)$. From (3.4.2), (3.4.34) and Proposition 1.2.1 we get

$$\Psi(y)=\int_0^y v^{-\frac{\beta}{\alpha}}L_2(v)^{-\frac{1}{\alpha}}\,dv \sim \frac{\alpha}{\alpha-\beta}\,y^{1-\frac{\beta}{\alpha}}\,L_2(y)^{-\frac{1}{\alpha}} = \frac{\alpha}{\alpha-\beta}\frac{y}{\psi(y)^{\frac{1}{\alpha}}},\quad y\to\infty, \qquad (3.4.35)$$

implying $\Psi(y)\in\mathrm{RV}(\frac{\alpha-\beta}{\alpha})$ and $\Psi^{-1}(y)\in\mathrm{RV}(\frac{\alpha}{\alpha-\beta})$ with $\frac{\alpha-\beta}{\alpha}>0$.

Theorem 3.4.1 *Suppose that* (3.4.1), (3.4.5), (3.4.6) *and* (C_2) *hold. Equation* (E_2) *possesses intermediate solutions* $x(t)\in\mathrm{ntr}-\mathrm{SV}$ *if and only if* (3.4.16) *holds, in which case any such solution* $x(t)$ *has the asymptotic behavior* $x(t)\sim X_1(t)$, $t\to\infty$, *where* $X_1(t)$ *is given by* (3.4.13).

Theorem 3.4.2 *Suppose that* (3.4.1), (3.4.5), (3.4.6) *and* (C_2) *hold. Equation* (E_2) *possesses intermediate solutions* $x(t) \in \mathrm{RV}(\rho)$ *with* $\rho \in (0, 1-\frac{\eta}{\alpha})$ *if and only if* (3.4.24) *holds, in which case* ρ *is given by* (3.2.22) *and any such solution* $x(t)$ *has the asymptotic behavior* $x(t) \sim X_2(t)$, $t \to \infty$, *where* $X_2(t)$ *is given by* (3.4.14).

Theorem 3.4.3 *Suppose that* (3.4.1), (3.4.5), (3.4.6) *and* (C_2) *hold. Equation* (E_2) *possesses intermediate solutions* $x(t) \in \mathrm{ntr}-\mathrm{RV}(1-\frac{\eta}{\alpha})$ *if and only if* (3.4.28) *holds, in which case any such solution* $x(t)$ *has the asymptotic behavior* $x(t) \sim X_3(t)$, $t \to \infty$, *where* $X_3(t)$ *is given by* (3.4.15).

Proof of the "only if" part of Theorems 3.4.1, 3.4.2, 3.4.3: Suppose that the equation (E_2) has an intermediate solution $x(t) \in \mathrm{RV}(\rho)$ with $\rho \in [0, 1-\frac{\eta}{\alpha}]$ defined on $[t_0, \infty)$. Since $\lim_{t\to\infty} p(t)\varphi(x'(t)) = 0$, integration of equation (E_2) on (t, ∞) using (3.4.2), (3.4.3) and (3.4.9) gives

$$(3.4.36) \qquad p(t)\varphi(x'(t)) = \int_t^\infty q(s)\,\psi(x(s))\,ds = \int_t^\infty s^{\sigma+\beta\rho}\,l_q(s)l_x(s)^\beta\,L_2(x(s))\,ds,$$

implying the convergence of the last integral in (3.4.36) i.e. implying that $\sigma + \beta\rho \leq -1$. We distinguish the two cases:

$$(a) \quad \sigma+\beta\rho = -1, \quad (b) \quad \sigma+\beta\rho < -1.$$

Assume that (a) holds. Multiplying (3.4.36) with $p(t)^{-1}$ we get

$$(3.4.37) \qquad \varphi(x'(t)) = p(t)^{-1}\,\xi(t), \quad \text{where} \quad \xi(t) = \int_t^\infty s^{-1}l_q(s)\,l_x(s)^\beta L_2(x(s))\,ds.$$

Clearly, $\xi(t) \in \mathrm{SV}$ and $\lim_{t\to\infty}\xi(t) = 0$. From (3.4.37), using (3.4.3) and (3.4.7) we have

$$(3.4.38) \quad x'(t) = \varphi^{-1}\left(p(t)^{-1}\xi(t)\right) = \varphi^{-1}(t^{-\eta}\,l_p(t)^{-1}\xi(t)) \sim \varphi^{-1}(t^{-\eta})\,l_p(t)^{-\frac{1}{\alpha}}\,\xi(t)^{\frac{1}{\alpha}},$$

as $t \to \infty$. Integrating (3.4.38) from t_0 to t and using (3.4.4) we get

$$(3.4.39) \quad x(t) \sim \int_{t_0}^t \varphi^{-1}(s^{-\eta})\,l_p(s)^{-\frac{1}{\alpha}}\,\xi(s)^{\frac{1}{\alpha}}\,ds = \int_{t_0}^t s^{-\frac{\eta}{\alpha}}\,L(s^{-\eta})\,l_p(s)^{-\frac{1}{\alpha}}\,\xi(s)^{\frac{1}{\alpha}}\,ds, \quad t\to\infty.$$

From (3.4.39) we find via Karamata's integration theorem that

$$(3.4.40) \qquad x(t) \sim \frac{\alpha}{\alpha-\eta}\,t^{1-\frac{\eta}{\alpha}}\,L(t^{-\eta})\,l_p(t)^{-\frac{1}{\alpha}}\,\xi(t)^{\frac{1}{\alpha}} \in \mathrm{RV}\left(1-\frac{\eta}{\alpha}\right), \quad t\to\infty.$$

Using (3.4.10) we rewrite (3.4.40) in the form

$$(3.4.41) \qquad x(t) \sim P(t)\,\xi(t)^{\frac{1}{\alpha}}, \quad t\to\infty.$$

Assume that (b) holds. Applying Proposition 1.2.1 to the last integral in (3.4.36) we have

$$p(t)\varphi(x'(t)) \sim \frac{t^{\sigma+\beta\rho+1}}{-(\sigma+\beta\rho+1)} l_q(t)\, l_x(t)^\beta L_2(x(t)), \quad t\to\infty. \tag{3.4.42}$$

Multiplying (3.4.42) with $p(t)^{-1}$ and using (3.4.3) we get

$$\varphi(x'(t)) \sim \frac{t^{\sigma+\beta\rho+1-\eta}}{-(\sigma+\beta\rho+1)}\, l_p(t)^{-1}\, l_q(t)\, l_x(t)^\beta L_2(x(t)), \quad t\to\infty.$$

Using Proposition 1.2.11, (3.4.7) and (3.4.4) we have

$$\begin{aligned} x'(t) &\sim \varphi^{-1}\left(t^{\sigma+\beta\rho+1-\eta}(-(\sigma+\beta\rho+1))^{-1}\, l_p(t)^{-1}\, l_q(t)\, l_x(t)^\beta L_2(x(t))\right) \\ &\sim \varphi^{-1}(t^{\sigma+\beta\rho+1-\eta})(-(\sigma+\beta\rho+1))^{-\frac{1}{\alpha}}\, l_p(t)^{-\frac{1}{\alpha}}\, l_q(t)^{\frac{1}{\alpha}}\, l_x(t)^{\frac{\beta}{\alpha}} L_2(x(t))^{\frac{1}{\alpha}} \qquad (3.4.43) \\ &= (-(\sigma+\beta\rho+1))^{-\frac{1}{\alpha}}\, t^{\frac{\sigma+\beta\rho+1-\eta}{\alpha}}\, L(t^{\sigma+\beta\rho+1-\eta})\, l_p(t)^{-\frac{1}{\alpha}}\, l_q(t)^{\frac{1}{\alpha}}\, l_x(t)^{\frac{\beta}{\alpha}} L_2(x(t))^{\frac{1}{\alpha}}, \end{aligned}$$

as $t\to\infty$. Integration of (3.4.43) on $[t_0,t]$ leads to

$$\begin{aligned} x(t) &\sim (-(\sigma+\beta\rho+1))^{-\frac{1}{\alpha}} \\ &\times \int_{t_0}^t s^{\frac{\sigma+\beta\rho+1-\eta}{\alpha}} L(s^{\sigma+\beta\rho+1-\eta}) l_p(s)^{-\frac{1}{\alpha}} l_q(s)^{\frac{1}{\alpha}} l_x(s)^{\frac{\beta}{\alpha}} L_2(x(s))^{\frac{1}{\alpha}} ds,\ t\to\infty. \end{aligned} \tag{3.4.44}$$

Since the above integral tends to infinity as $t\to\infty$ (note that $x(t)\to\infty, t\to\infty$), we consider the following two cases separately:

$$(b.1)\quad \frac{\sigma+\beta\rho+1-\eta}{\alpha} > -1, \quad (b.2)\quad \frac{\sigma+\beta\rho+1-\eta}{\alpha} = -1.$$

Assume that $(b.1)$ holds. Applying Proposition 1.2.1 to the integral in (3.4.44), we get as $t\to\infty$

$$\begin{aligned} x(t) &\sim \frac{\alpha}{\sigma+\beta\rho+1-\eta+\alpha}\,(-(\sigma+\beta\rho+1))^{-\frac{1}{\alpha}}\, t^{\frac{\sigma+\beta\rho+1-\eta+\alpha}{\alpha}} L(t^{\sigma+\beta\rho+1-\eta}) \\ & l_p(t)^{-\frac{1}{\alpha}}\, l_q(t)^{\frac{1}{\alpha}}\, l_x(t)^{\frac{\beta}{\alpha}} L_2(x(t))^{\frac{1}{\alpha}} \in \mathrm{RV}\left(\frac{\sigma+\beta\rho+1-\eta+\alpha}{\alpha}\right). \end{aligned} \tag{3.4.45}$$

Assume that $(b.2)$ holds. Then, (4.3.29) shows that $x(t)\in \mathrm{SV}$, that is $\rho=0$, and hence $\sigma=\eta-\alpha-1$. Since $\sigma+\beta\rho+1=\eta-\alpha$, (4.3.29) reduced to

$$x(t) \sim (\alpha-\eta)^{-\frac{1}{\alpha}} \int_{t_0}^t s^{-1}\, L(s^{-\alpha})\, l_p(s)^{-\frac{1}{\alpha}}\, l_q(s)^{\frac{1}{\alpha}} l_x(s)^{\frac{\beta}{\alpha}} L_2(x(s))^{\frac{1}{\alpha}}\, ds \in \mathrm{SV}, \tag{3.4.46}$$

as $t \to \infty$.

Let us now suppose that $x(t)$ is an intermediate solution of (E_2) belonging to $\mathrm{ntr} - \mathrm{SV}$. From the above observation this is possible only when the case $(b.2)$ holds, in which case $\rho = 0$, $\sigma = \eta - \alpha - 1$ and $x(t) = l_x(t)$ must satisfy the asymptotic behavior (3.4.46). Denote the right-hand side of (3.4.46) by $\mu(t)$. Then, $\mu(t) \to \infty$, $t \to \infty$ and satisfies

$$\begin{aligned}\mu'(t) &= (\alpha-\eta)^{-\frac{1}{\alpha}}\, t^{-1}\, L(t^{-\alpha})\, l_p(t)^{-\frac{1}{\alpha}}\, l_q(t)^{\frac{1}{\alpha}} l_x(t)^{\frac{\beta}{\alpha}} L_2(x(t))^{\frac{1}{\alpha}} \\ &= (\alpha-\eta)^{-\frac{1}{\alpha}}\, t^{-1}\, L(t^{-\alpha})\, l_p(t)^{-\frac{1}{\alpha}}\, l_q(t)^{\frac{1}{\alpha}}\, \psi(x(t))^{\frac{1}{\alpha}},\end{aligned}$$

where we use (3.4.2) in the last step. Since (3.4.46) is equivalent to $x(t) \sim \mu(t)$, $t \to \infty$, from the above using (3.4.18) we obtain

$$\frac{\mu'(t)}{\psi(\mu(t))^{\frac{1}{\alpha}}} \sim \varphi^{-1}\left(p(t)^{-1}\int_t^\infty q(s)ds\right),\ t \to \infty.$$

An integration of the last relation over $[t_0, t]$ gives

$$\int_{\mu(t_0)}^{\mu(t)} \frac{dv}{\psi(v)^{\frac{1}{\alpha}}} \sim \Psi(\mu(t)) \sim \int_{t_0}^t \varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)dr\right) ds,\ t \to \infty,$$

or

$$x(t) \sim \mu(t) \sim \Psi^{-1}\left(\int_{t_0}^t \varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)dr\right) ds\right),\ t \to \infty.$$

Thus, it has been shown that $x(t) \sim X_1(t)$, $t \to \infty$, where $X_1(t)$ is given by (3.4.13). Notice that the verification of (3.4.16) is included in the above discussions. This proves the "only if" part of Theorem 3.4.1.

Next, suppose that $x(t)$ is an intermediate solution of (E_2) belonging to $\mathrm{RV}(\rho)$, $\rho \in (0, 1-\frac{\eta}{\alpha})$. This is possible only when $(b.1)$ holds, in which case $x(t)$ must satisfy the asymptotic relation (3.4.45). Therefore,

$$\rho = \frac{\sigma+\beta\rho+1-\eta+\alpha}{\alpha} \quad \Rightarrow \quad \rho = \frac{\sigma+\alpha+1-\eta}{\alpha-\beta},$$

which justifies (3.2.22). An elementary calculation shows that

$$0 < \rho < 1 - \frac{\eta}{\alpha} \quad \Rightarrow \quad \eta - \alpha - 1 < \sigma < \frac{\beta}{\alpha}\eta - \beta - 1,$$

which determines the range (3.4.24) of σ. Since $\sigma + \beta\rho + 1 - \eta + \alpha = \alpha\rho$ and $-(\sigma+\beta\rho+1) = \alpha(1-\rho) - \eta$, (3.4.45) reduced to

$$\begin{aligned}x(t) &\sim \frac{t^\rho}{\rho(\alpha(1-\rho)-\eta)^{\frac{1}{\alpha}}}\, L(t^{\alpha(\rho-1)})\, l_p(t)^{-\frac{1}{\alpha}}\, l_q(t)^{\frac{1}{\alpha}}\, l_x(t)^{\frac{\beta}{\alpha}}\, L_2(x(t))^{\frac{1}{\alpha}} \\ &= \frac{t^{2-\rho+\frac{1}{\alpha}}}{\rho(\alpha(1-\rho)-\eta)^{\frac{1}{\alpha}}}\, \varphi^{-1}(t^{\alpha(\rho-1)})\, p(t)^{-\frac{1}{\alpha}}\, q(t)^{\frac{1}{\alpha}}\, \psi(x(t))^{\frac{1}{\alpha}},\ t \to \infty, \qquad (3.4.47)\end{aligned}$$

where we use (3.4.2), (3.4.3), (3.4.4) and (3.4.9) in the last step. From (3.4.47) using (3.4.35) we get

$$\Psi(x(t)) \sim \frac{\alpha}{\alpha-\beta}\,\frac{x(t)}{\psi(x(t))^{\frac{1}{\alpha}}} \sim \frac{\alpha}{\alpha-\beta}\,\frac{t^{2-\rho+\frac{1}{\alpha}}}{\rho(\alpha(1-\rho)-\eta)^{\frac{1}{\alpha}}}\,\varphi^{-1}(t^{\alpha(\rho-1)})\,p(t)^{-\frac{1}{\alpha}}\,q(t)^{\frac{1}{\alpha}},$$

as $t\to\infty$. Thus, we conclude that $x(t)$ enjoys the asymptotic formula $x(t)\sim X_2(t)$, $t\to\infty$, where $X_2(t)$ is given by (3.4.14). This proves the "only if" part of the Theorem 3.4.2.

Finally, suppose that $x(t)$ is an intermediate solution of (E$_2$) belonging to $\mathrm{ntr}-\mathrm{RV}(1-\frac{\eta}{\alpha})$. Then, the case (a) is the only possibility for $x(t)$, which means that $\rho = 1-\frac{\eta}{\alpha}$, $\sigma = \frac{\beta}{\alpha}\eta-\beta-1$ and (3.4.41) is satisfied by $x(t)$. Differentiation of $\xi(t)$, defined in (3.4.37), using (3.4.2), (3.4.3) and (3.4.9) leads to

$$\xi'(t) \sim -t^{-1}\,l_q(t)\,l_x(t)^{\beta}\,L_2(x(t)) \sim -q(t)\,\psi(x(t)),\ t\to\infty.$$

Noting that $x(t)\sim P(t)\xi(t)^{\frac{1}{\alpha}}$, $t\to\infty$ and using (3.4.8), one can transform the above relation into

$$\xi'(t) \sim -q(t)\,\psi(P(t)\xi(t)^{\frac{1}{\alpha}}) \sim -q(t)\psi(P(t))\xi(t)^{\frac{\beta}{\alpha}},\ t\to\infty.$$

So, we get the differential asymptotic relation for $\xi(t)$:

$$\xi(t)^{-\frac{\beta}{\alpha}}\,\xi'(t) \sim -q(t)\,\psi(P(t)),\ t\to\infty. \tag{3.4.48}$$

Due to fact that $\alpha-\beta>0$ and $\xi(t)\to 0$ as $t\to\infty$, the left-hand side of (3.4.48) can be integrated over (t,∞), assuring the integrability of $q(t)\,\psi(P(t))$ on (t,∞), which implies the convergence of the integral in (3.4.28). Integration of (3.4.48) on (t,∞) yields

$$\xi(t) \sim \left(\frac{\alpha-\beta}{\alpha}\int_t^{\infty} q(s)\,\psi(P(s))\,ds\right)^{\frac{\alpha}{\alpha-\beta}},\ t\to\infty. \tag{3.4.49}$$

Combining (4.3.32) with (3.4.41) gives us $x(t)\sim X_3(t)$, $t\to\infty$, where $X_3(t)$ is given by (3.4.15). This completes the "only if" part of the proof of Theorem 3.4.3. □

Proof of the "if" part of Theorems 3.4.1, 3.4.2, 3.4.3: Suppose that (3.4.16), (3.4.24) or (3.4.28) holds. From Lemmas 3.4.1, 3.4.2 and 3.4.3 it is known that each $X_i(t)$, $i=1,2,3$, defined by (3.4.13),(3.4.14) and (3.4.15), satisfies the asymptotic relation (3.4.12) for any $b\geq a$. We perform the simultaneous proof for $X_i(t)$,

$i = 1, 2, 3$ so the subscript $i = 1, 2, 3$ will be deleted in the rest of proof. By (3.4.12) there exists $T_0 > a$ such that

$$\int_{T_0}^{t} \varphi^{-1}\left(p(s)^{-1}\int_{s}^{\infty} q(r)\,\psi(X(r))\,dr\right) ds \leq 2X(t), \;\; t \geq T_0. \tag{3.4.50}$$

Let such a T_0 be fixed. We may assume that $X(t)$ is increasing on $[T_0, \infty)$. Since (3.4.12) is satisfied with $b = T_0$, there exists $T > T_0$ such that

$$\int_{T_0}^{t} \varphi^{-1}\left(p(s)^{-1}\int_{s}^{\infty} q(r)\,\psi(X(r))\,dr\right) ds \geq \frac{1}{2}X(t), \;\; t \geq T. \tag{3.4.51}$$

Applying Proposition 1.2.5 to the function $\psi(s) \in \mathrm{RV}(\beta)$, $\beta > 0$ we see that there exists a constant $A > 1$ such that

$$\psi(s_1) \leq A\psi(s_2) \quad \text{for each} \quad 0 \leq s_1 \leq s_2. \tag{3.4.52}$$

Now we choose positive constants m and M such that

$$m^{1-\frac{\beta}{\alpha}} \leq \frac{1}{4(2A)^{1/\alpha}}, \quad M^{1-\frac{\beta}{\alpha}} \geq 8(2A)^{1/\alpha}, \quad 2mX(T) \leq MX(T_0). \tag{3.4.53}$$

In addition, since $X(t) \to \infty$ as $t \to \infty$, from (1.2.1), for $\lambda > 0$ we have

$$\frac{\lambda^\beta}{2}\psi(X(t)) \leq \psi(\lambda X(t)) \leq 2\lambda^\beta \psi(X(t)), \quad \text{for all sufficiently large } t. \tag{3.4.54}$$

Also, since $Q(t) = 1/p(t)\int_t^\infty q(s)\psi(X(s))\,ds \to 0$ as $t \to \infty$, from (1.2.2), for $\lambda > 0$ we have

$$\frac{\lambda^{1/\alpha}}{2}\varphi^{-1}(Q(t)) \leq \varphi^{-1}(\lambda Q(t)) \leq 2\lambda^{1/\alpha}\varphi^{-1}(Q(t)), \quad \text{for all sufficiently large } t. \tag{3.4.55}$$

Define the integral operator $\mathcal{F}$ by

$$\mathcal{F}x(t) = x_0 + \int_{T_0}^{t} \varphi^{-1}\left(p(s)^{-1}\int_{s}^{\infty} q(r)\psi(x(r))\,dr\right) ds, \quad t \geq T_0, \tag{3.4.56}$$

where x_0 is constant such that

$$mX(T) \leq x_0 \leq \frac{M}{2}X(T_0), \tag{3.4.57}$$

and let it act on the set

$$\mathcal{X} := \{x(t) \in C[T_0, \infty) : mX(t) \leq x(t) \leq MX(t), \; t \geq T_0\}. \tag{3.4.58}$$

It is clear that $\mathcal{X}$ is a closed convex subset of the locally convex space $C[T_0,\infty)$ equipped with the topology of uniform convergence on compact subintervals of $[T_0,\infty)$.

Let $x(t)\in\mathcal{X}$. Using first (3.4.52) and (3.4.58) and then (3.4.54) and (3.4.57) we get

$$\begin{aligned}\mathcal{F}x(t) &\le x_0+\int_{T_0}^t \varphi^{-1}\left(Ap(s)^{-1}\int_s^\infty q(r)\psi(MX(r))\,dr\right)ds\\ &\le \frac{M}{2}X(T_0)+\int_{T_0}^t \varphi^{-1}\left(2AM^\beta p(s)^{-1}\int_s^\infty q(r)\psi(X(r))\,dr\right)ds,\quad t\ge T_0,\end{aligned}$$

from which, using (3.4.55), (3.4.50) and (3.4.53), it follows that

$$\begin{aligned}\mathcal{F}x(t) &\le \frac{M}{2}X(T_0)+2(2AM^\beta)^{1/\alpha}\int_{T_0}^t \varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)\psi(X(r))\,dr\right)ds\\ &\le \frac{M}{2}X(t)+4(2AM^\beta)^{1/\alpha}X(t)\le\frac{M}{2}X(t)+\frac{M}{2}X(t)=MX(t),\quad t\ge T_0.\end{aligned}$$

On the other hand, using (3.4.57) we have

$$\mathcal{F}x(t)\ge x_0\ge mX(T)\ge mX(t)\quad\text{for}\quad T_0\le t\le T,$$

and using (3.4.58),(3.4.52) and (3.4.54) we obtain

$$\begin{aligned}\mathcal{F}x(t) &\ge \int_{T_0}^t \varphi^{-1}\left(\frac{p(s)^{-1}}{A}\int_s^\infty q(r)\psi(mX(r))\,dr\right)ds\\ &\ge \int_{T_0}^t \varphi^{-1}\left(\frac{m^\beta\, p(s)^{-1}}{2A}\int_s^\infty q(r)\psi(X(r))\,dr\right)ds,\quad t\ge T.\end{aligned}$$

From the above using (3.4.55), (3.4.51) and (3.4.53) we conclude

$$\begin{aligned}\mathcal{F}x(t) &\ge \frac{1}{2}\left(\frac{m^\beta}{2A}\right)^{\frac{1}{\alpha}}\int_{T_0}^t \varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)\psi(X(r))\,dr\right)ds\\ &\ge \frac{1}{4}\left(\frac{m^\beta}{2A}\right)^{\frac{1}{\alpha}}X(t)\ge mX(t),\quad t\ge T.\end{aligned}$$

This shows that $\mathcal{F}x(t)\in\mathcal{X}$, that is, $\mathcal{F}$ maps $\mathcal{X}$ into itself.

Furthermore it can be verified (similarly to the proof of Theorem 3 in [11]) that $\mathcal{F}$ is a continuous mapping and that $\mathcal{F}(\mathcal{X})$ is relatively compact in $C[T_0,\infty)$.

Thus, all the hypotheses of the Schauder-Tychonoff fixed point theorem are fulfilled and so there exists a fixed point $x(t)\in\mathcal{X}$ of $\mathcal{F}$, which satisfies integral equation

$$x(t)=x_0+\int_{T_0}^t \varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)\psi(x(r))\,dr\right)ds,\quad t\ge T_0.$$

Differentiating the above twice shows that $x(t)$ is a solution of (E$_2$) on $[T_0,\infty)$. It is clear from (3.4.58) that $x(t)$ is an intermediate solution of (E$_2$).

Therefore, the existence of three types of intermediate solutions of (E$_2$) has been established. The proof of our main results will be completed with the verification that the intermediate solutions of (E$_2$) constructed above are actually regularly varying functions.

We define the function

$$J(t)=\int_{T_0}^{t}\varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)\psi(X(r))\,dr\right)ds,\quad t\geq T_0,$$

and put

$$l=\liminf_{t\to\infty}\frac{x(t)}{J(t)},\quad L=\limsup_{t\to\infty}\frac{x(t)}{J(t)}.$$

Since $x(t)\in\mathcal{X}$, it is clear that $0<l\leq L<\infty$. By Lemmas 3.4.1, 3.4.2 and 3.4.3 we have

$$J(t)\sim X(t),\quad t\to\infty. \tag{3.4.59}$$

Using Lemma 1.1.1 and (3.4.2) we see that

$$\begin{aligned}\liminf_{t\to\infty}\frac{\int_t^\infty q(s)\,\psi(x(s))\,ds}{\int_t^\infty q(s)\,\psi(X(s))\,ds}&\geq\liminf_{t\to\infty}\frac{\psi(x(t))}{\psi(X(t))}\\&=\liminf_{t\to\infty}\frac{x(t)^\beta L_2(x(t))}{X(t)^\beta L_2(X(t))}\geq\liminf_{t\to\infty}\left(\frac{x(t)}{X(t)}\right)^\beta\liminf_{t\to\infty}\frac{L_2\left(\frac{x(t)}{X(t)}X(t)\right)}{L_2(X(t))}.\end{aligned} \tag{3.4.60}$$

Since $m\leq\frac{x(t)}{X(t)}\leq M$, $t\geq T_0$, using the uniform convergence theorem ([2],Theorem 1.2.1) we conclude

$$\left|\frac{L_2\left(\frac{x(t)}{X(t)}X(t)\right)}{L_2(X(t))}-1\right|\leq\sup_{\lambda\in[m,M]}\left|\frac{L_2(\lambda X(t))}{L_2(X(t))}-1\right|\longrightarrow 0,\quad t\to\infty. \tag{3.4.61}$$

From (3.4.60), using (3.4.61) and (3.4.59) we get

$$\liminf_{t\to\infty}\frac{\int_t^\infty q(s)\,\psi(x(s))\,ds}{\int_t^\infty q(s)\,\psi(X(s))\,ds}\geq\left(\liminf_{t\to\infty}\frac{x(t)}{X(t)}\right)^\beta=\left(\liminf_{t\to\infty}\frac{x(t)}{J(t)}\right)^\beta=l^\beta. \tag{3.4.62}$$

Similarly, we conclude that

$$\limsup_{t\to\infty}\frac{\int_t^\infty q(s)\,\psi(x(s))\,ds}{\int_t^\infty q(s)\,\psi(X(s))\,ds}\leq L^\beta. \tag{3.4.63}$$

We denote $\hat{x}(t) = p(t)^{-1} \int_t^\infty q(s)\psi(x(s))ds$ and $\hat{X}(t) = p(t)^{-1} \int_t^\infty q(s)\psi(X(s))ds$. Using Lemma 1.1.1 and (3.4.4) we obtain

$$l \geq \liminf_{t\to\infty} \frac{x'(t)}{J'(t)} = \liminf_{t\to\infty} \frac{\varphi^{-1}(\hat{x}(t))}{\varphi^{-1}\left(\hat{X}(t)\right)} \geq \liminf_{t\to\infty} \left(\frac{\hat{x}(t)}{\hat{X}(t)}\right)^{\frac{1}{\alpha}} \liminf_{t\to\infty} \frac{L\left(\frac{\hat{x}(t)}{\hat{X}(t)}\hat{X}(t)\right)}{L(\hat{X}(t))}.$$

From (3.4.62) and (3.4.63) we have that $\frac{\hat{x}(t)}{\hat{X}(t)}$ is bounded. So, we can apply the Uniform convergence again, identically to (3.4.61), to get

$$l \geq \liminf_{t\to\infty} \left(\frac{\hat{x}(t)}{\hat{X}(t)}\right)^{\frac{1}{\alpha}} = \left(\liminf_{t\to\infty} \frac{\int_t^\infty q(s)\,\psi(x(s))\,ds}{\int_t^\infty q(s)\,\psi(X(s))\,ds}\right)^{\frac{1}{\alpha}}. \tag{3.4.64}$$

In view of (3.4.62) and (3.4.64) we have $l \geq l^{\frac{\beta}{\alpha}}$, implying that $l \geq 1$ because $\alpha > \beta$. If we argue similarly by taking the superior limits instead of the inferior limits, we are led to the inequality $L \leq L^{\frac{\beta}{\alpha}}$, which implies that $L \leq 1$. Thus we conclude that $l = L = 1$, i.e. $\lim_{t\to\infty} x(t)/J(t) = 1$. This combined with (3.4.59) shows that $x(t) \sim X(t)$, $t \to \infty$, which shows that $x(t)$ is a regularly varying function whose regularity index ρ is 0, $\frac{\sigma+\alpha+1-\eta}{\alpha-\beta}$, or $1 - \frac{\eta}{\alpha}$ according to whether $\sigma = \eta - \alpha - 1$, $\eta - \alpha - 1 < \sigma < \frac{\beta}{\alpha}\eta - \beta - 1$, or $\sigma = \frac{\beta}{\alpha}\eta - \beta - 1$. $\square$

3.5 Examples

Now, we present two examples that illustrate results presented in previous sections. First example illustrates Theorems 3.2.1-3.2.3.

Example 3.5.1 Consider the equation

$$(p(t)\,\varphi(|x'(t)|))' = q(t)\psi(x(t)), \quad t \geq t_0 > e, \tag{3.5.1}$$

where $p(t) = t^{2\alpha}(\log t)^{-\frac{2\alpha}{3}} \in \mathrm{RV}(2\alpha)$, $\varphi(s) = s^\alpha \in \mathcal{RV}(\alpha)$ and $\psi(s) = s^\beta \log s \in \mathrm{RV}(\beta)$,
$\alpha > \beta > 0$. So that $\eta = 2\alpha > \alpha$ and the functions $\varphi^{-1}(s)$ and $\psi(s)$ satisfy the additional requirements (3.2.7) and (3.2.8), respectively. Since, $\varphi^{-1}(p(t)^{-1}) = \left(\frac{\sqrt[3]{\log t}}{t}\right)^2$, applying Proposition 1.2.1 we have $\pi(t) \sim \frac{\sqrt[3]{(\log t)^2}}{t}$, $t \to \infty$.

(i) Suppose that

$$q(t) \sim \frac{\alpha}{3}\, t^{\beta-1}\, \frac{r(t)\,(\log t)^{\frac{\alpha}{3}-\beta-1}}{\log\frac{\log t}{t}}, \quad t \to \infty, \tag{3.5.2}$$

where $r(t)$ is continuous function on $[t_0, \infty)$ such that $\lim_{t\to\infty} r(t) = 1$. Here, $q(t) \in \mathrm{RV}(\beta - 1)$. Therefore, $\sigma = \frac{\beta}{\alpha}\eta - \beta - 1$ and

$$\int_{t_0}^{t} q(s)\psi(\pi(s))ds \sim \frac{\alpha}{3}\int_{t_0}^{t}(\log s)^{\frac{\alpha-\beta}{3}-1}\frac{ds}{s} \sim \frac{\alpha}{\alpha-\beta}(\log t)^{\frac{\alpha-\beta}{3}} \longrightarrow \infty, \quad t\to\infty,$$

implying that (3.2.15) holds. Therefore, by Theorem 3.2.1 there exist nontrivial regularly varying solutions of index $1 - \frac{\eta}{\alpha}$ of (3.5.1) and any such solution $x(t)$ has asymptotic behavior

$$x(t) \sim \frac{\log t}{t}, \quad t\to\infty.$$

If in (3.5.2) instead of " $\sim$ " one has " $=$ " and in particular

$$r(t) = \left(1 - \frac{1}{\log t}\right)^{\alpha-1}\left(1 + \frac{2}{\log t}\right),$$

then (3.5.1) possesses an exact solution $x(t) = \frac{\log t}{t}$.

(ii) Suppose that

$$q(t) \sim \frac{2\alpha}{3^{\alpha+1}}\, t^{\frac{2\alpha+\beta}{3}-1}\, \frac{r(t)}{(\log t)^{\frac{\alpha+\beta}{3}}\log\sqrt[3]{\frac{\log t}{t}}}, \quad t\to\infty, \tag{3.5.3}$$

where $r(t)$ is continuous function on $[t_0, \infty)$ such that $\lim_{t\to\infty} r(t) = 1$. It is clear that $q(t)$ is regularly varying function of index

$$\sigma = \frac{2\alpha+\beta}{3} - 1 \in \left(\frac{\beta}{\alpha}\eta - \beta - 1, \eta - \alpha - 1\right) = (\beta - 1, \alpha - 1)$$

and that $\rho = \dfrac{\sigma + \alpha + 1 - \eta}{\alpha - \beta} = -\dfrac{1}{3}$. By Theorem 3.2.2 there exist regularly varying solutions of index ρ of (3.5.1) and any such solution $x(t)$ has asymptotic behavior

$$\Psi(x(t)) \sim \frac{\alpha}{\alpha-\beta} t^{\frac{\beta-\alpha}{3\alpha}}(\log t)^{\frac{\alpha-\beta}{3\alpha}}\left(\log\sqrt[3]{\frac{\log t}{t}}\right)^{-\frac{1}{\alpha}}, \quad t\to\infty.$$

In the view of (3.2.31) we have

$$x(t)^{\frac{\alpha-\beta}{\alpha}}(\log x(t))^{-\frac{1}{\alpha}} \sim \left(\sqrt[3]{\frac{\log t}{t}}\right)^{\frac{\alpha-\beta}{\alpha}}\left(\log\sqrt[3]{\frac{\log t}{t}}\right)^{-\frac{1}{\alpha}}, \quad t\to\infty,$$

implying that

$$x(t) \sim \sqrt[3]{\frac{\log t}{t}}, \quad t \to \infty.$$

Observe that in (3.5.3) instead " $\sim$ " one has " $=$ " and

$$r(t) = \left(1 - \frac{3}{2\log t} + \frac{2}{\log^2 t}\right)\left(1 - \frac{1}{\log t}\right)^{\alpha-1},$$

then $x(t) = \sqrt[3]{\frac{\log t}{t}}$ is an exact solution.

(iii) Suppose that

$$q(t) \sim \frac{\alpha}{3^\alpha}\, t^{\alpha-1}\, \frac{r(t)\, (\log t)^{\frac{\beta}{3}-2\alpha}}{\log(\log t)^{-\frac{1}{3}}}, \quad t \to \infty, \tag{3.5.4}$$

where $r(t)$ is continuous function on $[t_0, \infty)$ such that $\lim\limits_{t\to\infty} r(t) = 1$. Then, $q(t) \in \mathrm{RV}(\alpha - 1)$, so that $\sigma = \eta - \alpha - 1$ and we see that

$$\begin{aligned}\int_t^\infty \varphi^{-1}\left(p(s)^{-1}\int_{t_0}^s q(r)dr\right) ds \ &\sim\ \frac{1}{3}\int_t^\infty (\log s)^{\frac{\beta}{3\alpha}-\frac{4}{3}}\left(\log(\log s)^{-\frac{1}{3}}\right)^{-\frac{1}{\alpha}}\frac{ds}{s}\\ &\sim\ \frac{\alpha}{\alpha-\beta}u^{\frac{\beta-\alpha}{3\alpha}}(\log u^{-\frac{1}{3}})^{-\frac{1}{\alpha}}\big|_{u=\log t} \longrightarrow 0,\end{aligned}$$

as $t \to \infty$, implying that (3.2.24) holds. Therefore, by Theorem 3.2.3 there exist nontrivial slowly varying solutions of (3.5.1), and any such solution $x(t)$ has asymptotic behavior

$$\Psi(x(t)) \sim \frac{\alpha}{\alpha-\beta}(\log t)^{\frac{\beta-\alpha}{3\alpha}}(\log(\log t)^{-\frac{1}{3}})^{-\frac{1}{\alpha}}, \quad t \to \infty.$$

In view of (3.2.31) we have

$$x(t)^{\frac{\alpha-\beta}{\alpha}}(\log x(t))^{-\frac{1}{\alpha}} \sim (\log t)^{-\frac{\alpha-\beta}{3\alpha}}(\log(\log t)^{-\frac{1}{3}})^{-\frac{1}{\alpha}}, \quad t \to \infty$$

implying that $x(t) \sim (\log t)^{-\frac{1}{3}}$, $t \to \infty$. If in (3.5.4) instead of " $\sim$ " one has " $=$ " and in particular $r(t) = 1 - \frac{2}{\log t}$, then (3.5.1) possesses an exact solution $x(t) = (\log t)^{-\frac{1}{3}}$.

In the following example we illustrate results of Theorems 3.4.1-3.4.3.

Example 3.5.2 Consider the equation

$$(p(t)\, \varphi(x'(t)))' + q(t)\psi(x(t)) = 0, \quad t \ge e = a,$$

where $p(t) = t^{\frac{\alpha}{2}}(\log t)^\alpha \in \mathrm{RV}(\frac{\alpha}{2})$, $\varphi(s) = s^\alpha \in \mathcal{RV}(\alpha)$ and $\psi(s) = s^\beta \log s \in \mathrm{RV}(\beta)$, $\alpha > \beta > 0$. So that $\eta = \frac{\alpha}{2} \in (0, \alpha)$, $P(t) \sim 2\sqrt{t}\,(\log t)^{-1}$ and the functions $\varphi^{-1}(s)$ and $\psi(s)$ satisfy additional requirements (3.4.5) and (3.4.6), respectively.

(i) Suppose that

$$q(t) \sim \frac{\alpha}{2^{\alpha+1}}\, t^{-1-\frac{\alpha}{2}}\, \frac{r(t)\,(\log t)^{\frac{\alpha-\beta}{2}}}{\log\sqrt{\log t}}, \quad t \to \infty, \tag{3.5.5}$$

where $r(t)$ is continuous function on $[a,\infty)$ such that $\lim_{t\to\infty} r(t) = 1$. Then, $q(t) \in \mathrm{RV}(-1-\frac{\alpha}{2})$, so that $\sigma = \eta - \alpha - 1$ and we see that

$$\begin{aligned}\int_a^t \varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)dr\right) ds &\sim \frac{1}{2}\int_a^t (\log s)^{-\frac{\alpha+\beta}{2\alpha}}\left(\log\sqrt{\log s}\right)^{-\frac{1}{\alpha}}\frac{ds}{s}\\ &\sim \frac{\alpha}{\alpha-\beta}(\log t)^{\frac{\alpha-\beta}{2\alpha}}(\log\sqrt{\log t})^{-\frac{1}{\alpha}} \longrightarrow \infty, \quad t\to\infty,\end{aligned}$$

implying that (3.4.16) holds. Therefore, by Theorem 3.4.1 there exist nontrivial slowly varying solutions of (E_2) , and any such solution $x(t)$ has asymptotic behavior

$$\Psi(x(t)) \sim \frac{\alpha}{\alpha-\beta}(\log t)^{\frac{\alpha-\beta}{2\alpha}}(\log\sqrt{\log t})^{-\frac{1}{\alpha}}, \quad t\to\infty.$$

In view of (3.4.35) we have

$$x(t)^{\frac{\alpha-\beta}{\alpha}}(\log x(t))^{-\frac{1}{\alpha}} \sim (\sqrt{\log t})^{\frac{\alpha-\beta}{\alpha}}(\log\sqrt{\log t})^{-\frac{1}{\alpha}}, \quad t\to\infty$$

implying that $x(t)\sim\sqrt{\log t}$, $t\to\infty$. If in (3.5.5) instead of " $\sim$ " one has " $=$ " and in particular $r(t) = 1 - \frac{1}{\log t}$, then (E_2) possesses an exact increasing nontrivial SV$-$solution $x(t) = \sqrt{\log t}$ on $[e,\infty)$.

(ii) Suppose that

$$q(t) \sim \frac{\alpha}{6\cdot 3^{\alpha}}\, t^{-\frac{\alpha}{6}-\frac{\beta}{3}-1}\, \frac{r(t)\,(\log t)^{\beta}}{\log\frac{\sqrt[3]{t}}{\log t}}, \quad t\to\infty, \tag{3.5.6}$$

where $r(t)$ is continuous function on $[a,\infty)$ such that $\lim_{t\to\infty} r(t) = 1$. It is clear that $q(t)$ is regularly varying function of index

$$\sigma = -\frac{\alpha}{6} - \frac{\beta}{3} - 1 \in \left(\eta-\alpha-1, \frac{\beta}{\alpha}\eta - \beta - 1\right) = (-1-\alpha/2, -1-\beta/2)$$

and that $\rho = \dfrac{\sigma+\alpha+1-\eta}{\alpha-\beta} = \dfrac{1}{3}$. By Theorem 3.4.2 there exist regularly varying solutions of index ρ of (E_2) and any such solution $x(t)$ has asymptotic behavior

$$\Psi(x(t)) \sim \frac{\alpha}{\alpha-\beta} t^{\frac{\alpha-\beta}{3\alpha}}(\log t)^{\frac{\beta}{\alpha}-1}\left(\log\frac{\sqrt[3]{t}}{\log t}\right)^{-\frac{1}{\alpha}}, \quad t\to\infty.$$

In view of (3.4.35) we have

$$x(t)^{\frac{\alpha-\beta}{\alpha}}(\log x(t))^{-\frac{1}{\alpha}} \sim \left(\frac{\sqrt[3]{t}}{\log t}\right)^{\frac{\alpha-\beta}{\alpha}}\left(\log \frac{\sqrt[3]{t}}{\log t}\right)^{-\frac{1}{\alpha}}, \quad t \to \infty,$$

implying that

$$x(t) \sim \frac{\sqrt[3]{t}}{\log t}, \quad t \to \infty.$$

Observe that in (3.5.6) instead " $\sim$ " one has " $=$ " and

$$r(t) = \left(1 - \frac{6}{\log t}\right)\left(1 + \frac{3}{\log t}\right)\left(1 - \frac{3}{\log t}\right)^{\alpha-1},$$

then $x(t) = \sqrt[3]{t}(\log t)^{-1}$ on $[e^6, \infty)$ is an exact increasing solution.

(iii) Suppose that

$$q(t) \sim \frac{\alpha}{2^\alpha}\, t^{-1-\frac{\beta}{2}}\, \frac{r(t)\,(\log t)^{2\beta-\alpha-1}}{\log \frac{\sqrt{t}}{\log^2 t}}, \quad t \to \infty, \tag{3.5.7}$$

where $r(t)$ is continuous function on $[a, \infty)$ such that $\lim_{t\to\infty} r(t) = 1$. Here, $q(t) \in \mathrm{RV}(-1 - \frac{\beta}{2})$. Therefore, $\sigma = \frac{\beta}{\alpha}\eta - \beta - 1$ and

$$q(t)\psi(P(t)) \sim \frac{\alpha}{2^{\alpha-\beta}} t^{-1}(\log t)^{\beta-\alpha-1}\frac{\log\frac{2\sqrt{t}}{\log t}}{\log\frac{\sqrt{t}}{\log^2 t}} \sim \frac{\alpha}{2^{\alpha-\beta}} t^{-1}(\log t)^{\beta-\alpha-1}, \quad t \to \infty,$$

from which it follows

$$\begin{aligned}\int_t^\infty q(s)\psi(P(s))ds &\sim \frac{\alpha}{2^{\alpha-\beta}}\int_t^\infty (\log s)^{\beta-\alpha-1}\,\frac{ds}{s}\\ &\sim \frac{1}{2^{\alpha-\beta}}\frac{\alpha}{\alpha-\beta}(\log t)^{\beta-\alpha} \longrightarrow 0, \quad t \to \infty,\end{aligned}$$

implying that (3.4.28) holds. Therefore, by Theorem 3.4.3 there exist non-trivial regularly varying solutions of index $1 - \frac{\eta}{\alpha} = \frac{1}{2}$ of (E_2) and any such solution $x(t)$ has asymptotic behavior

$$x(t) \sim 2\sqrt{t}(\log t)^{-1}\left(\frac{\alpha-\beta}{\alpha}\frac{1}{2^{\alpha-\beta}}\frac{\alpha}{\alpha-\beta}(\log t)^{\beta-\alpha}\right)^{\frac{1}{\alpha-\beta}} \sim \frac{\sqrt{t}}{\log^2 t}, \quad t \to \infty.$$

If in (3.5.7) instead of " $\sim$ " one has " $=$ " and in particular

$$r(t) = \left(1 - \frac{4}{\log t}\right)^{\alpha-1}\left(1 - \frac{8}{\log t}\right),$$

then (E_2) possesses an exact increasing solution $x(t) = \sqrt{t}(\log t)^{-2}$ on $[e^8, \infty)$.

Chapter 4

Asymptotic behavior of positive solutions of fourth order quasilinear differential equation

The main objective in this chapter is to acquire as detailed information as possible about the existence and asymptotic behavior of all positive solutions of fourth order quasilinear differential equation

$$\text{(E)} \qquad \big(p(t)|x''(t)|^{\alpha-1}\, x''(t)\big)'' + q(t)|x(t)|^{\beta-1}\, x(t) = 0, \quad t \ge a > 0, \quad \alpha > \beta > 0,$$

under two different conditions:

$$(\mathrm{C}_1) \qquad \int_a^\infty \frac{t}{p(t)^{\frac{1}{\alpha}}}\, dt = \infty \quad \wedge \quad \int_a^\infty \left(\frac{t}{p(t)}\right)^{\frac{1}{\alpha}} dt = \infty \quad \wedge \quad \int_a^\infty \frac{dt}{p(t)^{\frac{1}{\alpha}}} < \infty,$$

and

$$(\mathrm{C}_2) \qquad \int_a^\infty \frac{dt}{p(t)^{\frac{1}{\alpha}}} = \infty.$$

We note that the condition (C_2) implies

$$(4.0.1) \qquad \int_a^\infty \frac{t}{p(t)^{\frac{1}{\alpha}}}\, dt = \infty \quad \text{and} \quad \int_a^\infty \left(\frac{t}{p(t)}\right)^{\frac{1}{\alpha}} dt = \infty.$$

Oscillation as well as the existence and asymptotic behavior of nonoscillatory solutions of the equation (E) under both conditions (C_1) and (C_2) have been already discussed in [64] and [73]. Under both conditions (C_1) and (C_2) the four types of primitive solutions of (E) are obtained and necessary and sufficient condition for their existence are given. However, we establish here that there exists two types of intermediate solutions of (E) under both conditions (C_1) and (C_2). Therefore, sufficient condition for the existence of these solutions is obtained assuming that the coefficients of (E) is positive continuous functions. We further restrict the coefficients of (E) to generalized regularly varying functions to get not only the desired necessary and sufficient conditions, but also the asymptotic formulas that describe the behaviour of these solutions at infinity.

All of the results in this chapter are original and are published in [37] and [55].

4.1 Classification of positive solutions of (E) under the condition (C_1)

We assume that $p, q : [a, \infty) \to (0, \infty)$ are continuous functions and that (C_1) holds.

Definition 4.1.1 *By a solution of (E) we mean a function $x(t) : [T, \infty) \to \mathbb{R}$, $T \geq a$, such that $x(t)$ and $p(t)|x''(t)|^{\alpha-1}x''(t)$ is twice continuously differentiable on $[T, \infty)$ and satisfies the equation (E) at every point of $[T, \infty)$.*

We begin by classifying the set of all possible positive solutions of (E) under the condition (C_1) according to their asymptotic behavior as $t \to \infty$. There a crucial role is played by the following four functions

$$\varphi_1(t) = 1,\ \ \varphi_2(t) = \int_a^t\int_s^\infty \frac{1}{p(r)^{\frac{1}{\alpha}}}\,drds,\ \ \varphi_3(t) = t,\ \ \varphi_4(t) = \int_a^t\int_a^s \left(\frac{r}{p(r)}\right)^{\frac{1}{\alpha}} drds,$$

which are the particular solutions of the unperturbed differential equation

$$(p(t)|x''(t)|^{\alpha-1}x''(t))'' = 0.$$

It is to be noted that the functions define above satisfy the dominance relation

$$\varphi_1(t) \prec \varphi_2(t) \prec \varphi_3(t) \prec \varphi_4(t), \quad t \to \infty. \tag{4.1.1}$$

Let $x(t)$ be a positive solution of (E). It is known (see [73]) that $x(t)$ satisfies either

$$x'(t) > 0, \quad x''(t) > 0, \quad (p(t)|x''(t)|^{\alpha-1}x''(t))' > 0 \quad \text{for all large } t, \tag{4.1.2}$$

or

$$x'(t) > 0, \quad x''(t) < 0, \quad (p(t)|x''(t)|^{\alpha-1}x''(t))' > 0 \quad \text{for all large } t. \tag{4.1.3}$$

Since (E) implies that $(p(t)|x''(t)|^{\alpha-1}x''(t))'$ is decreasing and positive, there exists a finite limit $\lim_{t\to\infty}(p(t)|x''(t)|^{\alpha-1}x''(t))' = \omega_3 \geq 0$.

Solutions satisfying (4.1.2). First let $x(t)$ satisfy (4.1.2) on $[t_0,\infty)$. Since $x'(t)$ is positive and increasing, we see that $x'(t) \geq x'(t_0)$, $t \geq t_0$, which by integration gives $x(t) \to \infty$, $t \to \infty$.

Suppose that $\omega_3 > 0$. Then, since $(p(t)\,x''(t)^\alpha)' \sim \omega_3,\ t \to \infty$, integrating this relation on $[t_0,t]$, we obtain

$$x''(t) \sim \omega_3^{\frac{1}{\alpha}} \left(\frac{t}{p(t)}\right)^{\frac{1}{\alpha}}, \quad t \to \infty,$$

from which, integrating twice on $[t_0,t]$ and using the condition (C_1), we find that

$$x(t) \sim \omega_3^{\frac{1}{\alpha}} \int_{t_0}^{t}\int_{t_0}^{s} \left(\frac{r}{p(r)}\right)^{\frac{1}{\alpha}} drds, \quad t \to \infty,$$

i.e., $x(t) \sim \omega_3^{\frac{1}{\alpha}}\,\varphi_4(t)$ as $t \to \infty$.

Suppose that $\omega_3 = 0$. Then, since $p(t)x''(t)^\alpha$ is positive and increasing, we have $\lim_{t\to\infty} p(t)x''(t)^\alpha = \omega_2 \in (0,\infty]$. If $\omega_2 > 0$ is finite, then rewriting the relation $p(t)x''(t)^\alpha \sim \omega_2,\ t \to \infty$ as $x''(t) \sim (\omega_2/p(t))^{\frac{1}{\alpha}},\ t \to \infty$, and integrating this from t_0 to t, we conclude with the help of (C_1) that $x'(t)$ tends to a finite limit $\omega_1 > 0$ as $t \to \infty$, which clearly implies that $x(t) \sim \omega_1 t,\ t \to \infty$. On the other hand, if $\omega_2 = \infty$, we first integrate (E) on $[t,\infty)$ and then on $[t_0,t]$ to obtain

$$x''(t) = \frac{1}{p(t)^{\frac{1}{\alpha}}}\left(c_2 + \int_{t_0}^{t}\int_{s}^{\infty} q(r)x(r)^\beta\,drds\right)^{\frac{1}{\alpha}}, \quad t \geq t_0, \tag{4.1.4}$$

where $c_2 = p(t_0)x''(t_0)^\alpha > 0$. Integrating the above twice on $[t_0,t]$ then yields

$$x(t) = c_0 + c_1(t-t_0) + \int_{t_0}^{t}\int_{t_0}^{s} \frac{1}{p(r)^{\frac{1}{\alpha}}}\left(c_2 + \int_{t_0}^{r}\int_{u}^{\infty} q(v)x(v)^\beta\,dudv\right)^{\frac{1}{\alpha}} drds, \tag{4.1.5}$$

for $t \geq t_0$, where $c_1 = x'(t_0) > 0$ and $c_0 = x(t_0) > 0$. Since

$$\int_{t_0}^{t}\int_{s}^{\infty} q(r)x(r)^\beta drds = O(t), \quad t \to \infty,$$

the condition (C_1) implies from (4.1.4) that $\lim_{t\to\infty} x'(t) = \infty$. Using the L' Hospital's rule, we easily see from (4.1.5) that $\lim_{t\to\infty} x(t)/\varphi_4(t) = 0$, or equivalently $\varphi_3(t) \prec x(t) \prec \varphi_4(t)$ as $t \to \infty$.

It follows from above observation that there are three types of possible asymptotic behavior for positive solutions $x(t)$ of (E) satisfying (4.1.2)

$$x(t) \sim k_3\varphi_3(t), \quad \text{or} \quad \varphi_3(t) \prec x(t) \prec \varphi_4(t), \quad \text{or} \quad x(t) \sim k_4\varphi_4(t), \quad \text{as} \quad t \to \infty,$$

where k_3 and k_4 are some positive constants.

Solutions satisfying (4.1.3). Let $x(t)$ satisfy (4.1.3) on $[t_0, \infty)$. It is necessary that $\omega_3 = 0$, so that we have

$$\text{(4.1.6)} \qquad -\Big(p(t)(-x''(t))^\alpha\Big)' = \int_t^\infty q(s)x(s)^\beta ds, \quad t \geq t_0.$$

Moreover, since $p(t)(-x''(t))^\alpha$ and $x'(t)$ are positive and decreasing , there exist finite limits $\lim_{t\to\infty} p(t)(-x''(t))^\alpha = \omega_2 \geq 0$ and $\lim_{t\to\infty} x'(t) = \omega_1 \geq 0$. Using this fact and integrating (4.1.6) twice on $[t, \infty)$, we obtain

$$\text{(4.1.7)} \qquad x'(t) = \omega_1 + \int_t^\infty \left[\frac{1}{p(s)}\Big(\omega_2 + \int_s^\infty (r-s)q(r)x(r)^\beta\, dr\Big)\right]^{\frac{1}{\alpha}} ds, \quad t \geq t_0,$$

which, integrated on $[t_0, t]$, gives

$$\text{(4.1.8)} \quad x(t) = c_0 + \omega_1(t-t_0) + \int_{t_0}^t \int_s^\infty \left[\frac{1}{p(r)}\Big(\omega_2 + \int_r^\infty (u-r)q(u)x(u)^\beta\, du\Big)\right]^{\frac{1}{\alpha}} dr ds,$$

for $t \geq t_0$, where $c_0 = x(t_0) > 0$. From (4.1.8) it follows that if $\omega_1 > 0$, then $x(t) \sim \omega_1 t$ as $t \to \infty$, regardless of the values of $\omega_2 \geq 0$, and that if $\omega_1 = 0$ and $\omega_2 > 0$, then $x(t) \sim \omega_2^{\frac{1}{\alpha}} \varphi_2(t)$, $t \to \infty$. It may happen that $\omega_1 = \omega_2 = 0$, in which case there are two possibilities: either $x(t)$ tends to a finite limit or $x(t)$ grows to infinity as $t \to \infty$. In the latter case it is clear that $\varphi_1(t) \prec x(t) \prec \varphi_2(t)$ as $t \to \infty$.

Thus it follows that the asymptotic behavior of positive solutions $x(t)$ of (E) satisfying (4.1.3) falls into one of the following four cases:

$$x(t) \sim k_1\varphi_1(t), \ \text{or} \ \varphi_1(t) \prec x(t) \prec \varphi_2(t), \ \text{or} \ x(t) \sim k_2\varphi_2(t), \ \text{or} \ x(t) \sim k_3\varphi_3(t),$$

as $t \to \infty$, where $k_i,\ i = 1, 2, 3$ are some positive constants.

Positive solutions $x(t)$ of (E) having the asymptotic behavior

$$x(t) \sim k_1\varphi_1(t), \quad x(t) \sim k_2\varphi_2(t), \quad x(t) \sim k_3\,\varphi_3(t), \quad x(t) \sim k_4\,\varphi_4(t), \quad \text{as} \quad t \to \infty,$$

for some positive constants $k_i,\ i = 1, 2, 3, 4$, are collectively called *primitive positive solutions* of equation (E), while the solutions which are not primitive are referred to as *intermediate solutions* of equation (E). It is convenient to divide the set of intermediate solutions into the following two types

$$\text{(I}_1\text{)} \qquad \varphi_1(t) \prec x(t) \prec \varphi_2(t), \quad t \to \infty,$$

$$(\mathrm{I}_2) \qquad \varphi_3(t) \prec x(t) \prec \varphi_4(t), \quad t \to \infty.$$

As regards the primitive solutions of equation (E), the existence of four types of primitive solutions has been completely characterized for both sublinear and superlinear case of (E) with continuous coefficients $p(t)$ and $q(t)$ as the following theorems proven in [64] and [73] show. In view of relation (4.1.1) primitive solutions of type $x(t) \sim k_1\varphi_1(t),\ t \to \infty$ and $x(t) \sim k_4\varphi_4(t),\ t \to \infty$ are often referred to as *minimal* and *maximal* solutions of (E) , respectively. Sufficient and necessary conditions for the existence of these solutions were proven under the condition (4.0.1), regardless of convergence or divergence of the integral $\int_a^\infty 1/p(t)^{\frac{1}{\alpha}}\, dt$.

Theorem 4.1.1 *Let $p(t), q(t) \in C[a,\infty)$. Equation (E) has a positive solution $x(t)$ satisfying $x(t) \sim k_1\varphi_1(t),\ t \to \infty$ if and only if*

$$\int_a^\infty t \left(\frac{1}{p(t)} \int_t^\infty (s-t)\, q(s)\, ds \right)^{\frac{1}{\alpha}} dt < \infty. \tag{4.1.9}$$

Theorem 4.1.2 *Let $p(t), q(t) \in C[a,\infty)$. Equation (E) has a positive solution $x(t)$ satisfying $x(t) \sim k_4\varphi_4(t),\ t \to \infty$ if and only if*

$$\int_a^\infty q(t)\, \varphi_4(t)^\beta\, dt < \infty. \tag{4.1.10}$$

The other two types of primitive solutions of (E) exists only under additional assumption that the integral $\int_a^\infty 1/p(t)^{\frac{1}{\alpha}}\, dt$ is convergent i.e. under the condition (C_1) .

Theorem 4.1.3 *Let $p(t), q(t) \in C[a,\infty)$ and* (C_1) *holds. Equation (E) has a positive solution $x(t)$ satisfying $x(t) \sim k_2\varphi_2(t),\ t \to \infty$ if and only if*

$$\int_a^\infty t\, q(t)\, \varphi_2(t)^\beta\, dt < \infty. \tag{4.1.11}$$

Theorem 4.1.4 *Let $p(t), q(t) \in C[a,\infty)$ and* (C_1) *holds. Equation (E) has a positive solution $x(t)$ satisfying $x(t) \sim k_3\varphi_3(t),\ t \to \infty$ if and only if*

$$\int_a^\infty \left(\frac{1}{p(t)} \int_a^t \int_s^\infty r^\beta\, q(r)\, dr ds \right)^{\frac{1}{\alpha}} dt < \infty. \tag{4.1.12}$$

Also, the following sharp oscillation theorem for sub-half-linear equation (E) was proved in [64].

Theorem 4.1.5 *Suppose that $\alpha \geq 1 > \beta > 0$. Then equation (E) has a nonoscillatory solution if and only if* (4.1.10) *holds.*

4.2 Existence of intermediate solutions of (E) under the condition (C_1)

In this section we prove the existence of solutions of type (I_1) and (I_2) of equation (E) under assumption that coefficients $p(t)$ and $q(t)$ are positive continuous functions and that (C_1) holds.

Theorem 4.2.1 *Let $p, q \in C[a,\infty)$ and (C_1) holds. If (4.1.11) holds and if*

$$\int_a^\infty t\left(\frac{1}{p(t)}\int_t^\infty (s-t)\,q(s)\,ds\right)^{\frac{1}{\alpha}} dt = \infty,$$

then equation (E) has a positive solution $x(t)$ such that $1 \prec x(t) \prec \varphi_2(t)$, $t \to \infty$.

Proof. Choose $t_0 \geq a$ such that $\varphi_2(t) \geq 1$ for $t \geq t_0$ and

$$\int_{t_0}^\infty t\,q(t)\,\varphi_2(t)^\beta\,dt \leq 2^{-\beta}. \tag{4.2.1}$$

Define the set

$$\mathcal{X}_1 = \{x \in C[t_0,\infty) : 1 \leq x(t) \leq 2\varphi_2(t),\ t \geq t_0\}, \tag{4.2.2}$$

and the operator $\mathcal{G} : \mathcal{X}_1 \to C[t_0,\infty)$

$$\mathcal{G}x(t) := 1 + \int_{t_0}^t \int_s^\infty \left(\frac{1}{p(r)}\int_r^\infty (u-r)\,q(u)\,x(u)^\beta\,du\right)^{\frac{1}{\alpha}} drds, \quad t \geq t_0. \tag{4.2.3}$$

It is clear that $\mathcal{X}_1$ is a closed convex subset of the locally convex space $C[t_0,\infty)$ equipped with the topology of uniform convergence on compact subintervals of $[t_0,\infty)$. Using (4.2.1)– (4.2.3), we see that $x \in \mathcal{X}_1$ implies

$$\begin{aligned}
1 \leq \mathcal{G}x(t) \;&\leq\; 1 + 2^{\frac{\beta}{\alpha}} \int_{t_0}^t \int_s^\infty \frac{1}{p(r)^{\frac{1}{\alpha}}} \left(\int_{t_0}^\infty u\,q(u)\,\varphi(u)^\beta\,du\right)^{\frac{1}{\alpha}} dr\,ds \\
&\leq\; 1 + 2^{\frac{\beta}{\alpha}}\,2^{-\frac{\beta}{\alpha}} \int_{t_0}^t \int_s^\infty \frac{1}{p(r)^{\frac{1}{\alpha}}}\,dr\,ds = 1 + \varphi_2(t) \leq 2\,\varphi_2(t), \quad t \geq t_0.
\end{aligned}$$

This means that $\mathcal{G}$ maps $\mathcal{X}_1$ into itself. Furthermore, it can be shown that $\mathcal{G}$ is a continuous map such that $\mathcal{G}(\mathcal{X}_1)$ is relatively compact in $C[t_0,\infty)$. Therefore, by the Schauder-Tychonoff fixed point theorem there exists a function $x_1 \in \mathcal{X}_1$ satisfying the integral equation $x_1(t) = \mathcal{G}x_1(t)$ for $t \geq t_0$. It follows that $x_1(t)$ is a solution of (E) on $[t_0, \infty)$. It is easy to see that $x_1(t)$ has the following asymptotic properties:

$$\lim_{t\to\infty} x_1(t) \geq \lim_{t\to\infty} \int_{t_0}^{t}\!\!\int_{s}^{\infty} \left(\frac{1}{p(r)} \int_{r}^{\infty} (u-r)\, q(u)\, du\right)^{\frac{1}{\alpha}} dr\, ds = \infty$$

and

$$\begin{aligned} 0 \;&\leq\; \lim_{t\to\infty} \frac{x_1(t)}{\varphi_2(t)} = \lim_{t\to\infty} \left(\int_{t}^{\infty} (s-t)\, q(s)\, x_1(s)^{\beta}\, ds\right)^{\frac{1}{\alpha}} \\ &\leq\; 2^{\frac{\beta}{\alpha}} \left(\lim_{t\to\infty} \int_{t}^{\infty} s q(s)\, \varphi_2(s)^{\beta}\, ds\right)^{\frac{1}{\alpha}} = 0, \end{aligned}$$

which means that $x_1(t)$ satisfies $1 \prec x_1(t) \prec \varphi_2(t),\ t \to \infty$, that is, $x_1(t)$ is an intermediate solution of type (I_1) of (E). □

Theorem 4.2.2 *Let $p, q \in C[a,\infty)$ and (C_1) holds. If (4.1.10) holds and if*

$$\int_{a}^{\infty} \left(\frac{1}{p(t)} \int_{a}^{t}\int_{s}^{\infty} r^{\beta}\, q(r)\, dr ds\right)^{\frac{1}{\alpha}} dt = \infty,$$

then equation (E) has a positive solution $x(t)$ such that $t \prec x(t) \prec \varphi_4(t)$, $t \to \infty$.

Proof. Choose $t_0 \geq a$ such that $\varphi_4(t) \geq t$ for $t \geq t_0$ and

$$\int_{t_0}^{\infty} q(t)\, \varphi_4(t)^{\beta}\, dt \leq 2^{-\beta}. \tag{4.2.4}$$

Define the set

$$\mathcal{X}_2 = \{x \in C[t_0,\infty) : t \leq x(t) \leq 2\varphi_4(t),\ t \geq t_0\}, \tag{4.2.5}$$

and the integral operator $\mathcal{H} : \mathcal{X}_2 \to C[t_0,\infty)$

$$\mathcal{H}x(t) := t + \int_{t_0}^{t}\!\!\int_{t_0}^{s} \left(\frac{1}{p(r)} \int_{t_0}^{r}\!\!\int_{\tau}^{\infty} q(u)\, x(u)^{\beta}\, du\, d\tau\right)^{\frac{1}{\alpha}} dr ds, \quad t \geq t_0. \tag{4.2.6}$$

It is clear that $\mathcal{X}_2$ is a closed convex subset of the locally convex space $C[t_0,\infty)$ equipped with the topology of uniform convergence on compact subintervals of $[t_0,\infty)$. Using (4.2.4)–(4.2.6), we see that $x \in \mathcal{X}_2$ implies

$$\begin{aligned} t \leq \mathcal{H}x(t) \;&\leq\; t + 2^{\frac{\beta}{\alpha}} \int_{t_0}^{t}\!\!\int_{t_0}^{s} \frac{1}{p(r)^{\frac{1}{\alpha}}} \left(\int_{t_0}^{r}\!\!\int_{t_0}^{\infty} q(u)\, \varphi_4(u)^{\beta}\, du\right)^{\frac{1}{\alpha}} dr\, ds \\ &\leq\; t + 2^{\frac{\beta}{\alpha}} 2^{-\frac{\beta}{\alpha}} \int_{t_0}^{t}\!\!\int_{t_0}^{s} \frac{1}{p(r)^{\frac{1}{\alpha}}} (r-t_0)^{\frac{1}{\alpha}} dr\, ds = t + \varphi_4(t) \leq 2\varphi_4(t),\ t \geq t_0. \end{aligned}$$

This means that $\mathcal{H}$ maps $\mathcal{X}_2$ into itself. Furthermore, it can be shown that $\mathcal{H}$ is a continuous map such that $\mathcal{H}(\mathcal{X}_2)$ is relatively compact in $C[t_0,\infty)$. Therefore, by the Schauder-Tychonoff fixed point theorem there exists a function $x_2 \in \mathcal{X}_2$ satisfying the integral equation $x_2(t) = \mathcal{H}x_2(t)$ for $t \geq t_0$. It follows that $x_2(t)$ is a solution of (E) on $[t_0,\infty)$. It is easy to see that $x_2(t)$ has the following asymptotic properties:

$$\begin{aligned} \lim_{t\to\infty} \frac{x_2(t)}{t} &= 1 + \lim_{t\to\infty} \int_{t_0}^{t} \left(\frac{1}{p(s)} \int_{t_0}^{s} \int_{r}^{\infty} q(u)\, x_2(u)^{\beta}\, du\, dr \right)^{\frac{1}{\alpha}} ds \\ &\geq \lim_{t\to\infty} \int_{t_0}^{t} \left(\frac{1}{p(s)} \int_{t_0}^{s} \int_{r}^{\infty} q(u)\, u^{\beta}\, du\, dr \right)^{\frac{1}{\alpha}} ds = \infty \end{aligned}$$

and

$$\begin{aligned} 0 \;\leq\; \lim_{t\to\infty} \frac{x_2(t)}{\varphi_4(t)} &= \left(\lim_{t\to\infty} \frac{\int_{t_0}^{t}\int_{s}^{\infty} q(r)\, x_2(r)^{\beta}\, dr\, ds}{t} \right)^{\frac{1}{\alpha}} \\ \leq\; 2^{\frac{\beta}{\alpha}} \left(\lim_{t\to\infty} \frac{\int_{t_0}^{t}\int_{s}^{\infty} q(r)\, \varphi_4(r)^{\beta}\, dr\, ds}{t} \right)^{\frac{1}{\alpha}} &= 2^{\frac{\beta}{\alpha}} \left(\lim_{t\to\infty} \int_{t}^{\infty} q(s)\, \varphi_4(s)^{\beta}\, ds \right)^{\frac{1}{\alpha}} = 0, \end{aligned}$$

which means that $x_2(t)$ satisfies $t \prec x_2(t) \prec \varphi_4(t),\ t \to \infty$, that is, $x_2(t)$ is an intermediate solution of type (I_2) of (E). $\square$

4.3 Asymptotic behavior of intermediate solutions of (E) under the condition (C_1)

In this section we assumed that (C_1) holds and that functions $p(t)$ and $q(t)$ are generalized regularly varying of index η and σ with respect to $R(t)$, which is defined with

(4.3.1) $$R(t) = \int_{a}^{t} \left(\frac{s}{p(s)} \right)^{\frac{1}{\alpha}} ds,$$

and expressed with

(4.3.2) $$p(t) = R(t)^{\eta}\, l_p(t),\ l_p(t) \in \mathrm{SV}_R \ \text{ and } \ q(t) = R(t)^{\sigma}\, l_q(t),\ l_q(t) \in \mathrm{SV}_R,$$

and the intermediate solutions $x(t) \in \mathrm{RV}_R(\rho)$ of (E) are represented as

(4.3.3) $$x(t) = R(t)^{\rho}\, l_x(t),\ \ l_x(t)\ \in \mathrm{SV}_R.$$

From (4.3.1) and (4.3.2) we have that

$$t^{\frac{1}{\alpha}} = R'(t)R(t)^{\frac{\eta}{\alpha}}\, l_p(t)^{\frac{1}{\alpha}}. \tag{4.3.4}$$

Integrating (4.3.4) from a to t and using the generalized Karamata integration theorem (Proposition 1.2.10) we have

$$\frac{t^{\frac{1}{\alpha}+1}}{\frac{1}{\alpha}+1} \sim \frac{R(t)^{\frac{\eta}{\alpha}+1}}{\frac{\eta}{\alpha}+1}\, l_p(t)^{\frac{1}{\alpha}}, \quad t \to \infty,$$

implying

$$t \sim \left(\frac{\alpha+\eta}{\alpha+1}\right)^{-\frac{\alpha}{\alpha+1}} R(t)^{\frac{\alpha+\eta}{\alpha+1}}\, l_p(t)^{\frac{1}{\alpha+1}}, \quad t \to \infty. \tag{4.3.5}$$

From above relations we get

$$R'(t) \sim \left(\frac{\alpha+\eta}{\alpha+1}\right)^{-\frac{1}{\alpha+1}} R(t)^{\frac{1-\eta}{\alpha+1}}\, l_p(t)^{-\frac{1}{\alpha+1}}, \quad t \to \infty. \tag{4.3.6}$$

We can rewrite (4.3.6) in the form

$$1 \sim \left(\frac{\alpha+\eta}{\alpha+1}\right)^{\frac{1}{\alpha+1}} R'(t)\, R(t)^{\frac{\eta-1}{\alpha+1}}\, l_p(t)^{\frac{1}{\alpha+1}}, \quad t \to \infty. \tag{4.3.7}$$

First, express the condition (C_1) in the terms of regular variation. Using (4.3.2), (4.3.5) and (4.3.7) we have

$$\int_t^\infty \frac{ds}{p(s)^{\frac{1}{\alpha}}} \sim \left(\frac{\alpha+\eta}{\alpha+1}\right)^{\frac{1}{\alpha+1}} \int_t^\infty R'(s)\, R(s)^{-\frac{\alpha+\eta}{\alpha(\alpha+1)}}\, l_p(s)^{-\frac{1}{\alpha(\alpha+1)}}\, ds, \quad t \to \infty$$

and

$$\int_a^t \frac{s}{p(s)^{\frac{1}{\alpha}}}\, ds \sim \left(\frac{\alpha+\eta}{\alpha+1}\right)^{\frac{1-\alpha}{\alpha+1}} \int_a^t R'(s)\, R(s)^{\frac{(\alpha+\eta)(\alpha-1)}{\alpha(\alpha+1)}}\, l_p(s)^{\frac{\alpha-1}{\alpha(\alpha+1)}}\, ds, \quad t \to \infty.$$

For condition (C_1) to hold it is necessary that

$$\alpha^2 - \eta \le 0 \quad \wedge \quad 2\alpha^2 + \alpha\,\eta - \eta \ge 0.$$

In what follows we limit ourselves to the case where

$$\alpha^2 - \eta < 0 \quad \wedge \quad 2\alpha^2 + \alpha\,\eta - \eta > 0, \tag{4.3.8}$$

excluding the other possibilities because of computational difficulty. Note that (4.3.8) holds for $\eta > \alpha^2$ if $\alpha \geq 1$, and for η satisfying $\alpha^2 < \eta < 2\alpha^2/(1-\alpha)$ if $\alpha < 1$. We introduce the notation:

$$(4.3.9)\quad m_1(\alpha,\eta) = \frac{2\alpha^2 + \alpha\eta - \eta}{\alpha(\alpha+1)}, \quad m_2(\alpha,\eta) = \frac{\alpha+\eta}{\alpha+1}, \quad m_3(\alpha,\eta) = \frac{2\alpha+\eta+1}{\alpha+1}.$$

It is clear that $0 < m_1(\alpha,\eta) < m_2(\alpha,\eta) < m_3(\alpha,\eta) = m_2(\alpha,\eta)+1$. In all proofs constants $m_i(\alpha,\eta)$, $i = 1,2,3$ will be abbreviated to m_i.

Now, we state a lemma which will be frequently used in our later discussions. The proof of this lemma follows directly using (4.3.7) and the generalized Karamata integration theorem.

Lemma 4.3.1 *Let $f(t) = R(t)^{\mu}\, L_f(t)$, $L_f(t) \in \mathrm{SV}_R$. Then,*

(i) If $\mu + m_2(\alpha,\eta) > 0$,

$$\int_a^t f(s)\, ds \sim \frac{m_2(\alpha,\eta)^{\frac{1}{\alpha+1}}}{\mu + m_2(\alpha,\eta)}\, R(t)^{\mu+m_2(\alpha,\eta)}\, L_f(t)\, l_p(t)^{\frac{1}{\alpha+1}}, \quad t \to \infty;$$

(ii) If $\mu + m_2(\alpha,\eta) < 0$,

$$\int_t^{\infty} f(s)\, ds \sim \frac{m_2(\alpha,\eta)^{\frac{1}{\alpha+1}}}{-(\mu + m_2(\alpha,\eta))}\, R(t)^{\mu+m_2(\alpha,\eta)}\, L_f(t)\, l_p(t)^{\frac{1}{\alpha+1}}, \quad t \to \infty;$$

(iii) If $\mu + m_2(\alpha,\eta) = 0$, then

$$\int_a^t f(s)\, ds \sim m_2(\alpha,\eta)^{\frac{1}{\alpha+1}} \int_a^t R'(s)R(s)^{-1}\, L_f(s) l_p(s)^{\frac{1}{\alpha+1}}\, ds \in SV_R,$$

$$\int_t^{\infty} f(s)\, ds \sim m_2(\alpha,\eta)^{\frac{1}{\alpha+1}} \int_t^{\infty} R'(s)R(s)^{-1}\, L_f(s) l_p(s)^{\frac{1}{\alpha+1}}\, ds \in SV_R.$$

In order to make an in depth analysis of intermediate solutions of type (I_1) and (I_2) of (E) under the condition (C_1), we need a fair knowledge of the structure of the functions $\varphi_1(t)$, $\varphi_2(t)$, $\varphi_3(t)$ and $\varphi_4(t)$ regarded as generalized regularly varying functions. It is clear that $\varphi_1(t) \in \mathrm{SV}_R$. From (4.3.5) it follows that $\varphi_3(t) \in \mathrm{RV}_R\,(m_2(\alpha,\eta))$. Using (4.3.2) and applying Lemma 4.3.1 twice, we get

$$\begin{aligned} \varphi_2(t) &\sim \int_a^t\!\!\int_s^{\infty} R(r)^{-\frac{\eta}{\alpha}}\, l_p(r)^{-\frac{1}{\alpha}}\, dr ds \\ (4.3.10)\qquad &\sim \frac{m_2(\alpha,\eta)^{\frac{2}{\alpha+1}}}{m_1(\alpha,\eta)(m_2(\alpha,\eta) - m_1(\alpha,\eta))}\, R(t)^{m_1(\alpha,\eta)}\, l_p(t)^{\frac{\alpha-1}{\alpha(\alpha+1)}},\ t \to \infty, \end{aligned}$$

which shows that $\varphi_2(t) \in \mathrm{RV}_R(m_1(\alpha,\eta))$. Further, another application of Lemma 4.3.1 yields

$$(4.3.11) \qquad \varphi_4(t) \sim \int_a^t R(s)\,ds \sim \frac{m_2(\alpha,\eta)^{\frac{1}{\alpha+1}}}{m_3(\alpha,\eta)} R(t)^{m_3(\alpha,\eta)}\, l_p(t)^{\frac{1}{\alpha+1}},\ t\to\infty,$$

implying $\varphi_4(t) \in \mathrm{RV}_R(m_3(\alpha,\eta))$.

4.3.1 Intermediate regularly varying solutions of type (I_1)

The first subsection is devoted to the study of the existence and asymptotic behavior of generalized regularly varying solutions of type (I_1) of equation (E) under the condition (C_1) with $p(t)$ and $q(t)$ satisfying (4.3.2). We seek such solutions $x(t)$ of (E) expressed in the form (4.3.3). Since

$$\lim_{t\to\infty}(p(t)|x''(t)|^{\alpha-1}x''(t))' = \lim_{t\to\infty} p(t)|x''(t)|^{\alpha-1}x''(t) = \lim_{t\to\infty} x'(t) = 0,\ \lim_{t\to\infty} x(t) = \infty,$$

integrating of equation (E) first three times on $[t,\infty)$ and then once on $[t_0,t]$ gives

$$(4.3.12) \qquad x(t) = x(t_0) + \int_{t_0}^t\!\!\int_s^\infty \left(\frac{1}{p(r)}\int_r^\infty (u-r)q(u)x(u)^\beta\,du\right)^{\frac{1}{\alpha}} drds,\quad t\ge t_0.$$

Conversely, if $x(t)$ is a positive continuous function satisfying (4.3.12) and $\lim_{t\to\infty} x(t) = \infty$, then it is a solution of (E) such that $\varphi_1(t) \prec x(t) \prec \varphi_2(t),\ t\to\infty$. Intermediate solutions of type (I_1) are constructed by solving the integral equation (4.3.12) for some constants $t_0 \ge a$ and $x(t_0) > 0$ using Schauder-Tychonoff fixed point theorem as our main tool. Denoting by $\mathcal{G}x(t)$ the right-hand side of (4.3.12), in order to find a fixed point of $\mathcal{G}$ it is crucial to choose a closed convex subset $\mathcal{X} \subset C[t_0,\infty)$ on which $\mathcal{G}$ is a self-map. However, since our goal here is to establish asymptotic behavior of these solutions, a subset X must be constructed in a different way compared to the proofs of Theorem 4.2.1 and Theorem 4.2.2, where the primary objective was the existence of intermediate solutions. It will be shown that such a choice of $\mathcal{X}$ is possible by solving the integral asymptotic relation

$$(4.3.13) \qquad x(t) \sim \int_b^t\!\!\int_s^\infty \left(\frac{1}{p(r)}\int_r^\infty (u-r)q(u)x(u)^\beta\,du\right)^{\frac{1}{\alpha}} dr\,ds,\quad t\to\infty,$$

for some $b \ge t_0$, which can be considered as an approximation (at infinity) of (4.3.12) in the sense that it is satisfied by all possible intermediate solutions of (E). It is a merit of theory of regular variation that ensures the solvability of (4.3.13) in the framework of generalized Karamata functions. Thus, we first show that the generalized regularly varying functions $X_i(t), i = 1, 2, 3$ defined respectively by

$$X_1(t) = \left(\frac{\alpha - \beta}{\alpha} \int_a^t s \left(\frac{1}{p(s)} \int_s^\infty (r-s)\, q(r)\, dr \right)^{\frac{1}{\alpha}} ds \right)^{\frac{\alpha}{\alpha-\beta}}, \tag{4.3.14}$$

$$X_2(t) = \left(\left(\frac{m_2(\alpha,\eta)}{\alpha} \right)^2 \frac{p(t)\, q(t)\, R(t)^{2\alpha}}{\rho^\alpha\, (m_1(\alpha,\eta) - \rho)\, (m_2(\alpha,\eta) - \rho)^\alpha\, (m_3(\alpha,\eta) - \rho)} \right)^{\frac{1}{\alpha-\beta}}, \tag{4.3.15}$$

$$X_3(t) = \varphi_2(t) \left(\frac{\alpha - \beta}{\alpha} \int_t^\infty s\, q(s)\, \varphi_2(s)^\beta\, ds \right)^{\frac{1}{\alpha-\beta}} \tag{4.3.16}$$

satisfy the asymptotic relation (4.3.13).

Lemma 4.3.2 *Suppose that*

$$\sigma = -2\alpha - \eta \quad \textit{and} \quad \int_a^\infty t \left(\frac{1}{p(t)} \int_t^\infty (s-t)\, q(s)\, ds \right)^{\frac{1}{\alpha}} dt = \infty \tag{4.3.17}$$

holds. The function $X_1(t) \in \mathrm{ntr} - \mathrm{SV}_R$ *given by* (4.3.14) *satisfies the asymptotic relation* (4.3.13) *for any* $b \geq a$.

Proof. First note that $\sigma = -2\alpha - \eta$ satisfies $\sigma + m_2 = -\alpha m_3$ and $\sigma + 2m_2 = -\alpha m_1$. We integrate $q(t) = R(t)^\sigma l_q(t)$ twice on $[t, \infty)$. Applying Lemma 4.3.1 and using (4.3.2) and (4.3.5), we obtain

$$\int_t^\infty q(s)\, ds \sim \frac{m_2^{\frac{1}{\alpha+1}}}{\alpha m_3} R(t)^{\sigma + m_2} l_p(t)^{\frac{1}{\alpha+1}}\, l_q(t),$$

and

$$\int_t^\infty (s-t) q(s)\, ds = \int_t^\infty \int_s^\infty q(r) dr ds \sim \frac{m_2^{\frac{2}{\alpha+1}}}{\alpha^2 m_1 m_3} R(t)^{\sigma + 2m_2} l_p(t)^{\frac{2}{\alpha+1}} l_q(t),$$

from which it readily follows that

$$t \left(\frac{1}{p(t)} \int_t^\infty (s-t) q(s)\, ds \right)^{\frac{1}{\alpha}} \sim \left(\frac{m_2^{2-\alpha}}{\alpha^2 m_1 m_3} \right)^{\frac{1}{\alpha}} R'(t) R(t)^{-1} l_p(t)^{\frac{1}{\alpha}} l_q(t)^{\frac{1}{\alpha}},$$

as $t \to \infty$. Integration on the last relation from a to t then yields

$$\begin{aligned}(4.3.18) \qquad & \int_a^t s\left(\frac{1}{p(s)}\int_s^\infty (r-s)\,q(r)\,dr\right)^{\frac{1}{\alpha}} ds \\ & \sim \left(\frac{m_2^{2-\alpha}}{\alpha^2 m_1 m_3}\right)^{\frac{1}{\alpha}} \int_a^t R'(s)R(s)^{-1}\,l_p(s)^{\frac{1}{\alpha}}\,l_q(s)^{\frac{1}{\alpha}}\,ds,\ t\to\infty,\end{aligned}$$

so that

$$X_1(t) \sim \left(\frac{(\alpha-\beta)m_2^{\frac{2-\alpha}{\alpha}}}{\alpha^{1+\frac{2}{\alpha}}\,(m_1\,m_3)^{\frac{1}{\alpha}}}\int_a^t R'(s)\,R(s)^{-1}\,l_p(s)^{\frac{1}{\alpha}}\,l_q(s)^{\frac{1}{\alpha}}\,ds\right)^{\frac{\alpha}{\alpha-\beta}}, \quad t\to\infty.$$

This shows that $X_1(t) \in \mathrm{SV}_R$. Next, we integrate $q(t)X_1(t)^\beta$ twice on $[t,\infty)$. Applying Lemma 4.3.1 as above, we see that

$$\left(\int_t^\infty (s-t)\,q(s)\,X_1(s)^\beta\,ds\right)^{\frac{1}{\alpha}} \sim \left(\frac{m_2^{\frac{2}{\alpha+1}}}{\alpha^2\,m_1 m_3}\right)^{\frac{1}{\alpha}} R(t)^{\frac{\sigma+2m_2}{\alpha}}\,l_p(t)^{\frac{2}{\alpha(\alpha+1)}}\,l_q(t)^{\frac{1}{\alpha}}\,X_1(t)^{\frac{\beta}{\alpha}},$$

as $t\to\infty$. Integrating the above relation multiplied by $p(t)^{-\frac{1}{\alpha}}$ first on $[t,\infty)$ and then on $[b,t]$, for any $b\geq a$, we conclude via Lemma 4.3.1 that

$$\begin{aligned}&\int_b^t\!\!\int_s^\infty\!\left(\frac{1}{p(r)}\int_r^\infty (u-r)\,q(u)\,X_1(u)^\beta\,du\right)^{\frac{1}{\alpha}} dr ds \sim \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}}\left(\frac{m_2^{2-\alpha}}{\alpha^2 m_1 m_3}\right)^{\frac{1}{\alpha-\beta}} \\ &\times \int_b^t R'(s)\,R(s)^{-1}\,l_p(s)^{\frac{1}{\alpha}}\,l_q(s)^{\frac{1}{\alpha}}\left(\int_a^s R'(r)\,R(r)^{-1}\,l_p(r)^{\frac{1}{\alpha}} l_q(r)^{\frac{1}{\alpha}}\,dr\right)^{\frac{\beta}{\alpha-\beta}} ds \\ =\ & \left(\frac{(\alpha-\beta)m_2^{\frac{2-\alpha}{\alpha}}}{\alpha^{1+\frac{2}{\alpha}}\,(m_1 m_3)^{\frac{1}{\alpha}}}\int_a^t R'(s)\,R(s)^{-1}\,l_p(s)^{\frac{1}{\alpha}}\,l_q(s)^{\frac{1}{\alpha}}\,ds\right)^{\frac{\alpha}{\alpha-\beta}} = X_1(t), \quad t\to\infty.\end{aligned}$$

This proves that $X_1(t)$ satisfies the asymptotic relation (4.3.13) for any $b\geq a$. □

Lemma 4.3.3 *Suppose that*

$$(4.3.19) \qquad -2\alpha-\eta < \sigma < -\beta\, m_1(\alpha,\eta) - 2\, m_2(\alpha,\eta)$$

holds and let ρ *be defined by*

$$(4.3.20) \qquad \rho = \frac{\sigma+2\alpha+\eta}{\alpha-\beta}.$$

The function $X_2(t) \in \mathrm{RV}(\rho)$ *given by* (4.3.15) *satisfies the asymptotic relation* (4.3.13) *for any* $b \geq a$.

Proof. (The constant $\lambda(\alpha,\eta,\rho)$ will be abbreviated as λ.)

Note that the function $X_2(t)$ given by (4.3.15) can be expressed in the form

$$X_2(t) \sim \lambda^{-\frac{1}{\alpha-\beta}} \left(\frac{m_2}{\alpha}\right)^{\frac{2}{\alpha-\beta}} R(t)^{\rho} \, (l_p(t)\, l_q(t))^{\frac{1}{\alpha-\beta}}, \; t \to \infty, \tag{4.3.21}$$

where

$$\lambda = \rho^{\alpha} \, (m_1-\rho) \, (m_2-\rho)^{\alpha} \, (m_3-\rho)\,.$$

Using (4.3.21) and (4.3.20) and applying Lemma 4.3.1 twice, we find that

$$\int_t^{\infty} q(s)\, X_2(s)^{\beta}\, ds \sim \frac{\lambda^{-\frac{\beta}{\alpha-\beta}} \left(\frac{m_2}{\alpha}\right)^{\frac{2\beta}{\alpha-\beta}} m_2^{\frac{1}{\alpha+1}}}{\alpha(m_3-\rho)} R(t)^{\alpha(\rho-m_3)} \, (l_p(t)l_q(t))^{\frac{\beta}{\alpha-\beta}} \, l_q(t)l_p(t)^{\frac{1}{\alpha+1}},$$

and

$$\begin{aligned}&\int_t^{\infty}\int_s^{\infty} q(r)\, X_2(r)^{\beta}\, dr ds \\ &\sim \frac{\lambda^{-\frac{\beta}{\alpha-\beta}} \left(\frac{m_2}{\alpha}\right)^{\frac{2\beta}{\alpha-\beta}} m_2^{\frac{2}{\alpha+1}}}{\alpha^2(m_1-\rho)(m_3-\rho)} R(t)^{\alpha(\rho-m_1)} \, (l_p(t)l_q(t))^{\frac{\beta}{\alpha-\beta}} \, l_q(t)l_p(t)^{\frac{2}{\alpha+1}}, \quad t\to\infty.\end{aligned}$$

We now raise the last relation to the exponent $1/\alpha$ and integrate it first on $[t,\infty)$ and then on $[b,t]$ for any $b \geq a$. As a result of application of Lemma 4.3.1, we obtain for $t \to \infty$

$$\begin{aligned}\int_t^{\infty} \left(\frac{1}{p(s)}\int_s^{\infty} (r-s)\, q(r)\, X_2(r)^{\beta}\, dr\right)^{\frac{1}{\alpha}} ds &\sim \frac{\lambda^{-\frac{\beta}{\alpha(\alpha-\beta)}} \left(\frac{m_2}{\alpha}\right)^{\frac{2\beta}{\alpha(\alpha-\beta)}} m_2^{\frac{2}{\alpha(\alpha+1)}} m_2^{\frac{1}{\alpha+1}}}{(m_2-\rho)(\alpha^2(m_1-\rho)(m_3-\rho))^{\frac{1}{\alpha}}} \\ &\times R(t)^{\rho-m_2} \, (l_p(t)l_q(t))^{\frac{\beta}{\alpha(\alpha-\beta)}} \, l_q(t)^{\frac{1}{\alpha}} \, l_p(t)^{\frac{2}{\alpha(\alpha+1)}} \, l_p(t)^{-\frac{1}{\alpha}} \, l_p(t)^{\frac{1}{\alpha+1}},\end{aligned}$$

and

$$\begin{aligned}&\int_b^t\int_s^{\infty} \left(\frac{1}{p(r)}\int_r^{\infty} (u-r)\, q(u)\, X_2(u)^{\beta}\, du\right)^{\frac{1}{\alpha}} dr\, ds \\ &\sim \frac{\lambda^{-\frac{\beta}{\alpha(\alpha-\beta)}} \left(\frac{m_2}{\alpha}\right)^{\frac{2\beta}{\alpha(\alpha-\beta)}} m_2^{\frac{2}{\alpha(\alpha+1)}} m_2^{\frac{2}{\alpha+1}}}{\rho(m_2-\rho)(\alpha^2(m_1-\rho)(m_3-\rho))^{\frac{1}{\alpha}}} R(t)^{\rho} \, (l_p(t)l_q(t))^{\frac{\beta}{\alpha(\alpha-\beta)}} \\ &\times l_q(t)^{\frac{1}{\alpha}} \, l_p(t)^{\frac{2}{\alpha(\alpha+1)}} \, l_p(t)^{-\frac{1}{\alpha}} \, l_p(t)^{\frac{2}{\alpha+1}} = X_2(t).\end{aligned}$$

This completes the proof of Lemma 4.3.3. □

Lemma 4.3.4 *Suppose that*

$$\sigma = -\beta\, m_1(\alpha,\eta) - 2\, m_2(\alpha,\eta) \quad \text{and} \quad \int_a^{\infty} tq(t)\, \varphi_2(t)^{\beta}\, dt < \infty \tag{4.3.22}$$

holds. The function $X_3(t) \in \mathrm{ntr}-\mathrm{RV}_R(m_1(\alpha,\eta))$ given by (4.3.16) satisfies the asymptotic relation (4.3.13) for any $b \geq a$.

Proof. Suppose that (4.3.22) holds. Using (4.3.2), (4.3.5), (4.3.7) and (4.3.10) we see that

$$tq(t)\,\varphi_2(t)^\beta \sim \frac{m_2^{\frac{2\beta-\alpha}{\alpha+1}}}{(m_1(m_2-m_1))^\beta}\,R(t)^{-m_2}\,l_p(t)^{\frac{\beta(\alpha-1)+\alpha}{\alpha(\alpha+1)}}\,l_q(t), \quad t\to\infty,$$

so that applying (iii) of Lemma 4.3.1 we have

$$\text{(4.3.23)} \qquad \int_t^\infty s\,q(s)\,\varphi_2(s)^\beta\,ds \sim \frac{m_2^{\frac{2\beta-\alpha+1}{\alpha+1}}}{(m_1(m_2-m_1))^\beta} \times \int_t^\infty R'(s)\,R(s)^{-1}\,l_p(s)^{\frac{\beta(\alpha-1)+2\alpha}{\alpha(\alpha+1)}}\,l_q(s)\,ds \in \mathrm{SV}_R, \quad t\to\infty.$$

This, combined with (4.3.16), gives the following expression for $X_3(t)$:

$$X_3(t) \sim \left(\frac{(\alpha-\beta)m_2}{\alpha(m_1(m_2-m_1))^\alpha}\right)^{\frac{1}{\alpha-\beta}} R(t)^{m_1}\,l_p(t)^{\frac{\alpha-1}{\alpha(\alpha+1)}} \times \left(\int_t^\infty R'(s)\,R(s)^{-1}\,l_p(s)^{\frac{\beta(\alpha-1)+2\alpha}{\alpha(\alpha+1)}}\,l_q(s)\,ds\right)^{\frac{1}{\alpha-\beta}} \in \mathrm{RV}_R\,(m_1)\,,\ t\to\infty.$$

Next, we integrate $q(t)\,X_3(t)^\beta$ twice on $[t,\infty)$ and raise the result to the exponent $1/\alpha$. Since $q(t)X_3(t)^\beta \in \mathrm{RV}_R(\beta m_1+\sigma) = \mathrm{RV}_R(-2m_2)$ (cf.(4.3.22)), repeated application of Lemma 4.3.1 yields

$$\left(\int_t^\infty\int_s^\infty q(r)\,X_3(r)^\beta\,dr\,ds\right)^{\frac{1}{\alpha}} \sim m_2^{\frac{2\beta-\alpha+1}{(\alpha-\beta)(\alpha+1)}}\left(\frac{\alpha-\beta}{\alpha(m_1(m_2-m_1))^\beta}\right)^{\frac{1}{\alpha-\beta}} \times \left(\int_t^\infty R'(s)\,R(s)^{-1}\,l_p(s)^{\frac{\beta(\alpha-1)+2\alpha}{\alpha(\alpha+1)}}\,l_q(s)\,ds\right)^{\frac{1}{\alpha-\beta}} \in \mathrm{SV}_R, \quad t\to\infty.$$

Multiplying the above by $p(t)^{-\frac{1}{\alpha}}$ and integrating it first on $[t,\infty)$ and then on $[b,t]$ for any fixed $b\geq a$, we conclude via Lemma 4.3.1 that

$$\int_t^\infty\left(\frac{1}{p(s)}\int_s^\infty (r-s)q(r)X_3(r)^\beta dr\right)^{\frac{1}{\alpha}}ds \sim m_1 m_2^{\frac{\beta+1}{(\alpha-\beta)(\alpha+1)}}\left(\frac{\alpha-\beta}{\alpha(m_1(m_2-m_1))^\alpha}\right)^{\frac{1}{\alpha-\beta}} \times R(t)^{m_1-m_2}l_p(t)^{-\frac{1}{\alpha(\alpha+1)}}\left(\int_t^\infty R'(s)R(s)^{-1}l_p(s)^{\frac{\beta(\alpha-1)+2\alpha}{\alpha(\alpha+1)}}l_q(s)\,ds\right)^{\frac{1}{\alpha-\beta}}$$

and

$$\int_b^t\int_s^\infty\left(\frac{1}{p(r)}\int_r^\infty (u-r)\,q(u)\,X_3(u)^\beta\,du\right)^{\frac{1}{\alpha}} drds \sim \left(\frac{(\alpha-\beta)m_2}{\alpha(m_1(m_2-m_1))^\alpha}\right)^{\frac{1}{\alpha-\beta}} \times R(t)^{m_1}l_p(t)^{\frac{\alpha-1}{\alpha(\alpha+1)}}\left(\int_t^\infty R'(s)\,R(s)^{-1}\,l_p(s)^{\frac{\beta(\alpha-1)+2\alpha}{\alpha(\alpha+1)}}\,l_q(s)\,ds\right)^{\frac{1}{\alpha-\beta}} = X_3(t),$$

as $t \to \infty$. This completes the proof of Lemma 4.3.4. $\square$

Since $\varphi_1(t) \prec x(t) \prec \varphi_2(t),\ t \to \infty$, the regularity index ρ of $x(t)$ must satisfy

$$0 \le \rho \le m_1(\alpha, \eta).$$

If $\rho = 0$, then since $x(t) = l_x(t) \to \infty,\ t \to \infty$, $x(t)$ is a member of $\mathrm{ntr}-\mathrm{SV}_R$, while if $\rho = m_1(\alpha,\eta)$, then since $x(t)/R(t)^{m_1(\alpha,\eta)} = l_x(t) \to 0,\ t \to \infty$, $x(t)$ is a member of $\mathrm{ntr}-\mathrm{RV}_R(m_1(\alpha,\eta))$. If $0 < \rho < m_1(\alpha,\eta)$, then $x(t)$ is a member of $\mathrm{RV}_R(\rho)$ and satisfies $x(t) \to \infty$ and $x(t)/R(t)^{m_1(\alpha,\eta)} \to 0$ as $t \to \infty$. Thus the set of all generalized regularly varying solutions of type (I_1) is naturally divided into the three disjoint classes

$$\mathrm{ntr}-\mathrm{SV}_R \quad \text{or} \quad \mathrm{RV}_R(\rho) \ \text{ with } \ \rho \in (0\ ,\ m_1(\alpha,\eta)) \quad \text{or} \quad \mathrm{ntr}-\mathrm{RV}_R\,(m_1(\alpha,\eta))\,.$$

Our aim is to establish necessary and sufficient conditions for each of the above classes to have a member and furthermore to show that the asymptotic behavior of all members of each class is governed by a unique explicit formula describing the growth order at infinity accurately.

Theorem 4.3.1 *Let $p(t) \in \mathrm{RV}_R(\eta)$, $q(t) \in \mathrm{RV}_R(\sigma)$ and (C_1) hold. Equation* (E) *has intermediate solutions $x(t) \in \mathrm{ntr}-\mathrm{SV}_R$ satisfying (I_1) if and only if* (4.3.17) *holds. The asymptotic behavior of any such solution $x(t)$ is governed by the unique formula $x(t) \sim X_1(t)$, $t \to \infty$, where $X_1(t)$ is given by* (4.3.14).

Theorem 4.3.2 *Let $p(t) \in \mathrm{RV}_R(\eta)$, $q(t) \in \mathrm{RV}_R(\sigma)$ and (C_1) hold. Equation* (E) *has intermediate solutions $x(t) \in \mathrm{RV}_R(\rho)$ with $\rho \in (0\ ,\ m_1(\alpha,\eta))$ if and only if* (4.3.19) *holds, in which case ρ is given by* (4.3.20) *and the asymptotic behavior of any such solution $x(t)$ is governed by the unique formula $x(t) \sim X_2(t)$, $t \to \infty$, where $X_2(t)$ is given by* (4.3.15).

Theorem 4.3.3 *Let $p(t) \in \mathrm{RV}_R(\eta)$, $q(t) \in \mathrm{RV}_R(\sigma)$ and (C_1) hold. Equation* (E) *has intermediate solutions $x(t) \in \mathrm{ntr}-\mathrm{RV}_R\,(m_1(\alpha,\eta))$ satisfying (I_1) if and only if* (4.3.22) *holds. The asymptotic behavior of any such solution $x(t)$ is governed by the unique formula $x(t) \sim X_3(t)$, $t \to \infty$, where $X_3(t)$ is given by* (4.3.16).

Proof of the "only if" part of Theorems 4.3.1, 4.3.2 and 4.3.3: Suppose that (E) has a type-(I_1) intermediate solution $x(t) \in \mathrm{RV}_R(\rho)$ on $[t_0, \infty)$. Clearly, $\rho \in [0, m_1]$. Using (4.3.2), (4.3.3) and (4.3.7), we obtain from (E)

$$\begin{aligned}(4.3.24)\qquad -\,(p(t)(-x''(t))^{\alpha})' &= \int_t^{\infty} q(s)x(s)^{\beta}\,ds \\ &\sim m_2^{\frac{1}{\alpha+1}} \int_t^{\infty} R'(s)R(s)^{\sigma+\beta\rho+m_2-1} l_p(s)^{\frac{1}{\alpha+1}} l_q(s) l_x(s)^{\beta}\,ds,\ t \to \infty.\end{aligned}$$

The convergence of the last integral in (4.3.24) means that $\sigma+\beta\rho+m_2\le 0$. But the possibility $\sigma+\beta\rho+m_2=0$ is precluded, because if this were the case the last integral in (4.3.24) would be an SV_R function, which is not integrable on $[t_0,\infty)$ by (i) of Lemma 4.3.1. This would contradict the fact that the left-hand side of (4.3.24) is integrable on $[t_0,\infty)$. It follows that $\sigma+\beta\rho+m_2<0$. Then, integration of (4.3.24) on $[t,\infty)$ with application of Lemma 4.3.1 gives

$$\begin{aligned} p(t)(-x''(t))^{\alpha} &\sim \frac{m_2^{\frac{1}{\alpha+1}}}{-(\sigma+\beta\rho+m_2)}\int_t^\infty R(s)^{\sigma+\beta\rho+m_2}\,l_p(s)^{\frac{1}{\alpha+1}}\,l_q(s)\,l_x(s)^\beta\,ds \\ &\sim \frac{m_2^{\frac{2}{\alpha+1}}}{-(\sigma+\beta\rho+m_2)}\int_t^\infty R'(s)\,R(s)^{\sigma+\beta\rho+2m_2-1}\,l_p(s)^{\frac{2}{\alpha+1}}\,l_q(s)\,l_x(s)^\beta\,ds,\ t\to\infty, \end{aligned} \tag{4.3.25}$$

where (4.3.7) has been used in the last step. Noting that the last integral is convergent, we distinguish the two cases:

$$(a)\ \ \sigma+\beta\rho+2m_2=0\ \ \text{and}\ \ (b)\ \ \sigma+\beta\rho+2m_2<0.$$

Assume that (a) holds. From (4.3.25) and (4.3.7) we have

$$\begin{aligned} -x''(t) &\sim m_2^{\frac{1-\alpha}{\alpha(\alpha+1)}}R(t)^{-\frac{\eta}{\alpha}}\,l_p(t)^{-\frac{1}{\alpha}}\left(\int_t^\infty R'(s)\,R(s)^{-1}\,l_p(s)^{\frac{2}{\alpha+1}}\,l_q(s)\,l_x(s)^\beta\,ds\right)^{\frac{1}{\alpha}} \\ &\sim m_2^{\frac{1}{\alpha(\alpha+1)}}R'(t)R(t)^{m_1-m_2-1}l_p(t)^{-\frac{1}{\alpha(\alpha+1)}}\left(\int_t^\infty R'(s)\,R(s)^{-1}l_p(s)^{\frac{2}{\alpha+1}}l_q(s)l_x(s)^\beta ds\right)^{\frac{1}{\alpha}}, \end{aligned}$$

as $t\to\infty$. Integrability of $x''(t)$ on $[t,\infty)$, given that $m_1-m_2<0$, allows us to integrate the previous relation on $[t,\infty)$, implying

$$\begin{aligned} x'(t) &\sim \frac{m_2^{\frac{1}{\alpha(\alpha+1)}}}{m_2-m_1}\,R(t)^{m_1-m_2}\,l_p(t)^{-\frac{1}{\alpha(\alpha+1)}} \\ &\quad\times\left(\int_t^\infty R'(s)R(s)^{-1}l_p(s)^{\frac{2}{\alpha+1}}l_q(s)l_x(s)^\beta\,ds\right)^{\frac{1}{\alpha}},\quad t\to\infty. \end{aligned} \tag{4.3.26}$$

Since the right-hand side of (4.3.26) is not integrable on $[t_0,\infty)$, due to the fact that $m_1<0$ (see Lemma 4.3.1-(i)) and $x(t)$ grows to ∞ as $t\to\infty$, integration of

(4.3.26) on $[t_0, t]$ then shows that

$$
\begin{aligned}
x(t) &\sim \frac{m_2^{\frac{1}{\alpha}}}{m_1(m_2 - m_1)} R(t)^{m_1} l_p(t)^{\frac{\alpha-1}{\alpha(\alpha+1)}} \\
&\times \left(\int_t^\infty R'(s)\, R(s)^{-1}\, l_p(s)^{\frac{2}{\alpha+1}}\, l_q(s)\, l_x(s)^\beta\, ds \right)^{\frac{1}{\alpha}} \sim m_2^{\frac{1-\alpha}{\alpha(\alpha+1)}} \varphi_2(t) \\
&\times \left(\int_t^\infty R'(s)\, R(s)^{-1}\, l_p(s)^{\frac{2}{\alpha+1}}\, l_q(s)\, l_x(s)^\beta\, ds \right)^{\frac{1}{\alpha}} \in \mathrm{RV}_R(m_1),\ t \to \infty.
\end{aligned} \tag{4.3.27}
$$

Assume next that (b) holds. From (4.3.25) we find via the generalized Karamata integration theorem that

$$
\begin{aligned}
-x''(t) &\sim \left(\frac{m_2^{\frac{2}{\alpha+1}}}{(\sigma + \beta\rho + m_2)\,(\sigma + \beta\rho + 2m_2)} \right)^{\frac{1}{\alpha}} R(t)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}} \\
&\times l_p(t)^{\frac{1-\alpha}{\alpha(\alpha+1)}} l_q(t)^{\frac{1}{\alpha}}\, l_x(t)^{\frac{\beta}{\alpha}} \sim \left(\frac{m_2^{\frac{\alpha+2}{\alpha+1}}}{(\sigma + \beta\rho + m_2)\,(\sigma + \beta\rho + 2m_2)} \right)^{\frac{1}{\alpha}} \\
&\times R'(t)\, R(t)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2-1}\, l_p(t)^{\frac{1}{\alpha(\alpha+1)}} l_q(t)^{\frac{1}{\alpha}} l_x(t)^{\frac{\beta}{\alpha}}, \quad t \to \infty.
\end{aligned} \tag{4.3.28}
$$

The integrability of $-x''(t)$ on $[t_0, \infty)$ implies that $(\sigma + \beta\rho + 2m_2 - \eta)/\alpha + m_2 \le 0$. But the equality is not allowed here. In fact, if the equality holds, then by (4.3.8)

$$
\sigma + \beta\rho + 2m_2 = \eta - \alpha m_2 = \frac{\eta - \alpha^2}{\alpha + 1} > 0,
$$

which contradicts the assumption (b). Therefore, from (4.3.28) integrated over $[t, \infty)$ we have

$$
\begin{aligned}
x'(t) \sim & \left(\frac{m_2^{\frac{\alpha+2}{\alpha+1}}}{(\sigma + \beta\rho + m_2)\,(\sigma + \beta\rho + 2m_2)} \right)^{\frac{1}{\alpha}} \frac{\alpha}{-(\sigma + \beta\rho + (\alpha+2)m_2 - \eta)} \\
& \times R(t)^{\frac{\sigma+\beta\rho+(\alpha+2)m_2-\eta}{\alpha}}\, l_p(t)^{\frac{1}{\alpha(\alpha+1)}} l_q(t)^{\frac{1}{\alpha}}\, l_x(t)^{\frac{\beta}{\alpha}},\ t \to \infty.
\end{aligned} \tag{4.3.29}
$$

Since $x'(t)$ is not integrable on $[t_0, \infty)$ (note that $x(t) \to \infty, t \to \infty$), it follows that

$$
\frac{\sigma + \beta\rho + (\alpha+2)m_2 - \eta}{\alpha} + m_2 = \frac{\sigma + \beta\rho + 2\alpha + \eta}{\alpha} \ge 0,
$$

and integration of (4.3.29) on $[t_0, t]$ leads to

$$x(t) \sim \left(\frac{m_2^2}{(\sigma+\beta\rho+m_2)\,(\sigma+\beta\rho+2m_2)}\right)^{\frac{1}{\alpha}} \frac{\alpha}{-(\sigma+\beta\rho+(\alpha+2)m_2-\eta)}$$
$$\times \int_{t_0}^{t} R'(s)\,R(s)^{\frac{\sigma+\beta\rho+\alpha+\eta}{\alpha}} l_p(s)^{\frac{1}{\alpha}} l_q(s)^{\frac{1}{\alpha}} l_x(s)^{\frac{\beta}{\alpha}}\,ds,\ t\to\infty, \tag{4.3.30}$$

where $\frac{\sigma+\beta\rho+\alpha+\eta}{\alpha} \geq -1$ because of the divergence of the last integral as $t \to \infty$. We distinguish the two cases:

$$(b.1)\quad \frac{\sigma+\beta\rho+\alpha+\eta}{\alpha} = -1 \quad \text{and} \quad (b.2)\quad \frac{\sigma+\beta\rho+\alpha+\eta}{\alpha} > -1.$$

Assume that $(b.1)$ holds. Then, (4.3.30) shows that $x(t) \in \mathrm{SV}_R$, that is, $\rho = 0$, and hence $\sigma = -2\alpha - \eta$. Since

$$\sigma+\beta\rho+m_2 = -\alpha m_3,\ \sigma+\beta\rho+2m_2 = -\alpha m_1,\ \sigma+\beta\rho+(\alpha+2)m_2-\eta = \alpha m_2,$$

(4.3.30) reduce to

$$x(t) \sim \left(\frac{m_2^{2-\alpha}}{\alpha^2 m_1 m_3}\right)^{\frac{1}{\alpha}} \int_{t_0}^{t} R'(s)\,R(s)^{-1}\,l_p(s)^{\frac{1}{\alpha}} l_q(s)^{\frac{1}{\alpha}}\,l_x(s)^{\frac{\beta}{\alpha}}\,ds, \quad t\to\infty. \tag{4.3.31}$$

Assume that $(b.2)$ holds. Applying Proposition 1.2.10 to the integral in (4.3.30), we get

$$x(t) \sim \left(\frac{m_2^2}{(\sigma+\beta\rho+m_2)\,(\sigma+\beta\rho+2m_2)}\right)^{\frac{1}{\alpha}} \frac{\alpha}{-(\sigma+\beta\rho+(\alpha+2)m_2-\eta)}$$
$$\times \frac{\alpha}{\sigma+\beta\rho+2\alpha+\eta} R(t)^{\frac{\sigma+\beta\rho+2\alpha+\eta}{\alpha}}\,l_p(t)^{\frac{1}{\alpha}}\,l_q(t)^{\frac{1}{\alpha}}\,l_x(t)^{\frac{\beta}{\alpha}}, \quad t\to\infty, \tag{4.3.32}$$

which implies that $x(t) \in \mathrm{RV}_R(\frac{\sigma+\beta\rho+2\alpha+\eta}{\alpha})$.

Let us now suppose that $x(t)$ is an intermediate solution of type (I_1) of (E) belonging to $\mathrm{ntr}-\mathrm{SV}_R$. From the above observations this is possible only when the case $(b.1)$ holds, in which case $\rho = 0$, $\sigma = -2\alpha-\eta$ and $x(t) = l_x(t)$ must satisfy the asymptotic behavior (4.3.31) as $t\to\infty$. Put

$$\mu(t) = H \int_{t_0}^{t} R'(s)\,R(s)^{-1}\,l_p(s)^{\frac{1}{\alpha}} l_q(s)^{\frac{1}{\alpha}}\,l_x(s)^{\frac{\beta}{\alpha}}\,ds, \qquad H = \left(\frac{m_2^{2-\alpha}}{\alpha^2 m_1 m_3}\right)^{\frac{1}{\alpha}}.$$

Noting that

$$\mu'(t) = H\,R'(t)\,R(t)^{-1}\,l_p(t)^{\frac{1}{\alpha}}\,l_q(t)^{\frac{1}{\alpha}}\,l_x(t)^{\frac{\beta}{\alpha}} \sim H\,R'(t)\,R(t)^{-1}\,l_p(t)^{\frac{1}{\alpha}}\,l_q(t)^{\frac{1}{\alpha}}\,\mu(t)^{\frac{\beta}{\alpha}},$$

we obtain the differential asymptotic relation

$$\mu(t)^{-\frac{\beta}{\alpha}}\,\mu'(t) \sim H R'(t)\,R(t)^{-1} l_p(t)^{\frac{1}{\alpha}}\,l_q(t)^{\frac{1}{\alpha}},\; t \to \infty.$$

Integrating the above from t_0 to t, we easily see that

$$x(t) \sim \mu(t) \sim \left(\frac{\alpha-\beta}{\alpha}\,H\int_{t_0}^{t} R'(s)\,R(s)^{-1}\,l_p(s)^{\frac{1}{\alpha}} l_q(s)^{\frac{1}{\alpha}}\,ds\right)^{\frac{\alpha}{\alpha-\beta}},\; t\to\infty,$$

which, in view of (4.3.18), is equivalent to

$$x(t) \sim \left(\frac{\alpha-\beta}{\alpha}\int_a^t s\left(\frac{1}{p(s)}\int_s^\infty (r-s)\,q(r)\,dr\right)^{\frac{1}{\alpha}} ds\right)^{\frac{\alpha}{\alpha-\beta}},\; t\to\infty. \tag{4.3.33}$$

Thus it has been shown that $x(t) \sim X_1(t)$, $t\to\infty$, where $X_1(t)$ is given by (4.3.14). Notice that the verification of (4.3.17) is included in the above discussions. This proves the "only if" part of Theorem 4.3.1.

Next, suppose that $x(t)$ is a solution of (E) belonging to $RV_R(\rho), \rho \in (0,m_1)$. This is possible only when $(b.2)$ holds, in which case $x(t)$ must satisfy the asymptotic relation (4.3.32). Therefore,

$$\rho = \frac{\sigma+\beta\rho+2\alpha+\eta}{\alpha} \quad\Rightarrow\quad \rho = \frac{\sigma+2\alpha+\eta}{\alpha-\beta},$$

which justifies (4.3.20). An elementary calculation shows that

$$0<\rho<m_1 \quad\Longrightarrow\quad -2\alpha-\eta<\sigma<-2\alpha-\eta+(\alpha-\beta)\,m_1 = -2m_2-\beta m_1,$$

which determines the range (4.3.19) of σ. Since

$$\sigma+\beta\rho+m_2 = \alpha(\rho-m_3), \qquad \sigma+\beta\rho+2m_2 = \alpha(\rho-m_1),$$

$$\sigma+\beta\rho+(\alpha+2)m_2-\eta = \alpha(\rho-m_2), \qquad \sigma+\beta\rho+2\alpha+\eta = \alpha\rho,$$

we conclude from (4.3.32) that $x(t)$ enjoys the asymptotic behavior $x(t)\sim X_2(t)$, $t\to\infty$, where $X_2(t)$ is given by (4.3.15). This proves the "only if" part of the Theorem 4.3.2.

Finally, suppose that $x(t)$ is an intermediate solution of type (I_1) of (E) belonging to $\mathrm{ntr}-\mathrm{RV}_R(m_1)$. Then, the case (a) is the only possibility for $x(t)$, which means that $\sigma = -\beta m_1 - 2m_2$ and (4.3.27) is satisfied by $x(t)$. Using $x(t) = R(t)^{m_1}\,l_x(t)$, (4.3.27) can be expressed as

$$l_x(t) \sim K\,l_p(t)^{\frac{\alpha-1}{\alpha(\alpha+1)}}\left(\int_t^\infty R'(s)\,R(s)^{-1}\,l_p(s)^{\frac{2}{\alpha+1}}\,l_q(s)\,l_x(s)^{\beta}\,ds\right)^{\frac{1}{\alpha}},\; t\to\infty, \tag{4.3.34}$$

where $K = m_2^{\frac{1}{\alpha}}/m_1(m_2 - m_1)$. Define $\nu(t)$ by

$$\nu(t) = \int_t^\infty R'(s)\, R(s)^{-1}\, l_p(s)^{\frac{2}{\alpha+1}}\, l_q(s)\, l_x(s)^\beta\, ds.$$

Then, noting that $l_x(t) \sim K\, l_p(t)^{\frac{\alpha-1}{\alpha(\alpha+1)}}\, \nu(t)^{\frac{1}{\alpha}}$ one can transform (4.3.34) into the following differential asymptotic relation for $\nu(t)$:

$$(4.3.35) \qquad -\nu(t)^{-\frac{\beta}{\alpha}}\, \nu'(t) \sim K^\beta\, R'(t)\, R(t)^{-1}\, l_p(t)^{\frac{(\alpha-1)\beta+2\alpha}{\alpha(\alpha+1)}}\, l_q(t),\ \ t \to \infty.$$

From (4.3.27), since $\lim_{t\to\infty} x(t)/\varphi_2(t) = 0$, we have $\lim_{t\to\infty} \nu(t) = 0$, implying that the left-hand side of (4.3.35) is integrable over $[t_0, \infty)$, so is the right-hand side. This, in view of (4.3.23), implies the convergence of the integral $\int_a^\infty tq(t)\varphi_2(t)^\beta\, dt$. Integrating (4.3.35) on $[t, \infty)$ and combining the result with (4.3.34), we find that

$$x(t) \sim K^{\frac{\alpha}{\alpha-\beta}}\, R(t)^{m_1}\, l_p(t)^{\frac{\alpha-1}{\alpha(\alpha+1)}} \left(\frac{\alpha-\beta}{\alpha} \int_t^\infty R'(s)\, R(s)^{-1} l_p(s)^{\frac{(\alpha-1)\beta+2\alpha}{\alpha(\alpha+1)}}\, l_q(s)\, ds \right)^{\frac{1}{\alpha-\beta}},$$

as $t \to \infty$, which due to (4.3.23) gives $x(t) \sim X_3(t)$, $t \to \infty$, where $X_3(t)$ is given by (4.3.16). This proves the "only if" part of the proof of Theorem 4.3.3. □

Proof of the "if" part of Theorems 4.3.1, 4.3.2 and 4.3.3: Suppose that (4.3.17) or (4.3.19) or (4.3.22) holds. From Lemmas 4.3.2, 4.3.3 and 4.3.4 it is known that $X_i(t)$, $i = 1, 2, 3$, defined by (4.3.14), (4.3.15) and (4.3.16) satisfy the asymptotic relation (4.3.13) for any $b \geq a$. We perform the simultaneous proof for $X_i(t)$, $i = 1, 2, 3$ so the subscripts $i = 1, 2, 3$ will be deleted in the rest of the proof. By (4.3.13) there exists $T_0 > a$ such that

$$(4.3.36) \qquad \int_{T_0}^t \int_s^\infty \left(\frac{1}{p(r)} \int_r^\infty (u-r)\, q(u)\, X(u)^\beta\, du \right)^{\frac{1}{\alpha}} drds \leq 2X(t),\ \ t \geq T_0.$$

Let such a T_0 be fixed. We may assume that $X(t)$ is increasing on $[T_0, \infty)$. Since (4.3.13) holds with $b = T_0$, there exists $T_1 > T_0$ such that

$$(4.3.37) \qquad \int_{T_0}^t \int_s^\infty \left(\frac{1}{p(r)} \int_r^\infty (u-r)\, q(u)\, X(u)^\beta\, du \right)^{\frac{1}{\alpha}} drds \geq \frac{X(t)}{2},\ \ t \geq T_1.$$

Choose positive constants m and M so that

$$(4.3.38) \qquad m^{1-\frac{\beta}{\alpha}} \leq \frac{1}{2},\ \ \ M^{1-\frac{\beta}{\alpha}} \geq 4,\ \ \ 2m\, X(T_1) \leq M\, X(T_0).$$

Define the integral operator

$$(4.3.39) \quad \mathcal{G}x(t) = x_0 + \int_{T_0}^t \int_s^\infty \left(\frac{1}{p(r)} \int_r^\infty (u-r)\, q(u)\, x(u)^\beta\, du \right)^{\frac{1}{\alpha}} drds,\ \ \ t \geq T_0,$$

where x_0 is a constant such that

$$m\,X(T_1) \le x_0 \le \frac{M}{2}\,X(T_0), \tag{4.3.40}$$

and let it act on set

$$\mathcal{X} = \{x \in C[T_0,\infty) : m\,X(t) \le x(t) \le M\,X(t),\ t \ge T_0\}. \tag{4.3.41}$$

It is clear that $\mathcal{X}$ is a closed, convex subset of the locally convex space $C[T_0,\infty)$ equipped with the topology of uniform convergence on compact subintervals of $[T_0,\infty)$.

It can be shown that $\mathcal{G}$ is a continuous self-map on $\mathcal{X}$ and that the set $\mathcal{G}(\mathcal{X})$ is relatively compact in $C[T_0,\infty)$.

(i) $\mathcal{G}(\mathcal{X}) \subset \mathcal{X}$. Let $x(t) \in \mathcal{X}$. Using (4.3.36), (4.3.38), (4.3.40) and (4.3.41) we get

$$\begin{aligned}\mathcal{G}x(t) &\le \frac{M}{2}\,X(T_0) + M^{\frac{\beta}{\alpha}} \int_{T_0}^{t}\!\int_s^\infty \left(\frac{1}{p(r)}\int_r^\infty (u-r)\,q(u)\,X(u)^\beta\,du\right)^{\frac{1}{\alpha}} drds\\ &\le \frac{M}{2}\,X(t) + 2M^{\frac{\beta}{\alpha}}\,X(t) \le \frac{M}{2}\,X(t) + \frac{M}{2}\,X(t) = M\,X(t), \quad t \ge T_0.\end{aligned}$$

On the other hand, using (4.3.37), (4.3.38), (4.3.40) and (4.3.41) we have

$$\mathcal{G}x(t) \ge x_0 \ge m\,X(T_1) \ge m\,X(t), \quad T_0 \le t \le T_1,$$

and

$$\begin{aligned}\mathcal{G}x(t) &\ge m^{\frac{\beta}{\alpha}} \int_{T_0}^{t}\!\int_s^\infty \left(\frac{1}{p(r)}\int_r^\infty (u-r)\,q(u)\,X(u)^\beta\,du\right)^{\frac{1}{\alpha}} drds\\ &\ge m^{\frac{\beta}{\alpha}}\,\frac{X(t)}{2} \ge m\,X(t),\ t \ge T_1.\end{aligned}$$

This shows that $\mathcal{G}x(t) \in \mathcal{X}$, that is, $\mathcal{G}$ maps $\mathcal{X}$ into itself.

(ii) $\mathcal{G}(\mathcal{X})$ *is relatively compact.* The inclusion $\mathcal{G}(\mathcal{X}) \subset \mathcal{X}$ ensures that $\mathcal{G}(\mathcal{X})$ is locally uniformly bounded on $[T_0,\infty)$. From the inequality

$$0 \le (\mathcal{G}x)'(t) \le M^{\frac{\beta}{\alpha}} \int_t^\infty \left(\frac{1}{p(s)}\int_s^\infty (r-s)q(r)X(r)^\beta\,dr\right)^{\frac{1}{\alpha}} ds, \quad t \ge T_0,$$

holding for all $x \in \mathcal{X}$ it follows that $\mathcal{G}(\mathcal{X})$ is locally equicontinuous on $[T_0,\infty)$. Then, the relative compactness of $\mathcal{G}(\mathcal{X})$ follows from the Arzela-Ascoli lemma.

(iii) $\mathcal{G}$ *is continuous on* $\mathcal{X}$. Let $\{x_n(t)\}$ be a sequence in $\mathcal{X}$ converging to $x(t)$ in $\mathcal{X}$ uniformly on any compact subinterval of $[T_0,\infty)$. From (4.3.39) we have

$$|\mathcal{G}x_n(t)-\mathcal{G}x(t)| \le \int_{T_0}^{t}\int_{s}^{\infty} \frac{1}{p(r)^{\frac{1}{\alpha}}} G_n(r)\,drds, \quad t \ge T_0,$$

where

$$G_n(t) = \left| \left(\int_t^\infty (s-t)\,q(s)\,x_n(s)^\beta\,ds \right)^{\frac{1}{\alpha}} - \left(\int_t^\infty (s-t)\,q(s)\,x(s)^\beta\,ds \right)^{\frac{1}{\alpha}} \right|.$$

Using the inequality $|x^\lambda - y^\lambda| \le |x-y|^\lambda$, $x,y \in \mathbb{R}^+$ holding for $\lambda \in (0,1)$, we see that if $\alpha \ge 1$, then

$$G_n(t) \le \left(\int_t^\infty (s-t) q(s) |x_n(s)^\beta - x(s)^\beta| ds \right)^{\frac{1}{\alpha}}.$$

On the other hand, using the mean value theorem, if $\alpha < 1$ we get

$$G_n(t) \le \frac{1}{\alpha} \left(M^\beta \int_t^\infty (s-t) q(s) X(s)^\beta ds \right)^{\frac{\alpha-1}{\alpha}} \int_t^\infty (s-t) q(s) |x_n(s)^\beta - x(s)^\beta| ds.$$

Thus, using that $q(t)\big|x_n(t)^\beta - x(t)^\beta| \to 0$ as $n \to \infty$ at each point $t \in [T_0,\infty)$ and $q(t)\big|x_n(t)^\beta - x(t)^\beta| \le M^\beta q(t) X(t)^\beta$ for $t \ge T_0$, while $q(t)X(t)^\beta$ is integrable on $[T_0,\infty)$, the uniform convergence $G_n(t) \to 0$ on $[T_0,\infty)$ follows by the application of the Lebesgue dominated convergence theorem. We conclude that $\mathcal{G}x_n(t) \to \mathcal{G}x(t)$ uniformly on any compact subinterval of $[T_0,\infty)$ as $n \to \infty$, which proves the continuity of $\mathcal{G}$.

Thus, all the hypotheses of the Schauder-Tychonoff fixed point theorem are fulfilled and so there exists a fixed point $x(t) \in \mathcal{X}$ of $\mathcal{G}$, which satisfies integral equation

$$x(t) = x_0 + \int_{T_0}^{t}\int_{s}^{\infty} \left(\frac{1}{p(r)} \int_r^\infty (u-r)\,q(u)\,x(u)^\beta\,du \right)^{\frac{1}{\alpha}} drds, \quad t \ge T_0. \tag{4.3.42}$$

Differentiating the above four times shows that $x(t)$ is a solution of (E) on $[T_0,\infty)$, which due to (4.3.41) is an intermediate solution of type (I_1). Therefore, the proof of our main results will be completed with the verification that the intermediate solutions of (E) constructed above are actually regularly varying functions with respect to $R(t)$. We define the function

$$J(t) = \int_{T_0}^{t}\int_{s}^{\infty} \left(\frac{1}{p(r)} \int_r^\infty (u-r)\,q(u)\,X(u)^\beta\,du \right)^{\frac{1}{\alpha}} drds,\ t \ge T_0,$$

and put
$$l = \liminf_{t\to\infty} \frac{x(t)}{J(t)}, \quad L = \limsup_{t\to\infty} \frac{x(t)}{J(t)}.$$
By Lemmas 4.3.2, 4.3.3 and 4.3.4 we have $X(t) \sim J(t)$, $t \to \infty$. Since, $x(t) \in \mathcal{X}$, it is clear that $0 < l \leq L < \infty$. We first consider L. Applying Lemma 1.1.1 four times, we obtain

$$\begin{aligned}
L &\leq \limsup_{t\to\infty} \frac{x'(t)}{J'(t)} \leq \limsup_{t\to\infty} \frac{x''(t)}{J''(t)} = \limsup_{t\to\infty} \frac{\left(\int_t^\infty (s-t)q(s)x(s)^\beta\, ds\right)^{\frac{1}{\alpha}}}{\left(\int_t^\infty (s-t)q(s)X(s)^\beta\, ds\right)^{\frac{1}{\alpha}}} \\
&= \left(\limsup_{t\to\infty} \frac{\int_t^\infty (s-t)q(s)x(s)^\beta\, ds}{\int_t^\infty (s-t)q(s)X(s)^\beta\, ds}\right)^{\frac{1}{\alpha}} \leq \left(\limsup_{t\to\infty} \frac{\int_t^\infty q(s)x(s)^\beta\, ds}{\int_t^\infty q(s)X(s)^\beta\, ds}\right)^{\frac{1}{\alpha}} \\
&\leq \left(\limsup_{t\to\infty} \frac{q(t)x(t)^\beta}{q(t)X(t)^\beta}\right)^{\frac{1}{\alpha}} = \left(\limsup_{t\to\infty} \frac{x(t)}{X(t)}\right)^{\frac{\beta}{\alpha}} = \left(\limsup_{t\to\infty} \frac{x(t)}{J(t)}\right)^{\frac{\beta}{\alpha}} = L^{\frac{\beta}{\alpha}},
\end{aligned}$$

where we have used $X(t) \sim J(t)$, $t \to \infty$, in the last step. Since $\beta/\alpha < 1$, the inequality $L \leq L^{\frac{\beta}{\alpha}}$ implies that $L \leq 1$. Similarly, repeated application of Lemma 1.1.1 to l leads to $l \geq 1$, from which it follows that $L = l = 1$, that is,

$$\lim_{t\to\infty} \frac{x(t)}{J(t)} = 1 \qquad \Longrightarrow \qquad x(t) \sim J(t) \sim X(t),\ t \to \infty.$$

Therefore it is concluded that if $p(t) \in \mathrm{RV}_R(\eta)$ and $q(t) \in \mathrm{RV}_R(\sigma)$, then the type-($\mathrm{I}_1$) solution $x(t)$ under consideration is a member of $\mathrm{RV}_R(\rho)$, where

$$\rho = 0 \quad \text{or} \quad \rho = \frac{2\alpha + \sigma + \eta}{\alpha - \beta} \in (0, m_1) \quad \text{or} \quad \rho = m_1,$$

according to whether the pair (η, σ) satisfies (4.3.17), (4.3.19) or (4.3.22), respectively. Needless to say, any such solution $x(t)$ in $\mathrm{RV}_R(\rho)$ enjoys one and the same asymptotic behavior (4.3.14), (4.3.15) or (4.3.16) according as $\rho = 0$, $\rho \in (0, m_1)$ or $\rho = m_1$. This completes the "if" parts of Theorems 4.3.1, 4.3.2 and 4.3.3. □

4.3.2 Intermediate regularly varying solutions of type (I_2)

Let us turn our attention to the study of intermediate solutions of type (I_2) of equation (E) under the condition (C_1), that is, those solutions $x(t)$ such that $\varphi_3(t) \prec x(t) \prec \varphi_4(t)$ as $t \to \infty$. As in the preceding subsection use is made of the expressions (4.3.2) and (4.3.3) for the coefficients $p(t)$, $q(t)$ and the solutions $x(t)$.

Let $x(t)$ be an intermediate solution of type (I_2) of (E) defined on $[t_0, \infty)$. Integrating (E) first from t to ∞ and then three times on $[t_0, t]$, we obtain

$$x(t) = c_0 + c_1(t - t_0) + \int_{t_0}^{t} (t-s) \frac{1}{p(s)^{\frac{1}{\alpha}}} \left(c_2 + \int_{t_0}^{s} \int_{r}^{\infty} q(u) x(u)^{\beta} \, du dr \right)^{\frac{1}{\alpha}} ds, \tag{4.3.43}$$

for $t \geq t_0$, where $c_0 = x(t_0)$, $c_1 = x'(t_0)$ and $c_2 = (p(t)\, x''(t)^{\alpha})'|_{t=t_0}$. From (4.3.43) we easily see that $x(t)$ satisfies the integral asymptotic relation

$$x(t) \sim \int_{b}^{t} (t-s) \left(\frac{1}{p(s)} \int_{t_0}^{s} \int_{r}^{\infty} q(u) x(u)^{\beta} \, du \, dr \right)^{\frac{1}{\alpha}} ds, \; t \to \infty, \tag{4.3.44}$$

for any $b \geq a$. This type of asymptotic relation will play a central role in constructing the intermediate solutions of type (I_2) of (E) by solving the integral equation (4.3.43) for some positive constants t_0 and c_i, $i = 0, 1, 2$. Therefore, first we show that the generalized regularly varying functions $Y_i(t)$, $i = 1, 2, 3$ defined respectively by

$$Y_1(t) = t \left(\frac{\alpha - \beta}{\alpha} \int_{a}^{t} \left(\frac{1}{p(s)} \int_{a}^{s} \int_{r}^{\infty} u^{\beta} \, q(u) \, du \, dr \right)^{\frac{1}{\alpha}} ds \right)^{\frac{\alpha}{\alpha - \beta}}, \tag{4.3.45}$$

(4.3.46)
$$Y_2(t) = \left(\left(\frac{m_2(\alpha, \eta)}{\alpha} \right)^2 \frac{p(t) \; q(t) \; R(t)^{2\alpha}}{\rho^{\alpha} \; (\rho - m_1(\alpha, \eta)) \; (\rho - m_2(\alpha, \eta))^{\alpha} \; (m_3(\alpha, \eta) - \rho)} \right)^{\frac{1}{\alpha - \beta}},$$

$$Y_3(t) = \varphi_4(t) \; \left(\frac{\alpha - \beta}{\alpha} \int_{t}^{\infty} q(s) \; \varphi_4(s)^{\beta} \; ds \right)^{\frac{1}{\alpha - \beta}}, \tag{4.3.47}$$

satisfies the asymptotic relation (4.3.44) for any $b \geq a$.

Lemma 4.3.5 *Suppose that*

$$\sigma = -\alpha - (\beta + 1) \, m_2(\alpha, \eta) \; \text{and} \int_{a}^{\infty} \left(\frac{1}{p(t)} \int_{a}^{t} \int_{s}^{\infty} r^{\beta} \, q(r) \, dr ds \right)^{\frac{1}{\alpha}} dt = \infty. \tag{4.3.48}$$

holds. The function $Y_1(t) \in \mathrm{ntr} - \mathrm{RV}_R(m_2(\alpha, \eta))$ *given by* (4.3.45) *satisfies the asymptotic relation* (4.3.44) *for any* $b \geq a$.

Proof. The proof needs the expression for $Y_1(t)$ in terms of $R(t)$, $l_p(t)$ and $l_q(t)$. To that end denote by $\Psi(t) = R'(t) \, R(t)^{-1} \, l_p(t)^{\frac{\beta+1}{\alpha(\alpha+1)}} \, l_q(t)^{\frac{1}{\alpha}}$. Let b be any constant such that $b \geq a$. Using (4.3.5) and (4.3.7) and applying Karamata's integration theorem, we first compute

$$\int_t^\infty s^\beta q(s)\,ds \sim m_2^{\frac{1-\alpha\beta}{\alpha+1}} \int_t^\infty R'(s)R(s)^{\sigma+\beta m_2+m_2-1}\,l_p(s)^{\frac{\beta+1}{\alpha+1}}\,l_q(s)\,ds$$

$$\sim m_2^{\frac{1-\alpha\beta}{\alpha+1}} \int_t^\infty R'(s)R(s)^{-\alpha-1} l_p(s)^{\frac{\beta+1}{\alpha+1}} l_q(s)\,ds \sim \frac{m_2^{\frac{1-\alpha\beta}{\alpha+1}}}{\alpha} R(t)^{-\alpha}\,l_p(t)^{\frac{\beta+1}{\alpha+1}}\,l_q(t),$$

as $t \to \infty$. Integrating the above on $[b,t]$ with the help of (4.3.7), we obtain

$$\begin{aligned}\left(\frac{1}{p(t)}\int_b^t\int_s^\infty r^\beta q(r)drds\right)^{\frac{1}{\alpha}} &\sim \frac{m_2^{\frac{2-\alpha\beta}{\alpha(\alpha+1)}}}{(\alpha^2(m_2-m_1))^{\frac{1}{\alpha}}} R(t)^{m_2-m_1-\frac{\eta}{\alpha}}\,l_p(t)^{\frac{-\alpha+\beta+1}{\alpha(\alpha+1)}}\,l_q(t)^{\frac{1}{\alpha}}\\ &\sim \frac{m_2^{\frac{2-\alpha\beta+\alpha}{\alpha(\alpha+1)}}}{(\alpha^2(m_2-m_1))^{\frac{1}{\alpha}}} R'(t)\,R(t)^{-1}\,l_p(t)^{\frac{\beta+1}{\alpha(\alpha+1)}}\,l_q(t)^{\frac{1}{\alpha}},\end{aligned} \tag{4.3.49}$$

from which it follows that, for any $b \geq a$,

$$\int_b^t \left(\frac{1}{p(s)}\int_b^s\int_r^\infty u^\beta\,q(u)\,du\,dr\right)^{\frac{1}{\alpha}} ds \sim \frac{m_2^{\frac{2-\alpha\beta+\alpha}{\alpha(\alpha+1)}}}{(\alpha^2(m_2-m_1))^{\frac{1}{\alpha}}}\int_b^t \Psi(s)\,ds, \tag{4.3.50}$$

as $t \to \infty$. Combining (4.3.5) with (4.3.50) then shows that $Y_1(t)$ can be expressed in the form

$$Y_1(t) \sim \left[\left(\frac{\alpha-\beta}{\alpha}\right)^\alpha \frac{m_2^{2-\alpha}}{\alpha^2(m_2-m_1)}\right]^{\frac{1}{\alpha-\beta}} R(t)^{m_2}\,l_p(t)^{\frac{1}{\alpha+1}}\left(\int_b^t \Psi(s)\,ds\right)^{\frac{\alpha}{\alpha-\beta}} \tag{4.3.51}$$

as $t \to \infty$.

To verify the relation (4.3.44) for $Y_1(t)$ we have to compute the repeated integral of $q(t)Y_1(t)^\beta$ on the right-hand side of (4.3.44). The computation is similar to that carried out above to derive (4.3.51) as a result of repeated integration of $t^\beta q(t)$. In fact, using (4.3.51) and denoting its constant multiplier by C, we integrate $q(t)Y_1(t)^\beta$ first on $[t,\infty)$ and then on $[b,t]$ to obtain

$$\begin{aligned}&\left(\frac{1}{p(t)}\int_b^t\int_s^\infty q(r)\,Y_1(r)^\beta\,dr\,ds\right)^{\frac{1}{\alpha}}\\ &\sim \left(\frac{C^\beta m_2^{\frac{\alpha+2}{\alpha+1}}}{\alpha^2(m_2-m_1)}\right)^{\frac{1}{\alpha}} R'(t)R(t)^{-1}l_p(t)^{\frac{\beta+1}{\alpha(\alpha+1)}}l_q(t)^{\frac{1}{\alpha}}\left(\int_b^t\Psi(s)\,ds\right)^{\frac{\beta}{\alpha-\beta}}\\ &= \left(\frac{C^\beta m_2^{\frac{\alpha+2}{\alpha+1}}}{\alpha^2(m_2-m_1)}\right)^{\frac{1}{\alpha}}\Psi(t)\left(\int_b^t\Psi(s)\,ds\right)^{\frac{\beta}{\alpha-\beta}},\ t\to\infty,\end{aligned}$$

which, integrating further on $[b,t]$, yields

$$\int_b^t \left(\frac{1}{p(s)}\int_b^s\int_r^\infty q(u)\,Y_1(u)^\beta\,du\,dr\right)^{\frac{1}{\alpha}} ds$$
$$\sim \frac{\alpha-\beta}{\alpha}\left(\frac{C^\beta m_2^{\frac{\alpha+2}{\alpha+1}}}{\alpha^2(m_2-m_1)}\right)^{\frac{1}{\alpha}}\left(\int_b^t \Psi(s)\,ds\right)^{\frac{\alpha}{\alpha-\beta}}, \quad t\to\infty.$$

Our final step is to integrate the above relation again on $[b,t]$:

$$\int_b^t (t-s)\left(\frac{1}{p(s)}\int_b^s\int_r^\infty q(u)Y_1(u)^\beta\,dudr\right)^{\frac{1}{\alpha}} ds$$
$$\sim CR(t)^{m_2}\,l_p(t)^{\frac{1}{\alpha+1}}\left(\int_b^t \Psi(s)\,ds\right)^{\frac{\alpha}{\alpha-\beta}} = Y_1(t),\ t\to\infty.$$

This proves that $Y_1(t)$ satisfies the asymptotic relation (4.3.44). □

Lemma 4.3.6 *Suppose that*

$$-\alpha-(\beta+1)\,m_2(\alpha,\eta) < \sigma < -\beta\,m_3(\alpha,\eta) - m_2(\alpha,\eta) \tag{4.3.52}$$

holds and let ρ be defined by (4.3.20). *The function $Y_2(t)\in \mathrm{RV}(\rho)$ given by* (4.3.46) *satisfies the asymptotic relation* (4.3.44) *for any $b\ge a$.*

Proof. Putting $\lambda=\rho^\alpha(\rho-m_1)(\rho-m_2)^\alpha(m_3-\rho)$, we express $Y_2(t)$ in the form

$$Y_2(t)\sim C\,R(t)^\rho\,l_p(t)^{\frac{1}{\alpha-\beta}}l_q(t)^{\frac{1}{\alpha-\beta}},\quad C=\left(\frac{1}{\lambda}\left(\frac{m_2}{\alpha}\right)^2\right)^{\frac{1}{\alpha-\beta}}.$$

We integrate $q(t)Y_2(t)^\beta$ twice: first on $[t,\infty)$ and then on $[b,t],\ b\ge a$. Since

$$q(t)Y_2(t)^\beta\sim C^\beta m_2^{\frac{1}{\alpha+1}}\,R'(t)R(t)^{-\alpha(m_3-\rho)-1}\,l_p(t)^{\frac{\alpha(\beta+1)}{(\alpha-\beta)(\alpha+1)}}\,l_q(t)^{\frac{\alpha}{\alpha-\beta}},\ t\to\infty$$

we see that

$$\int_t^\infty q(s)Y_2(s)^\beta ds\sim \frac{C^\beta m_2^{\frac{1}{\alpha+1}}}{\alpha(m_3-\rho)}R(t)^{-\alpha(m_3-\rho)}l_p(t)^{\frac{\alpha(\beta+1)}{(\alpha-\beta)(\alpha+1)}}l_q(t)^{\frac{\alpha}{\alpha-\beta}}$$
$$\sim \frac{C^\beta m_2^{\frac{2}{\alpha+1}}}{\alpha(m_3-\rho)}R'(t)\,R(t)^{\alpha(\rho-m_1)-1}\,l_p(t)^{\frac{\alpha\beta+2\alpha-\beta}{(\alpha-\beta)(\alpha+1)}}l_q(t)^{\frac{\alpha}{\alpha-\beta}},$$

and

$$(4.3.53)\qquad \int_b^t \int_s^\infty q(r)\, Y_2(r)^\beta\, drds$$
$$\sim \frac{C^\beta m_2^{\frac{2}{\alpha+1}}}{\alpha^2(m_3-\rho)(\rho-m_1)} R(t)^{\alpha(\rho-m_1)}\, l_p(t)^{\frac{\alpha\beta+2\alpha-\beta}{(\alpha-\beta)(\alpha+1)}}\, l_q(t)^{\frac{\alpha}{\alpha-\beta}}, \quad t\to\infty.$$

Since (4.3.53) implies

$$\left(\frac{1}{p(t)}\int_b^t \int_s^\infty q(r) Y_2(r)^\beta\, drds\right)^{\frac{1}{\alpha}}$$
$$\sim \frac{C^{\frac{\beta}{\alpha}}\, m_2^{\frac{\alpha+2}{\alpha(\alpha+1)}}}{(\alpha^2(m_3-\rho)(\rho-m_1))^{\frac{1}{\alpha}}} R'(t)\, R(t)^{\rho-m_2-1} l_p(t)^{\frac{\beta+1}{(\alpha-\beta)(\alpha+1)}} l_q(t)^{\frac{1}{\alpha-\beta}},\ t\to\infty,$$

integrating the last relation twice on $[b,t]$, we conclude that

$$\int_b^t (t-s)\left(\frac{1}{p(s)}\int_b^s \int_r^\infty q(u)\, Y_2(u)^\beta\, du\, dr\right)^{\frac{1}{\alpha}} ds$$
$$\sim \quad \frac{C^{\frac{\beta}{\alpha}} m_2^{\frac{2}{\alpha}}}{(\alpha^2(m_3-\rho)(\rho-m_1))^{\frac{1}{\alpha}}(\rho-m_2)\rho} R(t)^\rho l_p(t)^{\frac{1}{\alpha-\beta}}\, l_q(t)^{\frac{1}{\alpha-\beta}} = Y_2(t),\ t\to\infty.$$

This proves that $Y_2(t)$ satisfies the asymptotic relation (4.3.44). □

Lemma 4.3.7 *Suppose that*

$$(4.3.54)\qquad \sigma = -\beta\, m_3(\alpha,\eta) - m_2(\alpha,\eta) \quad \textit{and} \quad \int_a^\infty q(t)\,\varphi_4(t)^\beta\, dt < \infty.$$

holds. The function $Y_3(t) \in \mathrm{ntr}-\mathrm{RV}_R(m_3(\alpha,\eta))$ given by (4.3.47) satisfies the asymptotic relation (4.3.44) for any $b \geq a$.

Proof. Suppose that (4.3.54) holds. Using (4.3.7) and (4.3.11) we easily see that

$$(4.3.55)\qquad \int_t^\infty q(s)\varphi_4(s)^\beta\, ds \sim \frac{m_2^{\frac{\beta+1}{\alpha+1}}}{m_3^\beta}\int_t^\infty R'(s)R(s)^{-1} l_p(s)^{\frac{\beta+1}{\alpha+1}} l_q(s)\, ds,\ t\to\infty.$$

To simplify expressions we denote by $\Psi(t) = R'(t)\, R(t)^{-1}\, l_p(t)^{\frac{\beta+1}{\alpha+1}}\, l_q(t)$. Combining the above with (4.3.47), we obtain the following asymptotic representation for $Y_3(t)$ in terms of $R(t)$, $l_p(t)$ and $l_q(t)$:

$$(4.3.56)\quad Y_3(t) \sim \left(\frac{(\alpha-\beta)\, m_2}{\alpha\, m_3^\alpha}\right)^{\frac{1}{\alpha-\beta}} R(t)^{m_3}\, l_p(t)^{\frac{1}{\alpha+1}} \left(\int_t^\infty \Psi(s)\, ds\right)^{\frac{1}{\alpha-\beta}}, \quad t\to\infty\,.$$

Using (4.3.56), we compute

$$\int_t^\infty q(s)\,Y_3(s)^\beta\,ds$$
$$\sim \left(\frac{(\alpha-\beta)\,m_2^{\frac{\alpha(\beta+1)}{\beta(\alpha+1)}}}{\alpha\,m_3^\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \int_t^\infty R'(s)\,R(s)^{-1}\,l_p(s)^{\frac{\beta+1}{\alpha+1}}\,l_q(s)\left(\int_s^\infty \Psi(r)\,dr\right)^{\frac{\beta}{\alpha-\beta}} ds$$
$$= \left(\frac{(\alpha-\beta)\,m_2^{\frac{\alpha(\beta+1)}{\beta(\alpha+1)}}}{\alpha\,m_3^\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \int_t^\infty \Psi(s)\left(\int_s^\infty \Psi(r)\,dr\right)^{\frac{\beta}{\alpha-\beta}} ds$$
$$= \left(\frac{(\alpha-\beta)m_2^{\frac{\beta+1}{\alpha+1}}}{\alpha m_3^\beta}\right)^{\frac{\alpha}{\alpha-\beta}} \left(\int_t^\infty \Psi(s)\,ds\right)^{\frac{\alpha}{\alpha-\beta}},\ t\to\infty.$$

Next we integrate the above relation on $[b,t]$, $b \geq a$, multiply it by $1/p(t)$ and raise the result to the power $1/\alpha$. Then we find that

$$\left(\frac{1}{p(t)}\int_b^t\int_s^\infty q(r)\,Y_3(r)^\beta\,drds\right)^{\frac{1}{\alpha}} \tag{4.3.57}$$
$$\sim \left(\frac{(\alpha-\beta)m_2^{\frac{\beta+1}{\alpha+1}}}{\alpha m_3^\beta}\right)^{\frac{1}{\alpha-\beta}} m_2^{-\frac{1}{\alpha+1}} R(t)^{\frac{m_2-\eta}{\alpha}}\,l_p(t)^{-\frac{1}{\alpha+1}}\left(\int_t^\infty \Psi(s)\,ds\right)^{\frac{1}{\alpha-\beta}}$$
$$\sim \left(\frac{(\alpha-\beta)m_2^{\frac{\beta+1}{\alpha+1}}}{\alpha m_3^\beta}\right)^{\frac{1}{\alpha-\beta}} R'(t)\left(\int_t^\infty \Psi(s)\,ds\right)^{\frac{1}{\alpha-\beta}},\quad t\to\infty.$$

Integrating (4.3.57) twice on $[b,t]$ leads to the desired conclusion that $Y_3(t)$ satisfies the integral asymptotic relation (4.3.44). □

Since $\varphi_3(t) \in \mathrm{RV}_R(m_2(\alpha,\eta))$ and $\varphi_4(t) \in \mathrm{RV}_R(m_3(\alpha,\eta))$ ((4.3.5) and (4.3.11)), the regularity index ρ of $x(t)$ must satisfy $m_2(\alpha,\eta) \leq \rho \leq m_3(\alpha,\eta)$. If $\rho = m_2(\alpha,\eta)$, then since $x(t)/R(t)^{m_2(\alpha,\eta)} = l_x(t) \to \infty,\ t\to\infty$, $x(t)$ is a member of $\mathrm{ntr}-\mathrm{RV}_R(m_2(\alpha,\eta))$, while if $\rho = m_3(\alpha,\eta)$, then $x(t)/R(t)^{m_3(\alpha,\eta)} \to 0,\ t\to\infty$, and so $x(t)$ is a member of $\mathrm{ntr}-\mathrm{RV}_R(m_3(\alpha,\eta))$. If $m_2(\alpha,\eta) < \rho < m_3(\alpha,\eta)$, then $x(t)$ belongs to $\mathrm{RV}_R(\rho)$ and clearly satisfies $x(t)/R(t)^{m_2(\alpha,\eta)} \to \infty$ and $x(t)/R(t)^{m_3(\alpha,\eta)} \to 0$ as $t\to\infty$. Therefore, it is natural to divide the totality of intermediate solutions of type (I_2) of (E) into the following three classes

$$\mathrm{ntr}-\mathrm{RV}_R(m_2(\alpha,\eta)),\quad \mathrm{RV}_R(\rho),\ \rho\in(m_2(\alpha,\eta),m_3(\alpha,\eta)),\quad \mathrm{ntr}-\mathrm{RV}_R(m_3(\alpha,\eta)).$$

Our purpose is to show that, for each of the above classes, necessary and sufficient conditions for the membership are established and that the asymptotic behavior at infinity of all members of each class is determined precisely by a unique explicit formula.

Theorem 4.3.4 *Let* $p(t) \in \mathrm{RV}_R(\eta)$, $q(t) \in \mathrm{RV}_R(\sigma)$ *and* (C_1) *holds. Equation* (E) *has intermediate solutions* $x(t) \in \mathrm{ntr} - \mathrm{RV}_R\,(m_2(\alpha,\eta))$ *satisfying* (I_2) *if and only if* (4.3.48) *holds. The asymptotic behavior of any such solution* $x(t)$ *is governed by the unique formula* $x(t) \sim Y_1(t), t \to \infty$, *where function* $Y_1(t)$ *is given by* (4.3.45).

Theorem 4.3.5 *Let* $p(t) \in \mathrm{RV}_R(\eta)$, $q(t) \in \mathrm{RV}_R(\sigma)$ *and* (C_1) *holds. Equation* (E) *has intermediate solutions* $x(t) \in \mathrm{RV}_R(\rho)$ *with* $\rho \in (m_2(\alpha,\eta)\ ,\ m_3(\alpha,\eta))$ *if and only if* (4.3.52) *holds, in which case* ρ *is given by* (4.3.20) *and the asymptotic behavior of any such solution* $x(t)$ *is governed by the unique formula* $x(t) \sim Y_2(t), t \to \infty$, *where function* $Y_2(t)$ *is given by* (4.3.46).

Theorem 4.3.6 *Let* $p(t) \in \mathrm{RV}_R(\eta)$, $q(t) \in \mathrm{RV}_R(\sigma)$ *and* (C_1) *holds. Equation* (E) *has intermediate solutions* $x(t) \in \mathrm{ntr} - \mathrm{RV}_R\,(m_3(\alpha,\eta))$ *satisfying* (I_2) *if and only if* (4.3.54) *holds. The asymptotic behavior of any such solution* $x(t)$ *is governed by the unique formula* $x(t) \sim Y_3(t)$, $t \to \infty$, *where function* $Y_3(t)$ *is given by* (4.3.47).

Proof of the "only if" part of Theorems 4.3.4, 4.3.5 and 4.3.6: Suppose that equation (E) has a type-(I_2) intermediate solution $x(t) \in \mathrm{RV}_R(\rho)$, $\rho \in [m_2, m_3]$, defined on $[t_0,\infty)$. We begin by integrating (E) on $[t,\infty)$. Using (4.3.2), (4.3.3) and (4.3.7), we have

$$(p(t)\,x''(t)^{\alpha})' = \int_t^{\infty} q(s)x(s)^{\beta} ds \tag{4.3.58}$$
$$\sim m_2^{\frac{1}{\alpha+1}} \int_t^{\infty} R'(s)R(s)^{\sigma+\beta\rho+m_2-1} l_p(s)^{\frac{1}{\alpha+1}} l_q(s) l_x(s)^{\beta}\, ds,\ t \to \infty.$$

To proceed further we distinguish the two cases:

$$(a)\ \sigma+\beta\rho+m_2-1=-1 \quad \text{and} \quad (b)\ \sigma+\beta\rho+m_2-1<-1.$$

Let case (a) hold. Integration of (4.3.58) on $[t_0,t]$ yields

$$x''(t) \sim m_2^{\frac{1-\alpha}{\alpha(\alpha+1)}}\, R(t)^{\frac{m_2-\eta}{\alpha}}\, l_p(t)^{-\frac{1}{\alpha+1}} \left(\int_t^{\infty} R'(s)\, R(s)^{-1}\, l_p(s)^{\frac{1}{\alpha+1}}\, l_q(s)\, l_x(s)^{\beta}\, ds\right)^{\frac{1}{\alpha}}$$
$$\sim m_2^{\frac{1}{\alpha(\alpha+1)}} R'(t) \left(\int_t^{\infty} R'(s)\, R(s)^{-1}\, l_p(s)^{\frac{1}{\alpha+1}}\, l_q(s)\, l_x(s)^{\beta}\, ds\right)^{\frac{1}{\alpha}},\ t \to \infty. \tag{4.3.59}$$

Integrating (4.3.59) twice over $[t_0, t]$, we obtain via Lemma 4.3.1 as $t \to \infty$

$$x'(t) \sim m_2^{\frac{1}{\alpha}} R'(t) R(t)^{m_2} l_p(t)^{\frac{1}{\alpha+1}} \left(\int_t^\infty R'(s)\, R(s)^{-1}\, l_p(s)^{\frac{1}{\alpha+1}}\, l_q(s)\, l_x(s)^\beta\, ds \right)^{\frac{1}{\alpha}}$$

and

$$x(t) \sim \frac{m_2^{\frac{1}{\alpha}}}{m_3} R(t)^{m_3}\, l_p(t)^{\frac{1}{\alpha+1}} \left(\int_t^\infty R'(s)\, R(s)^{-1}\, l_p(s)^{\frac{1}{\alpha+1}}\, l_q(s)\, l_x(s)^\beta\, ds \right)^{\frac{1}{\alpha}}. \tag{4.3.60}$$

Let case (b) hold. Then, from (4.3.58) it follows that

$$\begin{aligned}(p(t)\, x''(t)^\alpha)' &\sim \frac{m_2^{\frac{1}{\alpha+1}}}{-(\sigma + \beta\rho + m_2)} R(t)^{\sigma+\beta\rho+m_2} l_p(t)^{\frac{1}{\alpha+1}} l_q(t) l_x(t)^\beta \\ &\sim \frac{m_2^{\frac{2}{\alpha+1}}}{-(\sigma + \beta\rho + m_2)} R'(t) R(t)^{\sigma+\beta\rho+2m_2-1}\, l_p(t)^{\frac{2}{\alpha+1}}\, l_q(t) l_x(t)^\beta,\ t \to \infty,\end{aligned}$$

which, integrated on $[t_0, t]$, gives

$$\begin{aligned} p(t)\, x''(t)^\alpha &\sim \frac{m_2^{\frac{2}{\alpha+1}}}{-\,(\sigma + \beta\rho +\, m_2)} \\ &\times \int_{t_0}^t R'(s)\, R(s)^{\sigma+\beta\rho+2m_2-1}\, l_p(s)^{\frac{2}{\alpha+1}}\, l_q(s)\, l_x(s)^\beta\, ds, \quad t \to \infty. \end{aligned} \tag{4.3.61}$$

The divergence of the last integral as $t \to \infty$ implies $\sigma + \beta\rho + 2m_2 \geq 0$, but the equality should be precluded, because if this would be the case, integrating the asymptotic expression for $x''(t)$ following from (4.3.61), we would have

$$\begin{aligned} x'(t) \ &\sim\ m_2^{\frac{1}{\alpha(\alpha+1)}} \int_{t_0}^t R'(s)\, R(s)^{-\frac{\eta}{\alpha}+m_2-1}\, l_p(s)^{-\frac{1}{\alpha(\alpha+1)}} \\ &\times\ \left(\int_{t_0}^s R'(r)\, R(r)^{-1}\, l_p(r)^{\frac{2}{\alpha+1}}\, l_q(r)\, l_x(r)^\beta\, dr \right)^{\frac{1}{\alpha}} ds,\ t \to \infty, \end{aligned}$$

from which, because of the divergence of the last integral as $t \to \infty$, it leads to a contradiction

$$0 \leq -\frac{\eta}{\alpha} + m_2 = \frac{\alpha^2 - \eta}{\alpha(\alpha + 1)} < 0\,.$$

Thus it holds $\sigma + \beta\rho + 2m_2 > 0$. Then, noting that (4.3.61) is transformed into

$$\begin{aligned} x''(t) \ &\sim\ \left(\frac{m_2^{\frac{2}{\alpha+1}}}{-(\sigma + \beta\rho + m_2)\,(\sigma + \beta\rho + 2m_2)} \right)^{\frac{1}{\alpha}} \\ &\times\ R(t)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}}\, l_p(s)^{\frac{1-\alpha}{\alpha(\alpha+1)}} l_q(s)^{\frac{1}{\alpha}}\, l_x(s)^{\frac{\beta}{\alpha}},\ t \to \infty. \end{aligned} \tag{4.3.62}$$

To preform further integration of (4.3.62) we consider the following two cases separately:

$$(b.1)\ \frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2=0; \qquad (b.2)\ \frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2>0.$$

Suppose that $(b.1)$ holds. Since $\sigma+\beta\rho+m_2=-\alpha$ and $\sigma+\beta\rho+2m_2=\alpha(m_2-m_1)$, integrating (4.3.62) twice on $[t_0,t]$, we have

$$\begin{aligned}(4.3.63)\qquad x(t)\ &\sim\ \left(\frac{m_2^{2-\alpha}}{\alpha^2(m_2-m_1)}\right)^{\frac{1}{\alpha}} R(t)^{m_2}\, l_p(t)^{\frac{1}{\alpha+1}}\\ &\times\ \int_{t_0}^{t} R'(s)R(s)^{-1} l_p(s)^{\frac{1}{\alpha(\alpha+1)}}\, l_q(s)^{\frac{1}{\alpha}}\, l_x(s)^{\frac{\beta}{\alpha}}\,ds,\ t\to\infty,\end{aligned}$$

which means that $x(t)\in \mathrm{RV}_R(m_2)$ and that its regularly varying part $l_x(t)$ satisfies the relation

$$\begin{aligned}(4.3.64)\qquad l_x(t)\ &\sim\ \left(\frac{m_2^{2-\alpha}}{\alpha^2(m_2-m_1)}\right)^{\frac{1}{\alpha}} l_p(t)^{\frac{1}{\alpha+1}}\\ &\times\ \int_{t_0}^{t} R'(s)R(s)^{-1} l_p(s)^{\frac{1}{\alpha(\alpha+1)}}\, l_q(s)^{\frac{1}{\alpha}}\, l_x(s)^{\frac{\beta}{\alpha}}\,ds,\ t\to\infty.\end{aligned}$$

Suppose that $(b.2)$ holds. Integrating (4.3.62) twice from t_0 to t, we obtain

$$\begin{aligned}(4.3.65)\qquad x(t)\ &\sim\ \left(\frac{m_2^{\frac{2}{\alpha+1}}}{-(\sigma+\beta\rho+m_2)\,(\sigma+\beta\rho+2m_2)}\right)^{\frac{1}{\alpha}}\\ &\times\ \frac{R(t)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+2m_2}\, l_p(t)^{\frac{1}{\alpha}}\, l_q(t)^{\frac{1}{\alpha}}\, l_x(t)^{\frac{\beta}{\alpha}}}{\left(\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2\right)\left(\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+2m_2\right)},\ t\to\infty.\end{aligned}$$

This implies that $x(t)\in \mathrm{RV}\left(\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+2m_2\right)$. It is easy to see that

$$m_2<\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+2m_2=\frac{\sigma+\beta\rho+2\alpha+\eta}{\alpha}<m_3.$$

Now, let $x(t)$ be an intermediate solution of type (I_2) of (E) belonging to $\mathrm{RV}_R(m_2)$. Then, from the above observations it is clear that only the case (b.1) is admissible, so that $\sigma=-\alpha-(\beta+1)m_2$ and $x(t)$ must satisfy (4.3.63). Put

$$\mu(t)=\int_{t_0}^{t} R'(s)R(s)^{-1} l_p(s)^{\frac{1}{\alpha(\alpha+1)}}\, l_q(s)^{\frac{1}{\alpha}}\, l_x(s)^{\frac{\beta}{\alpha}}\,ds.$$

Then, we can convert (4.3.64) to the differential asymptotic relation for $\mu(t)$

$$\text{(4.3.66)} \quad \mu(t)^{-\frac{\beta}{\alpha}}\,\mu'(t) \sim \left(\frac{m_2^{2-\alpha}}{\alpha^2(m_2-m_1)}\right)^{\frac{\beta}{\alpha^2}} R'(t)\,R(t)^{-1}\,l_p(t)^{\frac{\beta+1}{\alpha(\alpha+1)}}\,l_q(t)^{\frac{1}{\alpha}},\ t\to\infty.$$

Since the left-hand side of (4.3.66) is not integrable on $[t_0,\infty)$ (note that $x(t)/t\to\infty$ as $t\to\infty$ and so $\mu(t)\to\infty$ as $t\to\infty$), so is the right-hand side, which in view of (4.3.50) means that

$$\int_a^\infty \left(\frac{1}{p(t)}\int_a^t\int_s^\infty r^\beta q(r)drds\right)^{\frac{1}{\alpha}} dt = \infty.$$

We now integrate (4.3.66) on $[t_0,t]$ to obtain

$$\mu(t) \sim \left\{\frac{\alpha-\beta}{\alpha}\left(\frac{m_2^{2-\alpha}}{\alpha^2(m_2-m_1)}\right)^{\frac{\beta}{\alpha^2}} \int_{t_0}^t R'(s)R(s)^{-1}l_p(s)^{\frac{\beta+1}{\alpha(\alpha+1)}}l_q(s)^{\frac{1}{\alpha}}\,ds\right\}^{\frac{\alpha}{\alpha-\beta}},$$

as $t\to\infty$, and this, combined with (4.3.63), shows that

$$\begin{aligned}
x(t) &\sim \left(\frac{m_2^{2-\alpha}}{\alpha^2(m_2-m_1)}\right)^{\frac{1}{\alpha}} R(t)^{m_2}l_p(t)^{\frac{1}{\alpha+1}}\\
&\times\left\{\frac{\alpha-\beta}{\alpha}\left(\frac{m_2^{2-\alpha}}{\alpha^2(m_2-m_1)}\right)^{\frac{\beta}{\alpha^2}} \int_{t_0}^t R'(s)R(s)^{-1}l_p(s)^{\frac{\beta+1}{\alpha(\alpha+1)}}l_q(s)^{\frac{1}{\alpha}}ds\right\}^{\frac{\alpha}{\alpha-\beta}}\\
&= \left[\left(\frac{\alpha-\beta}{\alpha}\right)^\alpha \frac{m_2^{2-\alpha}}{\alpha^2(m_2-m_1)}\right]^{\frac{1}{\alpha-\beta}} R(t)^{m_2}l_p(t)^{\frac{1}{\alpha+1}}\\
&\times\left(\int_{t_0}^t R'(s)R(s)^{-1}l_p(s)^{\frac{\beta+1}{\alpha(\alpha+1)}}l_q(s)^{\frac{1}{\alpha}}ds\right)^{\frac{\alpha}{\alpha-\beta}}\\
&\sim t\left(\frac{\alpha-\beta}{\alpha}\int_a^t\left(\frac{1}{p(s)}\int_a^s\int_r^\infty u^\beta q(u)dudr\right)^{\frac{1}{\alpha}}ds\right)^{\frac{\alpha}{\alpha-\beta}} = Y_1(t),\ \ t\to\infty.
\end{aligned}$$

This completes the "only if" part of the Theorem 4.3.4.

Next, let $x(t)$ be an intermediate solution of (E) belonging to $\mathrm{RV}_R(\rho)$ for some $\rho\in(m_2,m_3)$. Clearly, $x(t)$ falls into the case $(b.2)$ and hence satisfies the asymptotic relation (4.3.65). This means that

$$\rho = \frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+2m_2 = \frac{\sigma+\beta\rho+2\alpha+\eta}{\alpha} \quad\Longrightarrow\quad \rho=\frac{2\alpha+\eta+\sigma}{\alpha-\beta},$$

verifying that the regularity index ρ is given by (4.3.20). From the requirement $m_2 < \rho < m_3$ it follows that $-\alpha - (\beta+1)m_2 < \sigma < -\beta m_3 - m_2$, showing that the range of σ is given by (4.3.52). Since

$$\frac{\sigma + \beta\rho + 2m_2 - \eta}{\alpha} + m_2 = \rho - m_2, \quad \frac{\sigma + \beta\rho + 2m_2 - \eta}{\alpha} + 2m_2 = \rho,$$

$$-(\sigma + \beta\rho + m_2) = \alpha(m_3 - \rho), \quad \sigma + \beta\rho + 2m_2 = \alpha(\rho - m_1),$$

the relation (4.3.65) can be rewritten as

$$x(t) \sim \left(\frac{m_2^2 p(t) q(t) R(t)^{2\alpha}}{\alpha^2 \rho^\alpha (\rho - m_1)(\rho - m_2)^\alpha (m_3 - \rho)} \right)^{\frac{1}{\alpha}} x(t)^{\frac{\beta}{\alpha}},$$

from which it readily follows that $x(t)$ enjoys the asymptotic behavior (4.3.46). This proves the "only if" part of the Theorem 4.3.5.

Finally, let $x(t)$ is an intermediate solution of type (I_2) of (E) belonging to $\mathrm{RV}_R(m_3)$. Since only the case (a) is possible for $x(t)$, it satisfies (4.3.60), which implies $\rho = m_3$ and $\sigma = -\beta m_3 - m_2$. Letting

$$\nu(t) = \left(\int_t^\infty R'(s)\, R(s)^{-1}\, l_p(s)^{\frac{1}{\alpha+1}}\, l_q(s)\, l_x(s)^\beta\, ds \right)^{\frac{1}{\alpha}},$$

and using the relation $l_x(t) \sim (m_2^{\frac{1}{\alpha}}/m_3) l_p(t)^{\frac{1}{\alpha+1}} \nu(t)$, we convert (4.3.60) into the differential asymptotic relation

$$-\alpha \nu(t)^{\alpha-\beta-1}\, \nu'(t) \sim \frac{m_2^{\frac{\beta}{\alpha}}}{m_3^\beta} R'(t)\, R(t)^{-1}\, l_p(t)^{\frac{\beta+1}{\alpha+1}}\, l_q(t), \quad t \to \infty. \tag{4.3.67}$$

Since the left-hand side of (4.3.67) is integrable on $[t_0, \infty)$, so is the right-hand side, that is,

$$\int_{t_0}^\infty R'(t) R(t)^{-1} l_p(t)^{\frac{\beta+1}{\alpha+1}} l_q(t) dt < \infty,$$

which is equivalent to $\int_a^\infty q(t) \varphi_4(t)^\beta\, dt < \infty$ (see (4.3.55) in the proof of Lemma 4.3.7). Integrating (4.3.67) over $[t, \infty)$ then yields

$$\nu(t) \sim \left(\frac{(\alpha - \beta) m_2^{\frac{\beta}{\alpha}}}{\alpha m_3^\beta} \int_t^\infty R'(s)\, R(s)^{-1}\, l_p(s)^{\frac{\beta+1}{\alpha+1}}\, l_q(s)\, ds \right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty,$$

and this combined with (4.3.60) determines the precise asymptotic behavior of $x(t)$ as follows:

$$x(t) \sim \frac{m_2^{\frac{1}{\alpha}}}{m_3} R(t)^{m_3} l_p(t)^{\frac{1}{\alpha+1}} \left(\frac{(\alpha-\beta) m_2^{\frac{\beta}{\alpha}}}{\alpha m_3^{\beta}} \int_t^\infty R'(s) R(s)^{-1} l_p(s)^{\frac{\beta+1}{\alpha+1}} l_q(s)\, ds \right)^{\frac{1}{\alpha-\beta}}$$

$$\sim \varphi_4(t) \left(\frac{\alpha-\beta}{\alpha} \int_t^\infty q(s) \varphi_4(s)^\beta \, ds \right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty.$$

Thus the "only if" part of the Theorem 4.3.6 has been proved. □

Proof of the "if" part of Theorems 4.3.4, 4.3.5 and 4.3.6: Suppose that (4.3.48) or (4.3.52) or (4.3.54) holds. From Lemmas 4.3.5, 4.3.6 and 4.3.7 it is known that $Y_i(t)$, $i = 1, 2, 3$, defined by (4.3.45), (4.3.46) and (4.3.47) satisfy the asymptotic relation (4.3.44). We perform the simultaneous proof for $Y_i(t)$, $i = 1, 2, 3$ so the subscripts $i = 1, 2, 3$ will be deleted in the rest of the proof. By (4.3.44) there exists $T_0 > a$ such that

$$\int_{T_0}^t (t-s) \left(\frac{1}{p(s)} \int_{T_0}^s \int_r^\infty q(u)\, Y(u)^\beta \, du\, dr \right)^{\frac{1}{\alpha}} ds \le 2Y(t), \ \ t \ge T_0.$$

Let such a T_0 be fixed. We may assume that $Y(t)$ is increasing on $[T_0, \infty)$. Since (4.3.44) holds with $b = T_0$, there exists $T_1 > T_0$ such that

$$\int_{T_0}^t (t-s) \left(\frac{1}{p(s)} \int_{T_0}^s \int_r^\infty q(u)\, Y(u)^\beta \, du\, dr \right)^{\frac{1}{\alpha}} ds \ge \frac{Y(t)}{2}, \ \ t \ge T_1.$$

Choose positive constants k and K such that

$$k^{1-\frac{\beta}{\alpha}} \le \frac{1}{2}, \quad K^{1-\frac{\beta}{\alpha}} \ge 4, \quad 2k\, Y(T_1) \le K\, Y(T_0).$$

Considering the integral operator

$$\mathcal{H}y(t) = y_0 + \int_{T_0}^t (t-s) \left(\frac{1}{p(s)} \int_{T_0}^s \int_r^\infty q(u)\, y(u)^\beta \, du\, dr \right)^{\frac{1}{\alpha}} ds, \quad t \ge T_0,$$

where y_0 is a constant such that $k\, Y(T_1) \le y_0 \le \dfrac{K}{2} Y(T_0)$, we may verify that $\mathcal{H}$ is continuous self-map on the set

$$\mathcal{Y} = \{ y \in C[T_0, \infty) : k\, Y(t) \le y(t) \le K\, Y(t), \ t \ge T_0 \},$$

and that $\mathcal{H}$ sends $\mathcal{Y}$ into relatively compact subset of $C[T_0,\infty)$. Thus, $\mathcal{H}$ has a fixed point $y(t)\in\mathcal{Y}$, which generates a solution of equation (E) of type (I_2) satisfying above inequalities and thus yields that

$$0<\liminf_{t\to\infty}\frac{y(t)}{Y(t)}\le\limsup_{t\to\infty}\frac{y(t)}{Y(t)}<\infty.$$

Denoting

$$L(t)=\int_a^t(t-s)\left(\frac{1}{p(s)}\int_a^s\int_r^\infty q(u)\,Y(u)^\beta\,du\,dr\right)^{\frac{1}{\alpha}}ds$$

and using $Y(t)\sim L(t)$, $t\to\infty$ we get

$$0<\liminf_{t\to\infty}\frac{y(t)}{L(t)}\le\limsup_{t\to\infty}\frac{y(t)}{L(t)}<\infty.$$

Then, proceeding exactly as in the proof of the "if" part of Theorems 4.3.1-4.3.3, with application of Lemma 1.1.1, we conclude that $y(t)\sim L(t)\sim Y(t)$, $t\to\infty$. Therefore, $y(t)$ is a generalized regularly varying solution of (E) with requested regularity index and the asymptotic behavior (4.3.45), (4.3.46), (4.3.47) depending on if $q(t)\in\mathrm{RV}_R(\sigma)$ satisfies, respectively, (4.3.48) or (4.3.52) or (4.3.54). Thus, the "if part" of Theorems 4.3.4, 4.3.5 and 4.3.6 has been proved. □

4.4 Classification of positive solutions of (E) under the condition (C_2)

We assume that $p,q:[a,\infty)\to(0,\infty)$ are continuous functions and that (C_2) holds. In our asymptotic analysis of positive solutions of (E) a special role is played by the four functions

$$\psi_1(t)=1,\quad\psi_2(t)=t,\quad\psi_3(t)=\int_a^t\!\!\int_a^s\frac{1}{p(r)^{\frac{1}{\alpha}}}\,drds,\quad\psi_4(t)=\int_a^t\int_a^s\left(\frac{r}{p(r)}\right)^{\frac{1}{\alpha}}drds,$$

which are the particular solutions of the unperturbed differential equation

$$(p(t)|x''(t)|^{\alpha-1}x''(t))''=0.$$

It is to be noted that the functions define above satisfy the dominance relation

$$\psi_1(t)\prec\psi_2(t)\prec\psi_3(t)\prec\psi_4(t),\quad t\to\infty. \tag{4.4.1}$$

Let $x(t)$ be a positive solution of (E). It is known (see [73]) that $x(t)$ satisfies either

$$x'(t)>0,\quad x''(t)>0,\quad (p(t)|x''(t)|^{\alpha-1}x''(t))'>0\quad\text{for all large } t, \tag{4.4.2}$$

or

$$(4.4.3) \qquad x'(t) > 0, \quad x''(t) < 0, \quad (p(t)|x''(t)|^{\alpha-1}x''(t))' > 0 \quad \text{for all large } t.$$

Since (E) implies that $(p(t)|x''(t)|^{\alpha-1}x''(t))'$ is decreasing and positive, there exists a finite limit $\lim_{t\to\infty}(p(t)|x''(t)|^{\alpha-1}x''(t))' = \omega_3 \geq 0$.

Solutions satisfying (4.4.2). First let $x(t)$ satisfy (4.4.2) on $[t_0,\infty)$. Since $x'(t)$ is positive and increasing, we see that $x'(t) \geq x'(t_0)$, $t \geq t_0$, which by integration gives $x(t) \to \infty$, $t \to \infty$.

Suppose that $\omega_3 > 0$. Then, since $(p(t)\, x''(t)^\alpha)' \sim \omega_3$, $t \to \infty$, integrating this relation on $[t_0,t]$, we obtain

$$x''(t) \sim \omega_3^{\frac{1}{\alpha}} \left(\frac{t}{p(t)}\right)^{\frac{1}{\alpha}}, \quad t \to \infty,$$

from which, integrating twice on $[t_0,t]$ we find that

$$x(t) \sim \omega_3^{\frac{1}{\alpha}} \int_{t_0}^{t}\int_{t_0}^{s} \left(\frac{r}{p(r)}\right)^{\frac{1}{\alpha}} drds, \quad t \to \infty,$$

i.e., $x(t) \sim \omega_3^{\frac{1}{\alpha}}\,\psi_4(t)$ as $t \to \infty$.

Suppose that $\omega_3 = 0$. Then, since $p(t)x''(t)^\alpha$ is positive and increasing, we have $\lim_{t\to\infty} p(t)x''(t)^\alpha = \omega_2 \in (0,\infty]$. If ω_2 is finite, then integrating the relation $x''(t) \sim (\omega_2/p(t))^{\frac{1}{\alpha}}$, $t \to \infty$ twice on $[t_0,t]$, we obtain

$$x(t) \sim \omega_2^{\frac{1}{\alpha}} \int_{t_0}^{t}\int_{t_0}^{s} \frac{1}{p(r)^{\frac{1}{\alpha}}} drds, \quad t \to \infty,$$

i.e., $x(t) \sim \omega_2^{\frac{1}{\alpha}}\,\psi_3(t)$, $t \to \infty$. On the other hand, if $\omega_2 = \infty$, we first integrate (E) on $[t,\infty)$ and then on $[t_0,t]$ to obtain

$$(4.4.4) \qquad x''(t) = \frac{1}{p(t)^{\frac{1}{\alpha}}}\left(c_2 + \int_{t_0}^{t}\int_{s}^{\infty} q(r)x(r)^\beta\, drds\right)^{\frac{1}{\alpha}}, \quad t \geq t_0,$$

where $c_2 = p(t_0)x''(t_0)^\alpha > 0$. Integrating the above twice on $[t_0,t]$ then yields

$$(4.4.5) \quad x(t) = c_0 + c_1(t-t_0) + \int_{t_0}^{t}\int_{t_0}^{s} \frac{1}{p(r)^{\frac{1}{\alpha}}}\left(c_2 + \int_{t_0}^{r}\int_{u}^{\infty} q(v)x(v)^\beta\, dvdu\right)^{\frac{1}{\alpha}} drds,$$

for $t \geq t_0$, where $c_1 = x'(t_0) > 0$ and $c_0 = x(t_0) > 0$. Since $\int_{t_0}^{t}\int_{s}^{\infty} q(r)x(r)^\beta drds = O(t)$ as $t \to \infty$, the condition (C_2) implies from (4.4.4) that $\lim_{t\to\infty} x'(t) = \infty$.

Using L' Hospital's rule, we easily see from (4.4.5) that $\lim_{t\to\infty} x(t)/\psi_3(t) = \infty$ and $\lim_{t\to\infty} x(t)/\psi_4(t) = 0$, or equivalently $\psi_3(t) \prec x(t) \prec \psi_4(t)$ as $t \to \infty$.

It follows from above observation that there are three types of possible asymptotic behavior for positive solutions $x(t)$ of (E) satisfying (4.4.2)

$$x(t) \sim k_3\psi_3(t), \quad \text{or} \quad \psi_3(t) \prec x(t) \prec \psi_4(t), \quad \text{or} \quad x(t) \sim k_4\psi_4(t), \quad \text{as} \quad t \to \infty,$$

where k_3 and k_4 are some positive constants.

Solutions satisfying (4.4.3). Let $x(t)$ satisfy (4.4.3) on $[t_0, \infty)$. It is necessary that $\omega_3 = 0$, so that we have

$$-\Big(p(t)(-x''(t))^\alpha\Big)' = \int_t^\infty q(s)x(s)^\beta ds, \quad t \ge t_0. \tag{4.4.6}$$

Moreover, since $p(t)(-x''(t))^\alpha$ and $x'(t)$ are positive and decreasing, there exist finite limits $\lim_{t\to\infty} p(t)(-x''(t))^\alpha = \omega_2 \ge 0$ and $\lim_{t\to\infty} x'(t) = \omega_1 \ge 0$. In fact, it must be $\omega_2 = 0$, because otherwise, integration of the relation $x''(t) \sim (-\omega_2/p(t))^{\frac{1}{\alpha}}$, $t \to \infty$ leads to $x'(t) \sim -\omega_2^{\frac{1}{\alpha}} \int_{t_0}^t ds/p(s)^{\frac{1}{\alpha}}$, $t \to \infty$. Thus, we conclude with the help of (C_2) that $\lim_{t\to\infty} x'(t) = -\infty$, an impossibility. Using this fact and integrating (4.4.6) twice on $[t, \infty)$, we obtain

$$x'(t) = \omega_1 + \int_t^\infty \left(\frac{1}{p(s)} \int_s^\infty (r-s)q(r)x(r)^\beta \, dr\right)^{\frac{1}{\alpha}} ds, \quad t \ge t_0,$$

which, integrated on $[t_0, t]$, gives

$$x(t) = c_0 + \omega_1(t - t_0) + \int_{t_0}^t \int_s^\infty \left(\frac{1}{p(r)} \int_r^\infty (u-r)q(u)x(u)^\beta \, du\right)^{\frac{1}{\alpha}} dr ds, \quad t \ge t_0,$$

where $c_0 = x(t_0) > 0$. It follows that if $\omega_1 > 0$, then $x(t) \sim \omega_1 \psi_2(t)$, $t \to \infty$ and that if $\omega_1 = 0$, there are two possibilities: either $x(t)$ tends to a finite limit or $x(t)$ grows to infinity as $t \to \infty$. In the latter case it is clear that $\psi_1(t) \prec x(t) \prec \psi_2(t)$ as $t \to \infty$.

Thus it follows that the asymptotic behavior of positive solutions $x(t)$ of (E) satisfying (4.4.3) falls into one of the following three cases:

$$x(t) \sim k_1\psi_1(t), \quad \text{or} \quad \psi_1(t) \prec x(t) \prec \psi_2(t), \quad \text{or} \quad x(t) \sim k_2\psi_2(t), \quad \text{as} \quad t \to \infty,$$

where k_1 and k_2 are some positive constants.

Positive solutions $x(t)$ of (E) having the asymptotic behavior

$$x(t) \sim k_1\psi_1(t), \quad x(t) \sim k_2\psi_2(t), \quad x(t) \sim k_3\,\psi_3(t), \quad x(t) \sim k_4\,\psi_4(t), \quad \text{as} \quad t \to \infty,$$

for some positive constants $k_i,\ i = 1, 2, 3, 4$, are collectively called *primitive positive solutions* of equation (E), while the solutions which are not primitive are referred to as *intermediate solutions* of equation (E). It is convenient to divide the set of intermediate solutions into the following two types

$$(\mathrm{I}_3) \qquad \psi_1(t) \prec x(t) \prec \psi_2(t), \quad t \to \infty,$$

$$(\mathrm{I}_4) \qquad \psi_3(t) \prec x(t) \prec \psi_4(t), \quad t \to \infty.$$

As regards the primitive solutions of equation (E), the existence of four types of such solutions has been completely characterized for both sublinear and superlinear case of (E) with continuous coefficients $p(t)$ and $q(t)$ as the following theorems proven in [64] and [73] show. For primitive solutions of type $x(t) \sim k_1\psi_1(t),\ t \to \infty$ and $x(t) \sim k_4\psi_4(t),\ t \to \infty$ which in view of relation (4.4.1) are *minimal* and *maximal* solutions of (E) respectively, necessary and sufficient condition are given in Theorem 4.1.1 and Theorem 4.1.2, respectively. The other two types of primitive solutions of (E) exist only under the additional assumption that the integral $\int_a^\infty 1/p(t)^{\frac{1}{\alpha}}\,dt$ is divergent i.e. under the condition (C_2) .

Theorem 4.4.1 *Let $p(t), q(t) \in C[a,\infty)$ and (C_2) holds. Equation (E) has a positive solution $x(t)$ satisfying $x(t) \sim k_2\psi_2(t),\ t \to \infty$ if and only if*

$$\int_a^\infty \left(\frac{1}{p(t)}\int_t^\infty (s-t)\, s^\beta\, q(s)\, ds\right)^{\frac{1}{\alpha}} dt < \infty. \tag{4.4.7}$$

Theorem 4.4.2 *Let $p(t), q(t) \in C[a,\infty)$ and (C_2) holds. Equation (E) has a positive solution $x(t)$ satisfying $x(t) \sim k_3\psi_3(t),\ t \to \infty$ if and only if*

$$\int_a^\infty t\, q(t)\, \psi_3(t)^\beta\, dt < \infty. \tag{4.4.8}$$

4.5 Existence of positive intermediate solutions of (E) under the condition (C_2)

In this section we prove the existence of solutions of type (I_3) and (I_4) of equation (E) under assumption that coefficients $p(t)$ and $q(t)$ are positive continuous functions and that (C_2) holds.

Theorem 4.5.1 *Let $p(t),\ q(t) \in C[a,\infty)$ and (C_2) holds. If (4.4.7) holds and if*

$$\int_a^\infty t\left(\frac{1}{p(t)}\int_t^\infty (s-t)\, q(s)\, ds\right)^{\frac{1}{\alpha}} dt = \infty,$$

then equation (E) has a positive solution $x(t)$ such that $1 \prec x(t) \prec t,\ t \to \infty$.

Proof. Choose $t_0 \geq \max\{1, a\}$ such that

$$2^{\frac{\beta}{\alpha}} \int_{t_0}^{\infty} \left(\frac{1}{p(t)} \int_t^{\infty} (s-t)\, s^{\beta} q(s)\, ds \right)^{\frac{1}{\alpha}} dt \leq 1. \tag{4.5.1}$$

Define the set

$$\mathcal{X}_1 = \{x \in C[t_0, \infty) : 1 \leq x(t) \leq 2t,\ t \geq t_0\}, \tag{4.5.2}$$

and the operator $\mathcal{G} : \mathcal{X}_1 \to C[t_0, \infty)$

$$\mathcal{G}x(t) := 1 + \int_{t_0}^{t} \int_s^{\infty} \left(\frac{1}{p(r)} \int_r^{\infty} (u-r)\, q(u)\, x(u)^{\beta}\, du \right)^{\frac{1}{\alpha}} dr ds, \quad t \geq t_0. \tag{4.5.3}$$

It is clear that $\mathcal{X}_1$ is a closed convex subset of the locally convex space $C[t_0, \infty)$ equipped with the topology of uniform convergence on compact subintervals of $[t_0, \infty)$. Using (4.5.1)– (4.5.3), we see that $x \in \mathcal{X}_1$ implies

$$\begin{aligned} 1 \leq \mathcal{G}x(t) &\leq 1 + 2^{\frac{\beta}{\alpha}} \int_{t_0}^{t} \int_{t_0}^{\infty} \left(\frac{1}{p(r)} \int_r^{\infty} (u-r)\, q(u)\, u^{\beta}\, du \right)^{\frac{1}{\alpha}} dr\, ds \\ &\leq 1 + t \leq 2t, \quad t \geq t_0. \end{aligned}$$

This means that $\mathcal{G}$ maps $\mathcal{X}_1$ into itself. Furthermore, it can be shown that $\mathcal{G}$ is a continuous map such that $\mathcal{G}(\mathcal{X}_1)$ is relatively compact in $C[t_0, \infty)$. Therefore, by the Schauder-Tychonoff fixed point theorem there exists a function $x_1 \in \mathcal{X}_1$ satisfying the integral equation $x_1(t) = \mathcal{G}x_1(t)$ for $t \geq t_0$. It follows that $x_1(t)$ is a solution of (E) on $[t_0, \infty)$. It is easy to see that $x_1(t)$ has the following asymptotic properties:

$$\lim_{t\to\infty} x_1(t) \geq \lim_{t\to\infty} \int_{t_0}^{t} \int_s^{\infty} \left(\frac{1}{p(r)} \int_r^{\infty} (u-r)\, q(u)\, du \right)^{\frac{1}{\alpha}} dr\, ds = \infty$$

and

$$\begin{aligned} 0 \leq \lim_{t\to\infty} \frac{x_1(t)}{t} &= \lim_{t\to\infty} \int_t^{\infty} \left(\frac{1}{p(s)} \int_s^{\infty} (r-s)\, q(r)\, x_1(r)^{\beta}\, dr \right)^{\frac{1}{\alpha}} ds \\ &\leq 2^{\frac{\beta}{\alpha}} \lim_{t\to\infty} \int_t^{\infty} \left(\frac{1}{p(s)} \int_s^{\infty} (r-s) q(r)\, r^{\beta}\, dr \right)^{\frac{1}{\alpha}} ds = 0, \end{aligned}$$

which means that $x_1(t)$ satisfies $1 \prec x_1(t) \prec t,\ t \to \infty$, that is, $x_1(t)$ is an intermediate solution of type (I_3) of (E). $\square$

Theorem 4.5.2 *Let $p(t)$, $q(t) \in C[a,\infty)$ and (C_2) holds. If (4.1.10) holds and if*

$$\int_a^\infty t\,q(t)\,\psi_3(t)^\beta\,dt = \infty,$$

then equation (E) has a positive solution $x(t)$ such that $\psi_3(t) \prec x(t) \prec \psi_4(t), t \to \infty$.

Proof. Choose $t_0 \geq \max\{1, a\}$ such that

$$2^{\frac{\beta}{\alpha}} \int_{t_0}^\infty q(t)\,\psi_4(t)^\beta\,dt \leq 1. \tag{4.5.4}$$

Define the set

$$\mathcal{X}_2 = \{x \in C[t_0,\infty) : \psi_3(t) \leq x(t) \leq 2^{\frac{1}{\alpha}}\psi_4(t),\ t \geq t_0\}, \tag{4.5.5}$$

and the integral operator $\mathcal{H} : \mathcal{X}_2 \to C[t_0,\infty)$

$$\mathcal{H}x(t) := \int_{t_0}^t (t-s)\left[\frac{1}{p(s)}\left(1 + \int_{t_0}^s\int_r^\infty q(u)\,x(u)^\beta\,du\,dr\right)\right]^{\frac{1}{\alpha}} ds, \quad t \geq t_0. \tag{4.5.6}$$

It is clear that $\mathcal{X}_2$ is a closed convex subset of the locally convex space $C[t_0,\infty)$ equipped with the topology of uniform convergence on compact subintervals of $[t_0,\infty)$. Using (4.5.4)–(4.5.6), we see that $x \in \mathcal{X}_2$ implies

$$\begin{aligned}
\psi_3(t) \leq \mathcal{H}x(t) &\leq \int_{t_0}^t (t-s)\left[\frac{1}{p(s)}\left(1 + 2^{\frac{\beta}{\alpha}}\int_{t_0}^s\int_{t_0}^\infty q(u)\,\psi_4(u)^\beta\,du\,dr\right)\right]^{\frac{1}{\alpha}} ds \\
&\leq \int_{t_0}^t (t-s)\left(\frac{1+s}{p(s)}\right)^{\frac{1}{\alpha}} ds \leq 2^{\frac{1}{\alpha}}\psi_4(t), \quad t \geq t_0.
\end{aligned}$$

This means that $\mathcal{H}$ maps $\mathcal{X}_2$ into itself. Furthermore, it can be shown that $\mathcal{H}$ is a continuous map such that $\mathcal{H}(\mathcal{X}_2)$ is relatively compact in $C[t_0,\infty)$. Therefore, by the Schauder-Tychonoff fixed point theorem there exists a function $x_2 \in \mathcal{X}_2$ satisfying the integral equation $x_2(t) = \mathcal{H}x_2(t)$ for $t \geq t_0$. It follows that $x_2(t)$ is a solution of (E) on $[t_0,\infty)$. It is easy to see that $x_2(t)$ has the following asymptotic properties:

$$\begin{aligned}
\lim_{t\to\infty}\frac{x_2(t)}{\psi_3(t)} &= \lim_{t\to\infty}\left(1 + \int_{t_0}^t\int_s^\infty q(r)\,x_2(r)^\beta\,dr\,ds\right)^{\frac{1}{\alpha}} \\
&\geq \lim_{t\to\infty}\left(\int_{t_0}^t\int_s^\infty q(r)\,\psi_3(r)^\beta\,dr\,ds\right)^{\frac{1}{\alpha}} = \infty
\end{aligned}$$

and

$$\begin{aligned} 0 \;\leq\; \lim_{t\to\infty} \frac{x_2(t)}{\psi_4(t)} &= \left(\lim_{t\to\infty} \frac{1+\int_{t_0}^{t}\int_{s}^{\infty} q(r)\,x_2(r)^{\beta}\,dr\,ds}{t} \right)^{\frac{1}{\alpha}} \\ &= \left(\lim_{t\to\infty} \int_{t}^{\infty} q(s)\,x_2(s)^{\beta}\,ds \right)^{\frac{1}{\alpha}} \leq \left(2^{\frac{\beta}{\alpha}} \lim_{t\to\infty} \int_{t}^{\infty} q(s)\,\psi_4(s)^{\beta}\,ds \right)^{\frac{1}{\alpha}} = 0, \end{aligned}$$

which means that $x_2(t)$ satisfies $\psi_3(t) \prec x_2(t) \prec \psi_4(t),\ t \to \infty$, that is, $x_2(t)$ is an intermediate solution of type (I_4) of (E). $\square$

4.6 Asymptotic behavior of intermediate solutions of (E) under the condition (C_2)

In this section we assume that functions $p(t)$ and $q(t)$ are generalized regularly varying of index η and σ with respect to $R(t)$, which is defined with (4.3.1) and expressed with (4.3.2) and the intermediate solutions $x(t) \in \mathrm{RV}_R(\rho)$ of (E) are represented as (4.3.3)

First, we express the condition (C_2) in the terms of regular variation. Using (4.3.2), (4.3.5) and (4.3.7) we have

$$\int_a^t \frac{ds}{p(s)^{\frac{1}{\alpha}}} \sim \left(\frac{\alpha+\eta}{\alpha+1}\right)^{\frac{1}{\alpha+1}} \int_a^t R'(s)\,R(s)^{-\frac{\alpha+\eta}{\alpha(\alpha+1)}}\,l_p(s)^{-\frac{1}{\alpha(\alpha+1)}}\,ds, \quad t\to\infty.$$

For condition (C_2) to hold it is necessary that $\alpha^2-\eta \geq 0$. In what follows we limit ourselves to the case where

$$\alpha^2 - \eta > 0 \tag{4.6.1}$$

excluding the possibility $\alpha^2-\eta=0$ because of computational difficulty. Under the condition (C_2) introducing the notation (4.3.9) we have

$$0 < m_2(\alpha,\eta) < m_1(\alpha,\eta) < m_3(\alpha,\eta) = m_2(\alpha,\eta)+1.$$

In proofs of our main results constants $m_i(\alpha,\eta),\ i=1,2,3$ will be abbreviated to m_i.

In order to make an in depth analysis of intermediate solutions of type (I_3) and (I_4) of (E) we need a fair knowledge of the structure of the functions $\psi_1(t)$, $\psi_2(t)$, $\psi_3(t)$ and $\psi_4(t)$ regarded as generalized regularly varying functions. It is clear that

$\psi_1(t) \in \mathrm{SV}_R$. From (4.3.5) it follows that $\psi_2(t) \in \mathrm{RV}_R\,(m_2(\alpha,\eta))$. Using (4.3.2) and applying Lemma 4.3.1 twice, we get

$$\begin{aligned}\psi_3(t) &\sim \int_a^t\!\!\int_a^s R(r)^{-\frac{\eta}{\alpha}}\, l_p(r)^{-\frac{1}{\alpha}}\, dr ds \\ &\sim \frac{m_2(\alpha,\eta)^{\frac{2}{\alpha+1}}}{m_1(\alpha,\eta)(m_1(\alpha,\eta)-m_2(\alpha,\eta))}\, R(t)^{m_1(\alpha,\eta)}\, l_p(t)^{\frac{\alpha-1}{\alpha(\alpha+1)}},\ t\to\infty, \qquad (4.6.2)\end{aligned}$$

which shows that $\psi_3(t) \in \mathrm{RV}_R\,(m_1(\alpha,\eta))$. Further, another application of Lemma 4.3.1 yields

$$\psi_4(t) \sim \int_a^t R(s)\, ds \sim \frac{m_2(\alpha,\eta)^{\frac{1}{\alpha+1}}}{m_3(\alpha,\eta)}\, R(t)^{m_3(\alpha,\eta)}\, l_p(t)^{\frac{1}{\alpha+1}},\ t\to\infty, \qquad (4.6.3)$$

implying $\psi_4(t) \in \mathrm{RV}_R\,(m_3(\alpha,\eta))$.

4.6.1 Intermediate regularly varying solutions of type (I_3)

The first subsection is devoted to the study of the existence and asymptotic behavior of generalized regularly varying solutions of type (I_3) of equation (E) with $p(t)$ and $q(t)$ satisfying (4.3.2).

Let $x(t)$ be a solution of (E) on $[t_0,\infty)$ such that $1 \prec x(t) \prec t$ as $t\to\infty$. Integration of equation (E) first three times on $[t,\infty)$ and then once on $[t_0,t]$ gives

$$x(t) = x(t_0) + \int_{t_0}^t\!\!\int_s^\infty \left(\frac{1}{p(r)}\int_r^\infty (u-r)q(u)x(u)^\beta\, du\right)^{\frac{1}{\alpha}} dr ds,\quad t\ge t_0, \qquad (4.6.4)$$

and implies the integral asymptotic relation

$$x(t) \sim \int_b^t\!\!\int_s^\infty \left(\frac{1}{p(r)}\int_r^\infty (u-r)q(u)x(u)^\beta\, du\right)^{\frac{1}{\alpha}} dr\, ds,\quad t\to\infty, \qquad (4.6.5)$$

for any $b \ge a$. This type of asymptotic relation will play a central role in constructing the intermediate solutions of type (I_3) of (E) by solving the integral equation (4.6.4) for some positive constants t_0 and $x(t_0)$. Therefore, first we show that the generalized regularly varying functions $X_i(t)$, $i=1,2,3$ defined respectively by

$$X_1(t) = \left(\frac{\alpha-\beta}{\alpha}\int_a^t s\left(\frac{1}{p(s)}\int_s^\infty (r-s)\,q(r)\,dr\right)^{\frac{1}{\alpha}} ds\right)^{\frac{\alpha}{\alpha-\beta}}, \qquad (4.6.6)$$

(4.6.7)

$$X_2(t) = \left(\left(\frac{m_2(\alpha,\eta)}{\alpha}\right)^2 \frac{p(t)\,q(t)\,R(t)^{2\alpha}}{\rho^\alpha\ (m_2(\alpha,\eta)-\rho)^\alpha\ (m_1(\alpha,\eta)-\rho)\ (m_3(\alpha,\eta)-\rho)}\right)^{\frac{1}{\alpha-\beta}},$$

$$(4.6.8) \qquad X_3(t) = t\left(\frac{\alpha-\beta}{\alpha}\int_t^\infty \left(\frac{1}{p(s)}\int_s^\infty (r-s)r^\beta\, q(r)\, dr\right)^{\frac{1}{\alpha}} ds\right)^{\frac{\alpha}{\alpha-\beta}},$$

satisfy the integral asymptotic relation (4.6.5).

Lemma 4.6.1 *Suppose that*

$$(4.6.9) \qquad \sigma = -2\alpha-\eta \quad \text{and} \quad \int_a^\infty t\left(\frac{1}{p(t)}\int_t^\infty (s-t)\, q(s)\, ds\right)^{\frac{1}{\alpha}} dt = \infty.$$

holds. The function $X_1(t) \in \mathrm{ntr}-\mathrm{SV}_R$ *given by* (4.6.6) *satisfies the asymptotic relation* (4.6.5) *for any* $b \geq a$.

Proof. The proof is the same as the proof of Lemma 4.3.2 □

Lemma 4.6.2 *Suppose that*

$$(4.6.10) \qquad -2\alpha-\eta < \sigma < -\alpha-(\beta+1)m_2(\alpha,\eta)$$

holds and let ρ *be defined by* (4.3.20). *The function* $X_2(t) \in \mathrm{RV}(\rho)$ *given by* (4.6.7) *satisfies the asymptotic relation* (4.6.5) *for any* $b \geq a$.

Proof. The proof is the same as the proof of Lemma 4.3.3. □

Lemma 4.6.3 *Suppose that*

$$(4.6.11) \quad \sigma = -\alpha-(\beta+1)m_2(\alpha,\eta) \text{ and } \int_a^\infty \left(\frac{1}{p(t)}\int_t^\infty (s-t)\, s^\beta\, q(s)\, ds\right)^{\frac{1}{\alpha}} dt < \infty$$

holds. The function $X_3(t) \in \mathrm{ntr}-\mathrm{RV}_R\,(m_2(\alpha,\eta))$ *given by* (4.6.8) *satisfies the asymptotic relation* (4.6.5) *for any* $b \geq a$.

Proof. Let (4.6.11) hold. Using (4.3.2) and (4.3.5) and applying Lemma 4.3.1 we see that

$$\int_t^\infty s^\beta q(s)\, ds \sim m_2^{\frac{-\alpha\beta}{\alpha+1}}\int_t^\infty R(s)^{\sigma+\beta m_2}\, l_p(s)^{\frac{\beta}{\alpha+1}}\, l_q(s)\, ds$$
$$\sim \frac{m_2^{\frac{1-\alpha\beta}{\alpha+1}}}{-(\sigma+(\beta+1)m_2)} R(t)^{\sigma+(\beta+1)m_2} l_p(t)^{\frac{\beta+1}{\alpha+1}} l_q(t) = \frac{m_2^{\frac{1-\alpha\beta}{\alpha+1}}}{\alpha}\, R(t)^{-\alpha}\, l_p(t)^{\frac{\beta+1}{\alpha+1}}\, l_q(t),$$

from which it follows that

$$\left(\frac{1}{p(t)}\int_t^\infty\int_s^\infty r^\beta\, q(r)\, dr ds\right)^{\frac{1}{\alpha}} \sim \frac{m_2^{\frac{2-\alpha\beta}{\alpha(\alpha+1)}}}{(\alpha^2(m_1-m_2))^{\frac{1}{\alpha}}} R(t)^{m_2-m_1-\frac{\eta}{\alpha}}\, l_p(t)^{\frac{-\alpha+\beta+1}{\alpha(\alpha+1)}}\, l_q(t)^{\frac{1}{\alpha}}$$
$$\sim \frac{m_2^{\frac{2-\alpha\beta+\alpha}{\alpha(\alpha+1)}}}{(\alpha^2(m_1-m_2))^{\frac{1}{\alpha}}} R'(t)\, R(t)^{-1}\, l_p(t)^{\frac{\beta+1}{\alpha(\alpha+1)}}\, l_q(t)^{\frac{1}{\alpha}},$$

as $t \to \infty$, where we use (4.3.7) in the last step. Integrating the above on $[t, \infty)$ we obtain

$$\int_t^\infty \left(\frac{1}{p(s)} \int_s^\infty \int_r^\infty u^\beta q(u) du dr \right)^{\frac{1}{\alpha}} ds \tag{4.6.12}$$

$$\sim \frac{m_2^{\frac{2-\alpha\beta+\alpha}{\alpha(\alpha+1)}}}{(\alpha^2(m_1 - m_2))^{\frac{1}{\alpha}}} \int_t^\infty R'(s) R(s)^{-1} l_p(s)^{\frac{\beta+1}{\alpha(\alpha+1)}} l_q(s)^{\frac{1}{\alpha}} ds, \quad t \to \infty.$$

This, combined with (4.3.5) and (4.6.8), gives the following expression for $X_3(t)$:

$$X_3(t) \sim \left(\frac{\alpha - \beta}{\alpha} \right)^{\frac{\alpha}{\alpha-\beta}} \left(\frac{m_2^{2-\alpha}}{\alpha^2(m_1 - m_2)} \right)^{\frac{1}{\alpha-\beta}} R(t)^{m_2} l_p(t)^{\frac{1}{\alpha+1}}$$

$$\times \left(\int_t^\infty R'(s)\, R(s)^{-1}\, l_p(s)^{\frac{\beta+1}{\alpha(\alpha+1)}}\, l_q(s)^{\frac{1}{\alpha}}\, ds \right)^{\frac{\alpha}{\alpha-\beta}} \in \mathrm{RV}_R\,(m_2)\,,\ t \to \infty.$$

Next, we integrate $q(t)\, X_3(t)^\beta$ twice on $[t, \infty)$, multiply by $1/p(t)$ and raise the result to the exponent $1/\alpha$. Since $q(t) X_3(t)^\beta \in \mathrm{RV}_R(\sigma + m_2\beta) = \mathrm{RV}_R(-\alpha - m_2)$ (cf.(4.6.11)), repeated application of Lemma 4.3.1, with the help of (4.3.7), yields

$$\left(\frac{1}{p(t)} \int_t^\infty \int_s^\infty q(r)\, X_3(r)^\beta\, dr\, ds \right)^{\frac{1}{\alpha}} \sim \left(\frac{\alpha - \beta}{\alpha} \right)^{\frac{\beta}{\alpha-\beta}} \left(\frac{m_2^{\frac{2-\alpha\beta+\alpha}{\alpha+1}}}{\alpha^2(m_1 - m_2)} \right)^{\frac{1}{\alpha-\beta}}$$

$$\times R'(t)\, R(t)^{-1}\, l_p(t)^{\frac{\beta+1}{\alpha(\alpha+1)}} l_q(t)^{\frac{1}{\alpha}} \left(\int_t^\infty R'(s)\, R(s)^{-1}\, l_p(s)^{\frac{\beta+1}{\alpha(\alpha+1)}}\, l_q(s)^{\frac{1}{\alpha}}\, ds \right)^{\frac{\beta}{\alpha-\beta}},$$

as $t \to \infty$. Integrating the above relation first on $[t, \infty)$ and then on $[b, t]$ for any fixed $b \geq a$, we conclude via Lemma 4.3.1 that

$$\int_b^t \int_s^\infty \left(\frac{1}{p(r)} \int_r^\infty (u - r)\, q(u)\, X_3(u)^\beta du \right)^{\frac{1}{\alpha}} dr ds \sim \left(\frac{\alpha - \beta}{\alpha} \right)^{\frac{\alpha}{\alpha-\beta}} \left(\frac{m_2^{2-\alpha}}{\alpha^2(m_1 - m_2)} \right)^{\frac{1}{\alpha-\beta}}$$

$$\times R(t)^{m_2} l_p(t)^{\frac{1}{\alpha+1}} \left(\int_t^\infty R'(s) R(s)^{-1} l_p(s)^{\frac{\beta+1}{\alpha(\alpha+1)}} l_q(s)^{\frac{1}{\alpha}} ds \right)^{\frac{\alpha}{\alpha-\beta}} = X_3(t), \quad t \to \infty.$$

This completes the proof of Lemma 4.6.3. □

Since $\psi_1(t) \prec x(t) \prec \psi_2(t),\ t \to \infty$, the regularity index ρ of $x(t)$ must satisfy

$$0 \leq \rho \leq m_2(\alpha, \eta).$$

If $\rho = 0$, then since $x(t) = l_x(t) \to \infty,\ t \to \infty$, $x(t)$ is a member of $\mathrm{ntr} - \mathrm{SV}_R$, while if $\rho = m_2(\alpha, \eta)$, then since $x(t)/R(t)^{m_2(\alpha,\eta)} = l_x(t) \to 0,\ t \to \infty$, $x(t)$ is a member of $\mathrm{ntr}-\mathrm{RV}_R(m_2(\alpha,\eta))$. If $0<\rho<m_2(\alpha,\eta)$, then $x(t)$ is a member of $\mathrm{RV}_R(\rho)$ and satisfies $x(t)\to\infty$ and $x(t)/R(t)^{m_2(\alpha,\eta)}\to 0$ as $t\to\infty$. Thus the set of all generalized regularly varying solutions of type (I_3) is naturally divided into the three disjoint classes

$$\mathrm{ntr}-\mathrm{SV}_R \quad \text{or} \quad \mathrm{RV}_R(\rho) \ \text{ with } \ \rho\in(0\,,\,m_2(\alpha,\eta)) \quad \text{or} \quad \mathrm{ntr}-\mathrm{RV}_R\,(m_2(\alpha,\eta))\,.$$

Theorem 4.6.1 *Let $p(t)\in\mathrm{RV}_R(\eta)$, $q(t)\in\mathrm{RV}_R(\sigma)$ and (C_2) hold. Equation* (E) *has intermediate solutions $x(t)\in\mathrm{ntr}-\mathrm{SV}_R$ satisfying* (I_3) *if and only if* (4.6.9) *holds. The asymptotic behavior of any such solution $x(t)$ is governed by the unique formula $x(t)\sim X_1(t)$, $t\to\infty$, where the function $X_1(t)$ is given by* (4.6.6).

Theorem 4.6.2 *Let $p(t)\in\mathrm{RV}_R(\eta)$, $q(t)\in\mathrm{RV}_R(\sigma)$ and (C_2) hold. Equation* (E) *has intermediate solutions $x(t)\in\mathrm{RV}_R(\rho)$ with $\rho\in(0\,,\,m_2(\alpha,\eta))$ if and only if* (4.6.10) *holds, in which case ρ is given by* (4.3.20) *and the asymptotic behavior of any such solution $x(t)$ is governed by the unique formula $x(t)\sim X_2(t)$, $t\to\infty$, where the function $X_2(t)$ is given by* (4.6.7).

Theorem 4.6.3 *Let $p(t)\in\mathrm{RV}_R(\eta)$, $q(t)\in\mathrm{RV}_R(\sigma)$ and (C_2) hold. Equation* (E) *has intermediate solutions $x(t)\in\mathrm{ntr}-\mathrm{RV}_R\,(m_2(\alpha,\eta))$ satisfying* (I_3) *if and only if* (4.6.11) *holds. The asymptotic behavior of any such solution $x(t)$ is governed by the unique formula $x(t)\sim X_3(t)$, $t\to\infty$, where the function $X_3(t)$ is given by* (4.6.8).

Proof of the "only if" part of Theorems 4.6.1, 4.6.2 and 4.6.3: Suppose that (E) has a type-(I_3) intermediate solution $x(t)\in\mathrm{RV}_R(\rho)$ on $[t_0,\infty)$ with $\rho\in[0,m_2]$. From

$$-\left(p(t)(-x''(t))^{\alpha}\right)'=\int_t^\infty q(s)x(s)^\beta\,ds\sim\int_t^\infty R(s)^{\sigma+\beta\rho}l_q(s)l_x(s)^\beta\,ds, \tag{4.6.13}$$

as $t\to\infty$, the convergence of the last integral in (4.6.13) means that $\sigma+\beta\rho+m_2\le 0$. But the possibility $\sigma+\beta\rho+m_2=0$ is precluded, because if this were the case the last integral in (4.6.13) would be an SV_R- function, which is not integrable on $[t_0,\infty)$ by (i) of Lemma 4.3.1. This would contradict the fact that the left-hand side of (4.6.13) is integrable on $[t_0,\infty)$. It follows that $\sigma+\beta\rho+m_2<0$. Then, integration of (4.6.13) on $[t,\infty)$ with application of Lemma 4.3.1 gives

$$p(t)(-x''(t))^{\alpha}\sim\frac{m_2^{\frac{1}{\alpha+1}}}{-(\sigma+\beta\rho+m_2)}\int_t^\infty R(s)^{\sigma+\beta\rho+m_2}\,l_p(s)^{\frac{1}{\alpha+1}}\,l_q(s)\,l_x(s)^\beta\,ds, \tag{4.6.14}$$

as $t \to \infty$. Noting that the integral in (4.6.14) is convergent, we conclude that $\sigma + \beta\rho + 2m_2 \leq 0$. But the equality is not allowed here. In fact, if the equality holds, then the right- hand side of (4.6.14) is SV_R-function denoted by $h(t)$ so that

$$-x''(t) \sim \left(\frac{h(t)}{p(t)}\right)^{\frac{1}{\alpha}} = R(t)^{-\frac{\eta}{\alpha}} l_p(t)^{-\frac{1}{\alpha}} h(t)^{\frac{1}{\alpha}}, \quad t \to \infty.$$

But then, the integrability of $x''(t)$ on $[t_0, \infty)$ implies that $m_2 - \frac{\eta}{\alpha} = \frac{\alpha^2-\eta}{\alpha(\alpha+1)} \leq 0$, which contradicts the assumption (4.6.1). Thus it holds $\sigma+\beta\rho+2m_2 < 0$. Applying Lemma 4.3.1 in (4.6.14) first and then multiplying by $1/p(t)$ and raising the result on $1/\alpha$, using (4.3.7) we obtain

$$(4.6.15) \qquad \begin{aligned} -x''(t) &\sim \frac{m_2^{\frac{2}{\alpha(\alpha+1)}}}{((\sigma+\beta\rho+m_2)(\sigma+\beta\rho+2m_2))^{\frac{1}{\alpha}}} \\ &\times R(t)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}} l_p(t)^{\frac{1-\alpha}{\alpha(\alpha+1)}} l_q(t)^{\frac{1}{\alpha}} l_x(t)^{\frac{\beta}{\alpha}}, \quad t \to \infty. \end{aligned}$$

The integrability of $x''(t)$ on $[t_0, \infty)$ implies that $\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha} + m_2 \leq 0$. We distinguish the two cases:

$$(a) \quad \frac{\sigma+\beta\rho+2m_2-\eta}{\alpha} + m_2 = 0 \quad (b) \quad \frac{\sigma+\beta\rho+2m_2-\eta}{\alpha} + m_2 < 0.$$

Assume that (a) holds. Since $\sigma+\beta\rho+m_2 = -\alpha$ and $\sigma+\beta\rho+2m_2 = \alpha(m_2-m_1)$, integration of (4.6.15) first on $[t, \infty)$, then on $[t_0, t]$, with application of Lemma 4.3.1, shows that

$$(4.6.16) \qquad \begin{aligned} x(t) &\sim \left(\frac{m_2^{2-\alpha}}{\alpha^2(m_1-m_2)}\right)^{\frac{1}{\alpha}} R(t)^{m_2} l_p(t)^{\frac{1}{\alpha+1}} \int_t^\infty R'(s)\, R(s)^{-1} l_p(s)^{\frac{1}{\alpha(\alpha+1)}} l_q(s)^{\frac{1}{\alpha}} l_x(s)^{\frac{\beta}{\alpha}}\, ds \\ &\sim t \left(\frac{m_1^{\frac{\alpha+2}{\alpha+1}}}{\alpha^2(m_1-m_2)}\right)^{\frac{1}{\alpha}} \int_t^\infty R'(s) R(s)^{-1} l_p(s)^{\frac{1}{\alpha(\alpha+1)}} l_q(s)^{\frac{1}{\alpha}} l_x(s)^{\frac{\beta}{\alpha}} ds \in \mathrm{RV}_R(m_2), \end{aligned}$$

as $t \to \infty$.

Assume next that (b) holds. Integrating (4.6.15) on $[t, \infty)$, then on $[t_0, t]$, we find via Lemma 4.3.1 that

$$(4.6.17) \qquad \begin{aligned} x(t) &\sim \left(\frac{m_2^{\frac{\alpha+2}{\alpha+1}}}{(\sigma+\beta\rho+m_2)\,(\sigma+\beta\rho+2m_2)}\right)^{\frac{1}{\alpha}} \frac{\alpha}{-(\sigma+\beta\rho+(\alpha+2)m_2-\eta)} \\ &\times \int_{t_0}^t R(s)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2} l_p(s)^{\frac{1}{\alpha(\alpha+1)}} l_q(s)^{\frac{1}{\alpha}}\, l_x(s)^{\frac{\beta}{\alpha}} ds, \ t \to \infty. \end{aligned}$$

Because of the divergence of the last integral (note that $x(t) \to \infty, t \to \infty$), it follows that

$$\frac{\sigma + \beta\rho + 2m_2 - \eta}{\alpha} + 2m_2 = \frac{\sigma + \beta\rho + 2\alpha + \eta}{\alpha} \geq 0.$$

We distinguish the two cases:

$$(b.1) \quad \frac{\sigma + \beta\rho + 2\alpha + \eta}{\alpha} = 0 \quad \text{and} \quad (b.2) \quad \frac{\sigma + \beta\rho + 2\alpha + \eta}{\alpha} > 0.$$

Assume that $(b.1)$ holds. Then, (4.6.17) shows that $x(t) \in \mathrm{SV}_R$, that is, $\rho = 0$, and hence $\sigma = -2\alpha - \eta$. Since

$$\sigma + \beta\rho + m_2 = -\alpha m_3,\ \sigma + \beta\rho + 2m_2 = -\alpha m_1,\ \sigma + \beta\rho + (\alpha+2)m_2 - \eta = -\alpha m_2,$$

(4.6.17) reduce to

$$x(t) \sim \left(\frac{m_2^{2-\alpha}}{\alpha^2 m_1 m_3}\right)^{\frac{1}{\alpha}} \int_{t_0}^{t} R'(s)\, R(s)^{-1}\, l_p(s)^{\frac{1}{\alpha}} l_q(s)^{\frac{1}{\alpha}}\, l_x(s)^{\frac{\beta}{\alpha}}\, ds \in \mathrm{SV}_R, \tag{4.6.18}$$

as $t \to \infty$.

Assume that $(b.2)$ holds. Applying Lemma 4.3.1 to the integral in (4.6.17), we get

$$\begin{aligned} x(t) &\sim \left(\frac{m_2^2}{(\sigma + \beta\rho + m_2)\,(\sigma + \beta\rho + 2m_2)}\right)^{\frac{1}{\alpha}} \frac{\alpha}{-(\sigma + \beta\rho + (\alpha+2)m_2 - \eta)} \\ &\times \frac{\alpha}{\sigma + \beta\rho + 2\alpha + \eta}\, R(t)^{\frac{\sigma+\beta\rho+2\alpha+\eta}{\alpha}}\, l_p(t)^{\frac{1}{\alpha}}\, l_q(t)^{\frac{1}{\alpha}}\, l_x(t)^{\frac{\beta}{\alpha}}, \quad t \to \infty, \end{aligned} \tag{4.6.19}$$

which implies that $x(t) \in \mathrm{RV}_R(\frac{\sigma+\beta\rho+2\alpha+\eta}{\alpha})$.

Let us now suppose that $x(t)$ is an intermediate solution of type (I_3) of (E) belonging to $\mathrm{ntr}-\mathrm{SV}_R$. From the above observations this is possible only when the case $(b.1)$ holds, in which case $\rho = 0$, $\sigma = -2\alpha - \eta$ and $x(t) = l_x(t)$ must satisfy the asymptotic behavior (4.6.18) as $t \to \infty$. Put

$$\mu(t) = H \int_{t_0}^{t} R'(s)\, R(s)^{-1}\, l_p(s)^{\frac{1}{\alpha}} l_q(s)^{\frac{1}{\alpha}}\, l_x(s)^{\frac{\beta}{\alpha}}\, ds, \qquad H = \left(\frac{m_2^{2-\alpha}}{\alpha^2 m_1 m_3}\right)^{\frac{1}{\alpha}}.$$

Noting that

$$\mu'(t) = H\, R'(t)\, R(t)^{-1}\, l_p(t)^{\frac{1}{\alpha}}\, l_q(t)^{\frac{1}{\alpha}}\, l_x(t)^{\frac{\beta}{\alpha}} \sim H\, R'(t)\, R(t)^{-1}\, l_p(t)^{\frac{1}{\alpha}}\, l_q(t)^{\frac{1}{\alpha}}\, \mu(t)^{\frac{\beta}{\alpha}},$$

as $t \to \infty$, we obtain the differential asymptotic relation

$$\mu(t)^{-\frac{\beta}{\alpha}}\, \mu'(t) \sim H R'(t)\, R(t)^{-1} l_p(t)^{\frac{1}{\alpha}}\, l_q(t)^{\frac{1}{\alpha}},\ t \to \infty. \tag{4.6.20}$$

Since the left-hand side of (4.6.20) is not integrable on $[t_0,\infty)$ (note that $x(t)\to\infty$ as $t\to\infty$ and so $\mu(t)\to\infty$ as $t\to\infty$), so is the right-hand side, which in view of (4.3.18), means that

$$\int_a^\infty t\left(\frac{1}{p(t)}\int_t^\infty (s-t)\,q(s)\,ds\right)^{\frac{1}{\alpha}} dt = \infty.$$

We now integrate (4.6.20) from t_0 to t to obtain

$$x(t)\sim\mu(t)\sim\left(\frac{\alpha-\beta}{\alpha}\,H\int_{t_0}^t R'(s)\,R(s)^{-1}\,l_p(s)^{\frac{1}{\alpha}}l_q(s)^{\frac{1}{\alpha}}\,ds\right)^{\frac{\alpha}{\alpha-\beta}},\ t\to\infty,$$

which, in view of (4.3.18), is equivalent to

$$x(t)\sim\left(\frac{\alpha-\beta}{\alpha}\int_a^t s\left(\frac{1}{p(s)}\int_s^\infty (r-s)\,q(r)\,dr\right)^{\frac{1}{\alpha}} ds\right)^{\frac{\alpha}{\alpha-\beta}},\ t\to\infty.$$

Thus it has been shown that $x(t)\sim X_1(t)$, $t\to\infty$, where $X_1(t)$ is given by (4.6.6). This proves the "only if" part of Theorem 4.6.1.

Next, suppose that $x(t)$ is a solution of (E) belonging to $\mathrm{RV}_R(\rho), \rho\in(0,m_2)$. This is possible only when $(b.2)$ holds, in which case $x(t)$ must satisfy the asymptotic relation (4.6.19). Therefore,

$$\rho=\frac{\sigma+\beta\rho+2\alpha+\eta}{\alpha}\quad\Rightarrow\quad\rho=\frac{\sigma+2\alpha+\eta}{\alpha-\beta},$$

which justifies (4.3.20) and combined with $\rho\in(0,m_2)$ determines that the range of σ is

$$-2\alpha-\eta<\sigma<-\alpha-(\beta+1)m_2.$$

Since

$$\sigma+\beta\rho+m_2=\alpha(\rho-m_3),\qquad \sigma+\beta\rho+2m_2=\alpha(\rho-m_1),$$
$$\sigma+\beta\rho+(\alpha+2)m_2-\eta=\alpha(\rho-m_2),\qquad \sigma+\beta\rho+2\alpha+\eta=\alpha\rho,$$

we conclude from (4.6.19) that $x(t)$ enjoys the asymptotic behavior $x(t)\sim X_2(t)$, $t\to\infty$, where $X_2(t)$ is given by (4.6.7). This proves the "only if" part of the Theorem 4.6.2.

Finally, suppose that $x(t)$ is an intermediate solution of type (I_3) of (E) belonging to $\mathrm{ntr}-\mathrm{RV}_R(m_2)$. Then, the case (a) is the only possibility for $x(t)$, which means that $\sigma=-\alpha-(\beta+1)m_2$ and (4.6.16) is satisfied by $x(t)$. Using $x(t)=R(t)^{m_2}\,l_x(t)$, (4.6.16) can be expressed as

$$\text{(4.6.21)}\qquad l_x(t)\sim K\,l_p(t)^{\frac{1}{\alpha+1}}\int_t^\infty R'(s)\,R(s)^{-1}\,l_p(s)^{\frac{1}{\alpha(\alpha+1)}}\,l_q(s)^{\frac{1}{\alpha}}\,l_x(s)^{\frac{\beta}{\alpha}}\,ds,\ t\to\infty,$$

where $K = \big(m_2^{2-\alpha}/\alpha^2(m_1-m_2)\big)^{\frac{1}{\alpha}}$. Define $\nu(t)$ by

$$\nu(t) = \int_t^\infty R'(s)\,R(s)^{-1}\,l_p(s)^{\frac{1}{\alpha(\alpha+1)}}\,l_q(s)^{\frac{1}{\alpha}}\,l_x(s)^{\frac{\beta}{\alpha}}\,ds.$$

Then, noting that $l_x(t) \sim K\,l_p(t)^{\frac{1}{\alpha+1}}\,\nu(t),\ t\to\infty$, one can transform (4.6.21) into the following differential asymptotic relation for $\nu(t)$:

$$-\nu(t)^{-\frac{\beta}{\alpha}}\,\nu'(t) \sim K^{\frac{\beta}{\alpha}}\,R'(t)\,R(t)^{-1}\,l_p(t)^{\frac{\beta+1}{\alpha(\alpha+1)}}\,l_q(t)^{\frac{1}{\alpha}},\ \ t\to\infty. \tag{4.6.22}$$

From (4.6.16), since $\lim_{t\to\infty} x(t)/t = 0$, we have $\lim_{t\to\infty}\nu(t) = 0$, implying that the left-hand side of (4.6.22) is integrable over $[t_0,\infty)$, so is the right-hand side. This, in view of (4.6.12), implies the convergence of the integral

$$\int_a^\infty \left(\frac{1}{p(t)}\int_t^\infty (s-t)\,s^\beta\,q(s)\,ds\right)^{\frac{1}{\alpha}}\,dt.$$

Integrating (4.6.22) on $[t,\infty)$ and combining the result with (4.6.21), we find that

$$x(t) \sim K^{\frac{\alpha}{\alpha-\beta}}\,R(t)^{m_2}\,l_p(t)^{\frac{1}{\alpha+1}}\left(\frac{\alpha-\beta}{\alpha}\int_t^\infty R'(s)\,R(s)^{-1}l_p(s)^{\frac{\beta+1}{\alpha(\alpha+1)}}\,l_q(s)^{\frac{1}{\alpha}}\,ds\right)^{\frac{\alpha}{\alpha-\beta}},$$

as $t\to\infty$, which due to (4.6.12) gives $x(t)\sim X_3(t),\ t\to\infty$, where $X_3(t)$ is given by (4.6.8). This proves the "only if" part of the proof of Theorem 4.6.3. □

Proof of the "if" part of Theorems 4.6.1, 4.6.2 and 4.6.3 is the same as the proof of the "if" part of Theorems 4.3.1, 4.3.2 and 4.3.3. □

4.6.2 Intermediate regularly varying solutions of type (I_4)

Let us turn our attention to the study of intermediate solutions of type (I_4) of equation (E), that is, those solutions $x(t)$ such that $\psi_3(t) \prec x(t) \prec \psi_4(t)$ as $t\to\infty$. As in the preceding subsection use is made of the expressions (4.3.2) and (4.3.3) for the coefficients $p(t)$, $q(t)$ and the solutions $x(t)$.

Let $x(t)$ be an intermediate solution of type (I_4) of (E) defined on $[t_0,\infty)$. Integrating (E) first from t to ∞ and then three times on $[t_0,t]$, we obtain

$$\begin{aligned} x(t) &= c_0 + c_1(t-t_0) \\ &+ \int_{t_0}^t (t-s)\left(\frac{1}{p(s)}\left(c_2 + \int_{t_0}^s\int_r^\infty q(u)x(u)^\beta\,du dr\right)\right)^{\frac{1}{\alpha}} ds, \quad t\ge t_0, \end{aligned} \tag{4.6.23}$$

where $c_0 = x(t_0)$, $c_1 = x'(t_0)$ and $c_2 = (p(t)\,x''(t)^\alpha)'|_{t=t_0}$. From (4.6.23) we easily see that $x(t)$ satisfies the integral asymptotic relation

$$y(t) \sim \int_b^t (t-s)\left(\frac{1}{p(s)}\int_b^s\int_r^\infty q(u)\,y(u)^\beta\,du\,dr\right)^{\frac{1}{\alpha}} ds,\ t\to\infty, \tag{4.6.24}$$

for any $b \geq a$. We first prove that generalized regularly varying functions $Y_i(t)$, $i = 1, 2, 3$ defined respectively by

$$Y_1(t) = \psi_3(t) \left(\frac{\alpha - \beta}{\alpha} \int_a^t s\, q(s)\, \psi_3(s)^\beta\, ds \right)^{\frac{1}{\alpha-\beta}}, \tag{4.6.25}$$

(4.6.26)

$$Y_2(t) = \left(\left(\frac{m_2(\alpha,\eta)}{\alpha} \right)^2 \frac{p(t)\, q(t)\, R(t)^{2\alpha}}{\rho^\alpha\, (\rho - m_2(\alpha,\eta))^\alpha\, (\rho - m_1(\alpha,\eta))\, (m_3(\alpha,\eta) - \rho)} \right)^{\frac{1}{\alpha-\beta}},$$

$$Y_3(t) = \psi_4(t) \left(\frac{\alpha - \beta}{\alpha} \int_t^\infty q(s)\, \psi_4(s)^\beta\, ds \right)^{\frac{1}{\alpha-\beta}}, \tag{4.6.27}$$

satisfy the integral asymptotic relation of type (4.6.24).

Lemma 4.6.4 *Suppose that*

$$\sigma = -2m_2(\alpha,\eta) - \beta m_1(\alpha,\eta) \quad \text{and} \quad \int_a^\infty t\, q(t)\, \psi_3(t)^\beta\, dt = \infty \tag{4.6.28}$$

holds. The function $Y_1(t)$ *given by* (4.6.25) *satisfies the asymptotic relation* (4.6.24) *for any* $b \geq a$.

Proof. Let (4.6.28) hold. Using (4.3.2), (4.3.5) and (4.6.2), since $\sigma + \beta m_1 + m_2 = -m_2$, we obtain

$$tq(t)\, \psi_3(t)^\beta \sim \frac{m_2^{\frac{2\beta-\alpha}{\alpha+1}}}{(m_1(m_1 - m_2))^\beta}\, R(t)^{-m_2}\, l_p(t)^{\frac{\beta(\alpha-1)+\alpha}{\alpha(\alpha+1)}}\, l_q(t), \quad t \to \infty,$$

so that applying (iii) of Lemma 4.3.1 we have

$$\begin{aligned} \int_a^t s\, q(s)\, \psi_3(s)^\beta\, ds &\sim \frac{m_2^{\frac{2\beta-\alpha+1}{\alpha+1}}}{(m_1(m_1 - m_2))^\beta} \\ &\times \int_a^t R'(s)\, R(s)^{-1}\, l_p(s)^{\frac{\beta(\alpha-1)+2\alpha}{\alpha(\alpha+1)}}\, l_q(s)\, ds, \quad t \to \infty. \end{aligned} \tag{4.6.29}$$

This, combined with (4.6.2), gives the following expression for $Y_1(t)$:

$$\begin{aligned} Y_1(t) &\sim \left(\frac{(\alpha - \beta) m_2}{\alpha (m_1(m_1 - m_2))^\alpha} \right)^{\frac{1}{\alpha-\beta}} R(t)^{m_1}\, l_p(t)^{\frac{\alpha-1}{\alpha(\alpha+1)}} \\ &\times \left(\int_a^t R'(s)\, R(s)^{-1}\, l_p(s)^{\frac{\beta(\alpha-1)+2\alpha}{\alpha(\alpha+1)}}\, l_q(s)\, ds \right)^{\frac{1}{\alpha-\beta}} \in \mathrm{RV}_R(m_1), \quad t \to \infty. \end{aligned}$$

Next, we integrate $q(t)\,Y_1(t)^\beta$ first on $[t,\infty)$, then on $[b,t]$, for any $b \ge a$. Since $q(t)Y_1(t)^\beta \in \mathrm{RV}_R(\beta m_1 + \sigma) = \mathrm{RV}_R(-2m_2)$ (cf.(4.6.28)), application of Lemma 4.3.1 and (4.3.7) yields

$$\int_b^t \int_s^\infty q(r)\,Y_1(r)^\beta\,dr\,ds \sim \left(\frac{\alpha-\beta}{\alpha(m_1(m_1-m_2))^\alpha}\right)^{\frac{\beta}{\alpha-\beta}} m_2^{\frac{\alpha(2\beta-\alpha+1)}{(\alpha-\beta)(\alpha+1)}}$$
$$\times \int_b^t R'(s)R(s)^{-1} l_p(s)^{\frac{\beta(\alpha-1)+2\alpha}{\alpha(\alpha+1)}} l_q(s) \left(\int_a^s R'(r)R(r)^{-1} l_p(r)^{\frac{\beta(\alpha-1)+2\alpha}{\alpha(\alpha+1)}} l_q(r)dr\right)^{\frac{\beta}{\alpha-\beta}} ds$$
$$\sim \left(\frac{\alpha-\beta}{\alpha(m_1(m_1-m_2))^\beta}\right)^{\frac{\alpha}{\alpha-\beta}} m_2^{\frac{\alpha(2\beta-\alpha+1)}{(\alpha-\beta)(\alpha+1)}} \left(\int_a^t R'(s)R(s)^{-1} l_p(s)^{\frac{\beta(\alpha-1)+2\alpha}{\alpha(\alpha+1)}} l_q(s)ds\right)^{\frac{\alpha}{\alpha-\beta}},$$

as $t \to \infty$. Multiply the above by $1/p(t)$, raise the result to the exponent $1/\alpha$ and then integrate twice on $[b,t]$, for any $b \ge a$, we conclude via Lemma 4.3.1 that

$$\int_b^t (t-s)\left(\frac{1}{p(s)}\int_b^s \int_r^\infty q(u)\,Y_1(u)^\beta\,dudr\right)^{\frac{1}{\alpha}} ds \sim \left(\frac{(\alpha-\beta)m_2}{\alpha(m_1(m_1-m_2))^\alpha}\right)^{\frac{1}{\alpha-\beta}}$$
$$\times R(t)^{m_1}\, l_p(t)^{\frac{\alpha-1}{\alpha(\alpha+1)}} \left(\int_a^t R'(s)\,R(s)^{-1}\, l_p(s)^{\frac{\beta(\alpha-1)+2\alpha}{\alpha(\alpha+1)}}\, l_q(s)\,ds\right)^{\frac{1}{\alpha-\beta}} = Y_1(t),$$

as $t \to \infty$. This proves that $Y_1(t)$ satisfies the asymptotic relation (4.6.24). □

Lemma 4.6.5 *Suppose that*

$$(4.6.30) \qquad -2m_2(\alpha,\eta) - \beta m_1(\alpha,\eta) < \sigma < -m_2(\alpha,\eta) - \beta\, m_3(\alpha,\eta),$$

holds and let ρ be defined by (4.3.20). *The function $Y_2(t)$ given by* (4.6.26) *satisfies the asymptotic relation* (4.6.24) *for any $b \ge a$.*

Proof. The proof is the same as the proof of Lemma 4.3.6. □

Lemma 4.6.6 *Suppose that*

$$(4.6.31) \qquad \sigma = -m_2(\alpha,\eta) - \beta\, m_3(\alpha,\eta) \quad \text{and} \quad \int_a^\infty q(t)\,\psi_4(t)^\beta\,dt < \infty$$

holds. The function $Y_3(t)$ given by (4.6.27) *satisfies the asymptotic relation* (4.6.24) *for any $b \ge a$.*

Proof. The proof is the same as the proof of Lemma 4.3.7. □

Since $\psi_3(t) \in \mathrm{RV}_R(m_1(\alpha,\eta))$ and $\psi_4(t) \in \mathrm{RV}_R(m_3(\alpha,\eta))$ (cf.(4.6.2) and (4.6.3)), the regularity index ρ of $x(t)$ must satisfy $m_1(\alpha,\eta) \le \rho \le m_3(\alpha,\eta)$. If $\rho = m_1(\alpha,\eta)$, then since $x(t)/R(t)^{m_1(\alpha,\eta)} = l_x(t) \to \infty$, $t \to \infty$, $x(t)$ is a member of $\mathrm{ntr}-\mathrm{RV}_R(m_1(\alpha,\eta))$, while if $\rho = m_3(\alpha,\eta)$, then $x(t)/R(t)^{m_3(\alpha,\eta)} \to 0$, $t \to \infty$, and so $x(t)$ is a member of $\mathrm{ntr}-\mathrm{RV}_R(m_3(\alpha,\eta))$. If $m_1(\alpha,\eta) < \rho < m_3(\alpha,\eta)$, then $x(t)$ belongs to $\mathrm{RV}_R(\rho)$ and clearly satisfies $x(t)/R(t)^{m_1(\alpha,\eta)} \to \infty$ and $x(t)/R(t)^{m_3(\alpha,\eta)} \to 0$ as $t \to \infty$. Therefore, it is natural to divide the the totality of intermediate solutions of type (I_4) of (E) into the following three classes

$$\mathrm{ntr}-\mathrm{RV}_R(m_1(\alpha,\eta)) \text{ or } \mathrm{RV}_R(\rho),\ \rho \in (m_1(\alpha,\eta), m_3(\alpha,\eta)) \text{ or } \mathrm{ntr}-\mathrm{RV}_R(m_3(\alpha,\eta)).$$

Theorem 4.6.4 *Let $p(t) \in \mathrm{RV}_R(\eta)$, $q(t) \in \mathrm{RV}_R(\sigma)$ and (C_2) hold. Equation* (E) *has intermediate solutions $x(t) \in \mathrm{ntr}-\mathrm{RV}_R(m_1(\alpha,\eta))$ satisfying* (I_4) *if and only if* (4.6.28) *holds. The asymptotic behavior of any such solution $x(t)$ is governed by the unique formula $x(t) \sim Y_1(t)$, $t \to \infty$, where the function $Y_1(t)$ is given by* (4.6.25).

Theorem 4.6.5 *Let $p(t) \in \mathrm{RV}_R(\eta)$, $q(t) \in \mathrm{RV}_R(\sigma)$ and (C_2) hold. Equation* (E) *has intermediate solutions $x(t) \in \mathrm{RV}_R(\rho)$ with $\rho \in (m_1(\alpha,\eta)\ ,\ m_3(\alpha,\eta))$ if and only if* (4.6.30) *holds, in which case ρ is given by* (4.3.20) *and the asymptotic behavior of any such solution $x(t)$ is governed by the unique formula $x(t) \sim Y_2(t)$, $t \to \infty$, where the function $Y_2(t)$ is given by* (4.6.26).

Theorem 4.6.6 *Let $p(t) \in \mathrm{RV}_R(\eta)$, $q(t) \in \mathrm{RV}_R(\sigma)$ and (C_2) hold. Equation* (E) *has intermediate solutions $x(t) \in \mathrm{ntr}-\mathrm{RV}_R(m_3(\alpha,\eta))$ satisfying* (I_4) *if and only if* (4.6.31) *holds. The asymptotic behavior of any such solution $x(t)$ is governed by the unique formula $x(t) \sim Y_3(t)$, $t \to \infty$, where the function $Y_3(t)$ is given by* (4.6.27).

Proof of the "only if" part of Theorems 4.6.4, 4.6.5 and 4.6.6: Suppose that equation (E) has a type-(I_4) intermediate solution $x(t) \in \mathrm{RV}_R(\rho)$, $\rho \in [m_1, m_3]$, defined on $[t_0,\infty)$. We begin by integrating (E) on $[t,\infty)$. Using (4.3.2), (4.3.3) and (4.3.7), we have

$$(p(t)\, x''(t)^{\alpha})' = \int_t^\infty q(s)x(s)^\beta ds \sim \int_t^\infty R(s)^{\sigma+\beta\rho} l_q(s) l_x(s)^\beta\, ds, \quad t \to \infty. \tag{4.6.32}$$

To proceed further we distinguish the two cases:

$$(a)\ \sigma + \beta\rho + m_2 = 0 \quad \text{and} \quad (b)\ \sigma + \beta\rho + m_2 < 0.$$

Let case (a) hold. Integration of (4.6.32) on $[t_0, t]$ yields

$$x''(t) \sim m_2^{\frac{1-\alpha}{\alpha(\alpha+1)}}\, R(t)^{\frac{m_2-\eta}{\alpha}}\, l_p(t)^{-\frac{1}{\alpha+1}} \left(\int_t^\infty R'(s)\, R(s)^{-1}\, l_p(s)^{\frac{1}{\alpha+1}}\, l_q(s)\, l_x(s)^\beta\, ds \right)^{\frac{1}{\alpha}},$$

as $t \to \infty$. Integrating (4.6.33) twice over $[t_0, t]$, we obtain via Lemma 4.3.1 and (4.6.3) that

$$(4.6.33)\; x(t) \sim \frac{m_2^{\frac{1}{\alpha}}}{m_3} R(t)^{m_3}\, l_p(t)^{\frac{1}{\alpha+1}} \left(\int_t^\infty R'(s)\, R(s)^{-1}\, l_p(s)^{\frac{1}{\alpha+1}}\, l_q(s)\, l_x(s)^\beta\, ds \right)^{\frac{1}{\alpha}}$$
$$\sim \psi_4(t)\, m_2^{\frac{1}{\alpha(\alpha+1)}} \left(\int_t^\infty R'(s)\, R(s)^{-1}\, l_p(s)^{\frac{1}{\alpha+1}}\, l_q(s)\, l_x(s)^\beta\, ds \right)^{\frac{1}{\alpha}},\ t \to \infty.$$

Let case (b) hold. Then, integration of (4.6.32) on $[t_0, t]$ gives

$$(4.6.34)\qquad p(t)\, x''(t)^\alpha \sim \frac{m_2^{\frac{1}{\alpha+1}}}{-(\sigma + \beta\rho + m_2)} \times \int_{t_0}^t R(s)^{\sigma+\beta\rho+m_2}\, l_p(s)^{\frac{1}{\alpha+1}}\, l_q(s)\, l_x(s)^\beta\, ds, \quad t \to \infty.$$

The divergence of the last integral as $t \to \infty$ implies $\sigma + \beta\rho + 2m_2 \geq 0$. To preform further integration of (4.6.34) we consider the following two cases separately:

$$(b.1) \quad \sigma + \beta\rho + 2m_2 = 0; \qquad (b.2) \quad \sigma + \beta\rho + 2m_2 > 0.$$

Suppose that $(b.1)$ holds. Since $\sigma + \beta\rho + m_2 = -m_2$ and $-\frac{\eta}{\alpha} + m_2 = m_1 - m_2$, integrating (4.6.34) twice on $[t_0, t]$, we have

$$x(t) \sim \frac{m_2^{\frac{1}{\alpha}}}{m_1(m_1 - m_2)} R(t)^{m_1}\, l_p(t)^{\frac{\alpha-1}{\alpha(\alpha+1)}} \left(\int_{t_0}^t R'(s) R(s)^{-1} l_p(s)^{\frac{2}{\alpha+1}} l_q(s)\, l_x(s)^\beta\, ds \right)^{\frac{1}{\alpha}}$$
$$(4.6.35)\qquad \sim \psi_3(t)\, m_2^{\frac{1-\alpha}{\alpha(\alpha+1)}} \left(\int_{t_0}^t R'(s) R(s)^{-1} l_p(s)^{\frac{2}{\alpha+1}} l_q(s)\, l_x(s)^\beta\, ds \right)^{\frac{1}{\alpha}},$$

as $t \to \infty$, which means that $x(t) \in \mathrm{RV}_R(m_1)$ and that its regularly varying part $l_x(t)$ satisfies the relation

$$(4.6.36)\qquad l_x(t) \sim \frac{m_2^{\frac{1}{\alpha}}}{m_1(m_1 - m_2)}\, l_p(t)^{\frac{\alpha-1}{\alpha(\alpha+1)}} \times \left(\int_{t_0}^t R'(s) R(s)^{-1} l_p(s)^{\frac{2}{\alpha+1}} l_q(s)\, l_x(s)^\beta\, ds \right)^{\frac{1}{\alpha}}, \quad t \to \infty.$$

Suppose that $(b.2)$ holds. Applying first Lemma 4.3.1 in (4.6.34), then multiplying by $1/p(t)$, raising the result on $1/\alpha$ and integrating twice from t_0 to t, we obtain

$$
(4.6.37) \qquad x(t) \sim \left(\frac{m_2^2}{-(\sigma + \beta\rho + m_2)\,(\sigma + \beta\rho + 2m_2)} \right)^{\frac{1}{\alpha}}
\times \frac{R(t)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+2m_2} l_p(t)^{\frac{1}{\alpha}} l_q(t)^{\frac{1}{\alpha}} l_x(t)^{\frac{\beta}{\alpha}}}{\left(\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha} + m_2\right)\left(\frac{\sigma\beta\rho+2m_2-\eta}{\alpha} + 2m_2\right)}, \quad t \to \infty.
$$

This implies that $x(t) \in \mathrm{RV}\left(\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha} + 2m_2\right)$. It is easy to see that

$$
m_1 < \frac{\sigma + \beta\rho + 2m_2 - \eta}{\alpha} + 2m_2 = \frac{\sigma + \beta\rho + 2\alpha + \eta}{\alpha} < m_3.
$$

Now, let $x(t)$ be an intermediate solution of type (I_4) of (E) belonging to $\mathrm{RV}_R(m_1)$. Then, from the above observations it is clear that only the case (b.1) is admissible, so that $\sigma = -2m_2 - \beta m_1$ and $x(t)$ must satisfy (4.6.35). Put

$$
\mu(t) = \int_{t_0}^{t} R'(s)R(s)^{-1} l_p(s)^{\frac{2}{\alpha+1}} l_q(s)\, l_x(s)^{\beta}\, ds.
$$

Then, we can convert (4.6.36) to the differential asymptotic relation for $\mu(t)$

$$
(4.6.38) \qquad \mu(t)^{-\frac{\beta}{\alpha}}\, \mu'(t) \sim C^{\beta} R'(t)\, R(t)^{-1}\, l_p(t)^{\frac{\beta(\alpha-1)+2\alpha}{\alpha(\alpha+1)}}\, l_q(t),\ t \to \infty,
$$

where $C = m_2^{\frac{1}{\alpha}}/m_1(m_1 - m_2)$. From (4.6.35), since $\lim_{t\to\infty} x(t)/\psi_3(t) = \infty$, we have $\lim_{t\to\infty} \mu(t) = \infty$, implying that the left-hand side of (4.6.38) is not integrable on $[t_0, \infty)$, so is the right-hand side, that is,

$$
\int_{t_0}^{\infty} R'(t)\, R(t)^{-1}\, l_p(t)^{\frac{\beta(\alpha-1)+2\alpha}{\alpha(\alpha+1)}}\, l_q(t) dt = \infty,
$$

which, as shown in the proof of Lemma 4.6.4 (cf.(4.6.29)), is equivalent to

$$
\int_{a}^{\infty} t\, q(t)\, \psi_3(t)^{\beta}\, dt = \infty.
$$

We now integrate (4.6.38) on $[t_0, t]$ and in view of (4.6.29), we obtain

$$
\mu(t) \sim m_2^{\frac{\alpha-1}{\alpha+1}} \left(\frac{\alpha - \beta}{\alpha} \int_{t_0}^{t} s\, q(s)\, \psi_3(s)^{\beta}\, ds \right)^{\frac{\alpha}{\alpha-\beta}}, \quad t \to \infty,
$$

and this, combined with (4.6.35), shows that

$$
x(t) \sim \psi_3(t)\, m_2^{\frac{1-\alpha}{\alpha(\alpha+1)}}\, m_2^{\frac{\alpha-1}{\alpha(\alpha+1)}} \left(\frac{\alpha - \beta}{\alpha} \int_{a}^{t} s\, q(s)\, \psi_3(s)^{\beta}\, ds \right)^{\frac{1}{\alpha-\beta}} = Y_1(t), \quad t \to \infty.
$$

This completes the "only if" part of the Theorem 4.6.4.

Next, let $x(t)$ be an intermediate solution of (E) belonging to $\mathrm{RV}_R(\rho)$ for some $\rho \in (m_1, m_3)$. Clearly, $x(t)$ falls into the case $(b.2)$ and hence satisfies the asymptotic relation (4.6.37). This means that

$$\rho = \frac{\sigma + \beta\rho + 2m_2 - \eta}{\alpha} + 2m_2 = \frac{\sigma + \beta\rho + 2\alpha + \eta}{\alpha} \quad \Longrightarrow \quad \rho = \frac{\sigma + 2\alpha + \eta}{\alpha - \beta},$$

verifying that the regularity index ρ is given by (4.3.20). From the requirement $m_1 < \rho < m_3$ it follows that $-2m_2 - \beta m_1 < \sigma < -m_2 - \beta m_3$, showing that the range of σ is given by (4.6.30). Since

$$\frac{\sigma + \beta\rho + 2m_2 - \eta}{\alpha} + m_2 = \rho - m_2, \qquad \frac{\sigma + \beta\rho + 2m_2 - \eta}{\alpha} + 2m_2 = \rho,$$

$$-(\sigma + \beta\rho + m_2) = \alpha(m_3 - \rho), \quad \sigma + \beta\rho + 2m_2 = \alpha(\rho - m_1),$$

the relation (4.6.37) can be rewritten as

$$x(t) \sim \left(\frac{m_2^2\, p(t)\, q(t)\, R(t)^{2\alpha}}{\alpha^2 \rho^\alpha (\rho - m_2)^\alpha (\rho - m_1)(m_3 - \rho)} \right)^{\frac{1}{\alpha}} x(t)^{\frac{\beta}{\alpha}},$$

from which it readily follows that $x(t)$ enjoys the asymptotic behavior (4.6.26). This proves the "only if" part of the Theorem 4.6.5.

Finally, let $x(t)$ be an intermediate solution of type (I_4) of (E) belonging to $\mathrm{RV}_R(m_3)$. Since only the case (a) is possible for $x(t)$, it satisfies (4.6.33), which implies $\rho = m_3$ and $\sigma = -m_2 - \beta m_3$. Letting

$$\nu(t) = \left(\int_t^\infty R'(s)\, R(s)^{-1}\, l_p(s)^{\frac{1}{\alpha+1}}\, l_q(s)\, l_x(s)^\beta\, ds \right)^{\frac{1}{\alpha}},$$

and using the relation $l_x(t) \sim (m_2^{\frac{1}{\alpha}}/m_3) l_p(t)^{\frac{1}{\alpha+1}} \nu(t)$, we convert (4.6.33) into the differential asymptotic relation

$$-\alpha\nu(t)^{\alpha-\beta-1}\, \nu'(t) \sim \frac{m_2^{\frac{\beta}{\alpha}}}{m_3^\beta} R'(t)\, R(t)^{-1}\, l_p(t)^{\frac{\beta+1}{\alpha+1}}\, l_q(t), \quad t \to \infty. \tag{4.6.39}$$

Since the left-hand side of (4.6.39) is integrable on $[t_0, \infty)$ (note that $\lim_{t\to\infty} x(t)/\psi_4(t) = 0$ and so $\lim_{t\to\infty} \nu(t) = 0$), so is the right-hand side, that is,

$$\int_{t_0}^\infty R'(t) R(t)^{-1} l_p(t)^{\frac{\beta+1}{\alpha+1}} l_q(t) dt < \infty$$

which is equivalent to $\int_a^\infty q(t)\psi_4(t)^\beta\, dt < \infty$(see (4.3.55)). Integrating (4.6.39) over $[t,\infty)$, using (4.3.55), then yields

$$\nu(t) \sim m_2^{-\frac{1}{\alpha(\alpha+1)}} \left(\frac{\alpha-\beta}{\alpha}\int_t^\infty q(s)\,\psi_4(s)^\beta\,ds\right)^{\frac{1}{\alpha-\beta}}, \quad t\to\infty,$$

and this combined with (4.6.33) determines the precise asymptotic behavior of $x(t)$ as follows:

$$x(t) \sim \psi_4(t)\, m_2^{\frac{1}{\alpha(\alpha+1)}}\, m_2^{-\frac{1}{\alpha(\alpha+1)}} \left(\frac{\alpha-\beta}{\alpha}\int_t^\infty q(s)\,\psi_4(s)^\beta\,ds\right)^{\frac{1}{\alpha-\beta}} = Y_3(t), \quad t\to\infty.$$

Thus the "only if" part of the Theorem 4.6.6 has been proved. □
Proof of the " if" part of Theorems 4.6.4, 4.6.5 and 4.6.6 as the same as the proof of the "if" part of Theorems 4.3.4, 4.3.5 and 4.3.6. □

4.7 Asymptotic behavior of intermediate regularly varying solutions of (E) in the sense of Karamata

This final section is concerned with the equation (E) whose coefficients $p(t)$ and $q(t)$ are regularly varying functions (in the sense of Karamata). It is natural to expect that such equation may possess intermediate solutions which are regularly varying. Our purpose here is to show that this new problem can be embedded in the framework of generalized regularly varying functions, so that the results of the preceding section provide full information about the existence and the precise asymptotic behavior of regularly varying solutions of (E) in the sense of Karamata.

We assume that $p(t)$ and $q(t)$ are regularly varying functions of indices η and σ, respectively, i.e.,

$$p(t) = t^\eta l_p(t), \qquad q(t) = t^\sigma l_q(t), \qquad l_p(t), l_q(t) \in \mathrm{SV}, \tag{4.7.1}$$

and seek regularly varying solutions $x(t)$ of (E) expressed in the from

$$x(t) = t^\rho l_x(t), \qquad l_x(t) \in \mathrm{SV}. \tag{4.7.2}$$

First, we assume that $p(t)$ satisfies (C_1) implying $\alpha \le \eta \le \min\{\alpha+1, 2\alpha\}$, in which case $R(t)$ defined by (4.3.1) takes the form

$$R(t) = \int_a^t s^{\frac{1-\eta}{\alpha}} l_p(s)^{-\frac{1}{\alpha}} ds.$$

It is easy to see that

$$(4.7.3)\quad R(t) \in \mathrm{SV} \text{ if } \eta = \alpha + 1 \qquad \text{and} \qquad R(t) \in \mathrm{RV}\left(\frac{\alpha+1-\eta}{\alpha}\right) \text{ if } \eta < \alpha + 1.$$

An important remark is that the possibility $\eta = \alpha + 1$ should be excluded. If this equality holds, then $R(t)$ is slowly varying by (4.7.3), and this fact prevents $p(t)$ from being a generalized regularly varying function with respect to $R(t)$. In fact, if $p(t) \in \mathrm{RV}_R(\eta*)$ for some $\eta*$, then there exists $f(t) \in \mathrm{RV}(\eta*)$ such that $p(t) = f(R(t))$, which implies that $p(t) \in \mathrm{SV}$. But this contradicts the hypothesis that $p(t) \in \mathrm{RV}(\eta) = \mathrm{RV}(\alpha + 1)$. Thus, the case $\eta = \alpha + 1$ is impossible, and so η must be restricted to

$$(4.7.4)\qquad \alpha \le \eta < \alpha + 1 \text{ if } \alpha \ge 1, \qquad \alpha \le \eta < 2\alpha \text{ if } \alpha < 1,$$

in which case $R(t)$ satisfies

$$(4.7.5)\qquad R(t) \sim \frac{\alpha}{\alpha+1-\eta} t^{\frac{\alpha+1-\eta}{\alpha}} l_p(t)^{-\frac{1}{\alpha}} \in \mathrm{RV}\left(\frac{\alpha+1-\eta}{\alpha}\right), \quad t \to \infty.$$

Without loss of generality we may assume that $R(t)$ is monotone increasing. Let $R^{-1}(t)$ denote the inverse function of $R(t)$ (do not confuse this notation with $R(t)^{-1} = 1/R(t)$). Then, $R^{-1}(t)$ is a regularly varying function of index $\alpha/(\alpha + 1 - \eta)$, and so any regularly varying function $f(t) \in \mathrm{RV}(\lambda)$ is considered as a generalized regularly varying function of index $\alpha\lambda/(\alpha + 1 - \eta)$ with respect to $R(t)$, and conversely any generalized regularly varying function $f(t) \in \mathrm{RV}_R(\lambda^*)$ is regarded as an (ordinary) regularly varying function of index $\lambda = \lambda^*(\alpha + 1 - \eta)/\alpha$. It follows that

$$p(t) \in \mathrm{RV}_R\left(\frac{\alpha\,\eta}{\alpha+1-\eta}\right), \quad q(t) \in \mathrm{RV}_R\left(\frac{\alpha\,\sigma}{\alpha+1-\eta}\right), \quad x(t) \in \mathrm{RV}_R\left(\frac{\alpha\,\rho}{\alpha+1-\eta}\right).$$

Put

$$(4.7.6)\qquad \eta^* = \frac{\alpha\eta}{\alpha+1-\eta}, \quad \sigma^* = \frac{\alpha\sigma}{\alpha+1-\eta}, \quad \rho^* = \frac{\alpha\rho}{\alpha+1-\eta}.$$

Note that (4.7.4) implies

$$\alpha^2 - \eta^* < 0 \quad \wedge \quad 2\alpha^2 + \alpha\eta^* - \eta^* > 0,$$

and that the tree positive constants given by (4.3.9) are reduced to

$$m_1(\alpha, \eta^*) = \frac{2\alpha-\eta}{\alpha+1-\eta}, \quad m_2(\alpha, \eta^*) = \frac{\alpha}{\alpha+1-\eta}, \quad m_3(\alpha, \eta^*) = \frac{2\alpha-\eta+1}{\alpha+1-\eta}.$$

It turns out therefore that any intermediate regularly varying solution of type (I_1) of (E) is a member of one of the three classes

$$\mathrm{ntr}-\mathrm{SV}, \quad \mathrm{RV}(\rho),\ \rho \in \left(0, \frac{2\alpha-\eta}{\alpha}\right), \quad \mathrm{ntr}-\mathrm{RV}\left(\frac{2\alpha-\eta}{\alpha}\right),$$

while any intermediate regularly varying solution of type (I_2) of (E) belongs to one of the three classes

$$\mathrm{ntr}-\mathrm{RV}(1), \quad \mathrm{RV}(\rho),\ \rho \in \left(1, \frac{2\alpha-\eta+1}{\alpha}\right), \quad \mathrm{ntr}-\mathrm{RV}\left(\frac{2\alpha-\eta+1}{\alpha}\right).$$

Based on the above observations we are able to apply the theory of generalized regularly varying functions to the present situation, thereby establishing necessary and sufficient conditions for the existence of intermediate regularly varying solutions of (E) and determining the asymptotic behavior of all such solutions explicitly and accurately. First, we state the results on intermediate solutions of type (I_1) that can be derived as corollaries of Theorems 4.3.1, 4.3.2 and 4.3.3.

Theorem 4.7.1 *Assume that* $p(t) \in \mathrm{RV}(\eta)$, $q(t) \in \mathrm{RV}(\sigma)$ *and* (C_1) *holds. Equation* (E) *possess intermediate slowly varying solutions if and only if*

$$(4.7.7) \qquad \sigma = \eta - 2\alpha - 2 \quad \textit{and} \quad \int_a^\infty t \left(\frac{1}{p(t)} \int_t^\infty (s-t)\, q(s)\, ds\right)^{\frac{1}{\alpha}} dt = \infty.$$

Any such solution $x(t)$ *enjoys one and the same asymptotic behavior* $x(t) \sim X_1(t)$, $t \to \infty$, *where* $X_1(t)$ *is given by* (4.3.14).

Theorem 4.7.2 *Assume that* $p(t) \in \mathrm{RV}(\eta)$, $q(t) \in \mathrm{RV}(\sigma)$ *and* (C_1) *holds. Equation* (E) *possess intermediate regularly varying solutions belonging to* $\mathrm{RV}(\rho)$ *with* $\rho \in \left(0\ ,\ \frac{2\alpha-\eta}{\alpha}\right)$ *if and only if*

$$(4.7.8) \qquad \eta - 2\alpha - 2 < \sigma < \frac{\beta}{\alpha}\eta - 2\beta - 2,$$

in which case ρ *is given by*

$$(4.7.9) \qquad \rho = \frac{2\alpha-\eta+\sigma+2}{\alpha-\beta}$$

and any such solution $x(t)$ *enjoys one and the same asymptotic behavior*

$$(4.7.10) \quad x(t) \sim \left(\frac{t^{2\alpha+2}\, p(t)^{-1}\, q(t)}{\rho^\alpha\ (2\alpha-\eta-\alpha\,\rho)\ (1-\rho)^\alpha\ (2\alpha-\eta+1-\alpha\,\rho)}\right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty.$$

Theorem 4.7.3 *Assume that $p(t) \in \mathrm{RV}(\eta)$, $q(t) \in \mathrm{RV}(\sigma)$ and* (C_1) *holds. Equation* (E) *possess intermediate regularly varying solutions belonging to* $\mathrm{RV}\left(\frac{2\alpha-\eta}{\alpha}\right)$ *if and only if*

$$\sigma = \frac{\beta}{\alpha}\eta - 2\beta - 2 \quad \textit{and} \quad \int_a^\infty tq(t)\,\varphi(t)^\beta\,dt < \infty. \tag{4.7.11}$$

Any such solution $x(t)$ enjoys one and the same asymptotic behavior $x(t) \sim X_3(t)$, $t \to \infty$, where $X_3(t)$ is given by (4.3.16).

To prove Theorem 4.7.1 and 4.7.3 we need only to check that

$$\sigma^* = -2\alpha - \eta^* \quad \Longleftrightarrow \quad \sigma = \eta - 2\alpha - 2,$$

$$\sigma^* = \beta m_1(\alpha,\eta^*) - 2m_2(\alpha,\eta^*) \quad \Longleftrightarrow \quad \sigma = \frac{\beta}{\alpha}\eta - 2\beta - 2,$$

and to prove Theorem 4.7.2 it suffices to note that

$$\rho^* = \frac{2\alpha + \eta^* + \sigma^*}{\alpha - \beta} \quad \Longleftrightarrow \quad \rho = \frac{2\alpha + \sigma - \eta + 2}{\alpha - \beta},$$

and to combine the relation $R(t) \sim \frac{\alpha}{\alpha+1-\eta} t^{\frac{\alpha+1-\eta}{\alpha}} l_p(t)^{-\frac{1}{\alpha}}$, $t \to \infty$, with equality

$$\begin{aligned}
&(\rho^*)^\alpha(m_1(\alpha,\eta^*) - \rho^*)(m_2(\alpha,\eta^*) - \rho^*)^\alpha(m_3(\alpha,\eta^*) - \rho^*) \\
&= \frac{\alpha^{2\alpha}}{(\alpha+1-\eta)^{2\alpha+2}}\rho^\alpha(2\alpha - \eta - \alpha\rho)(1-\rho)^\alpha(2\alpha - \eta + 1 - \alpha\rho).
\end{aligned}$$

Similarly, we are able to gain a through knowledge of intermediate regularly varying solutions of type (I_2) of (E) from Theorems 4.3.4, 4.3.5 and 4.3.6.

Theorem 4.7.4 *Assume that $p(t) \in \mathrm{RV}(\eta)$, $q(t) \in \mathrm{RV}(\sigma)$ and* (C_1) *holds. Equation* (E) *possess intermediate regularly varying solutions of index* 1 *if and only if*

$$\sigma = \eta - \alpha - \beta - 2 \quad \textit{and} \quad \int_a^\infty \left(\frac{1}{p(t)}\int_a^t\int_s^\infty r^\beta\,q(r)\,drds\right)^{\frac{1}{\alpha}} dt = \infty. \tag{4.7.12}$$

The asymptotic behavior of any such solution $x(t)$ is governed by the unique formula $x(t) \sim Y_1(t)$, $t \to \infty$, where $Y_1(t)$ is given by (4.3.45).

Theorem 4.7.5 *Assume that $p(t) \in \mathrm{RV}(\eta)$, $q(t) \in \mathrm{RV}(\sigma)$ and* (C_1) *holds. Equation* (E) *possess intermediate regularly varying solutions belonging to* $\mathrm{RV}(\rho)$ *with* $\rho \in \left(1, \frac{2\alpha-\eta+1}{\alpha}\right)$ *if and only if*

$$\eta - \alpha - \beta - 2 < \sigma < \frac{\beta}{\alpha}\eta - \frac{\beta}{\alpha} - 2\beta - 1, \tag{4.7.13}$$

in which case ρ is given by (4.7.9) *and the asymptotic behavior of any such solution $x(t)$ is governed by the unique formula*

$$(4.7.14)\quad x(t) \sim \left(\frac{t^{2\alpha+2}\, p(t)^{-1}\, q(t)}{\rho^{\alpha}\ (\alpha\,\rho - 2\alpha + \eta)\ (\rho - 1)^{\alpha}\ (2\alpha - \eta + 1 - \alpha\,\rho)} \right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty.$$

Theorem 4.7.6 *Assume that $p(t) \in \mathrm{RV}(\eta)$, $q(t) \in \mathrm{RV}(\sigma)$ and* (C_1) *holds. Equation* (E) *possess intermediate regularly varying solutions of index $\frac{2\alpha-\eta+1}{\alpha}$ if and only if*

$$(4.7.15)\qquad \sigma = \frac{\beta}{\alpha}\eta - \frac{\beta}{\alpha} - 2\beta - 1 \quad \textit{and} \quad \int_a^{\infty} q(t)\,\psi(t)^{\beta}\,dt < \infty.$$

The asymptotic behavior of any such solution $x(t)$ is governed by the unique formula $x(t) \sim Y_3(t)$, $t \to \infty$, where $Y_3(t)$ is given by (4.3.47).

Above corollaries combined with Theorems 4.1.1-4.1.2 enable us to describe in full details the structure of RV-solutions of equation (E) with RV-coefficients. Denote with $\mathcal{R}$ the set of all regularly varying solutions of (E) and define the subsets

$$\mathcal{R}(\rho) = \mathcal{R} \cap \mathrm{RV}(\rho), \quad \mathrm{tr} - \mathcal{R}(\rho) = \mathcal{R} \cap \mathrm{tr} - \mathrm{RV}(\rho), \quad \mathrm{ntr} - \mathcal{R}(\rho) = \mathcal{R} \cap \mathrm{ntr} - \mathrm{RV}(\rho).$$

Corollary 4.7.1 *Let $p(t) \in \mathrm{RV}(\eta)$, $q(t) \in \mathrm{RV}(\sigma)$ and* (C_1) *holds.*

(i) *If $\sigma < \eta - 2\alpha - 2$, or $\sigma = \eta - 2\alpha - 2$ and $\int_a^{\infty} t \left(\frac{1}{p(t)} \int_t^{\infty} (s-t)\, q(s)\, ds \right)^{\frac{1}{\alpha}} dt < \infty$, then*

$$\mathcal{R} = \mathrm{tr} - \mathcal{R}(0) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha - \eta}{\alpha}\right) \cup \mathrm{tr} - \mathcal{R}(1) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha + 1 - \eta}{\alpha}\right).$$

(ii) *If $\sigma = \eta - 2\alpha - 2$ and $\int_a^{\infty} t \left(\frac{1}{p(t)} \int_t^{\infty} (s-t)\, q(s)\, ds \right)^{\frac{1}{\alpha}} dt = \infty$, then*

$$\mathcal{R} = \mathrm{ntr} - \mathcal{R}(0) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha - \eta}{\alpha}\right) \cup \mathrm{tr} - \mathcal{R}(1) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha + 1 - \eta}{\alpha}\right).$$

(iii) *If $\sigma \in \left(\eta - 2\alpha - 2, \frac{\beta}{\alpha}\eta - 2\beta - 2\right)$, then*

$$\mathcal{R} = \mathcal{R}\left(\frac{\sigma + 2\alpha + 2 - \eta}{\alpha - \beta}\right) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha - \eta}{\alpha}\right) \cup \mathrm{tr} - \mathcal{R}(1) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha + 1 - \eta}{\alpha}\right).$$

(iv) *If $\sigma = \frac{\beta}{\alpha}\eta - 2\beta - 2$ and $\int_a^{\infty} t\, q(t)\, \varphi_2(t)^{\beta}\, dt < \infty$, then*

$$\mathcal{R} = \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha - \eta}{\alpha}\right) \cup \mathrm{ntr} - \mathcal{R}\left(\frac{2\alpha - \eta}{\alpha}\right) \cup \mathrm{tr} - \mathcal{R}(1) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha + 1 - \eta}{\alpha}\right).$$

(v) *If* $\sigma = \frac{\beta}{\alpha}\eta - 2\beta - 2$ *and* $\int_a^\infty t\, q(t)\, \varphi_2(t)^\beta\, dt = \infty$, *or* $\sigma \in \left(\frac{\beta}{\alpha}\eta - 2\beta - 2, \eta - \alpha - \beta - 2\right)$, *or* $\sigma = \eta - \alpha - \beta - 2$ *and* $\int_a^\infty \left(\frac{1}{p(t)} \int_a^t \int_s^\infty r^\beta\, q(r)\, drds\right)^{\frac{1}{\alpha}} dt < \infty$, *then*

$$\mathcal{R} = \mathrm{tr} - \mathcal{R}(1) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha + 1 - \eta}{\alpha}\right).$$

(vi) *If* $\sigma = \eta - \alpha - \beta - 2$ *and* $\int_a^\infty \left(\frac{1}{p(t)} \int_a^t \int_s^\infty r^\beta\, q(r)\, drds\right)^{\frac{1}{\alpha}} dt = \infty$, *then*

$$\mathcal{R} = \mathrm{ntr} - \mathcal{R}(1) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha + 1 - \eta}{\alpha}\right).$$

(vii) *If* $\sigma \in \left(\eta - \alpha - \beta - 2, \frac{\beta}{\alpha}\eta - 2\beta - \frac{\beta}{\alpha} - 1\right)$, *then*

$$\mathcal{R} = \mathcal{R}\left(\frac{\sigma + 2\alpha + 2 - \eta}{\alpha - \beta}\right) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha + 1 - \eta}{\alpha}\right).$$

(viii) *If* $\sigma = \frac{\beta}{\alpha}\eta - 2\beta - \frac{\beta}{\alpha} - 1$ *and* $\int_a^\infty q(t)\, \varphi_4(t)^\beta\, dt < \infty$, *then*

$$\mathcal{R} = \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha + 1 - \eta}{\alpha}\right) \cup \mathrm{ntr} - \mathcal{R}\left(\frac{2\alpha + 1 - \eta}{\alpha}\right).$$

(ix) *If* $\sigma = \frac{\beta}{\alpha}\eta - 2\beta - \frac{\beta}{\alpha} - 1$ *and* $\int_a^\infty q(t)\, \varphi_4(t)^\beta\, dt = \infty$, *or* $\sigma > \frac{\beta}{\alpha}\eta - 2\beta - \frac{\beta}{\alpha} - 1$, *then*

$$\mathcal{R} = \varnothing.$$

Now, we assume that $p(t)$ satisfies (C_2), which implies $\eta \le \alpha$. In what follows we assume that $\eta < \alpha$, excluding the case $\eta = \alpha$ because of computational difficulty and the fact that integral

$$\int_a^\infty \frac{dt}{p(t)^{\frac{1}{\alpha}}} = \int_a^\infty t^{-\frac{\eta}{\alpha}} l_p(t)^{-\frac{1}{\alpha}} dt$$

might be either convergent or divergent. Using notation (4.7.6), from $\eta < \alpha$ we get $\alpha^2 - \eta^* > 0$ and that the tree positive constants given by (4.3.9) are reduced to

$$m_2(\alpha, \eta^*) = \frac{\alpha}{\alpha + 1 - \eta}, \quad m_1(\alpha, \eta^*) = \frac{2\alpha - \eta}{\alpha + 1 - \eta}, \quad m_3(\alpha, \eta^*) = \frac{2\alpha - \eta + 1}{\alpha + 1 - \eta}.$$

It turns out therefore that any intermediate regularly varying solution of type (I_3) of (E) is a member of one of the three classes

$$\mathrm{ntr} - \mathrm{SV}, \quad \mathrm{RV}(\rho),\ \rho \in (0, 1), \quad \mathrm{ntr} - \mathrm{RV}(1),$$

while any intermediate regularly varying solution of type (I_4) belongs to one of the three classes

$$\mathrm{ntr}-\mathrm{RV}\left(\frac{2\alpha-\eta}{\alpha}\right), \mathrm{RV}(\rho), \rho\in\left(\frac{2\alpha-\eta}{\alpha}, \frac{2\alpha-\eta+1}{\alpha}\right), \mathrm{ntr}-\mathrm{RV}\left(\frac{2\alpha-\eta+1}{\alpha}\right).$$

Based on the above observations we are able to apply results for generalized regularly varying solutions to the present situation, thereby establishing necessary and sufficient conditions for the existence of intermediate regularly varying solutions of (E) and determining the asymptotic behavior of all such solutions explicitly and accurately. First, we state the results on intermediate solutions of type (I_3) that can be derived as corollaries of Theorems 4.6.1, 4.6.2 and 4.6.3.

Theorem 4.7.7 *Assume that $q(t)\in\mathrm{RV}(\sigma)$, $p(t)\in\mathrm{RV}(\eta)$ and* (C_2) *holds. Equation* (E) *possess intermediate slowly varying solutions if and only if*

$$\sigma=\eta-2\alpha-2 \quad \textit{and} \quad \int_a^\infty t\left(\frac{1}{p(t)}\int_t^\infty (s-t)\,q(s)\,ds\right)^{\frac{1}{\alpha}} dt=\infty. \tag{4.7.16}$$

Any such solution $x(t)$ enjoys one and the same asymptotic behavior $x(t)\sim X_1(t)$, $t\to\infty$, where $X_1(t)$ is given by (4.6.6).

Theorem 4.7.8 *Assume that $q(t)\in\mathrm{RV}(\sigma)$, $p(t)\in\mathrm{RV}(\eta)$ and* (C_2) *holds. Equation* (E) *possess intermediate regularly varying solutions belonging to* $\mathrm{RV}(\rho)$ *with $\rho\in(0,1)$ if and only if*

$$\eta-2\alpha-2<\sigma<\eta-\alpha-\beta-2, \tag{4.7.17}$$

in which case ρ is given (4.7.9) *and any such solution $x(t)$ enjoys one and the same asymptotic behavior*

$$x(t)\sim\left(\frac{t^{2\alpha+2}\,p(t)^{-1}\,q(t)}{\rho^\alpha\,(1-\rho)^\alpha\,(2\alpha-\eta-\alpha\,\rho)\,(2\alpha-\eta+1-\alpha\,\rho)}\right)^{\frac{1}{\alpha-\beta}},\quad t\to\infty. \tag{4.7.18}$$

Theorem 4.7.9 *Assume that $q(t)\in\mathrm{RV}(\sigma)$, $p(t)\in\mathrm{RV}(\eta)$ and* (C_2) *holds. Equation* (E) *possess intermediate regularly varying solutions belonging to* $\mathrm{RV}(1)$ *if and only if*

$$\sigma=\eta-\alpha-\beta-2 \quad \textit{and} \quad \int_a^\infty\left(\frac{1}{p(t)}\int_t^\infty (s-t)\,s^\beta\,q(s)\,ds\right)^{\frac{1}{\alpha}} dt<\infty. \tag{4.7.19}$$

Any such solution $x(t)$ enjoys one and the same asymptotic behavior $x(t)\sim X_3(t)$, $t\to\infty$, where $X_3(t)$ is given by (4.6.8).

Similarly, we are able to gain a through knowledge of intermediate regularly varying solutions of type (I_4) of (E) from Theorems 4.6.4, 4.6.5 and 4.6.6.

Theorem 4.7.10 *Assume that* $q(t) \in \mathrm{RV}(\sigma)$, $p(t) \in \mathrm{RV}(\eta)$ *and* (C_2) *holds. Equation* (E) *possess intermediate regularly varying solutions of index* $\frac{2\alpha-\eta}{\alpha}$ *if and only if*

$$\sigma = \frac{\beta}{\alpha}\eta - 2\beta - 2 \quad \text{and} \quad \int_a^\infty t\, q(t)\, \psi_3(t)^\beta\, dt = \infty. \tag{4.7.20}$$

The asymptotic behavior of any such solution $x(t)$ *is governed by the unique formula* $x(t) \sim Y_1(t)$, $t \to \infty$, *where* $Y_1(t)$ *is given by* (4.6.25).

Theorem 4.7.11 *Assume that* $q(t) \in \mathrm{RV}(\sigma)$, $p(t) \in \mathrm{RV}(\eta)$ *and* (C_2) *holds. Equation* (E) *possess intermediate regularly varying solutions belonging to* $\mathrm{RV}(\rho)$ *with* $\rho \in \left(\frac{2\alpha-\eta}{\alpha}, \frac{2\alpha-\eta+1}{\alpha}\right)$ *if and only if*

$$\frac{\beta}{\alpha}\eta - 2\beta - 2 < \sigma < \frac{\beta}{\alpha}\eta - \frac{\beta}{\alpha} - 2\beta - 1, \tag{4.7.21}$$

in which case ρ *is given by* (4.7.9) *and the asymptotic behavior of any such solution* $x(t)$ *is governed by the unique formula*

$$x(t) \sim \left(\frac{t^{2\alpha+2}\, p(t)^{-1}\, q(t)}{\rho^\alpha\, (\rho-1)^\alpha\, (\alpha\,\rho - 2\alpha + \eta)\, (2\alpha - \eta + 1 - \alpha\,\rho)} \right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty. \tag{4.7.22}$$

Theorem 4.7.12 *Assume that* $q(t) \in \mathrm{RV}(\sigma)$, $p(t) \in \mathrm{RV}(\eta)$ *and* (C_2) *holds. Equation* (E) *possess intermediate regularly varying solutions of index* $\frac{2\alpha-\eta+1}{\alpha}$ *if and only if*

$$\sigma = \frac{\beta}{\alpha}\eta - \frac{\beta}{\alpha} - 2\beta - 1 \quad \text{and} \quad \int_a^\infty q(t)\, \psi_4(t)^\beta\, dt < \infty. \tag{4.7.23}$$

The asymptotic behavior of any such solution $x(t)$ *is governed by the unique formula* $x(t) \sim Y_3(t)$, $t \to \infty$, *where* $Y_3(t)$ *is given by* (4.6.27).

Above corollaries combined with Theorem 4.1.1, Theorem 4.1.2, Theorem 4.4.1 and Theorem 4.4.2 enable us to describe in full details the structure of RV-solutions of equation (E) with RV-coefficients. Denote with $\mathcal{R}$ the set of all regularly varying solutions of (E) and define the subsets

$$\mathcal{R}(\rho) = \mathcal{R} \cap \mathrm{RV}(\rho), \quad \mathrm{tr} - \mathcal{R}(\rho) = \mathcal{R} \cap \mathrm{tr} - \mathrm{RV}(\rho), \quad \mathrm{ntr} - \mathcal{R}(\rho) = \mathcal{R} \cap \mathrm{ntr} - \mathrm{RV}(\rho).$$

Corollary 4.7.2 *Let* $q(t) \in \mathrm{RV}(\sigma)$, $p(t) \in \mathrm{RV}(\eta)$ *and* (C_2) *holds.*

(i) *If* $\sigma < \eta - 2\alpha - 2$, *or* $\sigma = \eta - 2\alpha - 2$ *and* $\int_a^\infty t \left(\frac{1}{p(t)} \int_t^\infty (s-t)\, q(s)\, ds \right)^{\frac{1}{\alpha}} dt < \infty$, *then*

$$\mathcal{R} = \mathrm{tr} - \mathcal{R}(0) \cup \mathrm{tr} - \mathcal{R}(1) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha - \eta}{\alpha}\right) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha + 1 - \eta}{\alpha}\right).$$

(ii) *If* $\sigma = \eta - 2\alpha - 2$ *and* $\int_a^\infty t \left(\frac{1}{p(t)} \int_t^\infty (s-t)\, q(s)\, ds \right)^{\frac{1}{\alpha}} dt = \infty$, *then*

$$\mathcal{R} = \mathrm{ntr} - \mathcal{R}(0) \cup \mathrm{tr} - \mathcal{R}(1) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha - \eta}{\alpha}\right) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha + 1 - \eta}{\alpha}\right).$$

(iii) *If* $\sigma \in (\eta - 2\alpha - 2, \eta - \alpha - \beta - 2)$, *then*

$$\mathcal{R} = \mathcal{R}\left(\frac{\sigma + 2\alpha - \eta + 2}{\alpha - \beta}\right) \cup \mathrm{tr} - \mathcal{R}(1) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha - \eta}{\alpha}\right) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha + 1 - \eta}{\alpha}\right).$$

(iv) *If* $\sigma = \eta - \alpha - \beta - 2$ *and* $\int_a^\infty \left(\frac{1}{p(t)} \int_t^\infty (s-t)\, s^\beta\, q(s)\, ds \right)^{\frac{1}{\alpha}} dt < \infty$, *then*

$$\mathcal{R} = \mathrm{tr} - \mathcal{R}(1) \cup \mathrm{ntr} - \mathcal{R}(1) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha - \eta}{\alpha}\right) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha + 1 - \eta}{\alpha}\right).$$

(v) *If* $\sigma = \eta - \alpha - \beta - 2$ *and* $\int_a^\infty \left(\frac{1}{p(t)} \int_t^\infty (s-t)\, s^\beta\, q(s)\, ds \right)^{\frac{1}{\alpha}} dt = \infty$, *or* $\sigma \in \left(\eta - \alpha - \beta - 2, \frac{\beta}{\alpha}\eta - 2\beta - 2\right)$, *or* $\sigma = \frac{\beta}{\alpha}\eta - 2\beta - 2$ *and* $\int_a^\infty t\, q(t)\, \psi_3(t)^\beta\, dt < \infty$, *then*

$$\mathcal{R} = \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha - \eta}{\alpha}\right) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha + 1 - \eta}{\alpha}\right).$$

(vi) *If* $\sigma = \frac{\beta}{\alpha}\eta - 2\beta - 2$ *and* $\int_a^\infty t\, q(t)\, \psi_3(t)^\beta\, dt = \infty$, *then*

$$\mathcal{R} = \mathrm{ntr} - \mathcal{R}\left(\frac{2\alpha - \eta}{\alpha}\right) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha + 1 - \eta}{\alpha}\right).$$

(vii) *If* $\sigma \in \left(\frac{\beta}{\alpha}\eta - 2\beta - 2, \frac{\beta}{\alpha}\eta - 2\beta - \frac{\beta}{\alpha} - 1\right)$, *then*

$$\mathcal{R} = \mathcal{R}\left(\frac{\sigma + 2\alpha - \eta + 2}{\alpha - \beta}\right) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha + 1 - \eta}{\alpha}\right).$$

(viii) *If* $\sigma = \frac{\beta}{\alpha}\eta - 2\beta - \frac{\beta}{\alpha} - 1$ *and* $\int_a^\infty q(t)\, \psi_4(t)^\beta\, dt < \infty$, *then*

$$\mathcal{R} = \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha + 1 - \eta}{\alpha}\right) \cup \mathrm{ntr} - \mathcal{R}\left(\frac{2\alpha + 1 - \eta}{\alpha}\right).$$

(ix) *If* $\sigma = \frac{\beta}{\alpha}\eta - 2\beta - \frac{\beta}{\alpha} - 1$ *and* $\int_a^\infty q(t)\, \psi_4(t)^\beta\, dt = \infty$, *or* $\sigma > \frac{\beta}{\alpha}\eta - 2\beta - \frac{\beta}{\alpha} - 1$, *then*

$$\mathcal{R} = \varnothing.$$

Bibliography

[1] V.G. Avakumović, *Sur l'equation differentielle de Thomas-Fermi*, Publ. Inst. Math. Beograd, Vol. 1 (1947) 101 – 113.

[2] N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular Variation*, Encyclopedia of Mathematics and its Applications, 27, Cambridge University Press, 1987.

[3] M. Cecchi, Z. Došlá, M. Marini, *On the dynamics of the generalized Emden-Fowler equations*, Georgian Math. J. 7 (2000) 269 - 282.

[4] M. Cecchi, Z. Došlá, M. Marini, *On nonoscillatory solutions of differential equations with p-Laplacian*, Adv.Math. Sci. Appl. 11 (2001) 419 - 436.

[5] M. Cecchi, Z. Došlá, M. Marini, I. Vrkoč, *Integral conditions for nonoscillation of second order nonlinear differential equations*, Nonlinear Analysis 64 (2006) 1278 - 1289.

[6] M. Cecchi, Z. Došlá, M. Marini, *Intermediate solutions for EmdenFowler type equations: Continuous Versus Discrete*, Advances in Dynamical Systems and Applications, 3 (2008) 161 - 176.

[7] M. Cecchi, Z. Došlá, M. Marini, *On intermediate solutions and the Wronskian-for half-linear differential equations*, J. Math. Anal. Appl. 336 (2007) 905 - 918.

[8] Z. Došlá, M. Marini, *On super-linear EmdenFowler type differential equations*, J. Math. Anal. Appl. 416 (2014) 497 – 510.

[9] O. Došly, P. Řehák, *Half-linear differential equations*, Mathematics studies Vol. 202, Elsevier, North-Holland, 2005.

[10] D. Djurčić, *$\mathcal{O}$-regularly varying functions and some asymptotic relations*, Publications de L'institut mathematique Nouvelle serie, tome 61, 75 (1997), 44 – 52.

[11] A.Elbert, T.Kusano, *Oscillation and non-oscilation theorems for a class of second order quasilinear diferential equations*, Acta Math.Hung. 56 (3–4) (1990), 325 – 336.

[12] E. Fermi, *Un metodo statistico per la determinazione di alcune proprieta dell atomo*, Rend. Accad. Naz. Lincei 6 (1927), 602 – 607.

[13] O. Haupt and G. Aumann, *Differential- und Integralrechnung*, Walter de Gruyter, Berlin, 1938.

[14] H. Howard and V. Marić, *Regularity and nonoscillation of solutions of second order linear differential equations*, Bull. Classe Sci. Mat. Nat., Sci. Math., Acad. Serbe Sci. Arts, Vol. CXIV, No. 22 (1997), 85 – 98.

[15] J. Jaroš, T. Kusano, *Self-adjoint differential equations and generalized Karamata functions*, Bull. Classe Sci. Mat. Nat., Sci. Math., Acad. Serbe Sci. Arts, Vol. CXXIX, No. 29 (2004), 25 – 60.

[16] J. Jaroš, T. Kusano, *Existence and precise asymptotic behavior of strongly monotone solutions of systems of nonlinear differential equations*, Differ. Equ. Appl. 5 (2013), 185 – 204.

[17] J. Jaroš, T. Kusano, *Slowly varying solutions of a class of first order systems of nonlinear differential equations*, Acta Math.Univ.Comenian. 82 (2013), 265 – 284.

[18] J. Jaroš, T. Kusano, *Asymptotic behavior of positive solutions of a class of systems of second order nonlinear differential equations*, Electronic Journal of Qualitative Theory of Differential Equations No. 23 (2013), 1 – 23.

[19] J. Jaroš, T. Kusano, *On strongly monotone solutions of a class of cyclic systems of nonlinear differential equations*, J. Math. Anal. Appl. 417 (2014), 996 – 1017.

[20] J. Jaroš, T. Kusano, *Strongly increasing solutions of cyclic systems of second order differential equations with power-type nonlinearities*, Opuscula Math. 35, no. 1 (2015), 47 - 69.

[21] J. Jaroš, T. Kusano, J. Manojlović, *Asymptotic analysis of positive solutions of generalized Emden-Fowler differential equations in the framework of regular variation*, Cent. Eur. J. Math., 11(12) (2013) 2215 – 2233.

[22] J. Jaroš, T. Kusano, T.Tanigawa, *Nonoscillatory half-linear differential equations and generalized Karamata functions*, Nonlinear Anal. 64(4) (2006) 762 - 787.

[23] K. Kamo, H. Usami, *Oscillation theorems for fourth-order quasilinear ordinary differential equations*, Studia Scientiarium Mathematicarum Hungarica 39 (2002) 385 – 406.

[24] K. Kamo, H. Usami, *Asymptotic forms of weakly increasing positive solutions for quasilinear ordinary differential equations*, Electronic Journal of Differential Equations, Vol. 2007(2007), No. 126, 1 - 12.

[25] K. Kamo, H. Usami, *Characterization of slowly decaying positive solutions of second-order quasilinear ordinary differential equations with subhomogeneity*, Bull. Lond. Math. Soc. 42 (2010) 420 – 428.

[26] K. Kamo, H. Usami, *Nonlinear oscillations for fourth order quasilinear ordinary differential equations*, Acta Math. Hungar. 132(3) (2011) 207 – 222.

[27] J. Karamata, *Sur une mode de croissance réguliere de fonctions*, Math. (Cluj) 4(1930), 38 - 53.

[28] I. T. Kiguradze and T. A. Chanturia, *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1993.

[29] M. Kitano and T. Kusano,*On a class of second order quasilinear ordinary differential equations*, Hiroshima Math. J. 25 (1995), 321 – 355.

[30] T. Kusano, J.V. Manojlović, *Asymptotic analysis of Emden-Fowler differential equations in the framework of regular variation*, Annali di Matematica Pura ed Applicata, 190 (2011) 619 – 644.

[31] T. Kusano, J. Manojlović, *Asymptotic behavior of positive solutions of sublinear differential equations of Emden-Fowler type*, Comput. Math. Appl. 62 (2011), 551 – 565.

[32] T. Kusano, J. Manojlović, *Positive solutions of fourth order Emden-Fowler type differential equations in the framework of regular variation*, Applied Mathematics and Computation, Vol. 218 (2012), 6684 – 6701.

[33] T. Kusano, J. Manojlović, *Positive solutions of fourth order Thomas-Fermi type differential equations in the framework of regular variation*, Acta Applicandae Mathematicae, Vol. 121 (2012), 81 – 103.

[34] T. Kusano, J. Manojlović, *Precise asymptotic behavior of intermediate solutions of even order nonlinear differential equation in the framework of regular variation*, Moscow Mathematical Journal, Vol. 13, No.4, (2013), 649 - 666.

[35] T. Kusano, J. Manojlović, V. Marić, *Increasing solutions of Thomas-Fermi type differential equations - the sublinear case*, Bull. T. de Acad. Serbe Sci. Arts, Classe Sci. Mat. Nat., Sci. Math., Vol CXLIII, No.36, (2011), 21 – 36.

[36] T. Kusano, J. Manojlović, J. Milošević, *Intermediate solutions of second order quasilinear differential equations in the framework of regular variation*, Applied Mathematics and Computation, 219 (2013), 8178 – 8191.

[37] T. Kusano, J. Manojlović, J. Milošević, *Intermediate solutions of fourth order quasilinear differential equations in the framework of regular variation*, Applied Mathematics and Computation, 248 (2014), 246 – 272.

[38] T. Kusano, J. Manojlović, T.Tanigawa, *Sharp oscillation criteria for a class of fourth order nonlinear differential equations*, Rocky Mountain Journal of Mathematics, Vol. 41 (2011), 249-274.

[39] T. Kusano, J. Manojlović, T. Tanigawa, *Existance and asymptotic behavior of positive solutions of fourth order quasilinear differential equations*, Taiwanese Journal of Mathematics, 17(3) (2013), 999 – 1030.

[40] T. Kusano, V. Marić, T. Tanigawa, *An asymptotic analysis of positive solutions of generalized Thomas-Fermi differential equations - the sub-half-linear case*, Nonlinear Anal. 75 (2012), 2474 – 2485.

[41] T. Kusano, A. Ogata, H. Usami, *Oscillation theory for a class of second order quasilinear ordinary differential equations with application to partial differential equations*, Japan. J. Math. 19 (1993) 131 – 147.

[42] T. Kusano, A. Ogata, H. Usami, *On the oscillation of solutions of second order quasilinear ordinary differential equations*, Hiroshima Math. J. 23 (1993) 645 – 667.

[43] T. Kusano, T.Tanigawa, *On the sructure of positive solutions of a class of fourth order nonlinear differential equations*, Annali di Mat. Pura ed Appl. 185(4) (2006), 521–536.

[44] B. M. Levitan, *Some questions of the theory of almost periodic functions I*, Uspehi Matem. Nauk (N.S.), 2(5(21)) (1947) 133 – 192.

[45] J.V. Manojlović, V. Marić, *An asymptotic analysis of positive solutions of Thomas-Fermi type differential equations - sublinear case*, Memoirs on Differential Equations and Mathematical Physics, Vol. 57 (2012), 75 – 94.

[46] J. Manojlović, J. Milošević, *Sharp oscillation criteria for fourth-order sub-half-linear and super-half-linear differential equations*, Electronic Journal of Qualitative Theory of Differential Equations No. 32 (2008), 1 – 13.

[47] V. Marić, *Regular Variation and Differential Equations*, Lecture Notes in Mathematics 1726, Springer-Verlar, Berlin-Heidelberg, 2000.

[48] V. Marić, M.Tomić, *Asymptotic properties of solutions of the equation* $y'' = f(x)\varphi(y)$, Math. Z 149 (1976) 261 – 266.

[49] V. Marić, M.Tomić, *Regular variation and asymptotic properties of solutions of nonlinear differential equations*, Publ. Inst. Math. (Beograd) (N.S.) 21(35)(1977), 119 – 129.

[50] V. Marić, M.Tomić, *Asymptotics of solutions of a generalized Thomas-Fermi equation*, J. Differential Equations 35, No. 1(1980), 36 – 44.

[51] V. Marić, M. Tomić, *A classification of solutions of second order linear differential equations by means of regularly varying functions*, Publ. Inst. Math. (Beograd) 48 (62) (1990), 199 – 207.

[52] V. Marić, M. Tomić, *Slowly varying solutions of second order linear differential equations*, Publ. Inst. Math. (Beograd) 58(72) (1995) 129 - 136.

[53] S. Matucci, P. Řehák, *Asymptotics of decreasing solutions of coupled* $p-$*Laplacian systems in the framework of regular variation*, Annali di Mat. Pura ed Appl. 193 Issue 3 (2014) 837 – 858.

[54] J. Milošević, *Asymptotic behavior of increasing positive solutions of second order quasilinear ordinary differential equations in the framework of regular variation*, Advances in Difference Equations (2015) 2015:273.

[55] J. Milošević, J.V. Manojlović, *Asymptotic analysis of fourth order quasilinear differential equations in the framework of regular variation*, Taiwanese Journal of Mathematics, (accepted)

[56] J. Milošević, J.V. Manojlović, *Positive decreasing solutions of second order quasilinear ordinary differential equations in the framework of regular variation*, Filomat, (accepted)

[57] J.D. Mirzov, *Asymptotic properties of solutions of the systems of nonlinear nonautonomous ordinary differential equations*, (Russian), Maikop, Adygeja Publ. (1993).

[58] J.D. Mirzov, *Asymptotic properties of solutions of systems of nonlinear nonautonomous ordinary differential equations*, volume 14 of Folia Facultatis Scientiarium Naturalium Universitatis Masarykianae Brunensis, Mathematica, Masaryk University, Brno, 2004.

[59] M. Mizukami, M. Naito, H. Usami, *Asymptotic behavior of solutions of a class of second order quasilinear ordinary differential equations*, Hiroshima Math. J. 32 (2002), 51 – 58.

[60] M. Naito, *On the asymptotic behavior of nonoscillatory solutions of second order quasilinear ordinary differential equations*, J. Math. Anal. Appl. 381 (2011), 315 – 327.

[61] M. Naito, *A note on the existence of slowly growing positive solutions to second order quasilinear ordinary differential equations*, Memoirs on Differential Equations and Mathematical Physics, 57 (2012), 95 – 108.

[62] M. Naito, *A remark on the existence of slowly growing positive solutions to second order super-linear ordinary differential equations*, Nonlinear Differ. Equ. Appl. 20 (2013), 1759 – 1769.

[63] M. Naito, F. Wu, *On the existance of eventually positive solutions of fourth order quasilinear differential equations*, Nonlinear Analysis 57 (2004), 253 – 263.

[64] M. Naito, F. Wu, *A note on the existance and asymptotic behavior of nonoscillatory solutions of fourth order quasilinear differential equations*, Acta Math. Hungar 102(3) (2004), 177 – 202.

[65] P. Řehák, *Asymptotic behavior of increasing solutions to a system of n nonlinear differential equations*, Nonlinear Analysis, 77 (2013), 45 – 58.

[66] P. Řehák, *On decreasing solutions of second order nearly linear differential equations*, Boundary Value Problems 2014, 2014:62 (2014) 1 – 13.

[67] E. Seneta, *Regularly Varying Functions*, Lecture Notes in Mathematics, Vol. 508, Springer Verlag, Berlin-Heidelberg-New York, 1976.

[68] T. Tanigawa,*Existence and asymptotic behavior of positive solutions of second order quasilinear differential equations*, Adv. Math. Sci. Appl. 9 (1999) 907-938.

[69] L. H. Thomas, *The calculation of atomic fields*, Proc. Cambridge Phil. Soc. 23 (1927), 542 – 548.

[70] J.Wang,*On second order quasilinear oscillations*, Funkcial. Ekvac., 41(1) (1998) 25-54.

[71] J. S. W. Wong, *On the generalized EmdenFowler equation*, SIAM Rev., 17 (1975) 339-360.

[72] J. S. W. Wong, *A nonoscillation theorem for EmdenFowler equations*, J. Math. Anal. Appl., 274(2) (2002) 746-754.

[73] F. Wu, *Nonoscillatory solutions of fourth order quasilinear differential equations*, Funkcialaj Ekvacioj, 45 (2002), 71 – 88.

[74] F. Wu, *Existence of eventually positive solutions of fourth order quasilinear differential equations*, J. Math. Anal. Appl. 389 (2012) 632 - 646.

Biography

Jelena Milošević was born on January 8th, 1979, in Niš, Serbia. She completed Radoje Domanović Elementary school in Niš, and Bora Stanković Grammar school, also in Niš.

In 1997/98 she enrolled the Faculty of Philosophy in Niš, at the Department of Mathematics, and graduated in 2001 with a grade point average of 9.52/10. In 2001 she began graduate studies at Faculty of Sciences and Mathematics in Niš, at the Department of Mathematics, which she finished with a grade point average of 10/10, and received her MA in 2008 by defending her master's thesis entitled "Oscillatory properties of solutions of fourth order nonlinear differential equations".

Since 2001 Jelena has been teaching at Faculty of Sciences and Mathematics in Niš (previously known as Faculty of Philosophy in Niš), where she held numerous courses at the Department of Mathematics, as well as at the other departments. Some of the main courses were Mathematical Analysis, Differential Equations, Partial Differential Equations, Dynamical Systems, Numerical Analysis, Business Mathematics, and so on. Several times, she was selected as a best professor by students. She has also participated in the projects supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia.

Jelena published six papers in international journals with IF.

Publications

1. J. Manojlović, J. Milošević, *Sharp oscillation criteria for fourth order sub-half-linear and super-half-linear differential equations*, Electronic Journal of Qualitative Theory of Differential Equations, 32 (2008) 1-13. **(M23)**

2. K.Takasi, J.Manojlović, J.Milošević, *Intermediate solutions of second order quasilinear ordinary differential equations in the framework of regular variation*, Applied Mathematics and Computation , 219 (2013) 8178-8191. **(M21)**

3. K.Takasi, J.Manojlović, J.Miloević, *Intermediate solutions of fourth order*

quasilinear differential equations in the framework of regular variation, Applied Mathematics and Computation , 248 (2014) 246-272. **(M21)**

4. J. Milošević, J.V. Manojlović, *Asymptotic analysis of fourth order quasilinear differential equations in the framework of regular variation*, Taiwanese Journal of Mathematics, (accepted)**(M22)**

5. J. Milošević, *Asymptotic behavior of increasing positive solutions of second order quasilinear ordinary differential equations in the framework of regular variation*, Advances in Difference Equations (2015) 2015:273. **(M22)**

6. J. Milošević, J.V. Manojlović, *Positive decreasing solutions of second order quasilinear ordinary differential equations in the framework of regular variation*, Filomat, (accepted) **(M22)**

Универзитет у Нишу

ИЗЈАВА О АУТОРСТВУ

Изјављујем да је докторска дисертација, под насловом

Асимптотска анализа решења нелинеарних диференцијалних једначина и Караматине правилно променљиве функције

која је одбрањена на Природно - математичком факултету Универзитета у Нишу:

- резултат сопственог истраживачког рада;
- да ову дисертацију, ни у целини, нити у деловима, нисам пријављивао/ла на другим факултетима, нити универзитетима;
- да нисам повредио/ла ауторска права, нити злоупотребио/ла интелектуалну својину других лица.

Дозвољавам да се објаве моји лични подаци, који су у вези са ауторством и добијањем академског звања доктора наука, као што су име и презиме, година и место рођења и датум одбране рада, и то у каталогу Библиотеке, Дигиталном репозиторијуму Универзитета у Нишу, као и у публикацијама Универзитета у Нишу.

У Нишу, 09.09.2015. год.

Аутор дисертације: Мр. Јелена Милошевић

Потпис аутора дисертације:

____

Универзитет у Нишу

ИЗЈАВА О ИСТОВЕТНОСТИ ШТАМПАНОГ И ЕЛЕКТРОНСКОГ ОБЛИКА ДОКТОРСКЕ ДИСЕРТАЦИЈЕ

Име и презиме аутора: Јелена Милошевић

Наслов дисертације: Асимптотска анализа решења нелинеарних диференцијалних једначина и Караматине правилно промењљиве функције

Ментор: Проф. др Јелена Манојловић

Изјављујем да је штампани облик моје докторске дисертације истоветан електронском облику, који сам предао/ла за уношење у **Дигитални репозиторијум Универзитета у Нишу.**

У Нишу, 09.09.2015. год.

Потпис аутора дисертације:

____

Универзитет у Нишу

ИЗЈАВА О КОРИШЋЕЊУ

Овлашћујем Универзитетску библиотеку „Никола Тесла“ да, у Дигитални репозиторијум Универзитета у Нишу, унесе моју докторску дисертацију, под насловом:

Асимптотска анализа решења нелинеарних диференцијалних једначина и Караматине_ правилно промењиве функције____

Дисертацију са свим прилозима предала сам у електронском облику, погодном за трајно архивирање.

Моју докторску дисертацију, унету у Дигитални репозиторијум Универзитета у Нишу, могу користити сви који поштују одредбе садржане у одабраном типу лиценце Креативне заједнице (Creative Commons), за коју сам се одлучила.

1. Ауторство **(CC BY)**

2. Ауторство – некомерцијално **(CC BY-NC)**

3. Ауторство – некомерцијално – без прераде **(CC BY-NC-ND)**

4. Ауторство – некомерцијално – делити под истим условима **(CC BY-NC-SA)**

5. Ауторство – без прераде **(CC BY-ND)**

6. Ауторство – делити под истим условима **(CC BY-SA)**

(Молимо да подвучете само једну од шест понуђених лиценци; опис лиценци дат је у Упутству).

У Нишу, 09.09.2015. год.

Аутор дисертације: __Мр. Јелена Милошевић______________________________

Потпис аутора дисертације:

____