UNIVERSITY OF NIŠ
FACULTY OF SCIENCE AND MATHEMATICS
DEPARTMENT OF MATHEMATICS

Jovana T. Nikolov Radenković

PSEUDOINVERSES AND REVERSE ORDER LAW FOR MATRICES AND OPERATORS

PhD thesis

Niš, 2015

UNIVERZITET U NIŠU
PRIRODNO-MATEMATIČKI FAKULTET
DEPARTMAN ZA MATEMATIKU

Jovana T. Nikolov Radenković

PSEUDOINVERZI I ZAKON OBRNUTOG REDOSLEDA ZA MATRICE I OPERATORE

Doktorska disertacija

Niš, 2015

Supervisor:	**Dragana Cvetković-Ilić** Full Professor Faculty of Sciences and Mathematics University of Niš, Serbia
Members of Committee:	**Vladimir Rakočević** Full Professor Faculty of Sciences and Mathematics University of Niš, Serbia
	Ljiljana Petković Full Professor Faculty of Mechanical Engineering University of Niš, Serbia
	Yimin Wei Full Professor Fudan University Shanghai, P.R. China
	Vladimir Pavlović Associate Professor Faculty of Sciences and Mathematics University of Niš, Serbia

Прилог 4/1

ПРИРОДНО - МАТЕМАТИЧКИ ФАКУЛТЕТ
НИШ

КЉУЧНА ДОКУМЕНТАЦИЈСКА ИНФОРМАЦИЈА

Редни број, **РБР**:	
Идентификациони број, **ИБР**:	
Тип документације, **ТД**:	монографска
Тип записа, **ТЗ**:	текстуални / графички
Врста рада, **ВР**:	докторска дисертација
Аутор, **АУ**:	Јована Т. Николов Раденковић
Ментор, **МН**:	Драгана Цветковић-Илић
Наслов рада, **НР**:	Псеудоинверзи и закон обрнутог редоследа за матрице и операторе
Језик публикације, **ЈП**:	енглески
Језик извода, **ЈИ**:	српски
Земља публиковања, **ЗП**:	Србија
Уже географско подручје, **УГП**:	Србија
Година, **ГО**:	2015.
Издавач, **ИЗ**:	ауторски репринт
Место и адреса, **МА**:	Ниш, Вишеградска 33.
Физички опис рада, **ФО**: (поглавља/страна/ цитата/табела/слика/графика/прилога)	87 стр.
Научна област, **НО**:	математика
Научна дисциплина, **НД**:	математичка анализа
Предметна одредница/Кључне речи, **ПО**:	функционална анализа
УДК	517.983.24 512.643
Чува се, **ЧУ**:	библиотека
Важна напомена, **ВН**:	
Извод, **ИЗ**:	У овој дисертацији разматран је закон обрнутог реда за матрице и операторе. Представљени су оригинални резултати везани за закон обрнутог редоследа два и више ограничених оператора на Хилбертовом простору. Ови резултати предсављају проширење већ постојећих резултата за комплексне матрице. Испитивано је и постојање ермитских и Ре-ннд генералисаних инверза. Описани су скупови таквих генералисаних инверза и услови под којима они постоје.
Датум прихватања теме, **ДП**:	03.03.2014.
Датум одбране, **ДО**:	
Чланови комисије, **КО**: Председник:	
Члан:	
Члан:	
Члан:	
Члан, ментор:	

Прилог 4/2

ПРИРОДНО - МАТЕМАТИЧКИ ФАКУЛТЕТ
НИШ

KEY WORDS DOCUMENTATION

Accession number, **ANO**:	
Identification number, **INO**:	
Document type, **DT**:	monograph
Type of record, **TR**:	textual / graphic
Contents code, **CC**:	doctoral dissertation
Author, **AU**:	Jovana T. Nikolov Radenković
Mentor, **MN**:	Dragana Cvetković-Ilić
Title, **TI**:	Pseudoinverses and reverse order law for matrices and operators
Language of text, **LT**:	English
Language of abstract, **LA**:	English
Country of publication, **CP**:	Serbia
Locality of publication, **LP**:	Serbia
Publication year, **PY**:	2015
Publisher, **PB**:	author's reprint
Publication place, **PP**:	Niš, Višegradska 33.
Physical description, **PD**: (chapters/pages/ref./tables/pictures/graphs/appendixes)	87 p. ;
Scientific field, **SF**:	mathematics
Scientific discipline, **SD**:	mathematical analysis
Subject/Key words, **S/KW**:	functional analysis
UC	517.983.24 512.643
Holding data, **HD**:	library
Note, **N**:	
Abstract, **AB**:	In this dissertation, reverse order laws for matrices and operators are investigated. Some original results concerning the various types of reverse order laws for the product of two and more than two bounded Hilbert space operators are presented. These results extended some previous results on the settings of complex matrices. Also, the existence of Re-nnd and Hermitian generalized inverses is investigated. Sets of such generalized inverses are described and necessary and sufficient conditions for their existence are derived.
Accepted by the Scientific Board on, **ASB**:	03.03.2014.
Defended on, **DE**:	
Defended Board, **DB**: President:	
Member:	
Member:	
Member:	
Member, Mentor:	

Contents

Preface

In this dissertation, reverse order laws for generalized inverses of matrices and operators are investigated. Some original results concerning the various types of reverse order laws for product of two or more than two bounded Hilbert space operators are presented. These results extend previous results on the settings of complex matrices. Also, the existence of Re-nnd and Hermitian generalized inverses is investigated. Sets of all Re-nnd and Hermitian generalized inverses are described and necessary and sufficient conditions for their existence are derived.

In the first chapter we introduce basic concepts of the theory of generalized inverses. In Section 1.1 we give some standard notation and terminology. Section 1.2 contains a brief history of developments in the theory of generalized inverses, where we discuss some of the most notable results. In section 1.3 we discuss genesis of generalized inverses and we state Moore's and Penrose's definitions of generalized inverse of a complex matrix. Furthermore, we describe some classes of generalized inverses and define weighted Moore-Penrose inverse and Drazin inverse. In Section 1.4 the set of solutions of matrix equation $AXB = C$ is described using some classes of generalized inverses and a relation between solutions of system of linear equations $Ax = b$ and generalized inverses is established. This will lead to defining some important classes of generalized inverses. Section 1.5 contains representations of some classes of generalized inverses which are obtained as sets of solutions of appropriate matrix equations. These representations are the basic tool for finding equivalent conditions for various reverse order laws. In Section 1.6 we introduce main properties of generalized inverses of operators on Hilbert spaces.

In the second chapter of the dissertation we consider the problem of the reverse order law for matrices and operators. In Section 2.1 we present some of the most important results related to the reverse order law for generalized inverses in various settings. Section 2.2 consists of original results published in [27] on the reverse order law for $\{1, 2, 3\}$-inverse of a matrix. In section 2.3 we present our results from [26] regarding the reverse order law for reflexive generalized inverse of an operator. In Section 2.4 we consider the reverse order law for $\{1\}$-, $\{1, 2\}$-, $\{1, 3\}$- and $\{1, 4\}$-inverses of product of n operators on Hilbert space. These results are direct improvements of the results of Wei [101] for complex matrices. Namely, Wei [101] used singular value decomposition of product (P-SVD) of matrices which is not applicable to infinite dimensional setting. In Section 2.5 we consider the reverse order law for weighted generalized inverses. Results from this section are published in [64]. In Section 2.6 we consider some additive results of $\{1,3\}$-, $\{1,4\}$- and $\{1,2,3\}$- and $\{1,2,4\}$-inverses.

The third chapter consists of our original results, published in [66], where we investigated the existence of Hermitian and Re-nnd generalized inverses. We find equivalent conditions for the existence of such generalized inverses and we completely describe these sets.

Finally, in the fourth chapter we introduce and study a special Schur complement of operators on Hilbert space.

At the end, I would like to express my sincere gratitude to my supervisor Professor Dragana Cvetković-Ilić for great commitment during our joint research and writing of the PhD thesis. I am also thankful to the other members of the committee and all professors who supported me through my education. Also I would like to thank my friends whom I might have neglected, for their understanding and support. Most importantly, I am infinitely thankful to my family for all their love and encouragement. They always had patience for me, and were a constant source of support and care that gave me strength all these years.

Chapter 1

Introduction

1.1 Notations

By $\mathbb{C}$ we will denote the field of complex numbers. Let $\mathbb{C}^{m\times n}$ be the vector space of all $m \times n$ complex matrices over $\mathbb{C}$. $\mathbb{C}^n$ is the vector-space of all n-tuples of complex numbers over $\mathbb{C}$. If $A \in \mathbb{C}^{m\times n}$, then in respect to the standard basis of $\mathbb{C}^n$ and $\mathbb{C}^m$, A induces a linear transformation $A : \mathbb{C}^n \to \mathbb{C}^m$. $\mathcal{R}(A)$ denotes the range of A, that is linear span of the columns of A. The null-space of A, $\mathcal{N}(A) = \{x \in \mathbb{C}^n : Ax = 0\}$. We use $r(A)$ and A^* to denote the rank and the conjugate transpose matrix of a matrix A, respectively. I always denotes identity matrix. For square matrix A, the index of A is the least nonnegative integer k such that $r(A^{k+1}) = r(A^k)$ and it is denoted by $ind(A)$.

Let X, Y be Banach spaces and $\mathcal{L}(X, Y)$ the set of all linear bounded operators from X to Y. For a given $A \in \mathcal{L}(X, Y)$, the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range of operator A, respectively. M and N are algebraically complemented subspaces of X if $M \cap N = \{0\}$ and $M + N = X$. A linear mapping $P : X \to X$ is called a projection if $P^2 = P$. If P is a projection, then $I - P$ is also a projection, $\mathcal{R}(P) = \mathcal{N}(I-P)$, $\mathcal{N}(P) = \mathcal{R}(I-P)$ and $\mathcal{R}(P)$, $\mathcal{N}(P)$ are algebraically complemented. Conversely, if M and N are algebraically complemented subspaces of X, then there is a unique projection P such that $\mathcal{R}(P) = M$ and $\mathcal{N}(P) = N$. We say that P is a projection of X on M parallel to N (for interesting properties of projections see [52], [50]). If M and N are closed and algebraically complemented subspaces of X, then subspaces M and N are called topologically complemented or simply complemented. We say that X is topological sum of M and N and we write $X = M \oplus N$. Not every closed subspace of Banach space has a topological complement. If projection P is bounded, then $X = \mathcal{R}(P) \oplus \mathcal{N}(P)$. Conversely, if $X = M \oplus N$, then the projection P with range M and null-space N is bounded.

Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be complex Hilbert spaces and let $\mathcal{B}(\mathcal{H}, \mathcal{K})$ denote the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. For an operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, by A_l^{-1} (A_r^{-1}) we denote the left (right) inverse of A and by $\mathcal{B}_l^{-1}(\mathcal{H}, \mathcal{K})$ ($\mathcal{B}_r^{-1}(\mathcal{H}, \mathcal{K})$) the set of all left (right) invertible operators from the space $\mathcal{B}(\mathcal{H}, \mathcal{K})$. For given sets M, N, by MN or $M \cdot N$ we denote the set consisting of all products XY, where $X \in M$ and $Y \in N$. Unlike in Banach space, closed subspace of Hilbert space is always complemented.

1.2 Historical notes on generalized inverses

In Gauss' "Theoria combinations" (1821), which deals with method of least squares, several formulas appear which relate to expressions found in the modern theory of generalized inverses. Although generalized inverses were not a part of Gauss' vocabulary, equivalent expressions may be found in his writings. (See Robinson [79]). In the setting of integral and differential operators the concept of generalized inverses was considered by Fredholm [38] (1903) and Hurwitz [47] (1912), and in a paper on generalized Green's functions written in 1904 by Hilbert. Generalized inverses of differential operators were consequently studied by numerous authors, in particular, Myller (1906), Westfall (1909), Bounitzky [13] in 1909, Elliott (1928), and Reid (1931). (For a history of this subject see the excellent survey by Reid [78] and Ben-Israel and Greville [8].)

Moore was the first who give a systematic study of generalized inverses. In a paper given at the Fourteenth Western Meeting of the American Mathematical Society at the University of Chicago, April, 1920, E. H. Moore defined a unique inverse (called by him the "general reciprocal") for every finite matrix (square or rectangular) (see Definition 1.3.1). E. H. Moore established the existence and uniqueness of $A^{\dagger}$ for any A, and gave an explicit form for $A^{\dagger}$ in terms of the subdeterminants of A and A^{*}. His work received practically no attention in the next 30 years, mostly because it used very complicated notation. To illustrate the difficulty of reading the original Moore, and the need for translation, we restate the theorem:

Theorem 1.2.1 $[Moore, 1920]$

$$\mathfrak{U}^{C}\mathfrak{B}^{1\ II}\mathfrak{B}^{2\ II}\kappa^{12}\cdot)\cdot$$
$$\exists \mid \lambda^{21\ \mathit{type}\ \mathfrak{M}^{\cdot}_{\kappa*}\overline{\mathfrak{M}^{\cdot}_{\kappa}}} \ni \cdot S^{2}\kappa^{12}\lambda^{21} = \delta^{11}_{\mathfrak{M}^{\cdot}_{\kappa}} \cdot S^{1}\lambda^{21}\kappa^{12} = \delta^{22}_{\mathfrak{M}^{\cdot}_{\kappa*}}.$$

During that time generalized inverses were given for matrices by Siegel [84] in 1937, and for operators by Tseng [96, 93, 94, 95], Murray and von Neumann [61] in 1936, Atkinson [5, 6] and others.

Unaware of Moore's work, A. Bjerhammar and R. Penrose both independently investigated the pseudoinverse. In 1951 Bjerhamar recognized the least squares properties of certain generalized inverses and noted the relation between some generalized inverses and solutions to linear systems [9, 10, 11] . In 1955, Penrose [68] sharpened and extended Bjerhammar's results on linear systems, and showed that Moore's inverse, for a given matrix A, is the unique matrix X satisfying the following four equations:

$$AXA = A$$
$$XAX = X$$
$$(AX)^{*} = AX$$
$$(XA)^{*} = XA$$

The latter discovery has been so important that this unique inverse is now commonly called the Moore–Penrose inverse. Since 1955 the theory of generalized inverses and also the applications and computational methods have been developing rapidly. Many authors investigated various types of generalized inverses [77, 41, 7]. Also, generalized inverses which satisfy some of the four Penrose's equations have been considered. Generalized inverses of singular linear operators, elements in various algebraic and topological structures have been studied.

One of the books that made major impact on the subject is "Generalized inverses - Theory and Applications" by A. Ben-Israel and T.N.E. Greville (1974) [8]. It contains seven chapters which treat generalized inverses of finite matrices, while the eighth introduces generalized inverses of operators between Hilbert spaces. Another excellent book on generalized inverses is one by S. L. Campbell and C. D. Meyer (1979) [15]. It presents unified treatment of the theory of generalized inverses and contains many diverse applications where generalized inverses play important role. Proceedings on of the Advanced Seminar on Generalized inverses and Applications held at the University of Wisconsin-Madison in 1973 edited by M.Z. Nashed contain 14 papers dealing with basic theory of generalized inverses, applications to analysis and linear equations, numerical analysis and approximation methods, applications to statistics and econometrics, optimization, system theory, operations research. It contains an exhaustive bibliography which includes all related references up til 1975. One of recent publications on this subject is book by G. Wang, Y. Wei, S. Qiao [43]. It contains results from dozens of published papers since 1979 on generalized inverses in the areas of perturbation theory, condition numbers, recursive algorithms etc. V. Rakočević and D. Djordjević [34] presented various results regarding generalized inverses of linear bounded operators on Banach and Hilbert spaces, and generalized inverses of elements in Banach and C^*-algebras.

1.3 Generalized inverses of complex matrices

It is well-known that if A is a nonsingular square matrix, then there exists a unique matrix B, such that $AB = BA = I$, where I is the identity matrix. Then B is called the inverse of A and is denoted by A^{-1}. In practical problems we more often deal with singular or rectangular matrices so there was a need of defining new kind of inverse which would have some of the properties of the ordinary inverse of a matrix. The most familiar application of generalized inverses is to the solution of a system of linear equations, which appears in many pure and applied problems. If matrix $A \in \mathbb{C}^{n\times n}$ is regular, the system of linear equations $Ax = b$ has the unique solution $x = A^{-1}b$ for each b. In many cases, solution of a system of linear equations exists even when the inverse of the matrix A does not exists. For A singular or rectangular, there may sometimes be no solutions or an infinite number of solutions. If $b \in \mathcal{R}(A)$, i.e. vector b is some linear combination of columns of A, system has a solution and if $n - r(A) > 0$, there is a multiplicity of solutions. In general, solution of a system of linear equations exist even when the inverse A^{-1} does not exist. Sometimes, when solution of system of linear equations does not exist, we may be interested in some kind of pseudosolution, for example, the one which minimizes the error $\|Ax - b\|$. For different purposes, there exist different types of pseudoinverses.

By a generalized inverse of a given matrix A we call a matrix X such that

(a) exists for a wider class of matrices than the class of nonsingular matrices;

(b) has some of the properties of the usual inverse;

(c) reduces to the usual inverse when A is nonsingular.

Generalized inverses of matrices were first introduced by E. H. Moore. He constructed the unique inverse for every finite matrix, which he called the "general reciprocal". First publication on this subject is an abstract of his talk given at the Fourteenth Western Meeting of the American Mathematical Society at the University of Chicago, April, 1920. Because of the complexity of the notation of the original Moore paper, we will state the A.Ben-Israel and Charnes [7] interpretation of Moore definition of the generalized inverse:

Definition 1.3.1 [Moore] *If $A \in \mathbb{C}^{m\times n}$, then the generalized inverse of A is the unique matrix $A^\dagger$ such that*

a) $AA^\dagger = P_{R(A)}$;

b) $A^\dagger A = P_{R(A^\dagger)}$.

Unaware of Moore's results, in 1955. R.Penrose [68] gave algebraic definition of generalized inverse:

Definition 1.3.2 [Penrose] *If $A \in \mathbb{C}^{m\times n}$, then the generalized inverse of A is the unique matrix $A^\dagger$ such that*

(1) $AA^\dagger A = A$;

(2) $A^\dagger AA^\dagger = A^\dagger$;

(3) $(AA^\dagger)^* = AA^\dagger$;

(4) $(A^\dagger A)^* = A^\dagger A$.

These two definitions are equivalent, and the proof of the equivalence can be found in [14]. In honour of Moore and Penrose's contribution, the unique matrix from Definitions 1.3.1 and 1.3.2 is called the *Moore-Penrose inverse* of matrix A and is usually denoted by $A^\dagger$. Equations $(1) - (4)$ are called Penrose equations. Generalized inverses which satisfy some, but not all, of the four Penrose equations play an important role in solution of systems of linear equations. Since we will often deal with a number of different subsets of the set of four equations, we need a convenient notation for a generalized inverse satisfying certain specified equations.

Definition 1.3.3 *For any $A \in \mathbb{C}^{m\times n}$ let $A\{i,j,\ldots,k\}$ denote the set of all matrices $X \in \mathbb{C}^{n\times m}$ which satisfy equations $(i),(j),\ldots,(k)$ from among the Penrose equations (1)–(4). A matrix $X \in A\{i,j,\ldots,k\}$ is called an $\{i,j,\ldots,k\}$-inverse of A, and arbitrary $\{i,j,\ldots,k\}$-inverse of A is denoted by $A^{(i,j,\ldots,k)}$.*

Obviously, $A\{1,2,3,4\} = A^\dagger$. If matrix A is nonsingular square matrix, then A^{-1} satisfies all four Penrose equations and because of the uniqueness of the Moore-Penrose inverse, $A^\dagger = A^{-1}$. $\{1\}$-inverse is often called inner generalized inverse, $\{2\}$-inverse is outer and $\{1,2\}$-inverse is reflexive generalized inverse. $\{1,3\}$- and $\{1,4\}$-inverses are, because of their properties, also called least squares g-inverse and minimum norm g-inverses, respectively.

One generalization of the Moore-Penrose inverse is the, so called, weighted Moore-Penrose inverse. Let $A \in \mathbb{C}^{m\times n}$ and let $M \in \mathbb{C}^{m\times m}$, $N \in \mathbb{C}^{n\times n}$ be two positive definite matrices. Moore-Penrose inverse $A^\dagger$ transforms into the weighted Moore-Penrose inverse $A^\dagger_{M,N}$ after the replacement of the usual vector inner product in $\mathbb{C}^m$ and $\mathbb{C}^n$ by the following weighted inner products

$$\langle x,y\rangle_M = y^* M x, \quad x,y \in \mathbb{C}^m, \qquad \langle x,y\rangle_N = y^* N x, \quad x,y \in \mathbb{C}^n.$$

Definition 1.3.4 *Let $A \in \mathbb{C}^{m\times n}$ and let $M \in \mathbb{C}^{m\times m}$, $N \in \mathbb{C}^{n\times n}$ be two positive definite matrices. The unique matrix X which satisfies*

$$(1)\ AXA = A,\ (2)\ XAX = X,\ (3)\ (MAX)^* = MAX,\ (4)\ (NXA)^* = NXA,$$

is called the weighted Moore-Penrose inverse of A and it is denoted by $A^\dagger_{M,N}$.

Obviously for $M = I_m$ and $N = I_n$ the weighted Moore-Penrose inverse of A is the Moore-Penrose inverse of A.

In 1958, Drazin [37] introduced a pseudoinverse in associative rings and semigroups that now carries his name. The inverse was extensively studied and applied in matrix setting [15, 85, 75, 74], as well as in the setting of bounded linear operators and elements of Banach algebras [63]. Drazin inverse has a very desirable spectral property: The nonzero eigenvalues of the Drazin inverse are the reciprocals of the nonzero eigenvalues of the given matrix, and the corresponding generalized eigenvectors have the same grade. On the settings of matrices, Drazin inverse is only defined on the subset of square matrices:

Definition 1.3.5 *Let $A \in \mathbb{C}^{n\times n}$ and let k be the index of A. The unique matrix which satisfies*

(1^k) $A^{k+1}A^d = A^k$;

(2) $A^dAA^d = A^d$;

(5) $A^dA = AA^d$.

is called the Drazin inverse of A and it is denoted by A^d .

1.4 Generalized inverses and solutions of linear systems of equations

We have already mentioned the importance of generalized inverses in solving the linear system $Ax = b$ where $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$. The following theorem, proved by Penrose [68], is very important and useful result for describing general solution of given linear system and characterizing some sets of $\{i, j, \dots, k\}$-inverses.

Theorem 1.4.1 *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$ and $D \in \mathbb{C}^{m\times q}$. Then the matrix equation*

$$AXB = D \tag{1.1}$$

is consistent if and only if for some $A^{(1)}$, $B^{(1)}$,

$$AA^{(1)}DB^{(1)}B = D,$$

in which case the general solution is

$$X = A^{(1)}DB^{(1)} + Y - A^{(1)}AYBB^{(1)} \tag{1.2}$$

for arbitrary $Y \in \mathbb{C}^{n\times p}$.

Specializing Theorem 1.4.1 to ordinary system of linear equations gives

Corollary 1.4.1 *Let $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^n$. Then the system of linear equations $Ax = b$ is consistent if and only if for some $A^{(1)}$,*

$$AA^{(1)}b = b, \tag{1.3}$$

in which case the general solution is

$$x = A^{(1)}b + (I - A^{(1)}A)y \tag{1.4}$$

for arbitrary $y \in \mathbb{C}^n$.

The following theorem proved in [80] gives an alternative representation od $A\{1\}$.

Theorem 1.4.2 *Let $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times m}$. Then $X \in A\{1\}$ if and only if for all b such that $Ax = b$ is consistent, $x = Xb$ is a solution.*

Using arbitrary $\{1\}$-inverse, we can describe the set of all solutions when the system of linear equations $Ax = b$ is consistent. The linear system $Ax = b$ is consistent if and only if $b \in R(A)$. Otherwise, the residual vector $r = b - Ax$ is nonzero for all $x \in C^n$, and there is often interest in finding the approximate solution, i.e. vector x which makes the residual the closest to the zero in some sense. The most commonly used approximate solution of given system is the least-square solution. The least-square solution is the vector x which minimizes the Euclidean norm of the residual vector

$$\|b - Ax\|^2 = \sum_{i=1}^{m} \big|b_i - \sum_{j=1}^{n} a_{ij}x_j\big|^2.$$

The following theorem, proved in [8], establishes the relation between the $\{1, 3\}$-inverses and the least–squares solutions of $Ax = b$.

Theorem 1.4.3 *Let $A \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^m$. Then $\|b-Ax\|$ is the smallest for $x = A^{(1,3)}b$, where $A^{(1,3)} \in A\{1,3\}$. Conversely, if $X \in \mathbb{C}^{n\times m}$ has the property that, for all b, $\|b - Ax\|$ is smallest when $x = Xb$, then $X \in A\{1,3\}$.*

Corollary 1.4.2 *A vector x is a least-squares solution of $Ax = b$ if and only if $Ax = AA^{(1,3)}b$. Thus, the general least–squares solution is*

$$x = A^{(1,3)}b + (I_n - A^{(1,3)}A)y,$$

with $A^{(1,3)} \in A\{1,3\}$ and arbitrary $y \in \mathbb{C}^n$.

When the system $Ax = b$ has a multiplicity of solutions for x, there is a unique solution of minimum norm. The following theorem relates minimum-norm solutions of $Ax = b$ and $\{1,4\}$-inverses of A.

Theorem 1.4.4 *Let $A \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^m$. If $Ax = b$ is solvable, then the unique solution for which $\|x\|$ is the smallest is given by*

$$x = A^{(1,4)}b,$$

where $A^{(1,4)} \in A\{1,4\}$. Converesely, if $X \in \mathbb{C}^{n\times m}$ is such that, whenever $Ax = b$ is solvable, $x = Xb$ is the solution of minimum norm, then $X \in A\{1,4\}$.

The unique minimum-norm least-squares solution of $Ax = b$, and the Moore-Penrose inverse $A^\dagger$ of A, are related as follows.

Corollary 1.4.3 [*69*] *Let $A \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^m$. Then, among the least-squares solutions of $Ax = b$, $A^\dagger b$ is the one of minimum-norm. Conversely, if $X \in \mathbb{C}^{n\times m}$ has the property that, for all b, Xb is the minimum-norm least-squares solution of $Ax = b$, then $X = A^\dagger$.*

The minimum-norm least-squares solution, $x_0 = A^\dagger b$ (also called the approximate solution; e.g., Penrose [69]) of $Ax = b$, can thus be characterized by the following two inequalities:

$$\|Ax_0 - b\| \leq \|Ax - b\| \quad \text{for all} \quad x \in \mathbb{C}^n \tag{1.5}$$

and

$$\|x_0\| < \|x\|$$

for any $x \neq x_0$ which gives equality in (1.5).

1.5 Characterization of some classes of generalized inverses

In this section we restate some well known characterizations of some classes of generalized inverses that will be often used.

Bjerhammar [9] gave the following characterization of the set $A\{1\}$.

Theorem 1.5.1 *Let $A \in \mathbb{C}^{m\times n}$ and $A^{(1)} \in A\{1\}$. Then*

$$A\{1\} = \{A^{(1)} + Z - A^{(1)}AZAA^{(1)} : Z \in \mathbb{C}^{n\times m}\}.$$

The following theorems using Theorem 1.4.1 provide characterization of sets $A\{1,3\}$ and $A\{1,4\}$ (for the proof see [7]).

Theorem 1.5.2 *Let $A \in \mathbb{C}^{m\times n}$. The set $A\{1,3\}$ consists of all solutions of the equation*

$$AX = AA^{(1,3)},$$

where $A^{(1,3)}$ is arbitrary element of $A\{1,3\}$.

Theorem 1.5.3 *Let $A \in \mathbb{C}^{m\times n}$. The set $A\{1,4\}$ consists of all solutions of the equation*

$$XA = A^{(1,4)}A,$$

where $A^{(1,4)}$ is arbitrary element of $A\{1,4\}$.

As corollaries, we get the following:

Corollary 1.5.1 *Let $A \in \mathbb{C}^{m\times n}$, $A^{(1,3)} \in A\{1,3\}$. Then*

$$A\{1,3\} = \{A^{(1,3)} + (I - A^{(1,3)}A)Z : Z \in \mathbb{C}^{n\times m}\}.$$

Corollary 1.5.2 *Let $A \in \mathbb{C}^{m\times n}$, $A^{(1,4)} \in A\{1,4\}$. Then*

$$A\{1,4\} = \{A^{(1,4)} + Z(I - AA^{(1,4)}) : Z \in \mathbb{C}^{n\times m}\}.$$

The following lemma is an elementary but useful result which proof will be given because of the completeness.

Lemma 1.5.1 *Let $A \in \mathbb{C}^{n\times m}$. Then*

$$A\{1,2,3\} = \{A^{\dagger} + (I - A^{\dagger}A)ZAA^{\dagger} : Z \in \mathbb{C}^{m\times n}\}. \tag{1.6}$$

Proof. Let $X \in A\{1,2,3\}$. Since $X \in A\{1,3\}$, it follows that

$$X = A^{\dagger} + (I - A^{\dagger}A)Y$$

for some $Y \in \mathbb{C}^{m\times n}$. Since $X \in A\{2\}$, it has to be

$$(A^{\dagger} + (I - A^{\dagger}A)Y)A(A^{\dagger} + (I - A^{\dagger}A)Y) = A^{\dagger} + (I - A^{\dagger}A)Y$$

which is equivalent to

$$(I - A^{\dagger}A)Y(I - AA^{\dagger}) = 0. \tag{1.7}$$

Now, the set of the solutions of (1.7) is described by

$$S_Y = \{Z - (I - A^{\dagger}A)Z(I - AA^{\dagger}) : Z \in \mathbb{C}^{m\times n}\}.$$

So, it follows that

$$A\{1,2,3\} \subseteq \{A^{\dagger} + (I - A^{\dagger}A)ZAA^{\dagger} : Z \in \mathbb{C}^{m\times n}\}.$$

The opposite is obvious. □

By taking adjoint we obtain the analogous result in the case of $\{1,2,4\}-$ generalized inverses.

Lemma 1.5.2 *Let $A \in \mathbb{C}^{n\times m}$. Then*

$$A\{1,2,4\} = \{A^{\dagger} + A^{\dagger}AZ(I - AA^{\dagger}) : Z \in \mathbb{C}^{m\times n}\}. \tag{1.8}$$

Notice that all given representations can be extended to the case of bounded linear operators on Hilbert spaces under the assumption of operator regularity.

1.6 Generalized inverses of operators

We will now introduce generalized inverses for bounded linear operators on Banach and Hilbert spaces. Let X, Y be Banach spaces and let $\mathcal{L}(X,Y)$ denote the set of all linear bounded operators from X to Y.

Definition 1.6.1 *Let $A \in \mathcal{L}(X,Y)$. If there exists some $B \in \mathcal{L}(Y,X)$ such that $ABA = A$ holds, then B is an inner generalized inverse of A, and the operator A is inner regular.*
If $CAC = C$ holds for some $C \in \mathcal{L}(Y,X)$, $C \neq 0$, then C is outer generalized inverse of A. In this case, A is outer regular.
An operator $D \in \mathcal{L}(Y,X)$ is a reflexive generalized inverse of A, if D is both inner and outer generalized inverse of A.

If $B \in \mathcal{L}(Y, X)$ is inner generalized inverse of A, then BAB is a reflexive generalized inverse of A. Because of that, A has reflexive generalized inverse if and only if A is inner regular. So inner regularity implies outer regularity. The following theorems, proved in [36], give necessary and sufficient conditions for inner regularity of an operator.

Theorem 1.6.1 *If $B \in \mathcal{L}(Y, X)$ is an inner generalized inverse of $A \in \mathcal{L}(X, Y)$, then AB is a projection from Y on $R(A)$ and $N(A)$ are complemented subspaces of Y and X, respectively.*

Theorem 1.6.2 *Let $A \in \mathcal{L}(X, Y)$. If $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are closed and complemented subspaces of Y and X respectively, then A is inner regular.*

Corollary 1.6.1 *An operator $A \in \mathcal{L}(X, Y)$ is inner regular, if and only if $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are closed and complemented subspaces of Y and X, respectively.*

If A is inner regular, then its reflexive generalized inverse is uniquely determined by taking its appropriate range and null space. If T and S are closed subspaces of X and Y respectively, such that $X = T \oplus \mathcal{N}(A)$ and $Y = \mathcal{R}(A) \oplus S$, then there is unique reflexive inverse $B \in \mathcal{L}(Y, X)$ of A, such that $\mathcal{R}(B) = T$ and $\mathcal{N}(B) = S$.

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Closed spaces of a Hilbert space are always complemented, so iit follows that A is inner regular if and only if $\mathcal{R}(A)$ is closed. Since reflexive generalized inverse is uniquely determined by its range and null-space, there is unique $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $ABA = A$, $BAB = B$, $\mathcal{R}(B) = \mathcal{N}(A)^{\perp} = \mathcal{R}(A^*)$ and $\mathcal{N}(B) = \mathcal{R}(A)^{\perp} = \mathcal{N}(A^*)$. Such inverse is called the Moore-Penrose inverse of A. The usual notation for the Moore-Penrose inverse is $A^{\dagger}$. In this case $AA^{\dagger}$ is the projection from $\mathcal{K}$ onto $\mathcal{R}(A)$ parallel to $\mathcal{N}(A^*)$, and $A^{\dagger}A$ is the projection from $\mathcal{H}$ onto $\mathcal{R}(A^*)$ parallel to $\mathcal{N}(A)$. The existence and the uniqueness of the Moore-Penrose inverse of A can be described with the Penrose's equations.

Definition 1.6.2 *Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ have a closed range. Then there exists the unique operator $A^{\dagger} \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ (known as the Moore-Penrose) inverse of A satisfying following equations:*

(1) $AA^{\dagger}A = A$;

(2) $A^{\dagger}AA^{\dagger} = A^{\dagger}$;

(3) $(AA^{\dagger})^* = AA^{\dagger}$;

(4) $(A^{\dagger}A)^* = A^{\dagger}A$.

Now we will restate some well-known properties of the Moore-Penrose inverse.

Theorem 1.6.3 *[40] Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ have a closed range.*

(1) $(A^{\dagger})^{\dagger} = A$, $(A^*)^{\dagger} = (A^{\dagger})^*$;

(2) $(\lambda A)^\dagger = \lambda^{-1}A^\dagger$, $\lambda \in \mathbb{C}\ \{0\}$;

(3) $(A^*A)^\dagger = A^\dagger(A^\dagger)^*$;

(4) $A^* = A^\dagger AA^* = A^*AA^\dagger$;

(5) $A^\dagger = (A^*A)^\dagger A^* = A^*(AA^*)^\dagger$;

(6) $(UAV)^\dagger = V^*A^\dagger U^*$ *if* $U \in \mathcal{B}(\mathcal{K})$ *and* $V \in \mathcal{B}(\mathcal{H})$ *are unitary;*

(7) $\mathcal{R}(A) = \mathcal{R}(AA^\dagger) = \mathcal{R}(AA^*)$;

(8) $\mathcal{R}(A^\dagger) = \mathcal{R}(A^*) = \mathcal{R}(A^\dagger A) = \mathcal{R}(A^*A)$;

(9) $\mathcal{R}(A^\dagger) = \mathcal{R}(A^*) = \mathcal{R}(A^\dagger A) = \mathcal{R}(A^*A)$.

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Consider the equation

$$Ax = b. \tag{1.9}$$

If A is invertible, then $A^{-1}b$ is the unique solution of of (1.9). If A is not invertible and $b \in \mathcal{R}(A)$ then there exist a solution (possible several solutions) of the equation (1.9). If $b \in \mathcal{R}(A)$. In last two cases it is possible to use generalized inverses of A in order to obtain generalized solutions of (1.9), or pseudo solutions of (1.9). There are several classes of pseudo solutions of (1.9).

Definition 1.6.3 *A vector $x_0 \in \mathcal{K}$ is the best approximate solution of the equation* (1.9)*, if the following holds:*

$$\|Ax_0 - b\| = \min_{x \in X} \|Ax - b\|.$$

Theorem 1.6.4 [*34*] *Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces. Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ have a closed range and $b \in \mathcal{K}$. Then $x_0 = A^\dagger b$ is the best approximate solution of the linear equation $Ax = b$. Moreover, if M is the set of all best approximate solutions of the equation $Ax = b$, then $x_0 = \min\{\|x\| : x \in M\}$.*

Let $M \in \mathcal{B}(\mathcal{K})$ and $N \in \mathcal{B}(\mathcal{H})$ be positive (and invertible) operators. Then we can introduce new inner products:

$$\langle x, y\rangle_N = \langle Nx, y\rangle \quad \text{in } \mathcal{H}, \qquad \langle u, v\rangle_M = \langle Mu, v\rangle \quad \text{in } \mathcal{K}.$$

Then $\mathcal{H}$ and $\mathcal{K}$ become Hilbert spaces with respect to new inner products. Also, a linear operator $A : \mathcal{H} \to \mathcal{K}$ is bounded with respect to previous inner products if and only if it is bounded with respect to new inner products. If A is relatively regular, then there exists the unique operator $B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $ABA = A$, $BAB = B$ and AB is orthogonal projection with respect to $\langle \cdot, \cdot\rangle_M$ and BA is orthogonal projection with respect to $\langle \cdot, \cdot\rangle_N$. Hence the following result holds.

Theorem 1.6.5 *Let $M \in \mathcal{B}(\mathcal{K})$ and $N \in \mathcal{B}(\mathcal{H})$ be positive (and invertible) operators and let let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ have a closed range. Then there exists the unique operator $B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that the following hold matrix X which satisfies*

$$(1)\ ABA = A,\ (2)\ BAB = B,\ (3)\ (MAB)^* = MAB,\ (4)\ (NBA)^* = NBA.$$

Such B is called the weighted Moore-Penrose inverse of A with respect to the weights M and N and it is denoted by $A^{\dagger}_{M,N}$.

Chapter 2

Reverse order law

2.1 Recent results on the reverse order laws

A problem of finding generalized inverses of product of two matrices led to investigation of so-called "reverse order law". If A and B are two regular matrices of same size, the product AB is also regular, and the usual inverse of AB is

$$(AB)^{-1} = B^{-1}A^{-1}.$$

This equality is called the reverse order law and it cannot be trivially extended to generalized inverses of matrix product. Reverse order laws for generalized inverses have been investigated in the literature since 1960s. Greville in his paper [40] first gave necessary and sufficient condition for the reverse order law for the Moore-Penrose inverse of product of two matrices to hold:

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger} \quad \Leftrightarrow \quad \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \quad \text{and} \quad \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*). \tag{2.1}$$

This was followed by further equivalent conditions for (2.1). For example, Arghiriade [4] proved that $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ holds if and only if A^*ABB^* is range-Hermitian, i.e. $\mathcal{R}(A^*ABB^*) = \mathcal{R}(BB^*A^*A)$. Also, further research on this subject has branched in several directions:

- Cases with more than two matrices were studied;
- Different classes of generalized inverses were considered;
- For elements in different settings (operator algebras, C^*-algebras, rings etc.)

Many authors considered triple reverse order law

$$(ABC)^{\dagger} = C^{\dagger}B^{\dagger}A^{\dagger}$$

on the set of matrices (see [45, 92, 87]).

When we consider the reverse order law for K-inverse, where $K \subseteq \{1,2,3,4\}$, we actually consider the following inclusions:

$$BK \cdot AK \subseteq (AB)K,$$
$$(AB)K \subseteq BK \cdot AK,$$
$$(AB)K = BK \cdot AK.$$

Reverse order law for inner inverses of matrices were investigated by many authors (see [82, 83, 104]). Rao [76] gave necessary and sufficient conditions for inclusion

$$B\{1\}A\{1\} \subseteq (AB)\{1\}. \tag{2.2}$$

He proved that (2.2) holds if and only if A is of full column rank or B is of full row rank. Finding equivalent conditions for

$$(AB)\{1\} = B\{1\}A\{1\} \tag{2.3}$$

is more diffucult problem. Shinozaki and Sibuya [82] derived equivalent conditions for $(AB)\{1\} \subseteq B\{1\}A\{1\}$. Werner [104] proved the following:

Theorem 2.1.1 [*104*] *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. The following conditions are equivalent:*

(i) $B\{1\}A\{1\} \subseteq (AB)\{1\}$

(ii) *for each $\mathcal{M} \in \mathcal{N}_c(A)$ and each $\mathcal{S} \in \mathcal{R}_c(B)$, we have $\mathcal{R}(B) \subseteq \mathcal{M} \oplus [\mathcal{N}(A) \cap \mathcal{S}] \oplus [\mathcal{N}(A) \cap \mathcal{R}(B)]$*

(iii) *for each $\mathcal{M} \in \mathcal{N}_c(A)$ and each $\mathcal{S} \in \mathcal{R}_c(B)$, we have $\mathcal{R}(A^*) \subseteq \mathcal{S}^\perp \oplus [\mathcal{N}(B^*) \cap \mathcal{M}^\perp] \oplus [\mathcal{N}(B^*) \cap \mathcal{R}(A^*)]$*

(iv) *$\mathcal{N}(A) \subseteq \mathcal{R}(B)$ and/or $\mathcal{R}(B) \subseteq \mathcal{N}(A)$,*

where $\mathcal{N}_c(A)$ is the set of all direct complements of $\mathcal{N}(A)$ and $\mathcal{R}_c(B)$ is the set of all direct complements of $\mathcal{R}(B)$.

He also proved that (2.3) holds in each of the following cases in particular:

(i) A and B are both of full column rank

(ii) A and B are both of full row rank

(iii) A is nonsingular and/or B is nonsingular

but he didn't solve (2.3) completely.

Later, Wei [100] derived new necessary and sufficient conditions for inclusions (2.2) and (2.3) using product singular value decomposition of a matrix (P-SVD):

Theorem 2.1.2 [*100*] *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. The following conditions are equivalent:*

(*i*) $(AB)\{1\} = B\{1\}A\{1\}$

(*ii*) (*a*) $\mathcal{R}(B) \subseteq \mathcal{N}(A)$ *and* $n \geq \min\{m + R(B), p + r(A)\}$*, or*
(*b*) $\mathcal{N}(A) \subseteq \mathcal{R}(B)$*, and* $m = r(A)$ *or* $p = r(B)$

(*iii*) (*a*) $r(AB) = 0$ *and* $n \geq \min\{m + R(B), p + r(A)\}$*, or*
(*b*) $r(A) + r(B) - r(AB) = n$*, and m=r(A) or p=r(B)*

V. Pavlović and D.S. Cvetković-Ilić [67] studied the reverse order

$$(AB)\{1\} \subseteq B\{1\}A\{1\} \tag{2.4}$$

for $\{1\}$-inverses for operators on Hilbert spaces:

Theorem 2.1.3 *[67] Let regular operators* $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ *and* $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ *be given by* (2.22) *and let* AB *be regular. Then the following conditions are equivalent:*

(*i*) $(AB)\{1\} \subseteq B\{1\}A\{1\}$,

(*ii*) *One of the following conditions is satisfied:*

(*a*) $\dim\mathcal{N}(A^*) \leq \dim\mathcal{N}(B)$, $\dim\mathcal{N}(A_1^*) + \dim\mathcal{N}(A^*) \leq \dim\mathcal{N}(B^*)$ *and* $\dim\mathcal{N}(B^*) < \infty$,

(*b*) $\dim\mathcal{N}(A^*) \leq \dim\mathcal{N}(B)$, $\dim\mathcal{N}(A^*) \leq \dim\mathcal{N}(A_2'') + \dim\mathcal{N}(A_2)$ *and* $\dim\mathcal{N}(B^*) = \infty$,

(*c*) $\dim\mathcal{N}(B) \leq \dim\mathcal{N}(A^*)$, $\dim\mathcal{N}(B_1^*) + \dim\mathcal{N}(B) \leq \dim\mathcal{N}(A)$ *and* $\dim\mathcal{N}(A) < \infty$,

(*d*) $\dim\mathcal{N}(B) \leq \dim\mathcal{N}(A^*)$, $\dim\mathcal{N}(B) \leq \dim\mathcal{N}(B_2'') + \dim\mathcal{N}(B_2)$ *and* $\dim\mathcal{N}(A) = \infty$,

where $A_2'' = P_{\mathcal{N}(A_1^*)}A_2|_{\mathcal{R}(A_2^*)}$, $B_1 = P_{\mathcal{R}(B^*)}B^*|_{\mathcal{R}(A^*)}$, $B_2 = P_{\mathcal{R}(B^*)}B^*|_{\mathcal{N}(A)}$ *and* $B_2'' = P_{\mathcal{N}(B_1^*)}B_2|_{\mathcal{R}(B_2^*)}$.

Shinozaki and Sibuya [82] proved that

$$(AB)\{1,2\} \subseteq B\{1,2\}A\{1,2\} \tag{2.5}$$

always hold. De Pierro and Wei [70] also studied reverse order law for reflexive generalized inverses of matrices:

Theorem 2.1.4 *[70] Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{n\times p}$*. Then the following conditions are equivalent:*

(*i*) $B\{1,2\}A\{1,2\} \subseteq (AB)\{1,2\}$;

(*ii*) (*a*) $r_1 = 0$ *or* (*b*) $r_2 = 0$ *or* (*c*) $r_2 = n$ *or* (*d*) $r_1 = n$,

(*iii*) (*a*) $A = 0$ *or* (*b*) $B = 0$ *or* (*c*) $r(B) = n$ *or* (*d*) $r(A) = n$, *where constants r_1 and r_2 are described in the P-SVD of matrices A and B.*

De Pierro and Wei [70] also proved that (2.5) always holds, but they used different technique than Shinozaki and Sibuya [82].

Reverse order laws for least squares generalized inverses and minimum norm generalized inverses are also important as well. They are useful in both theoretical study and practical scientific computing. Wei and Guo [102] first studied reverse order law for these generalized inverses of matrices. We will restate some of their results.

Theorem 2.1.5 *[102] Suppose that $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then $B\{1,3\}\cdot A\{1,3\} \subseteq (AB)\{1,3\}$ if and only if the following two conditions hold:*

$$Z_{12} = 0 \text{ and } Z_{14} = 0$$

where submatrices Z_{12} and Z_{14} are described in the P-SVD of matrices A and B.

Equivalent condition for reverse inclusion is given in following theorem.

Theorem 2.1.6 *[102] Suppose that $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then $(AB)\{1,3\} \subseteq B\{1,3\} \cdot A\{1,3\}$ if and only if the following conditions hold:*

$$\begin{aligned} &dim(\mathcal{R}(Z_{14}) = dim(\mathcal{R}(Z_{12}, Z_{14})), \text{ and} \\ &0 \le \min\{p - r_2, m - r_1\} \le n - r_1 - r_2^2 - r(Z_{14}), \end{aligned}$$

where submatrices Z_{12}, Z_{14} and constants r_1, r_2, r_2^2 are described in the P-SVD of matrices A and B.

Reverse order law for $\{1,2,3\}$-inverses of matrices was considered by Xiong and Zheng [111]. Their techniques involved expressions for maximal and minimal ranks of the generalized Schur complement. Cvetković-Ilić and Harte [23] studied reverse order law for $\{1,2,3\}$-inverse in C^*-algebras. In Section 2.2 we will present these results and derive new conditions for reverse order law for $\{1,2,3\}$-inverses of matrices.

Reverse order law was studied also for generalized inverses on the set of bounded linear operators, C^*-algebras, rings with involution etc. [23, 20, 28]. Djordjevic [34] studied the reverse order law for the Moore-Penrose inverse for product of two operators using matrix form of a linear bounded operator induced by decompositions of Hilbert spaces; product of three operators was studied by Dinčić and Djordjević [33]. X. Liu et al. [57] studied the reverse order law for $\{1,2,3\}$- and $\{1,2,4\}$-inverses of product of two operators using some block-operator matrix techniques. Cvetković-Ilić and Nikolov [26] studied the reverse order law for $\{1,2\}$-inverse of product of two operators.

Reverse order rule for weighted Moore-Penrose inverse of the form

$$(AB)^{\dagger}_{M,L} = B^{\dagger}_{N,L} A^{\dagger}_{M,N}$$

was studied by Sun and Wei [86]:

Theorem 2.1.7 *[86] Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times l}$. Also, let $M \in \mathbb{C}^{m\times m}$, $N \in \mathbb{C}^{n\times n}$, $L \in \mathbb{C}^{l\times l}$ be positive definite Hermite matrices. Then*

$$(AB)^{\dagger}_{M,L} = B^{\dagger}_{N,L}A^{\dagger}_{M,N}$$

if and only if

$$\mathcal{R}(A^{\sharp}AB) \subseteq \mathcal{R}(B) \quad \text{and} \quad R(BB^{\sharp}A^{\sharp}) \subseteq R(A^{\sharp}),$$

where $A^{\sharp} = N^{-1}A^{*}M$ and $B^{\sharp} = L^{-1}B^{*}N$.

Greville [40] first studied the reverse order law for Drazin inverse. He showed that

$$(AB)^d = B^d A^d$$

holds if $AB = BA$. Tian [90] and Wang [97] studied the reverse order law for Drazin inverse of product od 2 and n matrices, respectively.

Tian [91] first studied the reverse order law for the Moore-Penrose inverse of product of n matrices:

$$(A_1A_2\cdots A_n)^{\dagger} = A_n^{\dagger}\cdots A_2^{\dagger}A_1^{\dagger}.$$

By using rank of matrices, he derived necessary and sufficient conditions for $A_n^{\dagger}A_{n-1}^{\dagger}\cdots A_1^{\dagger}$ to be $\{1\}$-, $\{1,2\}$-, $\{1,3\}$-, $\{1,4\}$-, $\{1,2,3\}$-, $\{1,2,4\}$-inverse or Moore-Penrose inverse of $A_1A_2\cdots A_n$. We will restate the main result from [91]:

Theorem 2.1.8 *[91] Let $A = A_1A_2\cdots A_n$ and $X = A_n^{\dagger}\cdots A_2^{\dagger}A_1^{\dagger}$, where $A_i \in \mathbb{C}^{s_i\times s_{i+1}}$, $i = 1,2,\dots n$. Then X is the Moore-Penrose inverse of A if and only of $A_1, A_2, \dots, A_n$ and A satisfy the following rank equality:*

$$r\begin{bmatrix} (-1)^n AA^*A & 0 & \cdots & 0 & AA_n^*A_n \\ 0 & & & A_{n-1}A_{n-1}^*A_{n-1} & A_{n-1}A_n \\ \vdots & & \iddots & & \\ 0 & A_2A_2^*A_2 & \iddots & & \\ A_1A_1^*A & A_1A_2 & & & \end{bmatrix} = r(A)+r(A_2)+\cdots r(A_{n-1}).$$

Wei studied reverse order law for $\{1\}$- and $\{1,2\}$-inverse and $\{1,3\}$- and $\{1,4\}$-inverse of product of n matrices [101, 58]. In Section 2.4 we will extend results from [101] to more general setting.

The importance of reverse order law problem is in its wide application in theoretic research and numerical computations in many areas, including the singular matrix problem, ill-posed problems, optimization problems, and statics problems (see for instance [8, 39, 77, 88, 103]).

2.2 Reverse order laws for $\{1,2,3\}$-generalized inverses

Xiong and Zheng [111] considered the reverse order law for $\{1,2,3\}$-inverses of the form:

$$B\{1,2,3\}A\{1,2,3\} \subseteq (AB)\{1,2,3\}. \tag{2.6}$$

Their methods involved expressions for maximal and minimal ranks of the generalized Schur complement. They conclude that (2.6) holds if and only if

$$B^*A^* = B^*A^*ABB^{(1,2,3)}A^{(1,2,3)} \quad \text{and} \quad r(B^{(1,2,3)}A^{(1,2,3)}) = r(AB),$$

for any $A^{(1,2,3)} \in A\{1,2,3\}$ and $B^{(1,2,3)} \in B\{1,2,3\}$ which are respectively equivalent to the following two rank identities:

$$\max_{B^{(1,2,3)},A^{(1,2,3)}} r(B^*A^* - B^*A^*ABB^{(1,2,3)}A^{(1,2,3)}) = 0$$

and

$$\max_{B^{(1,2,3)},A^{(1,2,3)}} r(B^{(1,2,3)}A^{(1,2,3)}) = \min_{B^{(1,2,3)},A^{(1,2,3)}} r(B^{(1,2,3)}A^{(1,2,3)}) = r(AB).$$

We will restate their results:

Theorem 2.2.1 *[111] Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times k}$. Then the following statements are equivalent:*

(*i*) $B\{1,2,3\}A\{1,2,3\} \subseteq (AB)\{1,2,3\}$;

(*ii*) $r(B, A^*AB) = r(B)$ *and*

$$r(AB) = \min\{r(A), r(B)\} = r(A) + r(B) - r\begin{pmatrix} A \\ B^* \end{pmatrix}.$$

Cvetković-Ilić and Harte [23] considered the reverse order law (2.6) for C^*-algebra case:

Theorem 2.2.2 *[23] If $a, b \in \mathcal{A}$ are such that a, b, ab and $a - abb^\dagger$ are regular, then the following conditions are equivalent:*

(1) $b\{1,2,3\}a\{1,2,3\} \subseteq (ab)\{1,2,3\}$,

(2) $bb^\dagger a^*ab = a^*ab$ *and* $(bb^\dagger - (abb^\dagger)^\dagger abb^\dagger)\mathcal{A}(aa^\dagger - (ab)(ab)^\dagger) = \{0\}$.

Notice that conditions given in [23] are purely algebraic.

Also, in [57] the reverse order law (2.6) for the linear bounded operators on Hilbert space is considered:

Theorem 2.2.3 [57] *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ be such that A, B, AB are regular operators and $AB \neq 0$. The following conditions are equivalent:*

(*i*) $B\{1,2,3\}A\{1,2,3\} \subseteq (AB)\{1,2,3\}$;

(*ii*) $\mathcal{R}(B) = \mathcal{R}(A^*AB) \oplus^{\perp} [\mathcal{R}(B) \cap \mathcal{N}(A)]$, $\mathcal{R}(AB) = \mathcal{R}(A)$.

In this section, we will present our results published in the paper [27] in which we derived purely algebraic necessary and sufficient conditions for

$$(AB)\{1,2,3\} \subseteq B\{1,2,3\} \cdot A\{1,2,3\} \tag{2.7}$$

on the set of matrices. We will also prove one unexpected fact, that (2.6) implies (2.7), i.e.

$$B\{1,2,3\} \cdot A\{1,2,3\} \subseteq (AB)\{1,2,3\} \Rightarrow B\{1,2,3\} \cdot A\{1,2,3\} = (AB)\{1,2,3\}.$$

By taking adjoint we obtain the analogous result in the case of $\{1,2,4\}-$ generalized inverses.

So, in the following theorem we give pure algebraic necessary and sufficient conditions for

$$(AB)\{1,2,3\} \subseteq B\{1,2,3\} \cdot A\{1,2,3\}.$$

Theorem 2.2.4 *Let $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times k}$. The following conditions are equivalent:*

(*i*) $(AB)\{1,2,3\} \subseteq B\{1,2,3\} \cdot A\{1,2,3\}$,

(*ii*) $(I - B^\dagger(B^\dagger(I - A^\dagger A))^\dagger)((AB)^\dagger - B^\dagger A^\dagger) = 0$

Proof. (*ii*) $\Rightarrow$ (*i*) : Suppose that (*ii*) holds. We need to prove that for arbitrary $(AB)^{(1,2,3)}$ there exist $A^{(1,2,3)}$ and $B^{(1,2,3)}$ such that $(AB)^{(1,2,3)} = B^{(1,2,3)} \cdot A^{(1,2,3)}$. Thus, given any $Z \in \mathbb{C}^{k\times n}$, we must show that there exist $X \in \mathbb{C}^{m\times n}$ and $Y \in \mathbb{C}^{k\times m}$ such that

$$\begin{aligned}&(AB)^\dagger + \big(I - (AB)^\dagger(AB)\big)ZAB(AB)^\dagger \\ &= \Big(B^\dagger + (I - B^\dagger B)YBB^\dagger\Big)\Big(A^\dagger + (I - A^\dagger A)XAA^\dagger\Big).\end{aligned} \tag{2.8}$$

Multiplying (2.8) by $B^\dagger B$ and $(I - B^\dagger B)$ from the left respectively, we get that (2.34) is equivalent to

$$(AB)^\dagger + \Big(B^\dagger B - (AB)^\dagger(AB)\Big)ZAB(AB)^\dagger = B^\dagger A^\dagger + B^\dagger(I - A^\dagger A)XAA^\dagger \tag{2.9}$$

and

$$\big(I - B^\dagger B\big)Z(AB)(AB)^\dagger = (I - B^\dagger B)YBB^\dagger\Big(A^\dagger + (I - A^\dagger A)XAA^\dagger\Big). \tag{2.10}$$

Since the condition (ii) is equivalent to

$$\Big(I - B^\dagger (B^\dagger(1 - A^\dagger A))^\dagger\Big)\Big((AB)^\dagger A - B^\dagger\Big) = 0, \tag{2.11}$$

we have that for given $Z \in \mathbb{C}^{k\times n}$ the equation (2.111) is solvable and the set of the solutions is described by

$$\begin{aligned} S_Z = \{ & S^\dagger\Big((AB)^\dagger - B^\dagger A^\dagger + \Big(B^\dagger B - (AB)^\dagger AB\Big)Z(AB)(AB)^\dagger\Big) \\ & + T - S^\dagger S T A A^\dagger : T \in \mathbb{C}^{m\times n}\}, \end{aligned}$$

where $S = B^\dagger(I - A^\dagger A)$. Substituting $X \in S_Z$ in equation (2.10), we get

$$\begin{aligned} & (I - B^\dagger B)Z(AB)(AB)^\dagger = (I - B^\dagger B)YBB^\dagger\Big(A^\dagger + \\ & (I - A^\dagger A)\big(S^\dagger(D + (B^\dagger B - (AB)^\dagger AB)Z(AB)(AB)^\dagger) + (I - S^\dagger S)TAA^\dagger\big)\Big) \end{aligned} \tag{2.12}$$

where $D = (AB)^\dagger - B^\dagger A^\dagger$. Since $B^\dagger S^\dagger D = D$ and by (2.11)

$$B^\dagger S^\dagger (B^\dagger B - (AB)^\dagger(AB)) = (B^\dagger B - (AB)^\dagger(AB)),$$

we get that (2.12) is equivalent with

$$(I - B^\dagger B)Z(AB)(AB)^\dagger = (I - B^\dagger B)YB\Big((AB)^\dagger + (B^\dagger B - (AB)^\dagger(AB))Z(AB)(AB)^\dagger\Big).$$

So we need to prove that for any $Z \in \mathbb{C}^{k\times n}$ there exist $Y \in \mathbb{C}^{k\times m}$ such that

$$(I - B^\dagger B)Z(AB)(AB)^\dagger = YB\Big((AB)^\dagger + (I - (AB)^\dagger(AB))Z(AB)(AB)^\dagger\Big),$$

which is true if and only if for any $Z \in \mathbb{C}^{k\times n}$

$$\begin{aligned} & (I - B^\dagger B)Z(AB)(AB)^\dagger\Big(B\Big((AB)^\dagger + (I - (AB)^\dagger(AB))Z(AB)(AB)^\dagger\Big)\Big)^\dagger \\ & \Big(B\Big((AB)^\dagger + (I - (AB)^\dagger(AB))Z(AB)(AB)^\dagger\Big)\Big) = (I - B^\dagger B)Z(AB)(AB)^\dagger \end{aligned}$$

or equivalently

$$P_{\mathcal{N}(B)} Z P_{\mathcal{R}(AB)} P_{\mathcal{N}(T)} = 0, \tag{2.13}$$

where $T = B\Big((AB)^\dagger + (I - (AB)^\dagger(AB))Z(AB)(AB)^\dagger\Big)$. Now, we will show that $P_{\mathcal{R}(AB)} P_{\mathcal{N}(T)} = 0$.

Take $x \in \mathcal{N}(T)$. Multiplying $B\Big((AB)^\dagger + (I - (AB)^\dagger(AB))Z(AB)(AB)^\dagger\Big)x = 0$ by $B^\dagger$ from the left we get

$$\Big((AB)^\dagger + (B^\dagger B - (AB)^\dagger(AB))Z(AB)(AB)^\dagger\Big)x = 0$$

which implies that $(AB)(AB)^\dagger x = 0$ i.e. $P_{\mathcal{R}(AB)}P_{\mathcal{N}(T)} = 0$. Now, we can conclude that the system of the equations (2.9) – (2.10) is solvalable which means that (2.8) is solvalable, i.e. (i) holds.

$(i) \Rightarrow (ii)$: If (i) holds, then for any $Z \in \mathbb{C}^{k\times n}$ there exist $X \in \mathbb{C}^{m\times n}$ and $Y \in \mathbb{C}^{k\times m}$ such that (2.8) holds which implies that (2.9) and (2.10) hold. Specially, for $Z = 0$ we get that (2.9) is equivalent to

$$(AB)^\dagger - B^\dagger A^\dagger = B^\dagger(I - A^\dagger A)XAA^\dagger,$$

so we get that the equation

$$(AB)^\dagger - B^\dagger A^\dagger = B^\dagger(I - A^\dagger A)X,$$

is solvalable which is satisfied if and only if the condition (ii) holds. □

In the following theorem we will prove unexpected fact:

$$B\{1,2,3\} \cdot A\{1,2,3\} \subseteq (AB)\{1,2,3\} \Rightarrow B\{1,2,3\} \cdot A\{1,2,3\} = (AB)\{1,2,3\}.$$

Theorem 2.2.5 *Let $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times k}$. The following conditions are equivalent:*

(i) $B\{1,2,3\} \cdot A\{1,2,3\} \subseteq (AB)\{1,2,3\}$,

(ii) $BB^\dagger A^*AB = A^*AB$ *and* $\big((ABB^\dagger)^\dagger ABB^\dagger = BB^\dagger$ *or* $(AB)(AB)^\dagger = AA^\dagger\big)$,

(iii) $B\{1,2,3\} \cdot A\{1,2,3\} = (AB)\{1,2,3\}$.

Proof. $(i) \Leftrightarrow (ii)$: Follows from Corollary 3.1 [23].

$(i) \Rightarrow (iii)$: Let $P = BB^\dagger$, $Q = B^\dagger B$ and $R = AA^\dagger$. We have that $A = A_1 + A_2$, where $A_1 = AP$ and $A_2 = A(I - P)$. To prove (iii), take arbitrary $X \in (AB)\{1,2,3\}$. We will show that there exist $Y \in B\{1,2,3\}$ and $Z \in A\{1,2,3\}$ such that $X = YZ$. Since $X \in (AB)\{1,2,3\}$, it is of the form $X = QX_1R + (I - Q)X_3R$, for some $X_1 \in \mathbb{C}^{k\times n}$ and $X_3 \in \mathbb{C}^{k\times n}$ such that $QX_1R \in (A_1B)\{1,2,3\}$ and $(I - Q)X_3A_1BX_1R = (I - Q)X_3R$.

Let $Z = BX_1R + A_2^\dagger$ and $Y = B^\dagger + (I - Q)X_3A_1$. By Lemma 2.5.2, it follows that $B\{1,2,3\} = \{B^\dagger + (I - Q)UP : U \in \mathbb{C}^{k\times m}\}$, so $Y \in B\{1,2,3\}$. To prove that $Z \in A\{1,2,3\}$, we can check that three Penrose equations are satisfied using that $A_2^*A_1 = 0$ which follows from the condition $BB^\dagger A^*AB = A^*AB$. Since $X = YZ$, it follows that $B\{1,2,3\} \cdot A\{1,2,3\} = (AB)\{1,2,3\}$.

$(iii) \Rightarrow (i)$: It is evident. □

In the following example, we show that the opposite is not true, i.e.

$$(AB)\{1,2,3\} \subseteq B\{1,2,3\} \cdot A\{1,2,3\} \not\Rightarrow B\{1,2,3\} \cdot A\{1,2,3\} \subseteq (AB)\{1,2,3\}.$$

Example. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $A^\dagger = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B^\dagger = B$.

It is easy to check that $BB^\dagger A^*AB \neq A^*AB$ which by Theorem 2.2.5 implies that

$$B\{1,2,3\} \cdot A\{1,2,3\} \nsubseteq (AB)\{1,2,3\}$$

while $(I - B^\dagger(B^\dagger(I - A^\dagger A))^\dagger)((AB)^\dagger - B^\dagger A^\dagger) = 0$ which by Theorem 3.2.1 implies that

$$(AB)\{1,2,3\} \subseteq B\{1,2,3\} \cdot A\{1,2,3\}. \quad \square$$

2.3 Reverse order laws for reflexive generalized inverse of operators

In this section we consider the reverse order law for $\{1,2\}-$inverse of product of two regular bounded linear operators on Hilbert spaces. The reverse order law for $\{1,2\}$-generalized inverses of the products of two matrices was studied by Pierro and Wei [70], where they obtained some results using the product singular value decomposition (P-SVD) of matrices which is in the case of bounded linear operators impossible. Using a completely different technique we improved the results from [70] for the case of regular bounded linear operators on Hilbert space. We give the necessary and sufficient conditions for

$$B\{1,2\} \cdot A\{1,2\} \subseteq (AB)\{1,2\},$$

and prove that $(AB)\{1,2\} \subseteq B\{1,2\} \cdot A\{1,2\}$ always holds.

We will suppose that $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ are Hilbert spaces. In the following theorem, for given regular operators $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$, we present necessary and sufficient conditions for

$$B\{1,2\} \cdot A\{1,2\} \subseteq (AB)\{1,2\} \tag{2.14}$$

to holds. It is interesting that for nonzero regular operators A and B which product AB is also regular, we have that (2.14) holds if and only if A is left or B is right invertible operator.

Theorem 2.3.1 *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ be such that A, B and AB are regular operators. The following conditions are equivalent:*

(i) $B\{1,2\} \cdot A\{1,2\} \subseteq (AB)\{1,2\}$,

(ii) $A = 0$ *or* $B = 0$ *or* $A \in \mathcal{B}_l^{-1}(\mathcal{H},\mathcal{K})$ *or* $B \in \mathcal{B}_r^{-1}(\mathcal{H},\mathcal{K})$.

Proof. $(i) \Rightarrow (ii)$: If (i) holds, then evidently $B^\dagger A^\dagger \in (AB)\{1,2\}$, so

$$ABB^\dagger A^\dagger AB = AB \tag{2.15}$$

and

$$B^\dagger A^\dagger ABB^\dagger A^\dagger = B^\dagger A^\dagger. \tag{2.16}$$

Since, for any $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, $A^\dagger + (I - A^\dagger A)XAA^\dagger \in A\{1,2\}$, we get

$$ABB^\dagger(A^\dagger + (I - A^\dagger A)XAA^\dagger)AB = AB,$$

which using (2.15) gives that

$$ABB^\dagger(I - A^\dagger A)XAB = 0, \tag{2.17}$$

holds for any $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.

Similarly, for any $Y \in \mathcal{B}(\mathcal{H}, \mathcal{L})$, $B^\dagger + B^\dagger BY(I - BB^\dagger) \in B\{1,2\}$, so

$$AB(B^\dagger + B^\dagger BY(I - BB^\dagger))A^\dagger AB = AB,$$

which using (2.15) gives that

$$ABY(I - BB^\dagger)A^\dagger AB = 0, \tag{2.18}$$

holds for any $Y \in \mathcal{B}(\mathcal{H}, \mathcal{L})$.

Since, for any $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $Y \in \mathcal{B}(\mathcal{H}, \mathcal{L})$, we have that

$$AB(B^\dagger + B^\dagger BY(I - BB^\dagger))(A^\dagger + (I - A^\dagger A)XAA^\dagger)AB = AB, \tag{2.19}$$

using (2.15), (2.17) and (2.18), we get that

$$ABY(I - BB^\dagger)(I - A^\dagger A)XAB = 0,$$

for any $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $Y \in \mathcal{B}(\mathcal{H}, \mathcal{L})$. Now,

$$AB = 0 \quad \text{or} \quad (I - BB^\dagger)(I - A^\dagger A) = 0. \tag{2.20}$$

Since, for any $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$,

$$\begin{aligned} &B^\dagger(A^\dagger + (I - A^\dagger A)XAA^\dagger)ABB^\dagger(A^\dagger + (I - A^\dagger A)XAA^\dagger) \\ &= B^\dagger(A^\dagger + (I - A^\dagger A)XAA^\dagger) \end{aligned}$$

using (2.16) we get that

$$\begin{aligned} &B^\dagger A^\dagger ABB^\dagger(I - A^\dagger A)XAA^\dagger + B^\dagger(I - A^\dagger A)XABB^\dagger A^\dagger \\ &+B^\dagger(I - A^\dagger A)XABB^\dagger(I - A^\dagger A)XAA^\dagger = B^\dagger(I - A^\dagger A)XAA^\dagger. \end{aligned} \tag{2.21}$$

Now by (2.20), we get that the first and the third term on the left side of (2.21) are zero, so

$$B^\dagger(I - A^\dagger A)X(ABB^\dagger A^\dagger - AA^\dagger) = 0,$$

for any $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Hence,

$$B^\dagger = B^\dagger A^\dagger A \quad \text{or} \quad ABB^\dagger A^\dagger = AA^\dagger.$$

Now, following (2.20) we have two cases:
Case 1. If $AB = 0$, then if $B^\dagger = B^\dagger A^\dagger A$ it follows that $B = 0$. If $ABB^\dagger A^\dagger = AA^\dagger$, we easily get that $A = 0$.
Case 2. If $(I - BB^\dagger)(I - A^\dagger A) = 0$, then if $B^\dagger = B^\dagger A^\dagger A$, it follows that $A^\dagger A = I$, i.e. A is left invertible. If $ABB^\dagger A^\dagger = AA^\dagger$, then multiplying $(I - BB^\dagger)(I - A^\dagger A) = 0$ by A from the left, we get

$$A = ABB^\dagger.$$

Now, since $A^\dagger A$ and $BB^\dagger$ commute, we get that $BB^\dagger = I$, i.e. B is right invertible.

$(ii) \Rightarrow (i)$: If A or B is zero, it is evident that (i) holds. Now, suppose that B is right invertible and let $B^{(1,2)} \in B\{1,2\}$ be arbitrary. Evidently, $B^{(1,2)}$ is a right inverse of B, i.e. $BB^{(1,2)} = I$. Then, for arbitrary $A^{(1,2)} \in A\{1,2\}$,

$$ABB^{(1,2)}A^{(1,2)}AB = AA^{(1,2)}AB = AB$$

and

$$B^{(1,2)}A^{(1,2)}ABB^{(1,2)}A^{(1,2)} = B^{(1,2)}A^{(1,2)}AA^{(1,2)} = B^{(1,2)}A^{(1,2)}.$$

If A is left invertible operator, for any $A^{(1,2)} \in A\{1,2\}$, we have that $A^{(1,2)}A = I$. Then, for arbitrary $A^{(1,2)} \in A\{1,2\}$ and $B^{(1,2)} \in B\{1,2\}$,

$$ABB^{(1,2)}A^{(1,2)}AB = ABB^{(1,2)}B = AB$$

and

$$B^{(1,2)}A^{(1,2)}ABB^{(1,2)}A^{(1,2)} = B^{(1,2)}BB^{(1,2)}A^{(1,2)} = B^{(1,2)}A^{(1,2)}. \quad \square$$

Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ be arbitrary regular operators. Using the following decompositions of the spaces $\mathcal{L}$, $\mathcal{H}$ and $\mathcal{K}$,

$$\mathcal{L} = \mathcal{R}(B^*) \oplus \mathcal{N}(B),\ \mathcal{H} = \mathcal{R}(B) \oplus \mathcal{N}(B^*),\ \mathcal{K} = \mathcal{R}(A) \oplus \mathcal{N}(A^*),$$

we have that the corresponding decompositions of A and B are given by

$$\begin{aligned} A &= \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}, \\ B &= \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}, \end{aligned} \tag{2.22}$$

where B_1 is an invertible operator and $\begin{bmatrix} A_1 & A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \mathcal{R}(A)$ is right invertible operator . In that case the operator AB is given by

$$AB = \begin{bmatrix} A_1B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}.$$

In the following lemma we present characterization of the sets $A\{1,2\}$, $B\{1,2\}$, and $(AB)\{1,2\}$ in the case when A and B are given by (2.22):

Lemma 2.3.1 *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ be regular operators given by* (2.22)*. Then*

(*i*) *arbitrary* $\{1,2\}$*-inverse of A is given by:*

$$A\{1,2\} \;=\; \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix},$$

where X_1 and X_3 satisfy that

$$A_1X_1 + A_2X_3 = I_{\mathcal{R}(A)},$$

and X_2 and X_4 are of the form

$$\begin{aligned} X_2 &= X_1A_1Z_1 + X_1A_2Z_2, \\ X_4 &= X_3A_1Z_1 + X_3A_2Z_2, \end{aligned}$$

for some operators $Z_1 \in \mathcal{B}(\mathcal{N}(A^*),\mathcal{R}(B))$ and $Z_2 \in \mathcal{B}(\mathcal{N}(A^*),\mathcal{N}(B^*))$.

(*ii*) *arbitrary* $\{1,2\}$*-inverse of B is given by:*

$$B^{(1,2)} \;=\; \begin{bmatrix} B_1^{-1} & U \\ V & VB_1U \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix},$$

where $U \in \mathcal{B}(\mathcal{N}(B^*),\mathcal{R}(B^*))$ and $V \in \mathcal{B}(\mathcal{R}(B),\mathcal{N}(B))$.

(*iii*) *if AB is regular, then arbitrary* $\{1,2\}$*-inverse of AB is given by:*

$$(AB)^{(1,2)} \;=\; \begin{bmatrix} (A_1B_1)^{(1,2)} & Y_2 \\ Y_3 & Y_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix},$$

where $(A_1B_1)^{(1,2)} \in (A_1B_1)\{1,2\}$ and Y_i, $i = \overline{2,4}$ satisfy the following system of the equations:

$$\begin{aligned} Y_2 &= (A_1B_1)^{(1,2)}A_1B_1Y_2, \\ Y_3 &= Y_3A_1B_1(A_1B_1)^{(1,2)}, \\ Y_4 &= Y_3A_1B_1Y_2. \end{aligned} \tag{2.23}$$

Proof. (*i*) Without loss of generality, we can suppose that arbitrary $\{1,2\}$-inverse of A is given by:

$$A^{(1,2)} \;=\; \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}.$$

By $AXA = A$ and $XAX = X$, we get that $X \in A\{1,2\}$ if and only if X_i, $i = \overline{1,4}$ satisfy the following equations

$$(A_1X_1 + A_2X_3)A_i = A_i, \quad i = \overline{1,2} \tag{2.24}$$

$$X_j(A_1X_1 + A_2X_3) = X_j, \quad j = \overline{1,3} \tag{2.25}$$

$$X_1(A_1X_2 + A_2X_4) = X_2, \quad X_3(A_1X_2 + A_2X_4) = X_4. \tag{2.26}$$

Since $S = \begin{bmatrix} A_1 & A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \mathcal{R}(A)$ is right invertible operator, there exists $S_r^{-1} : \mathcal{R}(A) \to \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix}$ such that $\begin{bmatrix} A_1 & A_2 \end{bmatrix} S_r^{-1} = I_{R(A)}$. Notice that (2.24) is equivalent to

$$\begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_3 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \end{bmatrix}. \tag{2.27}$$

Multiplying (2.94) by S_r^{-1} from the right, we get that (2.94) is equivalent with $\begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_3 \end{bmatrix} = I_{R(A)}$, i.e.

$$A_1X_1 + A_2X_3 = I_{\mathcal{R}(A)}. \tag{2.28}$$

Note, that for X_1 and X_3 which satisfy (2.28), (2.25) also holds. Condition (2.26) is equivalent to

$$\begin{bmatrix} X_1 \\ X_3 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} X_2 \\ X_4 \end{bmatrix} = \begin{bmatrix} X_2 \\ X_4 \end{bmatrix}$$

i.e.

$$(I - P) \begin{bmatrix} X_2 \\ X_4 \end{bmatrix} = 0,$$

where $P = \begin{bmatrix} X_1 \\ X_3 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \end{bmatrix}$. Since P is a projection,

$$\begin{bmatrix} X_2 \\ X_4 \end{bmatrix} = P \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix},$$

i.e.

$$\begin{bmatrix} X_2 \\ X_4 \end{bmatrix} = \begin{bmatrix} X_1A_1Z_1 + X_1A_2Z_2 \\ X_3A_1Z_1 + X_3A_2Z_2 \end{bmatrix},$$

where Z_1 and Z_2 are operators from the appropriate spaces.

(ii) Suppose that arbitrary $\{1,2\}$-inverse of B is given by:

$$B^{(1,2)} = \begin{bmatrix} S & U \\ V & W \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix}.$$

From $BB^{(1,2)}B = B$ it follows that $B_1SB_1 = B_1$ and since B_1 is invertible, $S = B_1^{-1}$. From $B^{(1,2)}BB^{(1,2)} = B^{(1,2)}$ we easily get $W = VB_1U$, where U and V are operators from the appropriate spaces.

(iii) Let arbitrary $\{1,2\}$-inverse of AB be given by:

$$(AB)^{(1,2)} = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix}.$$

From $AB(AB)^{(1,2)}AB = AB$, we get

$$A_1B_1Y_1A_1B_1 = A_1B_1, \tag{2.29}$$

and by $(AB)^{(1,2)}AB(AB)^{(1,2)} = (AB)^{(1,2)}$, we get

$$Y_1A_1B_1Y_1 = Y_1, \tag{2.30}$$
$$Y_1A_1B_1Y_2 = Y_2, \tag{2.31}$$
$$Y_3A_1B_1Y_1 = Y_3, \tag{2.32}$$
$$Y_3A_1B_1Y_2 = Y_4. \tag{2.33}$$

Now, by (2.29) and (2.30), we get that $Y_1 \in (A_1B_1)\{1,2\}$. Substituting $Y_1 = (A_1B_1)^{(1,2)}$ in (2.31), (2.32) and (2.33), we get (2.23). □

In the following theorem, we will prove that for given regular operators $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ such that AB is regular, the following inclusion

$$(AB)\{1,2\} \subseteq B\{1,2\} \cdot A\{1,2\}$$

always holds.

Theorem 2.3.2 *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ be regular operators such that AB is regular. Then*

$$(AB)\{1,2\} \subseteq B\{1,2\} \cdot A\{1,2\}.$$

Proof. Take arbitrary $(AB)^{(1,2)} \in (AB)\{1,2\}$. We will show that there exist $A^{(1,2)} \in A\{1,2\}$ and $B^{(1,2)} \in B\{1,2\}$ such that $(AB)^{(1,2)} = B^{(1,2)}A^{(1,2)}$. Without loss of generality, we can suppose that A and B are given by (2.22). By Lemma 2.3.1, we have that

$$(AB)^{(1,2)} = \begin{bmatrix} (A_1B_1)^{(1,2)} & Y_2 \\ Y_3 & Y_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix},$$

for $(A_1B_1)^{(1,2)} \in (A_1B_1)\{1,2\}$ and some Y_i, $i = \overline{2,4}$ which satisfy system (2.23). Since $\begin{bmatrix} A_1 & A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \mathcal{R}(A)$ is right invertible operator, there exists (in general case non unique) $\begin{bmatrix} X_1' \\ X_3' \end{bmatrix} : \mathcal{R}(A) \to \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix}$ such that $A_1X_1' + A_2X_3' = I_{\mathcal{R}(A)}$. Since B_1 is an invertible, we have that $(A_1B_1)(A_1B_1)^{(1,2)}A_1X_1' = A_1X_1'$. Let $X_3 = X_3'$ and $X_1 = B_1(A_1B_1)^{(1,2)}A_1X_1'$. Obviously, $A_1X_1 + A_2X_3 = I_{\mathcal{R}(A)}$. Now, let

$$C = \begin{bmatrix} X_1 & X_1A_1B_1Y_2 \\ X_3 & X_3A_1B_1Y_2 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix},$$
$$D = \begin{bmatrix} B_1^{-1} & U \\ Y_3A_1 & Y_3A_1B_1U \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix},$$

where $U = (A_1B_1)^{(1,2)}A_2$. We will show that $C \in A\{1,2\}$, $D \in B\{1,2\}$ and that $(AB)^{(1,2)} = DC$. By Lemma 2.3.1, we can check that $C \in A\{1,2\}$ and $D \in B\{1,2\}$. To prove that $(AB)^{(1,2)} = DC$, it is sufficient to show that the following system of the equations is satisfied:

$$\begin{aligned}
&(A_1B_1)^{(1,2)} = B_1^{-1}X_1 + UX_3,\\
&Y_2 = B_1^{-1}X_1A_1B_1Y_2 + UX_3A_1B_1Y_2,\\
&Y_3 = Y_3A_1X_1 + Y_3A_1B_1UX_3,\\
&Y_4 = Y_3A_1X_1A_1B_1Y_2 + Y_3A_1B_1UX_3A_1B_1Y_2.
\end{aligned}$$

The first equation is satisfied, since $X_1 = B_1(A_1B_1)^{(1,2)}(I - A_2X_3)$, while the other three equations are satisfied by (2.23). □

Remark. (**1**) It is interesting to note that by the first part of the proof of Theorem 2.3.1, we can conclude that

$$B\{1,2\} \cdot A\{1,2\} \subseteq (AB)\{1\}$$

if and only if

$$AB = 0 \quad \text{or} \quad (I - BB^\dagger)(I - A^\dagger A) = 0$$

i.e

$$AB = 0 \quad \text{or} \quad \mathcal{N}(A) \subseteq \mathcal{R}(B).$$

(**2**) Remark that since B_1 is invertible, $(A_1B_1)\{1,2\} = B_1^{-1}A_1\{1,2\}$, so for any $(A_1B_1)^{(1,2)} \in (AB)\{1,2\}$, there exists $A_1^{(1,2)} \in A_1\{1,2\}$ such that $(A_1B_1)^{(1,2)} = B_1^{-1}A_1^{(1,2)}$.

(**3**) Results from the Theorems 2.3.1 and 2.3.2 can be generalized to the C^*-algebra case.

2.4 Reverse order law for multiple operator product

The reverse order law for the Moore-Penrose inverse of a product of n matrices was considered by Tian [91]. By using rank formulas, he derived necessary and sufficient conditions for $A_n^\dagger A_{n-1}^\dagger \cdots A_1^\dagger$ to be $\{1\}$-, $\{1,2\}$-, $\{1,3\}$-, $\{1,4\}$-, $\{1,2,3\}$-, $\{1,2,4\}$-inverse of $A_1A_2\cdots A_n$. After that, by applying the multiple product singular value decomposition (P-SVD), Wei [101] obtained necessary and sufficient conditions for inclusions

$$\begin{aligned}
A_n\{1\} \cdot A_{n-1}\{1\} \cdots A_1\{1\} &\subseteq (A_1A_2\cdots A_n)\{1\},\\
A_n\{1,2\} \cdot A_{n-1}\{1,2\} \cdots A_1\{1,2\} &\subseteq (A_1A_2\cdots A_n)\{1,2\},\\
(A_1A_2\cdots A_n)\{1,2\} &\subseteq A_n\{1,2\} \cdot A_{n-1}\{1,2\} \cdots A_1\{1,2\}
\end{aligned}$$

on the set of matrices. Zheng and Xiong [113] derived necessary and sufficient conditions for inclusions

$$A_n\{1,2,3\} \cdot A_{n-1}\{1,2,3\} \cdots A_1\{1,2,3\} \subseteq (A_1A_2\cdots A_n)\{1,2,3\}$$

and

$$A_n\{1,2,4\} \cdot A_{n-1}\{1,2,4\} \cdots A_1\{1,2,4\} \subseteq (A_1A_2 \cdots A_n)\{1,2,4\}$$

on the set of matrices. Their techniques involved expressions for maximal and minimal ranks of the generalized Schur complement. Methods used in [91], [101] and [113] are not applicable to infinite dimensional settings. Using a completely different technique we improved the results from [70] for the case of regular bounded linear operators on Hilbert space. We also improved result for one side inclusion for the least square g-inverse from [58]. We derived new simple conditions involving only ranges of operators.

In this section, we present results from our paper [72] in which we derived necessary and sufficient conditions for the inclusions

$$A_n\{1\} \cdot A_{n-1}\{1\} \cdots A_1\{1\} \subseteq (A_1A_2 \cdots A_n)\{1\}, \tag{2.34}$$

$$A_n\{1,2\} \cdot A_{n-1}\{1,2\} \cdots A_1\{1,2\} \subseteq (A_1A_2 \cdots A_n)\{1,2\},$$

$$A_n\{1,2\} \cdot A_{n-1}\{1,2\} \cdots A_1\{1,2\} = (A_1A_2 \cdots A_n)\{1,2\},$$

$$A_n\{1,3\} \cdot A_{n-1}\{1,3\} \cdots A_1\{1,3\} \subseteq (A_1A_2 \cdots A_n)\{1,3\},$$

$$A_n\{1,4\} \cdot A_{n-1}\{1,4\} \cdots A_1\{1,4\} \subseteq (A_1A_2 \cdots A_n)\{1,4\}.$$

We also proved that if $A_1A_2 \cdots A_n \neq 0$, (2.34) implies

$$A_k\{1\} \cdot A_{k-1}\{1\} \cdots A_1\{1\} \subseteq (A_1A_2 \cdots A_k)\{1\} \quad \text{for} \quad k = 2,3,\ldots,n.$$

2.4.1 Reverse order law for $\{1,2\}$-inverses

By applying the multiple product singular value decomposition (P-SVD), Wei [101] obtained necessary and sufficient conditions for inclusion

$$A_n\{1,2\} \cdot A_{n-1}\{1,2\} \cdots A_1\{1,2\} \subseteq (A_1A_2 \cdots A_n)\{1,2\}$$

Theorem 2.4.1 [*101*] *Suppose that* $A_i \in \mathbb{C}^{s_i \times s_{i+1}}$, $i = 1,2,\ldots n$. *Then the following conditions are equivalent:*

(*i*) $A_n\{1,2\} \cdot A_{n-1}\{1,2\} \cdots A_1\{1,2\} \subseteq (A_1A_2 \cdots A_n)\{1,2\}$;

(*ii*)

(*a*) $r_n^1 > 0$, $m_1 \geq \ldots \geq m_n$ *and* $r_j = m_{j+1}$ *for* $j = 1,\ldots,n-1$, *or*

(*b*) $r_n^1 > 0$, $m_2 \leq \ldots \leq m_{n+1}$ *and* $r_j = m_{j+1}$ *for* $j = 2,\ldots,n$, *or*

(*c*) $r_n^1 > 0$ *and there exists an integer* q *with* $2 \leq q < n$
such that $m_1 \geq \ldots \geq m_q$ *and* $r_j = m_{j+1}$ *for* $j = 1,\ldots,q-1$,
$m_q \geq \ldots \geq m_{n+1}$ *and* $r_j = m_{j+1}$ *for* $j = q,\ldots,n$,

(*d*) *There exists an integer* q *with* $1 \leq q \leq n$, *such that* $r_q = 0$,

(iii)

(a) $r(A_1 \cdots A_n) > 0$ *and* A_j *are of full column rank for* $j = 1, \ldots, n-1$, *or*

(b) $r(A_1 \cdots A_n) > 0$ *and* A_j *are of full row rank for* $j = 2, \ldots, n$, *or*

(c) $r(A_1 \cdots A_n) > 0$ *and there exists an integer* q *with* $2 \leq q < n$ *such that* A_j *are of full column rank for* $j = 1, \ldots, q-1$ *and* A_j *are of full row rank for* $j = q, \ldots, n$,

(d) There exists an integer q *with* $1 \leq q \leq n$, *such that* $r(A_q) = 0$

where constants r_n^1, r_j, m_j, $j = 1, \ldots n$ *are defined in P-SVD of matrices* $A_1, A_2, \ldots, A_n$.

The proof of the following theorem analogous to the matrix case (see [Theorem 4.1, [101]]).

Theorem 2.4.2 *Let* $A_i \in \mathcal{B}(\mathcal{H}_{i+1}, \mathcal{H}_i)$, *be such that* A_i, $i = 1, 2, \ldots, n$ *and* $A_1 A_2 \cdots A_j$, $j = 2, 3, \ldots, n$, *are regular operators. Then*

$$(A_1 A_2 \cdots A_n)\{1,2\} \subseteq A_n\{1,2\} \cdot A_{n-1}\{1,2\} \cdots A_1\{1,2\}.$$

Proof. We will prove this relation by induction on n. For $n = 2$ it follows from Theorem 2.3.2 that $(AB)\{1,2\} \subseteq B\{1,2\} \cdot A\{1,2\}$. Now suppose that for $2 \leq k \leq n$ the assertion is true. For $k = n+1$, let $A_1 A_2 \cdots A_k = B$. By using again Theorem 2.3.2 we obtain $(BA_{n+1})\{1,2\} \subseteq A_{n+1}\{1,2\}B\{1,2\}$. From the assumption of the induction,

$$(A_1 \cdots A_n)\{1,2\} \subseteq A_n\{1,2\} \cdots A_1\{1,2\},$$

so we get

$$\begin{aligned} (A_1 \cdots A_n A_{n+1})\{1,2\} &\subseteq A_{n+1}\{1,2\}(A_1 \cdots A_n)\{1,2\} \\ &\subseteq A_{n+1}\{1,2\}A_n\{1,2\} \cdots A_1\{1,2\}. \quad \Box \end{aligned}$$

Now, we will derive necessary and sufficient conditions for the inclusion $A_n\{1,2\} \cdot A_{n-1}\{1,2\} \cdots A_1\{1,2\} \subseteq (A_1 A_2 \cdots A_n)\{1,2\}$.

Theorem 2.4.3 *Let* $A_i \in \mathcal{B}(\mathcal{H}_{i+1}, \mathcal{H}_i)$, *be such that* A_i, $i = 1, 2 \ldots, n$ *and all* $A_1 A_2 \cdots A_j$, $A_{j-1} A_j$, $j = 2, 3, \ldots, n$, *are regular operators. The following conditions are equivalent:*

(i) $A_n\{1,2\} \cdot A_{n-1}\{1,2\} \cdots A_1\{1,2\} = (A_1 A_2 \cdots A_n)\{1,2\}$,

(ii) $A_n\{1,2\} \cdot A_{n-1}\{1,2\} \cdots A_1\{1,2\} \subseteq (A_1 A_2 \cdots A_n)\{1,2\}$,

(iii) There exist an integer i, $1 \leq i \leq n$, *such that* $A_i = 0$,
or
$A_1 A_2 \cdots A_n \neq 0$ *and* $A_i \in \mathcal{B}_r^{-1}(\mathcal{H}_{i+1}, \mathcal{H}_i)$ *for* $i = 2, 3, \ldots, n$,
or
$A_1 A_2 \cdots A_n \neq 0$ *and* $A_i \in \mathcal{B}_l^{-1}(\mathcal{H}_{i+1}, \mathcal{H}_i)$ *for* $i = 1, 2, \ldots, n-1$,
or
$A_1 A_2 \cdots A_n \neq 0$ *and there exists an integer* k, $2 \leq k \leq n-1$, *such that* $A_i \in \mathcal{B}_l^{-1}(\mathcal{H}_{i+1}, \mathcal{H}_i)$ *for* $i = 1, 2, \ldots, k-1$, *and* $A_i \in \mathcal{B}_r^{-1}(\mathcal{H}_{i+1}, \mathcal{H}_i)$ *for* $i = k+1, k+2, \ldots, n$.

Proof. $(i) \Leftrightarrow (ii)$: Follows from Theorem 2.4.2.

$(ii) \Rightarrow (iii)$: We will prove this implication by induction on n.

Implication holds for $n = 2$ by Theorem 2.3.1.

Assume that $(ii) \Rightarrow (iii)$ holds for $n = k-1$; we will prove that it is true for $n = k$. Suppose that for $A_i \in \mathcal{B}(\mathcal{H}_{i+1}, \mathcal{H}_i)$, $i = 1, 2, \ldots, k$

$$A_k\{1,2\} \cdot A_{k-1}\{1,2\} \cdots A_1\{1,2\} \subseteq (A_1 A_2 \cdots A_k)\{1,2\} \tag{2.35}$$

is satisfied. Since from Theorem 2.4.2

$$(A_1 A_2 \cdots A_{k-1})\{1,2\} \subseteq A_{k-1}\{1,2\} \cdot A_{k-2}\{1,2\} \cdots A_1\{1,2\}, \tag{2.36}$$

we have from (3.23)

$$A_k\{1,2\} \cdot (A_1 A_2 \cdots A_{k-1})\{1,2\} \subseteq (A_1 A_2 \cdots A_k)\{1,2\}. \tag{2.37}$$

This is by Theorem 2.3.1 satisfied if and only if

$$\begin{aligned} & A_1 A_2 \cdots A_{k-1} = 0, \\ \text{or} \quad & A_k = 0, \\ \text{or} \quad & A_1 A_2 \cdots A_{k-1} \in \mathcal{B}_l^{-1}(\mathcal{H}_k, \mathcal{H}_1), \\ \text{or} \quad & A_k \in \mathcal{B}_r^{-1}(\mathcal{H}_{k+1}, \mathcal{H}_k). \end{aligned} \tag{2.38}$$

Now, according to (2.38), we have four cases:

Case 1. $A_1 A_2 \cdots A_{k-1} = 0$. Since $A_1 A_2 \cdots A_{k-1} A_k = 0$ it follows that $(A_1 A_2 \cdots A_{k-1} A_k)\{1,2\} = \{0\}$. Because of (ii) we have

$$A_k\{1,2\} \cdot A_{k-1}\{1,2\} \cdots A_1\{1,2\} = \{0\}. \tag{2.39}$$

Let $A_i^{(1,2)} \in A_i\{1,2\}$ $i = 1, 2, \ldots, k-1$ be arbitrary. Then from (2.39) we have

$$A_k^\dagger A_{k-1}^{(1,2)} \cdots A_1^{(1,2)} = 0. \tag{2.40}$$

Since for any $Z \in \mathcal{B}(\mathcal{H}_k, \mathcal{H}_{k+1})$, $A_k^\dagger + A_k^\dagger A_k Z(I_{\mathcal{H}_k} - A_k A_k^\dagger) \in A_k\{1,2\}$, we get

$$(A_k^\dagger + A_k^\dagger A_k Z(I_{\mathcal{H}_k} - A_k A_k^\dagger)) A_{k-1}^{(1,2)} A_{k-2}^{(1,2)} \cdots A_1^{(1,2)} = 0,$$

which using (2.40) gives that

$$A_k^\dagger A_k Z A_{k-1}^{(1,2)} A_{k-2}^{(1,2)} \cdots A_1^{(1,2)} = 0$$

holds for any $Z \in \mathcal{B}(\mathcal{H}_k, \mathcal{H}_{k+1})$. Now,

$$A_k = 0 \quad \text{or} \quad A_{k-1}^{(1,2)} \cdots A_1^{(1,2)} = 0. \tag{2.41}$$

If $A_k = 0$, then (iii) holds. Suppose that $A_k \neq 0$. Then $A_{k-1}^{(1,2)} A_{k-2}^{(1,2)} \cdots A_1^{(1,2)} = 0$ for arbitrary $A_i^{(1,2)} \in A_i\{1,2\}$ $i = 1, 2, \ldots, k-1$ so it follows

$$\begin{aligned}&A_{k-1}\{1,2\}A_{k-2}\{1,2\}\cdots A_1\{1,2\}=\{0\}\\&\subseteq(A_1A_2\cdots A_{k-2}A_{k-1})\{1,2\}.\end{aligned}\tag{2.42}$$

By the induction hypothesis, from (2.42) it follows that at least one of the following conditions is true:

(1) There exists $i\in\{1,2,\dots,k-1\}$ such that $A_i=0$,

(2) $A_i\in B_l^{-1}(\mathcal{H}_{i+1},\mathcal{H}_i)$, $i=1,2,\dots,k-2$,

(3) $A_i\in B_r^{-1}(\mathcal{H}_{i+1},\mathcal{H}_i)$, $i=2,3,\dots,k-1$,

(4) There exists $i\in\{1,2,\dots,k-1\}$ such that $A_j\in B_l^{-1}(\mathcal{H}_{j+1},\mathcal{H}_j)$ for $j=1,2,\dots,i-1$ and $A_j\in B_r^{-1}(\mathcal{H}_{j+1},\mathcal{H}_j)$, $j=i+1,i+2,\dots,k-1$.

If (1) holds, then (iii) is satisfied.
Suppose that (2) is true. Since $A_1A_2\cdots A_{k-1}=0$ we get that $A_{k-1}=0$ so (iii) holds.
If (3) is true, then from $A_1A_2\cdots A_{k-1}=0$ we get that $A_1=0$.
Suppose that (4) holds. Multiplying

$$A_1A_2\cdots A_{k-1}=0$$

by $A_{i-1}^\dagger A_{i-2}^\dagger\cdots A_1^\dagger$ from the left, we get

$$A_iA_{i+1}\cdots A_{k-1}=0.\tag{2.43}$$

Multiplying (2.43) by $A_{k-1}^\dagger A_{k-2}^\dagger\cdots A_{i+1}^\dagger$ from the right we get

$$A_i=0.$$

Hence, (iii) is satisfied.
Case 2. If $A_k=0$ then (iii) obviously holds.
Case 3. Suppose that $A_1A_2\cdots A_{k-1}\in\mathcal{B}_l^{-1}(\mathcal{H}_k,\mathcal{H}_1)$. Then $A_{k-1}\in\mathcal{B}_l^{-1}(\mathcal{H}_k,\mathcal{H}_{k-1})$. From Theorem 2.3.1 we have

$$(A_{k-1}A_k)\{1,2\}\subseteq A_k\{1,2\}A_{k-1}\{1,2\},$$

so it follows that

$$\begin{aligned}&(A_{k-1}A_k)\{1,2\}\cdot A_{k-2}\{1,2\}\cdots A_1\{1,2\}\\&\subseteq A_k\{1,2\}\cdot A_{k-1}\{1,2\}\cdots A_1\{1,2\}\\&\subseteq(A_1A_2\cdots A_k)\{1,2\}.\end{aligned}\tag{2.44}$$

By induction hypothesis, from (2.44) it follows that at least one of the following conditions is true:

(1′) There exists $i\in\{1,2,\dots,k-2\}$ such that $A_i=0$ or $A_{k-1}A_k=0$, (2′) $A_i \in B_l^{-1}(\mathcal{H}_{i+1}, \mathcal{H}_i)$, $i = 1, 2, \dots, k-2$,

(3′) $A_i \in B_r^{-1}(\mathcal{H}_{i+1}, \mathcal{H}_i)$, $i = 2, 3, \dots, k-2$, and $A_{k-1}A_k \in B_r^{-1}(\mathcal{H}_{k+1}, \mathcal{H}_{k-1})$,

(4′) There exists $i \in \{1, 2, \dots, k-1\}$ such that $A_j \in B_l^{-1}(\mathcal{H}_{j+1}, \mathcal{H}_j)$ for $j = 1, 2, \dots, i-1$ and $A_j \in B_r^{-1}(\mathcal{H}_{j+1}, \mathcal{H}_j)$, $j = i+1, i+2, \dots, k-2$ and $A_{k-1}A_k \in B_r^{-1}(\mathcal{H}_{k+1}, \mathcal{H}_{k-1})$.

Suppose that (1′) is true. If $A_i = 0$ for some $i \in \{1, 2, \dots, k-2\}$ then (iii) holds. If $A_{k-1}A_k = 0$ then, since $A_{k-1} \in \mathcal{B}_l^{-1}(\mathcal{H}_k, \mathcal{H}_{k-1})$ it follows that $A_k = 0$ so (iii) holds.
If (2′) holds, then $A_i \in B_l^{-1}(\mathcal{H}_{i+1}, \mathcal{H}_i)$, $i = 1, 2, \dots, k-1$, so (iii) is satisfied.
Suppose that (3′) holds. Then $A_{k-1} \in B_r^{-1}(\mathcal{H}_k, \mathcal{H}_{k-1})$. We will prove that $A_k \in B_r^{-1}(\mathcal{H}_{k+1}, \mathcal{H}_k)$. Let $x \in \mathcal{H}_k$ be arbitrary. Then $A_{k-1}x \in \mathcal{H}_{k-1}$. Since $\mathcal{R}(A_{k-1}A_k) = \mathcal{H}_{k-1}$, there exists $y \in \mathcal{H}_{k+1}$ such that $A_{k-1}A_k y = A_{k-1}x$. Multiplying last equality by $A_{k-1}^\dagger$ from the left and using $A_{k-1}^\dagger A_{k-1} = I_{\mathcal{H}_k}$ we get $A_k y = x$ i.e. $x \in \mathcal{R}(A_k)$. So it follows that $\mathcal{R}(A_k) = \mathcal{H}_k$, i.e. $A_k \in B_r^{-1}(\mathcal{H}_{k+1}, \mathcal{H}_k)$. Now we have $A_i \in B_r^{-1}(\mathcal{H}_{i+1}, \mathcal{H}_i)$, $i = 2, 3, \dots, k$, so (iii) is satisfied.
Suppose that (4′) holds. Now as in (3′) from $A_{k-1}A_k \in B_r^{-1}(\mathcal{H}_{k+1}, \mathcal{H}_{k-1})$ using $A_{k-1} \in \mathcal{B}_l^{-1}(\mathcal{H}_k, \mathcal{H}_{k-1})$ we obtain $A_{k-1} \in B_r^{-1}(\mathcal{H}_k, \mathcal{H}_{k-1})$ and $A_k \in B_r^{-1}(\mathcal{H}_{k+1}, \mathcal{H}_k)$. Now we have $A_j \in B_l^{-1}(\mathcal{H}_{j+1}, \mathcal{H}_j)$ for $j = 1, 2, \dots, i-1$ and $A_j \in B_r^{-1}(\mathcal{H}_{j+1}, \mathcal{H}_j)$, $j = i+1, i+2, \dots, k$, so (iii) is satisfied.
Case 4. Suppose that $A_k \in \mathcal{B}_r^{-1}(\mathcal{H}_{k+1}, \mathcal{H}_k)$. Then $A_k A_k^{(1,2)} = I_{\mathcal{H}_k}$ for arbitrary $A_k^{(1,2)} \in A_k\{1,2\}$. Let $A_i^{(1,2)} \in A_i\{1,2\}$, $i = 1, 2, \dots, k$ be arbitrary. Then

$$A_1A_2 \cdots A_k A_k^{(1,2)} A_{k-1}^{(1,2)} \cdots A_1^{(1,2)} A_1A_2 \cdots A_k = A_1A_2 \cdots A_k \tag{2.45}$$

and

$$\begin{aligned} &A_k^{(1,2)} A_{k-1}^{(1,2)} \cdots A_1^{(1,2)} A_1A_2 \cdots A_k A_k^{(1,2)} A_{k-1}^{(1,2)} \cdots A_1^{(1,2)} \\ &= A_k^{(1,2)} A_{k-1}^{(1,2)} \cdots A_1^{(1,2)}. \end{aligned} \tag{2.46}$$

Multiplying (2.45) by $A_k^{(1,2)}$ from the right and (2.46) by A_k from the left we get that

$$A_1A_2 \cdots A_{k-1} A_{k-1}^{(1,2)} A_{k-2}^{(1,2)} \cdots A_1^{(1,2)} A_1A_2 \cdots A_{k-1} = A_1A_2 \cdots A_{k-1} \tag{2.47}$$

and

$$\begin{aligned} &A_{k-1}^{(1,2)} A_{k-2}^{(1,2)} \cdots A_1^{(1,2)} A_1A_2 \cdots A_{k-1} A_{k-1}^{(1,2)} A_{k-2}^{(1,2)} \cdots A_1^{(1,2)} \\ &= A_{k-1}^{(1,2)} A_{k-2}^{(1,2)} \cdots A_1^{(1,2)}. \end{aligned} \tag{2.48}$$

Since (2.47) and (2.48) hold for arbitrary $A_i^{(1,2)} \in A_i\{1,2\}$, $i = 1, 2, \dots, k-1$, we obtain

$$A_{k-1}\{1,2\} \cdot A_{k-2}\{1,2\} \cdots A_1\{1,2\} \subseteq (A_1A_2 \cdots A_{k-1})\{1,2\}. \tag{2.49}$$

By the induction hypothesis, from (2.49) it follows that at least one of the following conditions is true:

(1″) There exists $i \in \{1, 2, \dots, k-1\}$ such that $A_i = 0$,

(2″) $A_i \in B_l^{-1}(\mathcal{H}_{i+1}, \mathcal{H}_i)$, $i = 1, 2, \dots, k-2$,

(3″) $A_i \in B_r^{-1}(\mathcal{H}_{i+1}, \mathcal{H}_i)$, $i = 2, 3, \dots, k-1$,

(4″) There exists $i \in \{1, 2, \dots, k-1\}$ such that $A_j \in B_l^{-1}(\mathcal{H}_{j+1}, \mathcal{H}_j)$ for $j = 1, 2, \dots, i-1$ and $A_j \in B_r^{-1}(\mathcal{H}_{j+1}, \mathcal{H}_j)$, $j = i+1, i+2, \dots, k-1$.

If (1″) holds, then (iii) is satisfied.
Suppose that (2″) is true. Then $A_i \in B_l^{-1}(\mathcal{H}_{i+1}, \mathcal{H}_i)$, $i = 1, 2, \dots, k-2$ and $A_k \in \mathcal{B}_r^{-1}(\mathcal{H}_{k+1}, \mathcal{H}_k)$, so (iii) obviously holds.
If (3″) is true, then $A_i \in B_r^{-1}(\mathcal{H}_{i+1}, \mathcal{H}_i)$, $i = 2, 3, \dots, k$, so (iii) is satisfied.
Suppose that (4″) holds. Then $A_j \in B_l^{-1}(\mathcal{H}_{j+1}, \mathcal{H}_j)$ for $j = 1, 2, \dots, i-1$ and $A_j \in B_r^{-1}(\mathcal{H}_{j+1}, \mathcal{H}_j)$, $j = i+1, i+2, \dots, k$, so (iii) holds.

$(iii) \Rightarrow (ii)$: If $A_1 A_2 \cdots A_n = 0$, then it is evident that (ii) holds. Suppose that $A_1 A_2 \cdots A_n \neq 0$ and let $A_i^{(1,2)} \in A_i\{1, 2\}$, $i = 1, 2, \dots, n$ be arbitrary.

If $A_i \in \mathcal{B}_r^{-1}(\mathcal{H}_{i+1}, \mathcal{H}_i)$ for $i = 2, 3, \dots, n$, then $A_i A_i^{(1,2)} = I_{\mathcal{H}_i}$ for $i = 2, 3, \dots, n$. Now,

$$\begin{aligned}
&A_1 A_2 \cdots A_{n-1} A_n A_n^{(1,2)} A_{n-1}^{(1,2)} \cdots A_1^{(1,2)} \\
&= A_1 A_2 \cdots A_{n-2} A_{n-1} A_{n-1}^{(1,2)} A_{n-2}^{(1,2)} \cdots A_1^{(1,2)} \\
&\vdots \\
&= A_1 A_2 A_2^{(1,2)} A_1^{(1,2)} \\
&= A_1 A_1^{(1,2)}.
\end{aligned} \tag{2.50}$$

From (2.50) it follows

$$\begin{aligned}
&A_1 A_2 \cdots A_n A_n^{(1,2)} A_{n-1}^{(1,2)} \cdots A_1^{(1,2)} A_1 A_2 \cdots A_n \\
&= A_1 A_2 \cdots A_n
\end{aligned}$$

and

$$\begin{aligned}
&A_n^{(1,2)} A_{n-1}^{(1,2)} \cdots A_1^{(1,2)} A_1 A_2 \cdots A_n A_n^{(1,2)} A_{n-1}^{(1,2)} \\
&= A_n^{(1,2)} A_{n-1}^{(1,2)} \cdots A_1^{(1,2)},
\end{aligned}$$

so $A_n^{(1,2)} A_{n-1}^{(1,2)} \cdots A_1^{(1,2)} \in (A_1 A_2 \cdots A_n)\{1, 2\}$.

If $A_i \in \mathcal{B}_l^{-1}(\mathcal{H}_{i+1}, \mathcal{H}_i)$ for $i = 1, 2, \dots, n-1$, then $A_i^{(1,2)} A_i = I_{\mathcal{H}_{i+1}}$ for $i = 1, 2, \dots, n$. Now,

$$\begin{aligned}
&A_n^{(1,2)} A_{n-1}^{(1,2)} \cdots A_2^{(1,2)} A_1^{(1,2)} A_1 A_2 \cdots A_{n-1} A_n \\
&= A_n^{(1,2)} A_{n-1}^{(1,2)} \cdots A_3^{(1,2)} A_2^{(1,2)} A_2 A_3 \cdots A_{n-1} A_n \\
&\vdots \\
&= A_n^{(1,2)} A_{n-1}^{(1,2)} A_{n-1} A_n \\
&= A_n^{(1,2)} A_n.
\end{aligned} \tag{2.51}$$

From (2.51) it follows

$$A_1 A_2 \cdots A_n A_n^{(1,2)} A_{n-1}^{(1,2)} \cdots A_1^{(1,2)} A_1 A_2 \cdots A_n = A_1 A_2 \cdots A_n$$

and

$$\begin{aligned}
&A_n^{(1,2)} A_{n-1}^{(1,2)} \cdots A_1^{(1,2)} A_1 A_2 \cdots A_n A_n^{(1,2)} A_{n-1}^{(1,2)} \cdots A_1^{(1,2)} \\
&= A_n^{(1,2)} A_{n-1}^{(1,2)} \cdots A_1^{(1,2)},
\end{aligned}$$

so $A_n^{(1,2)} A_{n-1}^{(1,2)} \cdots A_1^{(1,2)} \in (A_1 A_2 \cdots A_n)\{1,2\}$.

Assume now that there exists $2 \le k \le n-1$ such that $A_i \in \mathcal{B}_l^{-1}(\mathcal{H}_{i+1}, \mathcal{H}_i)$ for $i = 1, 2, \ldots, k-1$, and $A_i \in \mathcal{B}_r^{-1}(\mathcal{H}_{i+1}, \mathcal{H}_i)$ for $i = k+1, k+2, \ldots, n$. Now it follows that

$$A_i^{(1,2)} A_i = I_{\mathcal{H}_{i+1}} \quad \text{for} \quad i = 1, 2, \ldots, k-1, \tag{2.52}$$

and

$$A_i A_i^{(1,2)} = I_{\mathcal{H}_i} \quad \text{for} \quad i = k+1, k+2, \ldots, n. \tag{2.53}$$

From (2.52) it follows that

$$A_{k-1}^{(1,2)} A_{k-2}^{(1,2)} \cdots A_1^{(1,2)} A_1 A_2 \cdots A_{k-1} = I_{\mathcal{H}_k}, \tag{2.54}$$

and from (2.53)

$$A_{k+1} A_{k+2} \cdots A_n A_n^{(1,2)} A_{n-1}^{(1,2)} \cdots A_{(k+1)}^{(1,2)} = I_{\mathcal{H}_k}. \tag{2.55}$$

Now, from (2.54) and (2.55) we have

$$\begin{aligned}
&A_1 A_2 \cdots A_{k-1} A_k A_{k+1} \cdots A_n A_n^{(1,2)} A_{n-1}^{(1,2)} \cdots A_{(k+1)}^{(1,2)} A_k^{(1,2)} \cdot \\
&A_{k-1}^{(1,2)} \cdots A_1^{(1,2)} A_1 A_2 \cdots A_{k-1} A_k A_{k+1} \cdots A_n \\
&= A_1 A_2 \cdots A_k I_{\mathcal{H}_k} A_k^{(1,2)} I_{\mathcal{H}_k} A_k A_{k+1} \cdots A_n \\
&= A_1 A_2 \cdots A_n
\end{aligned}$$

and

$$\begin{aligned}
&A_n^{(1,2)} A_{n-1}^{(1,2)} \cdots A_1^{(1,2)} A_1 A_2 \cdots A_n A_n^{(1,2)} A_{n-1}^{(1,2)} \cdots A_1^{(1,2)} \\
&= A_n^{(1,2)} A_{n-1}^{(1,2)} \cdots A_{(k+1)}^{(1,2)} A_k^{(1,2)} A_{k-1}^{(1,2)} \cdots A_1^{(1,2)} A_1 A_2 \cdots A_{k-1} A_k \cdot \\
&A_{k+1} \cdots A_n A_n^{(1,2)} A_{n-1}^{(1,2)} \cdots A_{(k+1)}^{(1,2)} A_k^{(1,2)} A_{k-1}^{(1,2)} \cdots A_1^{(1,2)} \\
&= A_n^{(1,2)} A_{n-1}^{(1,2)} \cdots A_{(k+1)}^{(1,2)} A_k^{(1,2)} I_{\mathcal{H}_k} A_k I_{\mathcal{H}_k} A_k^{(1,2)} A_{k-1}^{(1,2)} \cdots A_1^{(1,2)} \\
&= A_n^{(1,2)} A_{n-1}^{(1,2)} \cdots A_1^{(1,2)},
\end{aligned}$$

so $A_n^{(1,2)}A_{n-1}^{(1,2)}\cdots A_1^{(1,2)} \in (A_1A_2\cdots A_n)\{1,2\}$. $\square$

2.4.2 Reverse order law for $\{1,3\}$- and $\{1,4\}$-inverses

The following elementary lemma will be often used in this section.

Lemma 2.4.1 *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be regular. Then*

$$X \in A\{1,3\} \Leftrightarrow A^*AX = A^*.$$

Wei [58] obtained necessary and sufficient conditions for inclusions

$$A_n\{1,3\}\cdot A_{n-1}\{1,3\}\cdots A_1\{1,3\} \subseteq (A_1A_2\cdots A_n)\{1,3\}$$

and

$$A_n\{1,4\}\cdot A_{n-1}\{1,4\}\cdots A_1\{1,4\} \subseteq (A_1A_2\cdots A_n)\{1,4\}$$

by applying the multiple product singular value decomposition (P-SVD). In following theorems, we generalize his result to the case of regular bounded linear operators on Hilbert spaces and we derive new simple conditions which involve only ranges of operators.

Theorem 2.4.4 *Let $A_i \in \mathcal{B}(\mathcal{H}_{i+1},\mathcal{H}_i)$, be such that A_i, $i = 1,2,\ldots,n$ and $A_1A_2\cdots A_n$, are regular operators. The following conditions are equivalent:*

(i) $A_n\{1,3\}\cdot A_{n-1}\{1,3\}\cdots A_1\{1,3\} \subseteq (A_1A_2\cdots A_n)\{1,3\}$,

(ii) $\mathcal{R}(A_k^*A_{k-1}^*\cdots A_1^*A_1A_2\cdots A_n) \subseteq \mathcal{R}(A_{k+1})$ *for* $k = 1,2,\ldots,n-1$.

Proof. $(i) \Rightarrow (ii)$: If $A_1A_2\cdots A_n = 0$, then

$$\mathcal{R}(A_k^*A_{k-1}^*\cdots A_1^*A_1A_2\cdots A_n) = \{0\} \subseteq \mathcal{R}(A_{k+1}),$$

for $k = 1,2,\ldots,n-1$, so (ii) holds.

Assume now that $A_1A_2\cdots A_n \neq 0$. Let $A_i^{(1,3)} \in A_i\{1,3\}$, $i = 1,2,\ldots,n$ be arbitrary. From (i) by Lemma 2.4.1 it follows that

$$(A_1A_2\cdots A_n)^*A_1A_2\cdots A_nA_n^{(1,3)}A_{n-1}^{(1,3)}\cdots A_1^{(1,3)} = A_1A_2\cdots A_n. \tag{2.56}$$

Let $i \in \{1,2,\ldots,n-1\}$ be arbitrary. Since, for arbitrary $X \in \mathcal{B}(\mathcal{H}_i,\mathcal{H}_{i+1})$, $A_i^{(1,3)} + (I_{\mathcal{H}_{i+1}} - A_i^{(1,3)}A_i)X \in A_i\{1,3\}$, from Lemma 2.4.1 it follows that

$$\begin{aligned}&(A_1A_2\cdots A_n)^*A_1A_2\cdots A_n\cdot\\ &A_n^{(1,3)}\cdots A_{i+1}^{(1,3)}(A_i^{(1,3)} + (I_{\mathcal{H}_{i+1}} - A_i^{(1,3)}A_i)X)A_{i-1}^{(1,3)}\cdots A_1^{(1,3)}\\ &= A_1\cdots A_n.\end{aligned} \tag{2.57}$$

Substracting (2.56) from (2.57) we get that

$$\begin{aligned}&(A_1A_2\cdots A_n)^*A_1A_2\cdots A_n\cdot\\&A_n^{(1,3)}\cdots A_{i+1}^{(1,3)}(I_{\mathcal{H}_{i+1}}-A_i^{(1,3)}A_i)XA_{i-1}^{(1,3)}\cdots A_1^{(1,3)}=0\end{aligned}\tag{2.58}$$

holds for arbitrary $i\in\{1,2,\ldots,n-1\}$.

From (2.58) it follows that

$$A_{i-1}^{(1,3)}\cdots A_1^{(1,3)}=0\tag{2.59}$$

or

$$(A_1A_2\cdots A_n)^*A_1A_2\cdots A_nA_n^{(1,3)}\cdots A_{i+1}^{(1,3)}(I_{\mathcal{H}_{i+1}}-A_i^{(1,3)}A_i)=0.\tag{2.60}$$

If (2.59) holds, then from (2.56) it follows that $A_1A_2\cdots A_n=0$, which is a contradiction, so (2.60) must hold arbitrary $i\in\{1,2,\ldots,n-1\}$.

Condition (ii) is equivalent to

$$\begin{aligned}&(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-k}A_{n-k+1}A_{n-k+1}^{(1,3)}\\&=(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-k},\quad k=1,2,\ldots,n-1.\end{aligned}\tag{2.61}$$

We will prove (2.61) by induction on k.

From (2.60) and (2.56) it follows that

$$\begin{aligned}&(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-1}A_nA_n^{(1,3)}\\&=(A_1A_2\cdots A_n)^*A_1A_2\cdots\cdots A_nA_n^{(1,3)}A_{n-1}^{(1,3)}A_{n-1}\\&=(A_1A_2\cdots A_n)^*A_1A_2\cdots\cdots A_nA_n^{(1,3)}A_{n-1}^{(1,3)}A_{n-2}^{(1,3)}A_{n-2}A_{n-1}\\&\vdots\\&=(A_1A_2\cdots A_n)^*A_1A_2\cdots\cdots A_nA_n^{(1,3)}A_{n-1}^{(1,3)}\cdots A_1^{(1,3)}A_1A_2\cdots A_{n-2}A_{n-1}\\&=(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-2}A_{n-1},\end{aligned}$$

so (2.61) holds for $k=1$.

Assume now that (2.61) holds for $k<l$ for some $l\leq n$, i.e.

$$\begin{aligned}&(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-k}A_{n-k+1}A_{n-k+1}^{(1,3)}\\&=(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-k},\quad k=1,2,\ldots,l-1.\end{aligned}\tag{2.62}$$

We will prove that (2.61) is true for $k=l$.

Using (2.62) we have

$$\begin{aligned}&(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-l}A_{n-l+1}A_{n-l+1}^{(1,3)}\\&=(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-l}A_{n-l+1}A_{n-l+2}A_{n-l+2}^{(1,3)}A_{n-l+1}^{(1,3)}\\&=(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-l}A_{n-l+1}A_{n-l+2}A_{n-l+3}A_{n-l+3}^{(1,3)}A_{n-l+2}^{(1,3)}A_{n-l+1}^{(1,3)}\\&\vdots\\&=(A_1A_2\cdots A_n)^*A_1A_2\cdots\cdots A_nA_n^{(1,3)}A_{n-1}^{(1,3)}\cdots A_{n-l+1}^{(1,3)},\end{aligned}$$

i.e.

$$\begin{aligned}&(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-l}A_{n-l+1}A_{n-l+1}^{(1,3)}\\&=(A_1A_2\cdots A_n)^*A_1A_2\cdots\cdots A_nA_n^{(1,3)}A_{n-1}^{(1,3)}\cdots A_{n-l+1}^{(1,3)}.\end{aligned}\tag{2.63}$$

Now, using (2.63) and (2.56) in the last step, we get

$$\begin{aligned}&(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-l}A_{n-l+1}A_{n-l+1}^{(1,3)}\\&=(A_1A_2\cdots A_n)^*A_1A_2\cdots\cdots A_nA_n^{(1,3)}A_{n-1}^{(1,3)}\cdots A_{n-l+1}^{(1,3)}\\&=(A_1A_2\cdots A_n)^*A_1A_2\cdots\cdots A_nA_n^{(1,3)}A_{n-1}^{(1,3)}\cdots A_{n-l+1}^{(1,3)}A_{n-l}^{(1,3)}A_{n-l}\\&=(A_1A_2\cdots A_n)^*A_1A_2\cdots\cdots A_nA_n^{(1,3)}A_{n-1}^{(1,3)}\cdots A_{n-l+1}^{(1,3)}A_{n-l}^{(1,3)}A_{n-l-1}^{(1,3)}A_{n-l-1}A_{n-l}\\&\vdots\\&=(A_1A_2\cdots A_n)^*A_1A_2\cdots\cdots A_nA_n^{(1,3)}A_{n-1}^{(1,3)}\cdots A_1^{(1,3)}A_1A_2\cdots A_{n-l}\\&=(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-l},\end{aligned}$$

so (2.61) holds for $k=l$.

$(ii)\Rightarrow(i)$: Let $A_i^{(1,3)}\in A_i\{1,3\}$, $i=1,2,\ldots,n$ be arbitrary. Condition (ii) is equivalent to

$$\begin{aligned}&(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-k}A_{n-k+1}A_{n-k+1}^{(1,3)}\\&=(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-k},\quad k=1,2,\ldots,n-1.\end{aligned}\tag{2.64}$$

Now, from (2.64) it follows

$$\begin{aligned}&(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-1}A_nA_n^{(1,3)}A_{n-1}^{(1,3)}\cdots A_1^{(1,3)}\\&=(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-2}A_{n-1}A_{n-1}^{(1,3)}A_{n-2}^{(1,3)}\cdots A_1^{(1,3)}\\&=(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-2}A_{n-2}^{(1,3)}\cdots A_1^{(1,3)}\\&\vdots\\&=(A_1A_2\cdots A_n)^*A_1A_1^{(1,3)}\\&=(A_1A_2\cdots A_n)^*.\end{aligned}$$

Now, from Lemma 2.4.1 it follows that

$$A_n^{(1,3)}A_{n-1}^{(1,3)}\cdots A_1^{(1,3)}\in(A_1A_2\cdots A_n)\{1,3\}$$

so (i) holds. □

The next result follows from Theorem 2.4.4 by taking adjoints:

Theorem 2.4.5 *Let $A_i\in\mathcal{B}(\mathcal{H}_{i+1},\mathcal{H}_i)$, be such that A_i, $i=1,2,\ldots,n$ and $A_1A_2\cdots A_n$, are regular operators. The following conditions are equivalent:*

(i) $A_n\{1,4\} \cdot A_{n-1}\{1,4\} \cdots A_1\{1,4\} \subseteq (A_1A_2 \cdots A_n)\{1,4\}$,

(ii) $\mathcal{R}(A_kA_{k-1} \cdots A_1A_1^*A_2^* \cdots A_n^*) \subseteq \mathcal{R}(A_{k+1}^*)$ *for* $k = 1, 2, \ldots, n-1$.

Notice that conditions given in Theorems 2.4.4 and 2.4.5 for $n = 2$ reduce to conditions obtained by Djordjević in [34]:

Theorem 2.4.6 *[34] Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ be such that A, B, AB have closed ranges. Then the following statements are equivalent:*

(1) $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$,

(2) $B\{1,3\} \cdot A\{1,3\} \subseteq (AB)\{1,3\}$,

(3) $B^\dagger A^\dagger \in (AB)\{1,3\}$,

(4) $B^\dagger A^\dagger \in (AB)\{1,2,3\}$.

Theorem 2.4.7 *[34] Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ be such that A, B, AB have closed ranges. Then the following statements are equivalent:*

(1) $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(B^*)$,

(2) $B\{1,4\} \cdot A\{1,4\} \subseteq (AB)\{1,4\}$,

(3) $B^\dagger A^\dagger \in (AB)\{1,4\}$,

(4) $B^\dagger A^\dagger \in (AB)\{1,2,4\}$.

2.4.3 Reverse order law for $\{1\}$-inverses

The following lemma is elementary but very useful result. It is completely analogous to the matrix case (see Theorem 1.5.1).

Theorem 2.4.8 *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be regular and $A^{(1)} \in A\{1\}$ be arbitrary. Then*

$$A\{1\} = \{A^{(1)} + X - A^{(1)}AXAA^{(1)} : X \in \mathcal{B}(\mathcal{K},\mathcal{H})\}.$$

In Theorem 2.1.1 Werner [70] proved that for arbitrary matrices $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$

$$B\{1\}A\{1\} \subseteq (AB)\{1\}$$

holds if and only if

$$\mathcal{N}(A) \subseteq \mathcal{R}(B) \quad \text{or} \quad \mathcal{R}(B) \subseteq \mathcal{N}(A).$$

The following theorem will be used in this section. It is a generalization of Werners result to the case of regular product of two regular operators on Hilbert spaces. The proof will be given because of the completeness.

Theorem 2.4.9 *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ be such that A, B and AB are regular operators. The following conditions are equivalent:*

(*i*) $B\{1\}A\{1\} \subseteq (AB)\{1\}$

(*ii*) $\mathcal{N}(A) \subseteq \mathcal{R}(B)$ *or* $\mathcal{R}(B) \subseteq \mathcal{N}(A)$.

Proof. (*i*) $\Rightarrow$ (*ii*): By Lemma 2.5.2, for arbitrary $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ and $Y \in \mathcal{B}(\mathcal{H},\mathcal{L})$, $A^{(1)} + (I_{\mathcal{H}} - A^{(1)}A)X \in A\{1\}$ and $B^{(1)} + Y(I_{\mathcal{H}} - BB^{(1)}) \in B\{1\}$. It follows that

$$AB(B^{(1)} + Y(I_{\mathcal{H}} - BB^{(1)}))(A^{(1)} + (I_{\mathcal{H}} - A^{(1)}A)X)AB = AB \tag{2.65}$$

holds for arbitrary $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ and $Y \in \mathcal{B}(\mathcal{H},\mathcal{L})$. Substituting $X = 0$ in (2.65) we get

$$AB(B^{(1)} + Y(I_{\mathcal{H}} - BB^{(1)}))A^{(1)}AB = AB. \tag{2.66}$$

Subtracting (2.66) from (2.65) we get

$$AB(B^{(1)} + Y(I_{\mathcal{H}} - BB^{(1)}))(I_{\mathcal{H}} - A^{(1)}A)XAB = 0. \tag{2.67}$$

Substituting $Y = 0$ in (2.67) we get

$$ABB^{(1)}(I_{\mathcal{H}} - A^{(1)}A)XAB = 0. \tag{2.68}$$

Subtracting (2.68) from (2.67) we get that

$$ABY(I_{\mathcal{H}} - BB^{(1)})(I_{\mathcal{H}} - A^{(1)}A)XAB = 0 \tag{2.69}$$

holds for arbitrary $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ and $Y \in \mathcal{B}(\mathcal{H},\mathcal{L})$. From (2.69) it follows that $AB = 0$ or $(I_{\mathcal{H}} - BB^{(1)})(I_{\mathcal{H}} - A^{(1)}A) = 0$, which is equivalent to (*ii*). (*ii*) $\Rightarrow$ (*i*): If $AB = 0$, then it is evident that (*i*) holds. Suppose that $\mathcal{N}(A) \subseteq \mathcal{R}(B)$ and let $A^{(1)} \in A\{1\}$ and $B^{(1)} \in B\{1\}$ be arbitrary. Since $\mathcal{N}(A) \subseteq \mathcal{R}(B)$, it follows that $BB^{(1)}(I_{\mathcal{H}} - A^{(1)}A) = I_{\mathcal{H}} - A^{(1)}A$ which is equivalent to

$$BB^{(1)}A^{(1)}A = BB^{(1)} + A^{(1)}A - I_{\mathcal{H}}. \tag{2.70}$$

Now, from (2.70) it follows that

$$ABB^{(1)}A^{(1)}AAB = AB. \quad \square$$

By applying the multiple product singular value decomposition (P-SVD), Wei [101] obtained necessary and sufficient conditions for inclusion $A_n\{1\} \cdot A_{n-1}\{1\} \cdots A_1\{1\} \subseteq (A_1A_2 \cdots A_n)\{1\}$:

Theorem 2.4.10 [*101*] *Suppose that* $A_i \in \mathbb{C}^{s_i \times s_{i+1}}$, $i = 1, 2, \ldots n$. *Then the following conditions are equivalent:*

(*i*) $A_n\{1\} \cdot A_{n-1}\{1\} \cdots A_1\{1\} \subseteq (A_1A_2 \cdots A_n)\{1\}$;

(*ii*)

(*a*) $r_n^1 = 0$, *or*

(*b*) $m_{i+1} - r_i = r_{i+1}^{j+1}$ *for* $i = 1, \dots, n-2$ *and*
$r_{i+1}^{j+1} = r_{i+2}^{j+1} = \cdots r_n^{j+1}$ *and* $r_j = m_{j+1}$ *for* $j = 2, \dots, n-2$

where constants r_j^i, r_j, m_j, $j = 1, \dots n$ *are defined in P-SVD of matrices* $A_1, \dots A_n$.

Now, we will present a result from [26] in which we derive necessary and sufficient conditions for the inclusion $A_n\{1\} \cdot A_{n-1}\{1\} \cdots A_1\{1\} \subseteq (A_1 A_2 \cdots A_n)\{1\}$.

Theorem 2.4.11 *Let* $A_i \in \mathcal{B}(\mathcal{H}_{i+1}, \mathcal{H}_i)$, $i = 1, 2, \dots, n$, *be such that* A_i, $i = 1, 2, \dots, n$ *and all* $A_1 A_2 \cdots A_j$, $j = 2, 3, \dots, n$, *are regular operators. The following conditions are equivalent:*

(*i*) $A_n\{1\} \cdot A_{n-1}\{1\} \cdots A_1\{1\} \subseteq (A_1 A_2 \cdots A_n)\{1\}$,

(*ii*) $A_1 A_2 \cdots A_n = 0$,
or
$\mathcal{N}(A_1 \cdots A_{j-1}) \subseteq \mathcal{R}(A_j)$ *for* $j = 2, 3, \dots, n$.

(*iii*) $A_1 A_2 \cdots A_n = 0$,
or
$A_k\{1\} \cdot A_{k-1}\{1\} \cdots A_1\{1\} \subseteq (A_1 A_2 \cdots A_k)\{1\}$ *for* $k = 2, 3, \dots, n$.

Proof. (*ii*) $\Rightarrow$ (*iii*) : If $A_1 A_2 \cdots A_n = 0$, it is evident that (*iii*) holds. Suppose that $A_1 A_2 \cdots A_n \neq 0$ and

$$\mathcal{N}(A_1 \cdots A_{j-1}) \subseteq \mathcal{R}(A_j) \quad \text{for} \quad j = 2, 3, \dots, n. \tag{2.71}$$

We will prove by induction on k that

$$A_k\{1\} \cdot A_{k-1}\{1\} \cdots A_1\{1\} \subseteq (A_1 A_2 \cdots A_k)\{1\} \tag{2.72}$$

holds for $k = 2, 3, \dots, n$. From (2.71) it follows that $\mathcal{N}(A_1) \subseteq \mathcal{R}(A_2)$ holds which using Theorem 3.2.1 implies $A_2\{1\}A_1\{1\} \subseteq (A_1 A_2)\{1\}$, so (2.72) holds for $k = 2$.

Suppose that (2.72) holds for $k = l - 1$, for some $l \in \{2, 3, \dots, n\}$, i.e.

$$A_{l-1}\{1\} \cdot A_{l-2}\{1\} \cdots A_1\{1\} \subseteq (A_1 A_2 \cdots A_{l-1})\{1\}. \tag{2.73}$$

We will prove that (2.72) holds for $k = l$. From (2.71) it follows that

$$\mathcal{N}(A_1 \cdots A_{l-1}) \subseteq \mathcal{R}(A_l). \tag{2.74}$$

From (2.74) and Theorem 3.2.1 we have

$$A_l\{1\} \cdot (A_1 A_2 \cdots A_{l-1})\{1\} \subseteq (A_1 A_2 \cdots A_{l-1} A_l)\{1\}. \tag{2.75}$$

Now, from (2.73) and (2.75) we get

$$A_l\{1\} \cdot A_{l-1}\{1\} \cdots A_1\{1\} \subseteq (A_1 A_2 \cdots A_l)\{1\},$$

so (2.72) holds for every $k \in \{2, 3, \ldots, n\}$.

$(iii) \Rightarrow (i)$: It is evident.

$(i) \Rightarrow (ii)$: Suppose that $A_1A_2 \cdots A_n \neq 0$. Let $j \in \{3, 4, \ldots, n\}$ and $i \in \{1, 2, \ldots, j-2\}$ be arbitrary. Then for arbitrary $X \in \mathcal{B}(\mathcal{H}_i, \mathcal{H}_{i+1})$ and $Y \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_{j+1})$, and arbitrary $A_i^{(1)} \in A_i\{1\}$ and $A_j^{(1)} \in A_j\{1\}$, by Lemma 2.5.2, $A_i^{(1)} + (I_{\mathcal{H}_{i+1}} - A_i^{(1)}A_i)X \in A_i\{1\}$ and $A_j^{(1)} + Y(I_{\mathcal{H}_j} - A_jA_j^{(1)}) \in A_j\{1\}$. From (i) it follows that

$$\begin{aligned} &A_1A_2 \cdots A_nA_n^{(1)} \cdots A_{j+1}^{(1)} \cdot \\ &(A_j^{(1)} + Y(I_{\mathcal{H}_j} - A_jA_j^{(1)}))A_{j-1}^{(1)} \cdots A_{i+1}^{(1)}(A_i^{(1)} + (I_{\mathcal{H}_{i+1}} - A_i^{(1)}A_i)X) \cdot \\ &A_{i-1}^{(1)} \cdots A_1^{(1)}A_1 \cdots A_n = A_1 \cdots A_n \end{aligned} \tag{2.76}$$

holds for arbitrary $X \in \mathcal{B}(\mathcal{H}_i, \mathcal{H}_{i+1})$ and $Y \in \mathcal{B}(\mathcal{H}_{j+1}, \mathcal{H}_j)$. Substituting $X = 0$ in (4.18) we get

$$\begin{aligned} &A_1A_2 \cdots A_nA_n^{(1)} \cdots A_{j+1}^{(1)} \cdot \\ &(A_j^{(1)} + Y(I_{\mathcal{H}_j} - A_jA_j^{(1)}))A_{j-1}^{(1)} \cdots A_{i+1}^{(1)}A_i^{(1)} \cdot \\ &A_{i-1}^{(1)} \cdots A_1^{(1)}A_1 \cdots A_n = A_1 \cdots A_n. \end{aligned} \tag{2.77}$$

Subtracting (4.19) from (4.18) we get that

$$\begin{aligned} &A_1A_2 \cdots A_nA_n^{(1)} \cdots A_{j+1}^{(1)} \cdot \\ &(A_j^{(1)} + Y(I_{\mathcal{H}_j} - A_jA_j^{(1)}))A_{j-1}^{(1)} \cdots A_{i+1}^{(1)}(I_{\mathcal{H}_{i+1}} - A_i^{(1)}A_i)X \cdot \\ &A_{i-1}^{(1)} \cdots A_1^{(1)}A_1 \cdots A_n = 0 \end{aligned} \tag{2.78}$$

holds for arbitrary $Y \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_{j+1})$.
Substituting $Y = 0$ in (3.12) we get

$$\begin{aligned} &A_1A_2 \cdots A_nA_n^{(1)} \cdots A_{j+1}^{(1)} \cdot \\ &A_j^{(1)}A_{j-1}^{(1)} \cdots A_{i+1}^{(1)}(I_{\mathcal{H}_{i+1}} - A_i^{(1)}A_i)X \cdot \\ &A_{i-1}^{(1)} \cdots A_1^{(1)}A_1 \cdots A_n = 0. \end{aligned} \tag{2.79}$$

Subtracting (3.13) from (3.12) we get that

$$\begin{aligned} &A_1A_2 \cdots A_nA_n^{(1)} \cdots A_{j+1}^{(1)} \cdot \\ &Y(I_{\mathcal{H}_j} - A_jA_j^{(1)})A_{j-1}^{(1)} \cdots A_{i+1}^{(1)}(I_{\mathcal{H}_{i+1}} - A_i^{(1)}A_i)X \cdot \\ &A_{i-1}^{(1)} \cdots A_1^{(1)}A_1 \cdots A_n = 0 \end{aligned} \tag{2.80}$$

holds for arbitrary $X \in \mathcal{B}(\mathcal{H}_i, \mathcal{H}_{i+1})$ and $Y \in \mathcal{B}(\mathcal{H}_{j+1}, \mathcal{H}_j)$.

From (3.14) it follows that

$$A_1A_2\cdots A_nA_n^{(1)}\cdots A_{j+1}^{(1)}=0 \tag{2.81}$$

or

$$(I_{\mathcal{H}_j}-A_jA_j^{(1)})A_{j-1}^{(1)}\cdots A_{i+1}^{(1)}(I_{\mathcal{H}_{i+1}}-A_i^{(1)}A_i)=0 \tag{2.82}$$

or

$$A_{i-1}^{(1)}\cdots A_1^{(1)}A_1\cdots A_n=0. \tag{2.83}$$

It is easy to see that both (2.81) and (2.83) using (3.10) imply $A_1A_2\cdots A_n=0$ which is a contradiction. So (2.82) must hold. We have proved

$$(I_{\mathcal{H}_j}-A_jA_j^{(1)})A_{j-1}^{(1)}\cdots A_{i+1}^{(1)}(I_{\mathcal{H}_{i+1}}-A_i^{(1)}A_i)=0 \tag{2.84}$$

holds for arbitrary $j\in\{3,4,\ldots,n\}$ and $i\in\{1,2,\ldots,j-2\}$, which is equivalent to

$$(I_{\mathcal{H}_j}-A_jA_j^{(1)})A_{j-1}^{(1)}\cdots A_{i+1}^{(1)}A_i^{(1)}A_i=(I_{\mathcal{H}_j}-A_jA_j^{(1)})A_{j-1}^{(1)}\cdots A_{i+1}^{(1)} \tag{2.85}$$

for arbitrary $j\in\{3,4,\ldots,n\}$ and $i\in\{1,2,\ldots,j-2\}$.

Let, $j\in\{2,3,\ldots,n\}$, $X\in\mathcal{B}(\mathcal{H}_{j-1},\mathcal{H}_j)$ and $Y\in\mathcal{B}(\mathcal{H}_j,\mathcal{H}_{j+1})$ be arbitrary. Then $A_{j-1}^{(1)}+(I_{\mathcal{H}_j}-A_{j-1}^{(1)}A_{j-1})X\in A_{j-1}\{1\}$ and $A_j^{(1)}+Y(I_{\mathcal{H}_j}-A_jA_j^{(1)})\in A_j\{1\}$. From (i) it follows that

$$\begin{aligned}&A_1A_2\cdots A_nA_n^{(1)}\cdots A_{j+1}^{(1)}\cdot\\&(A_j^{(1)}+Y(I_{\mathcal{H}_j}-A_jA_j^{(1)}))(A_{j-1}^{(1)}+(I_{\mathcal{H}_j}-A_{j-1}^{(1)}A_{j-1})X)\cdot\\&A_{j-2}^{(1)}\cdots A_1^{(1)}A_1\cdots A_n=A_1\cdots A_n\end{aligned} \tag{2.86}$$

holds for arbitrary $X\in\mathcal{B}(\mathcal{H}_{j-1},\mathcal{H}_j)$ and $Y\in\mathcal{B}(\mathcal{H}_j,\mathcal{H}_{j+1})$. Substituting $X=0$ in (2.86) we get

$$\begin{aligned}&A_1A_2\cdots A_nA_n^{(1)}\cdots A_{j+1}^{(1)}\cdot\\&(A_j^{(1)}+Y(I_{\mathcal{H}_j}-A_jA_j^{(1)}))A_{j-1}^{(1)}\cdot\\&A_{j-2}^{(1)}\cdots A_1^{(1)}A_1\cdots A_n=A_1\cdots A_n.\end{aligned} \tag{2.87}$$

Subtracting (2.87) from (2.86) we get that

$$\begin{aligned}&A_1A_2\cdots A_nA_n^{(1)}\cdots A_{j+1}^{(1)}\cdot\\&(A_j^{(1)}+Y(I_{\mathcal{H}_j}-A_jA_j^{(1)}))(I_{\mathcal{H}_j}-A_{j-1}^{(1)}A_{j-1})X\cdot\\&A_{j-2}^{(1)}\cdots A_1^{(1)}A_1\cdots A_n=0\end{aligned} \tag{2.88}$$

holds for arbitrary $Y\in\mathcal{B}(\mathcal{H}_j,\mathcal{H}_{j+1})$.
Substituting $Y=0$ in (2.88) we get

$$\begin{aligned}&A_1A_2\cdots A_nA_n^{(1)}\cdots A_{j+1}^{(1)}\cdot\\&A_j^{(1)}(I_{\mathcal{H}_j}-A_{j-1}^{(1)}A_{j-1})X\cdot\\&A_{j-2}^{(1)}\cdots A_1^{(1)}A_1\cdots A_n=0.\end{aligned}\tag{2.89}$$

Subtracting (2.89) from (2.88) we get that

$$\begin{aligned}&A_1A_2\cdots A_nA_n^{(1)}\cdots A_{j+1}^{(1)}\cdot\\&Y(I_{\mathcal{H}_j}-A_jA_j^{(1)})(I_{\mathcal{H}_j}-A_{j-1}^{(1)}A_{j-1})X\cdot\\&A_{j-2}^{(1)}\cdots A_1^{(1)}A_1\cdots A_n=0\end{aligned}\tag{2.90}$$

holds for arbitrary $X\in\mathcal{B}(\mathcal{H}_{j-1},\mathcal{H}_j)$ and $Y\in\mathcal{B}(\mathcal{H}_j,\mathcal{H}_{j+1})$.
As from (3.14), from (2.90) using $A_1A_2\cdots A_n\neq 0$ it follows that

$$(I_{\mathcal{H}_j}-A_jA_j^{(1)})(I_{\mathcal{H}_j}-A_{j-1}^{(1)}A_{j-1})=0\tag{2.91}$$

holds for arbitrary $j\in\{2,3,\ldots,n\}$.
By taking $j=2$ in (2.91) we get

$$(I_{\mathcal{H}_2}-A_2A_2^{(1)})(I_{\mathcal{H}_2}-A_1^{(1)}A_1)=0,$$

which is equivalent to

$$I_{\mathcal{H}_2}-A_1^{(1)}A_1=A_2A_2^{(1)}(I_{\mathcal{H}_2}-A_1^{(1)}A_1).\tag{2.92}$$

From (2.92) it follows that

$$\mathcal{N}(A_1)\subseteq\mathcal{R}(A_2).\tag{2.93}$$

Now, let $j\in\{3,4,\ldots,n\}$ be arbitrary . Using (2.85) and (2.91) we have

$$\begin{aligned}&(I_{\mathcal{H}_j}-A_jA_j^{(1)})A_{j-1}^{(1)}\cdots A_2^{(1)}A_1^{(1)}A_1A_2A_3\cdots A_{j-1}\\&=(I_{\mathcal{H}_j}-A_jA_j^{(1)})A_{j-1}^{(1)}\cdots A_2^{(1)}A_2A_3\cdots A_{j-1}\\&\vdots\\&=(I_{\mathcal{H}_j}-A_jA_j^{(1)})A_{j-1}^{(1)}A_{j-1}^{(1)}\\&=I_{\mathcal{H}_j}-A_jA_j^{(1)}.\end{aligned}$$

i.e.

$$(I_{\mathcal{H}_j}-A_jA_j^{(1)})A_{j-1}^{(1)}\cdots A_2^{(1)}A_1^{(1)}A_1A_2A_3\cdots A_{j-1}=I_{\mathcal{H}_j}-A_jA_j^{(1)}\tag{2.94}$$

or equivalently

$$\begin{aligned}&A_jA_j^{(1)}(I-A_{j-1}^{(1)}\cdots A_2^{(1)}A_1^{(1)}A_1A_2A_3\cdots A_{j-1})\\&=I-A_{j-1}^{(1)}\cdots A_2^{(1)}A_1^{(1)}A_1A_2A_3\cdots A_{j-1}.\end{aligned}\tag{2.95}$$

From (2.95) it follows that

$$\mathcal{N}(A_1A_2\cdots A_{j-1}) \subseteq \mathcal{R}(A_j) \tag{2.96}$$

holds for arbitrary $j \in \{3, 4, \ldots, n\}$. Now, from (2.93) and (2.96) it follows that

$$\mathcal{N}(A_1A_2\cdots A_{j-1}) \subseteq \mathcal{R}(A_j)$$

holds for arbitrary $j \in \{2, 3, \ldots, n\}$. $\square$

Corollary 2.4.1 *Let $A_i \in \mathcal{B}(\mathcal{H}_{i+1}, \mathcal{H}_i)$, $i = 1, 2, \ldots, n$, be such that A_i, $i = 1, 2, \ldots, n$ and all products $A_1A_2\cdots A_j$, $j = 2, 3, \ldots, n$, are regular operators. If $A_1A_2\cdots A_n \neq 0$, then the following conditions are equivalent:*

(i) $A_n\{1\} \cdot A_{n-1}\{1\} \cdots A_1\{1\} \subseteq (A_1A_2\cdots A_n)\{1\}$,

(ii) $A_k\{1\} \cdot A_{k-1}\{1\} \cdots A_1\{1\} \subseteq (A_1A_2\cdots A_k)\{1\}$ *for* $k = 2, 3, \ldots, n$.

Remark. 1. Throughout all the proofs we used the fact that for operators $A \in \mathcal{B}(\mathcal{K}, \mathcal{M})$ and $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$, where $\mathcal{H}$, $\mathcal{L}$, $\mathcal{K}$ and $\mathcal{M}$ are Hilbert spaces,

$$AXB = 0 \quad \text{for all} \quad X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$$

implies

$$A = 0 \quad \text{or} \quad B = 0.$$

This holds from the fact that $\mathcal{B}(\mathcal{H}, \mathcal{K})$ is a prime algebra.

2. Equivalent conditions for inclusions

$$(A_1A_2)\{1\} \subseteq A_2\{1\} \cdot A_1\{1\},$$

$$(A_1A_2)\{1,3\} \subseteq A_2\{1,3\} \cdot A_1\{1,3\},$$
$$(A_1A_2)\{1,4\} \subseteq A_2\{1,4\} \cdot A_1\{1,4\},$$
$$(A_1A_2\cdots A_n)\{1\} \subseteq A_n\{1\} \cdot A_{n-1}\{1\} \cdots A_1\{1\},$$
$$(A_1A_2\cdots A_n)\{1,3\} \subseteq A_n\{1,3\} \cdot A_{n-1}\{1,3\} \cdots A_1\{1,3\}$$

and

$$(A_1A_2\cdots A_n)\{1,4\} \subseteq A_n\{1,4\} \cdot A_{n-1}\{1,4\} \cdots A_1\{1,4\}$$

remain open problems for further investigation.

2.5 Reverse order law for weighted generalized inverses

Let $A \in \mathbb{C}^{m\times n}$ and let $M \in \mathbb{C}^{m\times m}$, $N \in \mathbb{C}^{n\times n}$ be two positive definite matrices. Recall that weighted Moore-Penrose inverse of A, denoted by $A^\dagger_{M,N}$, is the unique matrix X which satisfies

$$(1)\ AXA = A,\ (2)\ XAX = X,\ (3)\ (MAX)^* = MAX,\ (4)\ (NXA)^* = NXA.$$

For $A \in \mathbb{C}^{m\times n}$, the sets of least-squares weighted generalized($\{1, 3M\}$-inverse of A), minimum-norm weighted generalized inverses($\{1, 4N\}$-inverse of A), $\{1, 2, 3M\}$-inverse of A and $\{1, 2, 4N\}$-inverse of A, respectively are given by:

$$\begin{aligned}
&A\{1, 3M\} = \{X : AXA = A,\ (MAX)^* = MAX\},\\
&A\{1, 4N\} = \{X : AXA = A,\ (NXA)^* = NXA\},\\
&A\{1, 2, 3M\} = \{X : AXA = A,\ XAX = X,\ (MAX)^* = MAX\},\\
&A\{1, 2, 4N\} = \{X : AXA = A,\ XAX = X,\ (NXA)^* = NXA\}.
\end{aligned}$$

In [112] authors presented a necessary and sufficient conditions for the various type of the reverse order laws for the weighted generalized inverses to hold. In this section, we will present some results from [64]. We will derive new necessary and sufficient conditions for the reverse order laws for the weighted generalized inverses of matrices. The significant of our result is that the conditions given in this paper, specially for the $\{1, 2, 3M\}$ and $\{1, 2, 4N\}$-reverse order law are purely algebraic while the appropriate conditions given in [112] are mostly rank conditions. We present necessary and sufficient conditions for the following inclusions

$$\begin{aligned}
&B\{1, 3N\}A\{1, 3M\} \subseteq (AB)\{1, 3M\}\\
&B\{1, 4K\}A\{1, 4N\} \subseteq (AB)\{1, 4K\}\\
&B\{1, 2, 3N\}A\{1, 2, 3M\} \subseteq (AB)\{1, 2, 3M\}\\
&B\{1, 2, 4K\}A\{1, 2, 4N\} \subseteq (AB)\{1, 2, 4K\}
\end{aligned}$$

to hold, where $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times k}$ and M, N and K are three positive definite matrices of order m, n and k, respectively. Also, we consider reverse order law for the weighted $\{1, 3, 4\}$-inverses. We give necessary and sufficient conditions for

$$B\{1, 3N, 4K\} \cdot A\{1, 3M, 4N\} \subseteq (AB)\{1, 3M, 4K\},$$

and

$$(AB)\{1, 3M, 4K\} \subseteq B\{1, 3N, 4K\} \cdot A\{1, 3M, 4N\},$$

in the case when M, N, K are positive definite matrices of the appropriate size.

2.5.1 Reverse order law for weighted $\{1,3\}$, $\{1,4\}$, $\{1,2,3\}$ and $\{1,2,4\}$-inverses

Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times k}$ and let $M \in \mathbb{C}^{m\times m}$, $N \in \mathbb{C}^{n\times n}$ and $K \in \mathbb{C}^{k\times k}$ be three positive definite matrices. In this subsection, we give the necessary and sufficient conditions for the following inclusions to hold:

$$B\{1,3N\} \cdot A\{1,3M\} \subseteq (AB)\{1,3M\}, \tag{2.1}$$

$$B\{1,4K\} \cdot A\{1,4N\} \subseteq (AB)\{1,4N\}, \tag{2.2}$$

$$B\{1,2,3N\} \cdot A\{1,2,3M\} = (AB)\{1,2,3M\}, \tag{2.3}$$

$$B\{1,2,4K\} \cdot A\{1,2,4N\} = (AB)\{1,2,4N\}. \tag{2.4}$$

The results from this section are a generalization of the results from [60] and [23] to the case of weighted generalized inverses. First, we will state the characterization for the sets $A\{1,3M\}$ and $A\{1,4N\}$ which is given in [111]:

Lemma 2.5.1 [*111*] *Let $A \in \mathbb{C}^{m\times n}$, $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$ be such that M and N are positive definite matrices. For $G \in \mathbb{C}^{n\times m}$, we have:*

(*i*) $G \in A\{1,3M\} \Leftrightarrow A^*MAG = A^*M$,

(*ii*) $G \in A\{1,4N\} \Leftrightarrow GAN^{-1}A^* = N^{-1}A^*$.

Obviously, we can conclude that

$$A\{1,3M\} = \{A^\dagger_{M,I_n} + (I_n - A^\dagger_{M,I_n}A)Y : Y \in \mathbb{C}^{n\times m}\} \tag{2.97}$$

and

$$A\{1,4N\} = \{A^\dagger_{I_m,N} + Z(I_m - AA^\dagger_{I_m,N}) : Z \in \mathbb{C}^{n\times m}\}. \tag{2.98}$$

Now, we will give a similar characterization of the sets $A\{1,2,3M\}$ and $A\{1,2,4N\}$.

Theorem 2.5.1 *Let $A \in \mathbb{C}^{m\times n}$, $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$ be such that M and N are positive definite matrices. For $G \in \mathbb{C}^{n\times m}$, we have:*

(*i'*) $G \in A\{1,2,3M\} \Leftrightarrow A^*MAG = A^*M$ *and* $GAA^\dagger_{M,I_n} = G$.

(*ii'*) $G \in A\{1,2,4N\} \Leftrightarrow GAN^{-1}A^* = N^{-1}A^*$ *and* $A^\dagger_{I_m,N}AG = G$.

Proof. (i') If $G \in A\{1,2,3M\}$, then

$$A^*MAG = A^*(MAG)^* = A^*M$$

and

$$\begin{aligned} GAA^\dagger_{MI_n} &= GM^{-1}(MAG)M^{-1}(MAA^\dagger_{MI_n}) \\ &= GM^{-1}(MAG)^*M^{-1}(MAA^\dagger_{MI_n})^* \\ &= G. \end{aligned}$$

If we suppose that $A^*MAG = A^*M$ and $GAA^\dagger_{M,I_n} = G$, we have that

$$AGA = M^{-1}(MAA^\dagger_{M,I_n})^*AGA = M^{-1}(MAA^\dagger_{M,I_n})^*A = A.$$

Also,

$$GAG = GM^{-1}(A^\dagger_{M,I_n})^*A^*MAG = GM^{-1}(MAA^\dagger_{M,I_n})^* = G$$

and

$$(MAG)^* = G^*A^*M = G^*A^*MAG = MAG.$$

The statement (ii') can be proved on the similar way. □

Throughout the paper, we will use the following lemma:

Lemma 2.5.2 [8] *Let $A \in \mathbb{C}^{m\times n}$, $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$ be such that M and N are positive definite matrices. Then*

$$A^\dagger_{M,N} = N^{-\frac{1}{2}}(M^{\frac{1}{2}}AN^{-\frac{1}{2}})^\dagger M^{\frac{1}{2}}. \tag{2.99}$$

In the next theorem we present new necessary and sufficient conditions (3° and 4°) for (2.1) to holds.

Theorem 2.5.2 *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times k}$. If $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$ are positive definite matrices, then the following conditions are equivalent:*

(1°) $B\{1,3N\} \cdot A\{1,3M\} \subseteq (AB)\{1,3M\}$,

(2°) $BB^\dagger_{N,I_k}N^{-1}A^*MAB = N^{-1}A^*MAB$,

(3°) $B^\dagger_{N,I_k}A^\dagger_{M,N} \in (AB)\{1,3M\}$,

(4°) $B^\dagger_{N,I_k}A^\dagger_{M,N} \in (AB)\{1,2,3M\}$.

Proof. From Corollary 2.2 [111], we have that (1°) is equivalent with (2°). Now, let us prove that (3°) is equivalent with (1°) and that (4°) is equivalent with (1°).

Let $\tilde{A} = M^{\frac{1}{2}}AN^{-\frac{1}{2}}$ and $\tilde{B} = N^{\frac{1}{2}}B$. For $X \in \mathbb{C}^{n\times m}$ and $Y \in \mathbb{C}^{k\times n}$, denote by $\tilde{X} = N^{\frac{1}{2}}XM^{-\frac{1}{2}}$ and $\tilde{Y} = YN^{-\frac{1}{2}}$. It is easy to see that the following equivalences hold:

$$X \in A\{1,3M\} \Leftrightarrow \tilde{X} \in \tilde{A}\{1,3\},$$

$$Y \in B\{1,3N\} \Leftrightarrow \tilde{Y} \in \tilde{B}\{1,3\},$$
$$YX \in AB\{1,3M\} \Leftrightarrow \tilde{Y}\tilde{X} \in \tilde{A}\tilde{B}\{1,3\}.$$

Obviously,

$$B\{1,3N\} \cdot A\{1,3M\} \subseteq (AB)\{1,3M\} \Leftrightarrow \tilde{B}\{1,3\} \cdot \tilde{A}\{1,3\} \subseteq (\tilde{A}\tilde{B})\{1,3\},$$

so (1°) is equivalent to

$$\tilde{B}\{1,3\} \cdot \tilde{A}\{1,3\} \subseteq (\tilde{A}\tilde{B})\{1,3\}. \quad (2.100)$$

Now, by Theorem 3.1 [23], we get that (2.110) is equivalent with $\tilde{B}\tilde{B}^\dagger\tilde{A}^*\tilde{A}\tilde{B} = \tilde{A}^*\tilde{A}\tilde{B}$, which is by Lemma 2.5.2 equivalent with

$$BB^\dagger_{N,I_k}N^{-1}A^*MAB = N^{-1}A^*MAB.$$

Since $B^\dagger_{N,I_k}A^\dagger_{M,N} \in (AB)\{1,3M\}$ is equivalent with $\tilde{B}^\dagger\tilde{A}^\dagger \subseteq (\tilde{A}\tilde{B})\{1,3\}$ and $B^\dagger_{N,I_k}A^\dagger_{M,N} \in (AB)\{1,2,3M\}$ is equivalent with $\tilde{B}^\dagger\tilde{A}^\dagger \subseteq (\tilde{A}\tilde{B})\{1,2,3\}$, the proof follows by Theorem 3.1 [23]. □

A similar result in the case of weighted $\{1,4\}$-inverses follows from Theorem 3.2.1 by reversal of products:

Theorem 2.5.3 *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times k}$. If $N \in \mathbb{C}^{n\times n}$ and $K \in \mathbb{C}^{k\times k}$ are positive definite matrices, then the following conditions are equivalent:*

(1°°) $ABK^{-1}B^*NA^\dagger_{I_m,N}A = ABK^{-1}B^*N$,

(2°°) $B\{1,4K\} \cdot A\{1,4N\} \subseteq (AB)\{1,4K\}$,

(3°°) $B^\dagger_{N,K}A^\dagger_{I_m,N} \in (AB)\{1,4K\}$,

(4°°) $B^\dagger_{N,K}A^\dagger_{I_m,N} \in (AB)\{1,2,4K\}$.

Proof. Let $\tilde{A} = AN^{-\frac{1}{2}}$ and $\tilde{B} = N^{\frac{1}{2}}BK^{-\frac{1}{2}}$. For $X \in \mathbb{C}^{n\times m}$ and $Y \in \mathbb{C}^{k\times n}$, denote by $\tilde{X} = N^{\frac{1}{2}}X$ and $\tilde{Y} = K^{\frac{1}{2}}YN^{-\frac{1}{2}}$. Now, the proof is similar as the proof of Theorem 3.2.1 and follows from Theorem 3.2 [23]. □

Now, we will consider the reverse order law for the weighted $\{1,2,3\}$-inverses and weighted $\{1,2,4\}$–inverses. Remark that necessary and sufficient rank conditions for the reverse order law of $\{1,2,3\}$ and $\{1,2,4\}$-inverses are given in [111].

Theorem 2.5.4 *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times k}$. If $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$ are positive definite matrices, then the following conditions are equivalent:*

(1′) $B\{1,2,3N\} \cdot A\{1,2,3M\} \subseteq (AB)\{1,2,3M\}$,

(2′) $BB^{\dagger}_{N,I_k}N^{-1}A^*MAB = N^{-1}A^*MAB$ *and* $\Big((ABB^{\dagger}_{N,I_k})^{\dagger}_{M,N}ABB^{\dagger}_{N,I_k} = BB^{\dagger}_{N,I_k}$ *or* $AB(AB)^{\dagger}_{M,I_n} = AA^{\dagger}_{M,N}\Big)$.

Proof. Let $\tilde{A} = M^{\frac{1}{2}}AN^{-\frac{1}{2}}$ and $\tilde{B} = N^{\frac{1}{2}}B$. For $X \in \mathbb{C}^{n\times m}$, $Y \in \mathbb{C}^{k\times n}$ and $Z \in \mathbb{C}^{k\times m}$, denote by $\tilde{X} = N^{\frac{1}{2}}XM^{-\frac{1}{2}}$, $\tilde{Y} = YN^{-\frac{1}{2}}$ and $\tilde{Z} = ZM^{-\frac{1}{2}}$. Then $\tilde{A}\tilde{B}\tilde{Z}\tilde{A}\tilde{B} = \tilde{A}\tilde{B}$ if and only if $ABZAB = AB$ and $\tilde{Z}\tilde{A}\tilde{B}\tilde{Z} = \tilde{Z}$ if and only if $ZABZ = Z$. Also, $(\tilde{A}\tilde{B}\tilde{Z})^* = \tilde{A}\tilde{B}\tilde{Z}$ if and only if $(MABZ)^* = MABZ$.

Hence,

$$\tilde{Z} \in (\tilde{A}\tilde{B})\{1,2,3\} \Leftrightarrow Z \in (AB)\{1,2,3M\}. \tag{2.101}$$

Using Lemma 2.5.2 we can easily prove the following:

$$(\tilde{A}\tilde{B}\tilde{B}^{\dagger})^{\dagger}\tilde{A}\tilde{B}\tilde{B}^{\dagger} = \tilde{B}\tilde{B}^{\dagger} \Leftrightarrow (ABB^{\dagger}_{N,I_k})^{\dagger}_{M,N}ABB^{\dagger}_{N,I_k} = BB_{N,I_k}, \tag{2.102}$$

$$(\tilde{A}\tilde{B})(\tilde{A}\tilde{B})^{\dagger} = \tilde{A}\tilde{A}^{\dagger}, \tag{2.103}$$

$$\tilde{B}\tilde{B}^{\dagger}\tilde{A}^*\tilde{A}\tilde{B} = \tilde{A}^*\tilde{A}\tilde{B} \Leftrightarrow BB^{\dagger}_{N,I_k}N^{-1}A^*MAB = A^*MAB. \tag{2.104}$$

Now, the proof follows from (3.19) and (3.20) using Corrolary 3.1 [23]. □

The case of weighted $\{1,2,4\}$-inverses is treated completely analogously, and the corresponding result follows by taking adjoints, or by reversal of products:

Theorem 2.5.5 *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times k}$. If $N \in \mathbb{C}^{n\times n}$ and $K \in \mathbb{C}^{k\times k}$ are positive definite matrices, then the following conditions are equivalent:*

(1″) $B\{1,2,4K\} \cdot A\{1,2,4N\} \subseteq (AB)\{1,2,4N\}$,

(2″) $ABK^{-1}B^*NA^{\dagger}_{I_m,N}A = ABK^{-1}B^*N$ *and* $\Big((A^{\dagger}_{I_m,N}AB)(A^{\dagger}_{I_m,N}AB)^{\dagger}_{N,K} = A^{\dagger}_{I_m,N}A$ *or* $(AB)^{\dagger}_{I_m,K}(AB) = B^{\dagger}_{N,K}B\Big)$.

2.5.2 Reverse order law for weighted $\{1,3,4\}$-inverses

In this section we consider the reverse order law for the weighted $\{1,3,4\}$-inverses. We give necessary and sufficient conditions for

$$B\{1,3N,4K\} \cdot A\{1,3M,4N\} \subseteq (AB)\{1,3M,4K\},$$

and

$$(AB)\{1,3M,4K\} \subseteq B\{1,3N,4K\} \cdot A\{1,3M,4N\},$$

in the case when M, N, K are positive definite matrices of the appropriate size.

Also, we give a very short proof that

$$(AB)\{1,3,4\} \subseteq B\{1,3,4\} \cdot A\{1,3,4\}$$

is actually equivalent to

$$(AB)\{1,3,4\} = B\{1,3,4\} \cdot A\{1,3,4\}.$$

We will begin with auxiliary result:

Lemma 2.5.3 *Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{n\times k}$. If $M \in \mathbb{C}^{m\times m}$, $N \in \mathbb{C}^{n\times n}$ and $K \in \mathbb{C}^{k\times k}$ are positive definite matrices then:*

(a) $B^\dagger_{N,K}B(AB)^\dagger_{M,K}(AB) = (AB)^\dagger_{M,K}(AB)$,

(b) $(AB)(AB)^\dagger_{M,K}AA^\dagger_{M,N} = (AB)(AB)^\dagger_{M,K}$.

Proof. (a) By easy computation, we can show that $B^\dagger_{N,K}B(AB)^\dagger_{M,K}(AB) \in (AB)\{1,2\}$. Since $MABB^\dagger_{N,K}B(AB)^\dagger_{M,K}(AB) = MAB(AB)^\dagger_{M,K}$ is hermitian and

$$\begin{aligned}
&KB^\dagger_{N,K}B(AB)^\dagger_{M,K}(AB) = B^*(B^\dagger_{N,K})^*K(AB)^\dagger_{M,K}(AB)\\
&= B^*(B^\dagger_{N,K})^*(AB)^*((AB)^\dagger_{M,K})^*K\\
&= B^*A^*((AB)^\dagger_{M,K})^*K\\
&= (K(AB)^\dagger_{M,K}(AB))^*\\
&= K(AB)^\dagger_{M,K}(AB),
\end{aligned}$$

we have $B^\dagger_{N,K}B(AB)^\dagger_{M,K}(AB) = (AB)^\dagger_{M,K}(AB)$. The identity (b) can be proved similarly. $\square$

In the following theprem we give necessary and sufficinet condition for $B\{1,3N,4K\}\cdot A\{1,3M,4N\} \subseteq (AB)\{1,3M,4K\}$:

Theorem 2.5.6 *Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{n\times k}$ and let $M \in \mathbb{C}^{m\times m}$, $N \in \mathbb{C}^{n\times n}$ and $K \in \mathbb{C}^{k\times k}$ be positive definite matrices. The following conditions are equivalent:*

(a') $B\{1,3N,4K\} \cdot A\{1,3M,4N\} \subseteq (AB)\{1,3M,4K\}$,

(b') $(AB)^\dagger_{M,K} = B^\dagger_{N,K}A^\dagger_{M,N}$.

Proof. Let $\tilde{A} = M^{\frac{1}{2}}AN^{-\frac{1}{2}}$ and $\tilde{B} = N^{\frac{1}{2}}BK^{-\frac{1}{2}}$. For $X \in \mathbb{C}^{n\times m}$ and $Y \in \mathbb{C}^{k\times n}$ denote by $\tilde{X} = N^{\frac{1}{2}}AM^{-\frac{1}{2}}$ and $\tilde{Y} = K^{\frac{1}{2}}YN^{-\frac{1}{2}}$. It is easy to see that

$$X \in A\{1,3M,4N\} \Leftrightarrow \tilde{X} \in \tilde{A}\{1,3,4\},$$

$$Y \in B\{1,3N,4K\} \Leftrightarrow \tilde{Y} \in \tilde{B}\{1,3,4\},$$
$$YX \in AB\{1,3M,4K\} \Leftrightarrow \tilde{Y}\tilde{X} \in \tilde{A}\tilde{B}\{1,3,4\}$$

and

$$B\{1,3N,4K\}\cdot A\{1,3M,4N\}\subseteq (AB)\{1,3M,4K\}$$
$$\Leftrightarrow \tilde{B}\{1,3,4\}\cdot \tilde{A}\{1,3,4\}\subseteq (\tilde{A}\tilde{B})\{1,3,4\}. \quad (2.105)$$

We can easily prove the next equivalence

$$(AB)^{\dagger}_{M,K}=B^{\dagger}_{N,K}A^{\dagger}_{M,N}\Leftrightarrow (\tilde{A}\tilde{B})^{\dagger}=\tilde{B}^{\dagger}\tilde{A}^{\dagger}. \quad (2.106)$$

Now, the proof follows by (2.105) and (2.106) using Theorem 2.1 [23]. □

Remark that conditions which are equivalent with (b') can be found in [86].

In the following theore we give prove that $(AB)\{1,3,4\}\subseteq B\{1,3,4\}\cdot A\{1,3,4\}$ is actually equivalent to $(AB)\{1,3,4\}=B\{1,3,4\}\cdot A\{1,3,4\}$:

Theorem 2.5.7 *Let $A\in\mathbb{C}^{m\times n}, A\in\mathbb{C}^{n\times k}$ and let $M\in\mathbb{C}^{m\times m}$, $N\in\mathbb{C}^{n\times n}$ and $K\in\mathbb{C}^{k\times k}$ be positive definite matrices. The following conditions are equivalent:*

(a'') $(AB)\{1,3M,4K\}\subseteq B\{1,3N,4K\}\cdot A\{1,3M,4N\}$,

(b'') $(AB)\{1,3M,4K\}=B\{1,3N,4K\}\cdot A\{1,3M,4N\}$.

Proof. $(a'')\Rightarrow(b'')$: For every $Z\in\mathbb{C}^{k\times m}$, there exists $X\in\mathbb{C}^{n\times m}$ and $Y\in\mathbb{C}^{k\times n}$ such that

$$(AB)^{\dagger}_{M,K}+(I-(AB)^{\dagger}_{M,K}(AB))Z(I-(AB)(AB)^{\dagger}_{M,K})$$
$$=(B^{\dagger}_{N,K}+(I-B^{\dagger}_{N,K}B)Y(I-BB^{\dagger}_{N,K}))(A^{\dagger}_{M,N}+(I-A^{\dagger}_{M,N}A)Y(I-AA^{\dagger}_{M,N})).$$

Multiplying last equality by $B^{\dagger}_{N,K}B$ from the left and by $AA^{\dagger}_{M,N}$ from the right, by Lemma 2.5.3 we have

$$(AB)^{\dagger}_{M,K}+(B^{\dagger}_{N,K}B-(AB)^{\dagger}_{M,K}(AB))Z(AA^{\dagger}_{M,N}-(AB)(AB)^{\dagger}_{M,K})=B^{\dagger}_{N,K}A^{\dagger}_{M,N}.$$

For $Z=0$ we get $(AB)^{\dagger}_{M,K}=B^{\dagger}_{N,K}A^{\dagger}_{M,N}$ which implies $B\{1,3N,4K\}A\{1,3M,4N\}\subseteq (AB)\{1,3M,4K\}$.

$(b'')\Rightarrow(a'')$: It is obvious. □

Theorem 2.5.8 *Let $A\in\mathbb{C}^{m\times n}, B\in\mathbb{C}^{n\times k}$ and let $M\in\mathbb{C}^{m\times m}$, $N\in\mathbb{C}^{n\times n}$ and $K\in\mathbb{C}^{k\times k}$ be positive definite matrices. The following conditions are equivalent:*

(a°) $(AB)\{1,3M,4K\}\subseteq B\{1,3N,4K\}\cdot A\{1,3M,4N\}$,

(b°) $(AB)\{1,3M,4K\}=B\{1,3N,4K\}\cdot A\{1,3M,4N\}$,

(c°) $(AB)^{\dagger}_{M,K}=B^{\dagger}_{N,K}A^{\dagger}_{M,N}$ *and* $(B=A^{\dagger}_{M,N}AB$ *or* $A=ABB^{\dagger}_{N,K})$.

Proof. Let $\tilde{A} = M^{\frac{1}{2}}AN^{-\frac{1}{2}}$ and $\tilde{B} = N^{\frac{1}{2}}BK^{-\frac{1}{2}}$. For $X \in \mathbb{C}^{n\times m}$ and $Y \in \mathbb{C}^{k\times n}$ denote by $\tilde{X} = N^{\frac{1}{2}}XM^{-\frac{1}{2}}$ and $\tilde{Y} = K^{\frac{1}{2}}YN^{-\frac{1}{2}}$. It is easy to see that

$$X \in A\{1, 3M, 4N\} \Leftrightarrow \tilde{X} \in \tilde{A}\{1, 3, 4\},$$

$$Y \in B\{1, 3N, 4K\} \Leftrightarrow \tilde{Y} \in \tilde{B}\{1, 3, 4\},$$

$$YX \in AB\{1, 3M, 4K\} \Leftrightarrow \tilde{Y}\tilde{X} \in \tilde{A}\tilde{B}\{1, 3, 4\}.$$

Now, the proof follows by Theorem 2.3 from [23]. □

It is interesting to remark that using Theorem 2.4 [28], we can conclude that

$$(AB)\{1, 3M, 4K\} \subseteq B\{1, 3N, 4K\} \cdot A\{1, 3M, 4N\} \tag{2.107}$$

can be true only in the case when $m \leq n$.

2.6 Some additive results for generalized inverses

Penrose [68] in 1955 proved that in the matrix case, $A_i \in \mathbb{C}^{m\times n}$, $i = 1, 2\ldots, k$ are such that $A_i^*A_j = 0$ for $i \neq$, $i, j = 1, 2, \ldots k$, the following holds:

$$(A_1 + A_2 + \cdots + A_n)^\dagger = A_1^\dagger + A_2^\dagger + \cdots A_n^\dagger.$$

In this section we derived necessary and sufficient conditions for analogous equality for least squares and minimum-norm g-inverses of operators. Xiong et al. [113] derived necessary and sufficient conditions for $(A + B)\{1, 3\} = A\{1, 3\} + B\{1, 3\}$ and $(A + B)\{1, 4\} = A\{1, 4\} + B\{1, 4\}$ on the set of matrices.

Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be complex Hilbert spaces and $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. E_A and F_A stand for two orthogonal projectors $E_A = I_\mathcal{K} - AA^\dagger$ and $F_A = I_\mathcal{H} - A^\dagger A$.

Let $A_i \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $i = 1, 2\ldots, k$. In this section give necessary and sufficient conditions for the following inclusions

$$A_1\{1, 3\} + A_2\{1, 3\} + ... + A_k\{1, 3\} \subseteq (A_1 + A_2 + ... + A_k)\{1, 3\},$$

$$A_1\{1, 4\} + A_2\{1, 4\} + ... + A_k\{1, 4\} \subseteq (A_1 + A_2 + ... + A_k)\{1, 4\},$$

$$(A_1 + A_2 + ... + A_k)\{1, 3\} \subseteq A_1\{1, 3\} + A_2\{1, 3\} + ... + A_k\{1, 3\},$$

$$(A_1 + A_2 + ... + A_k)\{1, 4\} \subseteq A_1\{1, 4\} + A_2\{1, 4\} + ... + A_k\{1, 4\},$$

$$A_1\{1, 2, 3\} + A_2\{1, 2, 3\} + ... + A_k\{1, 2, 3\} \subseteq (A_1 + A_2 + ... + A_k)\{1, 2, 3\}$$

and

$$A_1\{1, 2, 4\} + A_2\{1, 2, 4\} + ... + A_k\{1, 2, 4\} \subseteq (A_1 + A_2 + ... + A_k)\{1, 2, 4\}.$$

It is well-known that for $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$,

$$\begin{array}{lcl} B \in A\{1,3\} & \Leftrightarrow & A^*AB = A^*, \\ B \in A\{1,4\} & \Leftrightarrow & BAA^* = A^*, \end{array} \tag{2.108}$$

and that sets of all $\{1,3\}$- and $\{1,4\}$-inverses of A are described by

$$\begin{array}{lcl} A\{1,3\} & = & \{A^\dagger + F_A V : \ V \in \mathcal{B}(\mathcal{K},\mathcal{H})\}, \\ A\{1,4\} & = & \{A^\dagger + V E_A : \ V \in \mathcal{B}(\mathcal{K},\mathcal{H})\}. \end{array} \tag{2.109}$$

Also, for $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$,

$$\begin{array}{l} B \in A\{1,2,3\} \Leftrightarrow (A^*AB = A^* \wedge BAA^\dagger = B), \\ B \in A\{1,2,4\} \Leftrightarrow (BAA^* = A^* \wedge A^\dagger AB = B), \end{array} \tag{2.110}$$

and sets of all $\{1,2,3\}$- and $\{1,2,4\}$-inverses of A are described by

$$\begin{array}{lcl} A\{1,2,3\} & = & \{A^\dagger + F_A V A A^\dagger : \ V \in \mathcal{B}(\mathcal{K},\mathcal{H})\}, \\ A\{1,2,4\} & = & \{A^\dagger + A^\dagger A V E_A : \ V \in \mathcal{B}(\mathcal{K},\mathcal{H})\}. \end{array} \tag{2.111}$$

Theorem 2.6.1 *Let $A_i \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $i = 1,2,\dots,k$, such that $A_1 + A_2 + ... + A_k$ is regular. The following conditions are equivalent:*

(i) $A_1\{1,3\} + A_2\{1,3\} + ... + A_k\{1,3\} \subseteq (A_1 + A_2 + ... + A_k)\{1,3\}$,

(ii) $A^*AF_{A_i} = 0$, $i = 1,2,\dots,k$, $\ A^*A\sum_{i=1}^k A_i^\dagger = A^*$, *where* $A = A_1 + A_2 + ... + A_k$.

Proof. $(ii) \Rightarrow (i)$: Suppose that (ii) holds. We need to prove that for arbitrary $A_i^{(1,3)} \in A_i\{1,3\}$, $i = 1,2,\dots,k$, it follows that $A_1^{(1,3)} + A_2^{(1,3)} + ... + A_k^{(1,3)} \in A\{1,3\}$. Thus, given any $V_i \in \mathcal{B}(\mathcal{K},\mathcal{H})$, $i = 1,2,\dots,k$, we must show that

$$A^*A\big(\sum_{i=1}^k A_i^\dagger + \sum_{i=1}^k F_{A_i}V_i\big) = A^*, \tag{2.112}$$

which is satisfied by (ii).

$(i) \Rightarrow (ii)$: If (i) holds, then for any $V_i \in \mathcal{B}(\mathcal{K},\mathcal{H})$, $i = 1,2,\dots,k$, we have that (2.112) holds. Specially, for $V_i = 0$, $i = 1,2,\dots,k$, by (2.112) we get that $A^*A\sum_{i=1}^k A_i^\dagger = A^*$. Similarly, if for any $i \in \{1,2,...,k\}$, we take that $V_i = F_{A_i}$ and that $V_j = 0$, $j \neq i$, by (2.112) we will get that $A^*AF_{A_i} = 0$. Hence, (ii) holds. □

In the following theorem we present the necessary and sufficient condition for

$$(A_1 + A_2 + ... + A_k)\{1,3\} \subseteq A_1\{1,3\} + A_2\{1,3\} + ... + A_k\{1,3\}.$$

Theorem 2.6.2 *Let $A_i \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $i = 1,2,\dots,k$, such that $A_1 + A_2 + ... + A_k$ is regular. The following conditions are equivalent:*

(i) $(A_1 + A_2 + ... + A_k)\{1,3\} \subseteq A_1\{1,3\} + A_2\{1,3\} + ... + A_k\{1,3\}$,

(ii) $CC^\dagger F_A = F_A$, $CC^\dagger(A^\dagger - \sum_{i=1}^k A_i^\dagger) = A^\dagger - \sum_{i=1}^k A_i^\dagger$, *where* $A = A_1 + A_2 + ... + A_k$ *and* $C = \begin{bmatrix} F_{A_1} & F_{A_2} & ... & F_{A_k} \end{bmatrix}$.

Proof. *(ii)* $\Rightarrow$ *(i)*: Suppose that *(ii)* holds. We need to prove that for arbitrary $V \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ there exist $V_i \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, $i = 1, 2, \dots, k$, such that

$$A^\dagger + F_A V = \sum_{i=1}^k A_i^\dagger + \sum_{i=1}^k F_{A_i} V_i,$$

i.e.

$$\begin{bmatrix} F_{A_1} & F_{A_2} & ... & F_{A_k} \end{bmatrix} \begin{bmatrix} V_1 \\ ... \\ V_k \end{bmatrix} = F_A V + A^\dagger - \sum_{i=1}^k A_i^\dagger. \tag{2.113}$$

Hence, to show *(i)* we need to prove that the equation (2.113) is solvalable for any $V \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ which holds if and only if

$$CC^\dagger\big(F_A V + A^\dagger - \sum_{i=1}^k A_i^\dagger\big) = F_A V + A^\dagger - \sum_{i=1}^k A_i^\dagger. \tag{2.114}$$

Obviously, (2.116) is satisfied by *(ii)*.

(i) $\Rightarrow$ *(ii)*: If *(i)* is satisfied, then (2.116) holds for any $V \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Taking $V = 0$ and $V = F_A$ in (2.116), we get that the both equalities from *(ii)* hold. $\square$

Theorem 2.6.3 *Let* $A_i \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $i = 1, 2, \dots, k$, *such that* $A_1 + A_2 + ... + A_k$ *is regular. The following conditions are equivalent:*

(i) $(A_1 + A_2 + ... + A_k)\{1,4\} \subseteq A_1\{1,4\} + A_2\{1,4\} + ... + A_k\{1,4\}$,

(ii) $DD^\dagger E_A = E_A$, $(A^\dagger - \sum_{i=1}^k A_i^\dagger)DD^\dagger = A^\dagger - \sum_{i=1}^k A_i^\dagger$, *where* $A = A_1 + A_2 + ... + A_k$ *and* $D = \begin{bmatrix} E_{A_1} & E_{A_2} & ... & E_{A_k} \end{bmatrix}$.

Proof. For arbitrary matrix $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ we have from (3.10) that $B \in A\{1,4\} \Leftrightarrow B^* \in A^*\{1,3\}$, so it follows that $(A\{1,4\})^* = A^*\{1,3\}$. Now, condition *(i)* is equivalent to

$$\begin{aligned}
&((A_1 + A_2 + ... + A_k)\{1,4\})^* \subseteq (A_1\{1,4\})^* + (A_2\{1,4\})^* + ... + (A_k\{1,4\})^* \\
&\Leftrightarrow (A_1 + A_2 + ... + A_k)^*\{1,3\} \subseteq A_1^*\{1,3\} + A_2^*\{1,3\} + ... + A_k^*\{1,3\} \\
&\Leftrightarrow (A_1^* + A_2^* + ... + A_k^*)\{1,3\} \subseteq A_1^*\{1,3\} + A_2^*\{1,3\} + ... + A_k^*\{1,3\}
\end{aligned}$$

which is from the Theorem 4.1.1 equivalent to

$$CC^\dagger F_{A^*} = F_{A^*},\ CC^\dagger((A^*)^\dagger - \sum_{i=1}^k (A_i^*)^\dagger) = (A^*)^\dagger - \sum_{i=1}^k (A_i^*)^\dagger, \tag{2.115}$$

where $A = A_1 + A_2 + ... + A_k$ and $C = \begin{bmatrix} F_{A_1^*} & F_{A_2^*} & ... & F_{A_k^*} \end{bmatrix}$. Since $F_{A^*} = E_A$ and $F_{A_i^*} = E_A$, it is easy to see that (2.115) is equivalent to (ii). □

In an analogous way, necessary and sufficient condition for the opposite inclusion can be derived from the Theorem 3.2.1.

Theorem 2.6.4 *Let $A_i \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $i = 1, 2, \ldots, k$, such that $A_1 + A_2 + ... + A_k$ is regular. The following conditions are equivalent:*

(i) $A_1\{1,4\} + A_2\{1,4\} + ... + A_k\{1,4\} \subseteq (A_1 + A_2 + ... + A_k)\{1,4\}$,

(ii) $AA^*E_{A_i} = 0$, $i = 1, 2, \ldots, k$, $(\sum_{i=1}^k A_i^\dagger)AA^* = A^*$, *where* $A = A_1 + A_2 + ... + A_k$.

Theorem 2.6.5 *Let $A_i \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $i = 1, 2, \ldots, k$, such that $A_1 + A_2 + ... + A_k$ is regular. The following conditions are equivalent:*

(i) $A_1\{1,2,3\} + A_2\{1,2,3\} + ... + A_k\{1,2,3\} \subseteq (A_1 + A_2 + ... + A_k)\{1,2,3\}$,

(ii) $A^*AF_{A_i} = 0$, $i = 1, 2, \ldots, k$, $A^*A\sum_{i=1}^k A_i^\dagger = A^*$, $\sum_{i=1}^k A_i^\dagger F_A = 0$, *and* $F_{A_i} = 0$ *or* $A_iA_i^\dagger F_A = 0$ *for every* $i \in \{1, \ldots k\}$ *where* $A = A_1 + A_2 + ... + A_k$.

Proof. $(ii) \Rightarrow (i)$: Suppose that (ii) holds. We need to prove that for arbitrary $A_i^{(1,2,3)} \in A_i\{1,2,3\}$, $i = 1, 2, \ldots, k$, it follows that $A_1^{(1,2,3)} + A_2^{(1,2,3)} + ... + A_k^{(1,2,3)} \in A\{1,2,3\}$. Thus, given any $V_i \in \mathcal{B}(\mathcal{K},\mathcal{H})$, $i = 1, 2, \ldots, k$, we must show that

$$A^*A\big(\sum_{i=1}^k A_i^\dagger + \sum_{i=1}^k F_{A_i}V_iA_iA_i^\dagger\big) = A^*, \tag{2.116}$$

and

$$\big(\sum_{i=1}^k A_i^\dagger + \sum_{i=1}^k F_{A_i}V_iA_iA_i^\dagger\big)(I - AA^\dagger) = 0, \tag{2.117}$$

which is satisfied by (ii).

$(i) \Rightarrow (ii)$: If (i) holds, then for any $V_i \in \mathcal{B}(\mathcal{K},\mathcal{H})$, $i = 1, 2, \ldots, k$, we have that (2.116) and (2.117) hold. Specially, for $V_i = 0$, $i = 1, 2, \ldots, k$, by (2.116) we get that $A^*A\sum_{i=1}^k A_i^\dagger = A^*$ and by (2.117) we get that $\sum_{i=1}^k A_i^\dagger F_A = 0$. Similarly, if for any $i \in \{1, 2, ..., k\}$, we take that $V_i = F_{A_i}$ and that $V_j = 0$, $j \neq i$, by (2.116) we will get that $A^*AF_{A_i} = 0$ and by (2.117) we will get that $F_{A_i} = 0$ or $A_iA_i^\dagger F_A = 0$. Hence, (ii) holds. □

Theorem 2.6.6 *Let $A_i \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $i = 1, 2, \ldots, k$, such that $A_1 + A_2 + ... + A_k$ is regular. The following conditions are equivalent:*

(i) $A_1\{1,2,4\} + A_2\{1,2,4\} + ... + A_k\{1,2,4\} \subseteq (A_1 + A_2 + ... + A_k)\{1,2,4\}$, *(ii)* $AA^*E_{A_i} = 0$, $i = 1, 2, \dots, k$, $(\sum_{i=1}^{k} A_i^\dagger)AA^* = A^*$, $E_A \sum_{i=1}^{k} A_i^\dagger = 0$, *and* $E_{A_i} = 0$ *or* $E_A A_i^\dagger A_i = 0$ *for every* $i \in \{1, \dots k\}$ *where* $A = A_1 + A_2 + ... + A_k$.

Proof. For arbitrary matrix $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ we have from (2.110) that $B \in A\{1,2,4\} \Leftrightarrow B^* \in A^*\{1,2,3\}$, so it follows that $(A\{1,2,4\})^* = A^*\{1,2,3\}$. Now, condition (i) is equivalent to

$$\begin{aligned}
&(A_1\{1,2,4\})^* + ... + (A_k\{1,2,4\})^* \subseteq ((A_1 + A_2 + ... + A_k)\{1,2,4\})^* \\
&\Leftrightarrow A_1^*\{1,2,3\} + ... + A_k^*\{1,2,3\} \subseteq (A_1 + A_2 + ... + A_k)^*\{1,2,3\} \\
&\Leftrightarrow A_1^*\{1,2,3\} + ... + A_k^*\{1,2,3\} \subseteq (A_1^* + A_2^* + ... + A_k^*)\{1,2,3\}
\end{aligned}$$

which is from the Theorem 4.1.4 equivalent to

$$\begin{aligned}
&AA^*F_{A_i^*} = 0, \quad i = 1, 2, \dots, k, \quad AA^* \sum_{i=1}^{k} (A_i^*)^\dagger = A, \\
&\sum_{i=1}^{k} (A_i^*)^\dagger F_{A^*} = 0, \\
&F_{A_i^*} = 0 \quad \text{or} \quad A_i^*(A_i^*)^\dagger F_{A^*} = 0, \quad \text{for every} \quad i \in \{1, \dots k\}
\end{aligned} \tag{2.118}$$

where $A = A_1 + A_2 + ... + A_k$. Since $F_{A^*} = E_A$ and $F_{A_i^*} = E_A$, it is easy to see that (2.118) is equivalent to (ii). $\square$

Chapter 3

Re-nnd and Hermitian generalized inverses

3.1 Hermitian generalized inverses

For complex matrix $A \in \mathbb{C}^{n\times m}$ $F_A = I - A^\dagger A$ and $E_A = I - AA^\dagger$ stand for two orthogonal projectors induced by A.

The Hermitian part of a matrix $A \in \mathbb{C}^{n\times n}$ is defined by $H(A) = \frac{1}{2}(A + A^*)$.

Matrix A is Re-nnd (Re-nonegative definite) if $H(A) \geq 0$. We will denote by $A_{re}^{(i,j,\ldots,k)}$ a Re-nnd $\{i, j, \ldots, k\}$-inverse of A. Set of all Re-nnd $\{i, j, \ldots, k\}$-inverses of A is denoted by $A_{re}\{i, j, \ldots, k\}$. By $A_{\geqslant}^{(i,j,\ldots,k)}$ we denote a nonnegative definite $\{i, j, \ldots, k\}$-inverse of A. $A_{\geqslant}\{i, j, \ldots, k\}$ stands for the set of all nonnegative definite $\{i, j, \ldots, k\}$-inverses of A.

The motivation for the results presented in this chapter was the paper [59], where some explicit conditions for the existence of the Hermitian $\{1, 3\}$, $\{1, 4\}$-inverses of a matrix A are given. We will restate some results from [59]. Main theorems in [59] are based on the following two results:

Theorem 3.1.1 [*59*] *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$ and $C \in \mathbb{C}^{p\times q}$ be given. The matrix equation*

$$AXB = C \tag{3.1}$$

has a Hermitian solution if and only if the pair of matrix equations

$$AXB = C \quad \text{and} \quad B^*XA^* = C^*$$

have a common solution. Provided a Hermitian solution exists, a representation of the general Hermitian solution of (3.1) *is of the form*

$$X_S = \frac{X + X^*}{2}, \tag{3.2}$$

where X is the representation of the general common solution of (3.2).

Theorem 3.1.2 *[59] Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$ and $C \in \mathbb{C}^{p\times q}$ be given. The matrix equation*

$$AXB = C \tag{3.3}$$

has a Hermitian least-squares solution if and only if the pair of matrix equations

$$AXB = C \quad \textit{and} \quad B^*XA^* = C^*$$

have a common least-squares solution. Provided a Hermitian solution exists, a representation of the general Hermitian solution of (3.3) is of the form

$$X_S = \frac{X + X^*}{2}, \tag{3.4}$$

where X is the representation of the general common least-squares solution of (3.4).

In following theorems necessary and sufficient conditions for the existence of Hermitian $\{1,3\}$- and $\{1,4\}$-inverses of a matrix $A \in \mathbb{C}^{n\times n}$ and their representations are given.

Theorem 3.1.3 *[59] Let $A \in \mathbb{C}^{n\times n}$. There exists Hermitian $\{1,3\}$-inverse of A if and only if*

$$(A^*)^2A = A^*A^2, \quad \textit{or} \quad AA^\dagger A^*A = A^2.$$

In this case, the general Hermitian $\{1,3\}$-inverse of A can be expressed as

$$A_h^{(1,3)} = A^\dagger + (A^\dagger)^* - \frac{1}{2}A^\dagger(A + A^*)(A^\dagger)^* + F_AHF_A,$$

where H is an arbitrary Hermitian matrix over a complex field.

Theorem 3.1.4 *[59] Let $A \in \mathbb{C}^{n\times n}$. There exists Hermitian $\{1,4\}$-inverse of A if and only if*

$$A^2A^* = A(A^*)^2, \quad \textit{or} \quad AA^*A^\dagger A = A^2.$$

In this case, the general Hermitian $\{1,4\}$-inverse of A can be expressed as

$$A_h^{(1,4)} = A^\dagger + (A^\dagger)^* - \frac{1}{2}(A^\dagger)^*(A + A^*)A^\dagger + E_AHE_A,$$

where H is an arbitrary Hermitian matrix over a complex field.

3.2 Re-nnd generalized inverses

In this section some results published in our paper [66] will be presented. We give some necessary and sufficient conditions for the existence of Re-nnd and nonnegative definite $\{1,3\}$ and $\{1,4\}$-inverses of a matrix $A \in \mathbb{C}^{n\times n}$ and completely describe these sets. Also, we prove that the existence of nonnegative definite $\{1,3\}$-inverse of a matrix A is equivalent with the existence of its nonnegative definite $\{1,2,3\}$-inverse and present the necessary and sufficient conditions for the existence of Re-nnd $\{1,3,4\}$-inverse of A.

In the following theorem, we present some necessary and sufficient conditions for the existence of Re-nnd $\{1,3\}$-inverses of a matrix $A \in \mathbb{C}^{n\times n}$ and completely described the set $A_{re}\{1,3\}$.

Theorem 3.2.1 *Let $A \in \mathbb{C}^{n\times n}$ Then the following statements are equivalent:*
(i) $A_{re}^{(1,3)}$ *exists;*
(ii) $(A^\dagger)^2A$ *is Re-nnd;*
(iii) $A^2A^\dagger$ *is Re-nnd;*
(iv) A^*A^2 *is Re-nnd.*

In this case, the set of all Re-nnd $\{1,3\}$-inverses of A is given by

$$\begin{aligned} A_{re}\{1,3\} = \{&A^\dagger - (A^\dagger)^* + ((A^\dagger)^2A)^* + \frac{1}{2}F_AUU^*F_A + F_AUJ^{\frac{1}{2}} \\ &+F_AVF_A : U,V \in \mathbb{C}^{n\times n}, V = -V^*\}, \end{aligned} \tag{3.5}$$

where $J = (A^\dagger)^2A + \big((A^\dagger)^2A\big)^*$.

Proof. Since arbitrary $A_{re}^{(1,3)}$ is a solution of the equation $AX = AA^\dagger$, the equivalence of (i) and (iii) follows by [Theorem 1, [42]]. Now, we will prove $(ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (ii)$.

$(ii) \Rightarrow (iii)$: If $(A^\dagger)^2A$ is Re-nnd, then

$$\begin{aligned} &A^\dagger A^\dagger A + A^\dagger A(A^\dagger)^* \geq 0 \\ &\Rightarrow AA^\dagger A^\dagger AA^* + AA^\dagger A(A^\dagger)^*A^* \geq 0 \\ &\Rightarrow AA^\dagger A^* + A^2A^\dagger \geq 0 \\ &\Leftrightarrow (A^2A^\dagger)^* + A^2A^\dagger \geq 0, \end{aligned}$$

so it follows that $A^2A^\dagger$ is also Re-nnd.

$(iii) \Rightarrow (iv)$: If $A^2A^\dagger$ is Re-nnd then $A^2A^\dagger + AA^\dagger A^* \geq 0$. Multiplying the last inequality A^* from the left and by A from the right, we get that $A^*A^2 + (A^*)^2A \geq 0$, so A^*A^2 is Re-nnd.

$(iv) \Rightarrow (ii)$: Suppose that A^*A^2 is Re-nnd. Then $A^*A^2 + (A^*)^2A \geq 0$. Multiplying the both sides of the last equality by $A^\dagger(A^\dagger)^*$, we get

$$\begin{aligned} &A^\dagger(A^\dagger)^*A^*A^2A^\dagger(A^\dagger)^* + A^\dagger(A^\dagger)^*(A^*)^2AA^\dagger(A^\dagger)^* \\ &= ((A^\dagger)^2A)^* + (A^\dagger)^2A \geq 0, \end{aligned}$$

so $(A^\dagger)^2A$ is Re-nnd.

To prove that the set of all Re-nnd $\{1,3\}$-inverses of A is described by (3.5), suppose that one of the conditions (i)-(iv) is satisfied. Observe that $X \in A_{re}\{1,3\}$ if and only if $X = A^\dagger + (I - A^\dagger A)Y$, for some $Y \in \mathbb{C}^{n\times n}$, such that

$$(I - A^\dagger A)Y + \big((I - A^\dagger A)Y\big)^* \geq -\big(A^\dagger + (A^\dagger)^*\big). \tag{3.6}$$

By Theorem 2.3 [89], we have that the general solution of (3.6) is given by

$$\begin{aligned} Y = &-\frac{1}{2}F_A\Big(A^\dagger + (A^\dagger)^* - \big(F_AU + J^{\frac{1}{2}}\big)\big(F_AU + J^{\frac{1}{2}}\big)^*\Big)(I + A^\dagger A) \\ &+ VF_A + A^\dagger AW, \end{aligned}$$

where $J = (A^\dagger)^2A + \big((A^\dagger)^2A\big)^*$ and $U, V, W \in \mathbb{C}^{n\times n}$ are arbitrary such that $V = -V^*$.

Let $T = J^{\frac{1}{2}}$. Since $J = A^\dagger AJA^\dagger A = T^2 = A^\dagger AT^2A^\dagger A = (A^\dagger AT)^2$ and $A^\dagger AT \geq 0$, we get that $T = A^\dagger AT = TA^\dagger A$. So,

$$\begin{aligned} (I - A^\dagger A)Y = &((A^\dagger)^2A)^* - (A^\dagger)^* + \frac{1}{2}(I - A^\dagger A)UU^*(I - A^\dagger A) \\ &+ (I - A^\dagger A)UJ^{\frac{1}{2}} + (I - A^\dagger A)V(I - A^\dagger A). \quad \square \end{aligned}$$

Remark 1. It is interesting to remark that applying Theorem 2.3 [89], we can obtain the different general form of the Re-nnd solutions of the equation $AX = B$ in the case when such solutions exist, than the one given in the recent literature. So, in 1992, Wu [105] studied the real-positive definite solution of $AX = B$, and Wu and Cain [106], the real-nonnegative solutions; Gross [42] gave a new derivation and a corrected version to some of Wu and Cain's results. He observed that any matrix of the form

$$X = A^\dagger B - (A^\dagger B)^* + A^\dagger AB^*(A^\dagger)^* + (I - A^\dagger A)Y(I - A^\dagger A),$$

for some Re-nd matrix $Y \in \mathbb{C}^{n\times n}$ is a Re-nnd solution of the equation $AX = B$ but the general form of such inverses is not given in his paper. The similar problem was considered in the paper of D.S. Cvetković-Ilić [21], where the general form of Re-nnd solutions of the equation $AXB = C$ is given but because of the complexity of that form it is not so much applicable. As we know the set of all Re-nnd solutions of the equation $AX = B$ in the case of bounded operators and in the settings of strongly *-reducing ring with involution was only described in the paper of Dajić et. al [30]. If we apply Theorem 7.3 [30] to the matrix case, we get that in the case when a Re-nnd solution of the equation $AX = B$ exists, the set of all such solutions is described by

$$X = X_0 + F_A S A^*(H(AB^*))^\dagger \Big(H(AB^*)(A^*)^\dagger + \frac{1}{4} A S^* F_A \Big) \\ + F_A T F_A \tag{3.7}$$

where $S, T \in \mathbb{C}^{n\times n}$, $T \in \mathbb{C}^{n\times n}$ is Re-nnd and $X_0 = A^\dagger B - (A^\dagger B)^* + A^\dagger A (A^\dagger B)^*$ is a particular real-positive solution.

Applying Theorem 2.3 [89] (similar as in the proof of Theorem 3.2.1), we obtain that the general Re-nnd solution of the equation $AX = B$ is given by

$$X = A^\dagger B - (I - A^\dagger A)(A^\dagger B)^* + \frac{1}{2}(I - A^\dagger A)UU^*(I - A^\dagger A) + \\ (I - A^\dagger A)UJ^{\frac{1}{2}} + (I - A^\dagger A)V(I - A^\dagger A), \tag{3.8}$$

where $J = A^\dagger B A^\dagger A + (A^\dagger B A^\dagger A)^* = 2A^\dagger A H(A^\dagger B) A^\dagger A$, and $U, V \in \mathbb{C}^{n\times n}$ are such that $V = -V^*$.

Remark that a general form of Re-nnd solution of the equation $AX = B$ given by (4.13) is different than the one given by (4.3). $\square$

The following lemma will be useful in proving the necessary and sufficient conditions for the existence of nonnegative definite $\{1,3\}$-inverse of A.

Lemma 3.2.1 *Let $A \in \mathbb{C}^{n\times n}$. If $(A^\dagger)^2 A \geqslant 0$ and $N((A^\dagger)^2 A) = N(A)$, then*

$$(A^2 A^\dagger)^\dagger = (A^\dagger)^*((A^\dagger)^2 A)^\dagger A^\dagger. \tag{3.9}$$

Proof. To prove that the Moore-Penrose inverse of $A^2 A^\dagger$ is given by (3.9), we will check that the four Moore-Penrose equations are satisfied: Let $X = (A^\dagger)^*((A^\dagger)^2 A)^\dagger A^\dagger$. Then,

$$\begin{aligned} A^2 A^\dagger X A^2 A^\dagger &= A(A^\dagger)^*((A^\dagger)^2 A)^\dagger A A^\dagger \\ &= AA^\dagger A (A^\dagger)^*((A^\dagger)^2 A)^\dagger A A^\dagger \\ &= A((A^\dagger)^2 A)^*((A^\dagger)^2 A)^\dagger A A^\dagger \\ &= AA^\dagger A A A^\dagger \\ &= A^2 A^\dagger, \end{aligned}$$

$$\begin{aligned} X A^2 A^\dagger X &= (A^\dagger)^*((A^\dagger)^2 A)^\dagger A^\dagger A (A^\dagger)^*((A^\dagger)^2 A)^\dagger A^\dagger \\ &= (A^\dagger)^*((A^\dagger)^2 A)^\dagger (A^\dagger)^2 A ((A^\dagger)^2 A)^\dagger A^\dagger \\ &= (A^\dagger)^*((A^\dagger)^2 A)^\dagger A^\dagger, \end{aligned}$$

$$\begin{aligned} A^2 A^\dagger X &= A(A^\dagger)^*((A^\dagger)^2 A)^\dagger A^\dagger \\ &= AA^\dagger A (A^\dagger)^*((A^\dagger)^2 A)^\dagger A^\dagger \\ &= A(A^\dagger)^2 A((A^\dagger)^2 A)^\dagger A^\dagger = \\ &= AA^\dagger A A^\dagger = \\ &= AA^\dagger, \end{aligned}$$

$$\begin{aligned} XA^2A^\dagger &= (AA^\dagger A^\dagger A((A^\dagger)^2A)^\dagger A^\dagger)^* \\ &= (A(A^\dagger)^2A((A^\dagger)^2A)^\dagger A^\dagger)^* = \\ &= (AA^\dagger AA^\dagger)^* = \\ &= AA^\dagger.\square \end{aligned}$$

Theorem 3.2.2 *Let* $A \in \mathbb{C}^{n\times n}$ *Then the following statements are equivalent:*
(i) $A_{\geqslant}^{(1,3)}$ *exists;*
(ii) $(A^\dagger)^2A \geqslant 0$ *and* $N((A^\dagger)^2A) = N(A)$.
Furthermore, the set of all nonnegative definite $\{1,3\}$*-inverses of* A *is given by*

$$A_{\geqslant}\{1,3\} = \{(A^2A^\dagger)^\dagger + F_AWF_A : W \in \mathbb{C}^{n\times n}, W \geqslant 0\}. \tag{3.10}$$

Proof. $(i) \Rightarrow (ii)$: Suppose that there exist $A_{\geqslant}^{(1,3)}$. By Theorem 3.2.1, it is represented by

$$A_{\geqslant}^{(1,3)} = A^\dagger - (A^\dagger)^* + ((A^\dagger)^2A)^* + \frac{1}{2}F_AUU^*F_A + F_AUJ^{\frac{1}{2}} + F_AVF_A, \tag{3.11}$$

where $J = (A^\dagger)^2A + \big((A^\dagger)^2A\big)^*$, $U, V \in \mathbb{C}^{n\times n}$ and $V = -V^*$. Since $A_{\geqslant}^{(1,3)}$ is Hemitian it follows that

$$\begin{aligned} &A^\dagger - (A^\dagger)^* + A^\dagger A(A^\dagger)^* + \frac{1}{2}F_AUU^*F_A + F_AUJ^{\frac{1}{2}} + F_AVF_A \\ &= (A^\dagger)^* - A^\dagger + (A^\dagger)^2A + \frac{1}{2}F_AUU^*F_A + J^{\frac{1}{2}}U^*F_A + F_AV^*F_A, \end{aligned}$$

i.e.

$$\begin{aligned} &2A^\dagger - 2(A^\dagger)^* + ((A^\dagger)^2A)^* - (A^\dagger)^2A + F_AUJ^{\frac{1}{2}} \\ &-J^{\frac{1}{2}}U^*F_A + 2F_AVF_A = 0. \end{aligned} \tag{3.12}$$

Multiplying the equation (3.12) by $A^\dagger A$ from the left side, we get

$$2A^\dagger - ((A^\dagger)^2A)^* - (A^\dagger)^2A - J^{\frac{1}{2}}U^*F_A = 0. \tag{3.13}$$

Now, multiplying the last equation by $A^\dagger A$ from the right side, we get

$$(A^\dagger)^2A = ((A^\dagger)^2A)^*, \tag{3.14}$$

which implies that $(A^\dagger)^2A \geqslant 0$. By (3.13), we have

$$J^{\frac{1}{2}}U^*F_A = 2A^\dagger F_A. \tag{3.15}$$

Since there exists $U \in \mathbb{C}^{n\times n}$ such that (3.15) holds, using the fact that $JJ^\dagger = (J^{\frac{1}{2}})(J^{\frac{1}{2}})^\dagger$, it follows that

$$JJ^\dagger A^\dagger F_A = A^\dagger F_A,$$

i.e.

$$JJ^\dagger A^\dagger = A^\dagger, \tag{3.16}$$

or equivalently

$$JJ^\dagger = A^\dagger A, \tag{3.17}$$

from which follows

$$N(A) = N((2(A^\dagger)^2A)^\dagger)) = N((2(A^\dagger)^2A)^*) = N((A^\dagger)^2A).$$

$(ii) \Rightarrow (i)$: Suppose that $(A^\dagger)^2A \geqslant 0$ and $N((A^\dagger)^2A) = N(A)$. By $N((A^\dagger)^2A) = N(A)$, it follows that $((A^\dagger)^2A)^\dagger((A^\dagger)^2A) = A^\dagger A$. Since $(A^\dagger)^2A$ is Hermitian, we have

$$((A^\dagger)^2A)((A^\dagger)^2A)^\dagger = ((A^\dagger)^2A)^\dagger((A^\dagger)^2A) = A^\dagger A,$$

which implies that

$$\begin{aligned} A(A^\dagger)^*((A^\dagger)^2A)^\dagger A^\dagger &= AA^\dagger A(A^\dagger)^*((A^\dagger)^2A)^\dagger A^\dagger \\ &= A((A^\dagger)^2A)((A^\dagger)^2A)^\dagger A^\dagger \\ &= AA^\dagger AA^\dagger = AA^\dagger. \end{aligned} \tag{3.18}$$

Now, it is easy to conclude that $(A^\dagger)^*((A^\dagger)^2A)^\dagger A^\dagger$ is a nonnegative definite $\{1,3\}$-inverse of A.

Now, suppose that one of the condition $(i)-(ii)$ holds and prove that the set $A_\geqslant\{1,3\}$ is described by (3.10). Let $A_\geqslant^{(1,3)} \in A_\geqslant\{1,3\}$ be arbitrary. Then $A_\geqslant^{(1,3)}$ is given by (3.11). Substituting (3.15) in (3.12), we get that $F_AVF_A = 0$, so

$$A_\geqslant^{(1,3)} = A^\dagger - (A^\dagger)^* + (A^\dagger)^2A + \frac{1}{2}F_AUU^*F_A + F_AUJ^{\frac{1}{2}}. \tag{3.19}$$

Now, substituting (3.15) in (3.19), we get

$$A_\geqslant^{(1,3)} = A^\dagger + (A^\dagger)^* - (A^\dagger)^2A + \frac{1}{2}F_AUU^*F_A, \tag{3.20}$$

where matrix $U \in \mathbb{C}^{n\times n}$ satisfies the equation (3.15). In the part $(ii) \Rightarrow (i)$ of the proof, it is shown that condition (ii) implies $JJ^\dagger = A^\dagger A$. According to Theorem **??**, matrix U which satisfies (3.15) must be of the form

$$U = 2F_A(A^\dagger)^*(J^{\frac{1}{2}})^\dagger + Y - F_AYJ^{\frac{1}{2}}(J^{\frac{1}{2}})^\dagger,$$

for some $Y \in \mathbb{C}^{n\times n}$. Using that $J^{\frac{1}{2}}(J^{\frac{1}{2}})^\dagger = JJ^\dagger = A^\dagger A$, we have that

$$\begin{aligned} &F_AUU^*F_A = 2F_A(A^\dagger)^*((A^\dagger)^2A)^\dagger A^\dagger F_A + F_AY(I - A^\dagger A)Y^*F_A = \\ &2F_A(A^\dagger)^*((A^\dagger)^2A)^\dagger A^\dagger - 2F_A(A^\dagger)^* + F_AY(I - A^\dagger A)Y^*F_A. \end{aligned}$$

so

$$\begin{aligned} A_{\geqslant}^{(1,3)} &= A^{\dagger} + F_A(A^{\dagger})^* + F_A(A^{\dagger})^*((A^{\dagger})^2A)^{\dagger}A^{\dagger} - F_A(A^{\dagger})^* \\ &\quad + \frac{1}{2}F_AY(I - A^{\dagger}A)Y^*F_A \\ &= A^{\dagger} + (A^{\dagger})^*((A^{\dagger})^2A)^{\dagger}A^{\dagger} - ((A^{\dagger})^2A)^*((A^{\dagger})^2A)^{\dagger}A^{\dagger} \\ &\quad + \frac{1}{2}F_AY(I - A^{\dagger}A)Y^*F_A \\ &= A^{\dagger} + (A^{\dagger})^*((A^{\dagger})^2A)^{\dagger}A^{\dagger} - A^{\dagger}AA^{\dagger} + \frac{1}{2}F_AY(I - A^{\dagger}A)Y^*F_A \\ &= (A^{\dagger})^*((A^{\dagger})^2A)^{\dagger}A^{\dagger} + \frac{1}{2}F_AY(I - A^{\dagger}A)Y^*F_A. \end{aligned}$$

Using Lemma 3.2.1, we get that

$$A_{\geqslant}^{(1,3)} = (A^2A^{\dagger})^{\dagger} + F_AZ(I - A^{\dagger}A)Z^*F_A, \tag{3.21}$$

for some $Z \in \mathbb{C}^{n\times n}$. Take $W = (Z(I - A^{\dagger}A))(Z(I - A^{\dagger}A))^*$ in (4.11), we have

$$A_{\geqslant}^{(1,3)} = (A^2A^{\dagger})^{\dagger} + F_AWF_A,$$

where $W \in \mathbb{C}^{n\times n}$ is nonnegative definite, so

$$A_{\geqslant}\{1,3\} \subseteq \{(A^2A^{\dagger})^{\dagger} + F_AWF_A, W \in \mathbb{C}^{n\times n},\ W \geqslant 0\}.$$

To prove the opposite inclusion, let $S \in \{(A^2A^{\dagger})^{\dagger} + F_AWF_A :\ W \in \mathbb{C}^{n\times n},\ W \geqslant 0\}$ be arbitrary. Using Lemma 3.2.1, we have that

$$S = (A^{\dagger})^*((A^{\dagger})^2A)^{\dagger}A^{\dagger} + F_AWF_A,$$

for some nonnegative definite $W \in \mathbb{C}^{n\times n}$. Obviously, $S \geqslant 0$. Using that

$$((A^{\dagger})^2A)((A^{\dagger})^2A)^{\dagger} = A^{\dagger}A,$$

we get

$$\begin{aligned} AS = A(A^{\dagger})^*((A^{\dagger})^2A)^{\dagger}A^{\dagger} &= AA^{\dagger}A(A^{\dagger})^*((A^{\dagger})^2A)^{\dagger}A^{\dagger} \\ = A((A^{\dagger})^2A)((A^{\dagger})^2A)^{\dagger}A^{\dagger} &= AA^{\dagger}, \end{aligned}$$

so $S \in A\{1,3\}$. Hence, $S \in A_{\geqslant}\{1,3\}$ and

$$\{(A^2A^{\dagger})^{\dagger} + F_AWF_A, W \in \mathbb{C}^{n\times n},\ W \geqslant 0\} \subseteq A_{\geqslant}\{1,3\}. \quad \square$$

Remark 2. Condition $N((A^{\dagger})^2A) = N(A)$ in item (ii) of Theorem 4.1.1 can be replaced with $r(A) = r(A^2)$. Suppose $(A^{\dagger})^2A \geqslant 0$ and $N((A^{\dagger})^2A) = N(A)$. In order to prove that $r(A) = r(A^2)$, it is sufficient to show that $R(A^*) \cap N(A^*) = \{0\}$: Let $y \in R(A^*) \cap N(A^*)$. Then $A^{\dagger}y = 0$ and $y = A^{\dagger}Ay$. Since $(A^{\dagger})^2Ay = A^{\dagger}y = 0$, we have that $Ay = 0$, i.e. $y = 0$. Hence, $r(A) = r(A^2)$. Suppose now that $(A^{\dagger})^2A \geqslant 0$ and $r(A) = r(A^2)$. By $r(A) = r(A^2)$, it follows that $R(A^*) \cap N(A^*) = \{0\}$, i.e. $R(A^{\dagger}A) \cap N(A^{\dagger}) = \{0\}$. Now, it follows that $N((A^{\dagger})^2A) = N(A^{\dagger}A) = N(A)$. $\square$

In an analogous way, similar result can be provided by Theorem 2.3 [89] in the case of $\{1,4\}$-inverses

Theorem 3.2.3 *Let $A \in \mathbb{C}^{n\times n}$ Then the following statements are equivalent:*
(i) $A_{re}^{(1,4)}$ *exists;*
(ii) $A(A^{\dagger})^2$ *is Re-nnd;*
(iii) $A^{\dagger}A^2$ *is Re-nnd;*
(iv) A^2A^* *is Re-nnd.*

In this case, the set of all Re-nnd $\{1,4\}$-inverses of A is given by

$$\begin{aligned} A_{re}\{1,4\} = \{&A^{\dagger} - (A^{\dagger})^* + (A(A^{\dagger})^2)^* + \frac{1}{2}E_AUU^*E_A \\ &+K^{\frac{1}{2}}U^*E_A + E_AVE_A : U,V \in \mathbb{C}^{n\times n}, V = -V^*\}, \end{aligned} \tag{3.22}$$

where $K = A(A^{\dagger})^2 + \left(A(A^{\dagger})^2\right)^*$.

Theorem 3.2.4 *Let $A \in \mathbb{C}^{n\times n}$ Then the following statements are equivalent:*
(i) $A_{\geqslant}^{(1,4)}$ *exists;*
(ii) $A(A^{\dagger})^2 \geqslant 0$ *and* $R(A) = R(A(A^{\dagger})^2)$
Furthermore, the set of all nonnegative definite $\{1,4\}$-inverses of A is given by

$$A_{\geqslant}\{1,4\} = \{(A^{\dagger}A^2)^{\dagger} + E_AWE_A : W \in \mathbb{C}^{n\times n}, W \geqslant 0\}.$$

Remark 3. It is easy to see that $A_{\geqslant}^{(1,3)}$ exists if and only if $A_{\geqslant}^{(1,2,3)}$ exists. If $A_{\geqslant}^{(1,3)}$ exists, then according to Theorem 4.1.1, $(A^2A^{\dagger})^{\dagger}$ is nonnegative-definite {1,3}-inverse of A. Using Lemma 3.2.1, $(A^2A^{\dagger})^{\dagger}$ is also $\{2\}$-inverse of A since

$$\begin{aligned} &(A^2A^{\dagger})^{\dagger}A(A^2A^{\dagger})^{\dagger} \\ &= (A^{\dagger})^*((A^{\dagger})^2A)^{\dagger}A^{\dagger}AA^{\dagger} \\ &= (A^{\dagger})^*((A^{\dagger})^2A)^{\dagger}A^{\dagger} \\ &= (A^2A^{\dagger})^{\dagger}. \end{aligned}$$

In an analogous way it can be proved that $A_{\geqslant}^{(1,4)}$ exists if and only if $A_{\geqslant}^{(1,2,4)}$ exists. □

In the next theorem, we present the necessary and sufficient conditions for the existence of Re-nnd $\{1,3,4\}$-inverse of A.

Theorem 3.2.5 *Let $A \in \mathbb{C}^{n\times n}$. Then the following statements are equivalent:*
(i) $A_{re}^{(1,3,4)}$ *exists.*
(ii) A^*A^2 *is Re-nnd and* $(I - AA^{\dagger})(A^*)^2(I - ((I - AA^{\dagger})A^*)^{\dagger}A^*) = 0$.

Proof. *(i)* ⇔ *(ii)*: $A_{re}^{(1,3,4)}$ exists if and only if the system of the matrix equations

$$A^*AX = A^* \quad XAA^* = A^*$$

has a common Re-nnd solution. According to [Theorem 2.1, [109]], this is satisfied if and only if A^*A^2 is Re-nnd and

$$r\begin{pmatrix} A^*AA^*A & 0 & A^*A^2+(A^*)^2A \\ A^*A & A^* & A \\ 0 & AA^* & -A^*A \end{pmatrix} = r(A) + r\begin{pmatrix} AA^* & A^*A \\ A^*AA^* & (A^*)^2A \end{pmatrix}. \tag{3.23}$$

Since

$$\begin{aligned} & r\begin{pmatrix} A^*AA^*A & 0 & A^*A^2+(A^*)^2A \\ A^*A & A^* & A \\ 0 & AA^* & -A^*A \end{pmatrix} = r\begin{pmatrix} 0 & -A^*AA^* & (A^*)^2A \\ A^*A & A^* & A \\ 0 & AA^* & -A^*A \end{pmatrix} \\ &= r\begin{pmatrix} 0 & 0 & 0 \\ A^*A & A^* & A \\ 0 & AA^* & -A^*A \end{pmatrix} = r\begin{pmatrix} A^*A & A^* & A \\ 0 & AA^* & -A^*A \end{pmatrix} \\ &= r\begin{pmatrix} 0 & A^* & A \\ -AA^*A & AA^* & -A^*A \end{pmatrix} = r\begin{pmatrix} 0 & A^* & A \\ -AA^*A & 0 & -A^*A \end{pmatrix} \\ &= r\begin{pmatrix} 0 & A^* & A \\ AA^*A & 0 & A^*A \end{pmatrix} = r\begin{pmatrix} 0 & A^*A & AA^*A \\ A^* & A & 0 \end{pmatrix} \end{aligned}$$

So (3.23) is satisfied if and only if

$$r\begin{pmatrix} 0 & A^*A & AA^*A \\ A^* & A & 0 \end{pmatrix} = r(A) + r\begin{pmatrix} AA^* & A^*A \end{pmatrix}. \tag{3.24}$$

From the Theorem 19 [62], we have

$$r\begin{pmatrix} 0 & X \\ Y & S \end{pmatrix} = r(S) + r(U) + r(V) + r(W), \tag{3.25}$$

where $W = (I - UU^-)XS^-Y(I - V^-V)$, $U = X(I - S^-S)$ and $V = (I - SS^-)Y$, for some arbitrary but fixed inner inverses U^-, S^- and V^- of U, S, V, respectively. Let $X = \begin{pmatrix} A^*A & AA^*A \end{pmatrix}$, $Y = A^*$ and $S = \begin{pmatrix} A & 0 \end{pmatrix}$. Since

$$\begin{aligned} U &= \begin{pmatrix} A^*A & AA^*A \end{pmatrix}(I - \begin{pmatrix} A^\dagger \\ 0 \end{pmatrix}\begin{pmatrix} A & 0 \end{pmatrix}) \\ &= \begin{pmatrix} A^*A & AA^*A \end{pmatrix}\begin{pmatrix} I - A^\dagger A & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & AA^*A \end{pmatrix}, \\ V &= (I - \begin{pmatrix} A & 0 \end{pmatrix}\begin{pmatrix} A^\dagger \\ 0 \end{pmatrix})A^* = E_AA^*, \\ W &= (I - \begin{pmatrix} 0 & AA^*A \end{pmatrix}\begin{pmatrix} 0 \\ (AA^*A)^\dagger \end{pmatrix})\begin{pmatrix} A^*A & AA^*A \end{pmatrix}\begin{pmatrix} A^\dagger \\ 0 \end{pmatrix}A^*(I - V^-V) \\ &= E_A(A^*)^2(I - (E_AA^*)^\dagger A^*), \end{aligned}$$

by (3.25), we get

$$r\begin{pmatrix} 0 & A^*A & AA^*A \\ A^* & A & 0 \end{pmatrix} = r(A) + r(AA^*A) + r(E_AA^*) + r(E_A(A^*)^2(I-(E_AA^*)^\dagger A^*)) \tag{3.26}$$

$$= \quad 2r(A) + r(E_AA^*) + r(E_A(A^*)^2(I-(E_AA^*)^\dagger A^*)).$$

Using Theorem 19 [62] we have

$$\begin{aligned} r\begin{pmatrix} AA^* & A^*A \end{pmatrix} &= r(AA^*) + r((I-AA^*(AA^*)^\dagger)A^*A) \\ &= r(A) + r(E_AA^*A). \end{aligned} \tag{3.27}$$

Now, from (3.26) and (3.27) it follows that (3.24) is equivalent to

$$E_A(A^*)^2(I-(E_AA^*)^\dagger A^*) = 0.\square$$

Chapter 4

Special Schur complement

The idea of the Schur complement goes back to Sylvester in 1851, while the term Schur complement was introduced by E. Haynsworth [46]. In the beginning, Schur complements were used in the theory of matrices.

Let

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{C}^{(n+m)\times(n+k)}, \tag{4.1}$$

where $A \in \mathbb{C}^{n\times n}$ is nonsingular. The Schur complement of A in M is the matrix

$$S = D - CA^{-1}B. \tag{4.2}$$

M.G. Krein [53] and W.N. Anderson and G.E. Trapp [1] extended the notion of Schur complements of matrices to shorted operators in Hilbert spaces. Trapp defined the generalized Schur complement by replacing the ordinary inverse A^{-1} with the generalized inverse. Thei importance of the study of the Schur complement lies in the wide range of its applications. Beside matrix theory, it plays an important role in electric network theory, multivariate statistics and some other fields [16, 71, 17, 2, 3]. Also there are many papers that deal with Schur complement from theoretical aspect [32, 29, 18, 25, 56, 81, 22].

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces. In this chapter we introduce and study a Schur complement of operators on Hilbert spaces. We will consider the inverse of the special Schur complement $CD^{-1}B$, where $B \in \mathcal{B}(\mathcal{K},\mathcal{H})$, $C \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $D \in \mathcal{B}(\mathcal{H})$ are such operators that $CD^{-1}B$ is invertibile. We will show that there always exist operators X and Y which belong to some special classes of generalized inverses of B and C respectively, such that $(CD^{-1}B)^{-1} = XDY$. We will give explicit expressions for such X and Y and present the inverse of $CD^{-1}B$ in terms of C, B, D and generalized inverses of B and C. Also, we will consider the special case when $B \in \mathcal{B}(\mathcal{K},\mathcal{H})$ and $D \in \mathcal{B}(\mathcal{H})$ is a positive and invertible operator such that $B^*D^{-1}B$ is invertible because of it importance in some problems of Electrical Engeneering. Some results from the paper of Xiong and Qin [108] are generalized to the operator case.

4.1 Inverse of a special Schur complement

In the next theorem, we will show that the inverse of $CD^{-1}B$ can be represented by

$$(CD^{-1}B)^{-1} = XDY,$$

where $X \in B\{1\}$ and $Y \in C\{1\}$.

Theorem 4.1.1 *Let $D \in \mathcal{B}(\mathcal{H})$ be invertible and let $C \in B(\mathcal{H}, \mathcal{K})$ and $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ be such that $CD^{-1}B$ is invertible. There exist $B^{(1)} \in B\{1\}$ and $C^{(1)} \in C\{1\}$ such that*

$$(CD^{-1}B)^{-1} = B^{(1)}DC^{(1)}. \tag{4.3}$$

Proof. Using the appropriate decompositions of the spaces $\mathcal{H}$ and $\mathcal{K}$, we can decompose operators C and B as follows:

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix} \to \begin{bmatrix} \mathcal{K} \\ \{0\} \end{bmatrix} \tag{4.4}$$

and

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K} \\ \{0\} \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}, \tag{4.5}$$

where B_1 and C_1 are invertible. In that case arbitrary $C^{(1)} \in C\{1\}$ and $B^{(1)} \in B\{1\}$ are represented by

$$C^{(1)} = \begin{bmatrix} C_1^{-1} & C_2 \\ C_3 & C_4 \end{bmatrix} : \begin{bmatrix} \mathcal{K} \\ \{0\} \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix} \tag{4.6}$$

and

$$B^{(1)} = \begin{bmatrix} B_1^{-1} & B_2 \\ B_3 & B_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{K} \\ \{0\} \end{bmatrix}, \tag{4.7}$$

for some operators C_i, B_i, $i = \overline{2,4}$ which are defined on the corresponding subspaces. Let D be given by

$$D = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \tag{4.8}$$

and D^{-1} by

$$D^{-1} = \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix}, \tag{4.9}$$

where W_i, $i = \overline{1,4}$ satisfy the following equations

$$\begin{aligned} &D_1W_1 + D_2W_3 = I,\ D_1W_2 + D_2W_4 = 0,\ D_3W_1 + D_4W_3 = 0, \\ &D_3W_2 + D_4W_4 = I,\ W_1D_1 + W_2D_3 = I,\ W_1D_2 + W_2D_4 = 0, \\ &W_3D_1 + W_4D_3 = 0,\ W_3D_2 + W_4D_4 = I. \end{aligned} \tag{4.10}$$

Now, it follows that

$$\begin{aligned} & CD^{-1}BB^{(1)}DC^{(1)} \\ = & C\begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix}\begin{bmatrix} I & B_1B_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}C^{(1)} \\ = & C\begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix}\begin{bmatrix} D_1+B_1B_2D_3 & D_2+B_1B_2D_4 \\ 0 & 0 \end{bmatrix}C^{(1)} \\ = & \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} W_1(D_1+B_1B_2D_3) & W_1(D_2+B_1B_2D_4) \\ W_3(D_1+B_1B_2D_3) & W_3(D_2+B_1B_2D_4) \end{bmatrix}\begin{bmatrix} C_1^{-1} & C_2 \\ C_3 & C_4 \end{bmatrix} \\ = & \begin{bmatrix} S & T \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K} \\ \{0\} \end{bmatrix} \to \begin{bmatrix} \mathcal{K} \\ \{0\} \end{bmatrix}, \end{aligned}$$

where $S = C_1W_1(D_1 + B_1B_2D_3)C_1^{-1} + C_1W_1(D_2 + B_1B_2D_4)C_3$ and $T = C_1W_1(D_1 + B_1B_2D_3)C_2 + C_1W_1(D_2 + B_1B_2D_4)C_4$.
Hence, $B^{(1)}DC^{(1)} = (CD^{-1}B)^{-1}$ if and only if $S = I$ i.e.

$$C_1W_1(D_1 + B_1B_2D_3)C_1^{-1} + C_1W_1(D_2 + B_1B_2D_4)C_3 = I. \tag{4.11}$$

Since $CD^{-1}B$ is invertible and

$$CD^{-1}B = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} W_1 & W_2 \\ W_3 & W_2 \end{bmatrix}\begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} = C_1W_1B_1, \tag{4.12}$$

it follows that $W_1 : \mathcal{R}(B) \to \mathcal{R}(C^*)$ is invertible. Using (4.10), it is obvious that (4.11) is satisfied for $C_3 = 0$ and $B_2 = B_1^{-1}W_1^{-1}W_2$. So for $C^{(1)} = \begin{bmatrix} C_1^{-1} & C_2 \\ 0 & C_4 \end{bmatrix}$ and $B^{(1)} = \begin{bmatrix} B_1^{-1} & B_1^{-1}W_1^{-1}W_2 \\ B_3 & B_4 \end{bmatrix}$, where C_2, C_4, B_3, B_4 are arbitrary, we have that (4.3) is satisfied. □

In the following result, we present the inverse of $CD^{-1}B$ as

$$(CD^{-1}B)^{-1} = XDY,$$

where $X \in B\{1,2\}$ and $Y \in C\{1,2\}$.

Theorem 4.1.2 *Let $D \in \mathcal{B}(\mathcal{H})$ be invertible and let $C \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{K},\mathcal{H})$ be such that $CD^{-1}B$ is invertible. There exist $B^{(1,2)} \in B\{1,2\}$ and $C^{(1,2)} \in C\{1,2\}$ such that*

$$(CD^{-1}B)^{-1} = B^{(1,2)}DC^{(1,2)}. \tag{4.13}$$

Proof. Suppose that operators B, C, D and D^{-1} are given as in the proof of Theorem 4.1.1. In that case, arbitrary $B^{(1,2)} \in B\{1,2\}$ and $C^{(1,2)} \in C\{1,2\}$ are represented by

$$B^{(1,2)} = \begin{bmatrix} B_1^{-1} & B_2 \\ B_3 & B_3B_1B_2 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{K} \\ \{0\} \end{bmatrix} \tag{4.14}$$

and

$$C^{(1,2)} = \begin{bmatrix} C_1^{-1} & C_2 \\ C_3 & C_3C_1C_2 \end{bmatrix} : \begin{bmatrix} \mathcal{K} \\ \{0\} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix}, \tag{4.15}$$

where B_i and C_i are arbitrary linear bounded operators on the corresponding subspaces.

Now, it follows that

$$\begin{aligned} CD^{-1}BB^{(1,2)}DC^{(1,2)} &= C \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix} \begin{bmatrix} I & B_1B_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix} C^{(1,2)} \\ &= \begin{bmatrix} S & T \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K} \\ \{0\} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{K} \\ \{0\} \end{bmatrix}, \end{aligned}$$

where $S = C_1W_1(D_1 + B_1B_2D_3)C_1^{-1} + C_1W_1(D_2 + B_1B_2D_4)C_3$ and $T = C_1W_1(D_1 + B_1B_2D_3)C_2 + C_1W_1(D_2 + B_1B_2D_4)C_3C_1C_2$.

Let $C_3 = 0$ and $B_2 = B_1^{-1}W_1^{-1}W_2$. Then for $C^{(1,2)} = \begin{bmatrix} C_1^{-1} & C_2 \\ 0 & 0 \end{bmatrix}$ and $B^{(1,2)} = \begin{bmatrix} B_1^{-1} & B_1^{-1}W_1^{-1}W_2 \\ B_3 & B_3W_1^{-1}W_2 \end{bmatrix}$, where C_2, B_3 are arbitrary, we have that (4.13) holds. □

The case when X and Y belong to the classes of $\{1,3\}$-inverses of B and C respectively, will be consider in the next result:

Theorem 4.1.3 *Let $D \in \mathcal{B}(\mathcal{H})$ be invertible and let $C \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{K},\mathcal{H})$ be such that $CD^{-1}B$ is invertible. There exist $B^{(1,3)} \in B\{1,3\}$ and $C^{(1,3)} \in C\{1,3\}$ such that*

$$(CD^{-1}B)^{-1} = B^{(1,3)}DC^{(1,3)}. \tag{4.16}$$

Proof. Suppose that B, C, D and D^{-1} are given as in the proof of Theorem 4.1.1. Arbitrary $B^{(1,3)} \in B\{1,3\}$ and $C^{(1,3)} \in C\{1,3\}$ are represented by

$$B^{(1,3)} = \begin{bmatrix} B_1^{-1} & 0 \\ B_3 & B_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{K} \\ \{0\} \end{bmatrix}$$

and

$$C^{(1,3)} = \begin{bmatrix} C_1^{-1} & 0 \\ C_3 & C_4 \end{bmatrix} : \begin{bmatrix} \mathcal{K} \\ \{0\} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix}$$

for some B_i and C_i, $i = \overline{3,4}$. Now,

$$\begin{aligned} CD^{-1}BB^{(1,3)}DC^{(1,3)} &= C \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix} C^{(1,3)} \\ &= \begin{bmatrix} S & T \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K} \\ \{0\} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{K} \\ \{0\} \end{bmatrix}, \end{aligned}$$

where $S = C_1W_1D_1C_1^{-1} + C_1W_1D_2C_3$ and $T = C_1W_1D_2C_4$.

Now, we can conclude that there exist $B^{(1,3)} \in B\{1,3\}$ and $C^{(1,3)} \in C\{1,3\}$ such that $(CD^{-1}B)^{-1} = B^{(1,3)}DC^{(1,3)}$ if and only if there exists $C_3 : \mathcal{K} \to \mathcal{N}(C)$ such that

$$C_1W_1D_1C_1^{-1} + C_1W_1D_2C_3 = I. \tag{4.17}$$

Since C_1 is invertible, (4.17) is equivalent to

$$W_1D_2C_3C_1 = I - W_1D_1. \tag{4.18}$$

Using (10) and (11) we get that (4.18) is equivalent to

$$-W_2D_4C_3C_1 = W_2D_3. \tag{4.19}$$

Since by (4.12) we have that W_1 is invertible, multiplying the third equation from (4.10) by W_2 from the left and by W_1^{-1} from the right we get $W_2D_3 = -W_2D_4W_3W_1^{-1}$, which implies that (4.19) is satisfied for $C_3 = W_3W_1^{-1}C_1^{-1}$. □

Theorem 4.1.4 *Let $D \in B(\mathcal{H})$ be invertible and let $C \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{K},\mathcal{H})$ be such that $CD^{-1}B$ is invertible. There exist $B^{(1,2,3)} \in B\{1,2,3\}$ and $C^{(1,2,3)} \in C\{1,2,3\}$ such that*

$$(CD^{-1}B)^{-1} = B^{(1,2,3)}DC^{(1,2,3)}. \tag{4.20}$$

Proof. Suppose that B, C, D and D^{-1} are given as in the proof of Theorem 4.1.1. Arbitrary $B^{(1,2,3)} \in B\{1,2,3\}$ and $C^{(1,2,3)} \in C\{1,2,3\}$ are given by

$$B^{(1,2,3)} = \begin{bmatrix} B_1^{-1} & 0 \\ B_3 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{K} \\ \{0\} \end{bmatrix}$$

and

$$C^{(1,2,3)} = \begin{bmatrix} C_1^{-1} & 0 \\ C_3 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K} \\ \{0\} \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix},$$

for some B_3 and C_3. Now,

$$CD^{-1}BB^{(1,2,3)}DC^{(1,2,3)} = C_1W_1D_1C_1^{-1} + C_1W_1D_2C_3 : \mathcal{K} \to \mathcal{K}$$

Hence, there exist $B^{(1,2,3)} \in B\{1,2,3\}$ and $C^{(1,2,3)} \in C\{1,2,3\}$ such that (4.20) holds if and only if there exists $C_3 : \mathcal{K} \to \mathcal{N}(C)$ such that

$$C_1W_1D_1C_1^{-1} + C_1W_1D_2C_3 = I, \tag{4.21}$$

which is equivalent with (4.19). So, the existence of such C_3 follows as in the proof of the Theorem 4.1.3. □

Remark 1. It is evident that Theorem 4.1.4 implies the validity of the Theorems 4.1.2 and 4.1.3. Beside that, we give the proofs of that theorems since from their proofs we can obtain the explicit forms for the generalized inverses of B and C which satisfy the appropriate formulas.

2. From the proofs of Theorem 4.1.3 and Theorem 4.1.4, we conclude that both formulas (4.16) and (4.20) holds for any $B^{(1,3)}$ and $B^{(1,2,3)}$, respectively, while $C^{(1,3)}$ and $C^{(1,2,3)}$ can not be arbitrary.

3. Taking adjoint, the similar results can be obtain in the case of $\{1,4\}$ and $\{1,2,4\}$-inverses.

Now, we will give some other explicit representations for the inverse of $CD^{-1}B$ in terms of the Moore-Penrose inverses of the operators B and C.

Theorem 4.1.5 *Let $D \in \mathcal{B}(\mathcal{H})$ be invertible and let $C \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{K},\mathcal{H})$ be such that $CD^{-1}B$ is invertible. Then*

$$(CD^{-1}B)^{-1} = B^\dagger D(C^\dagger + (I - C^\dagger C)D^{-1}M^\dagger C^\dagger), \tag{4.22}$$

where $M = C^\dagger CD^{-1}BB^\dagger$.

Proof. Suppose that B, C, D and D^{-1} are given as in the proof of Theorem 4.1.1. Since $\mathcal{R}(M) = C^\dagger R(CD^{-1}BB^\dagger) = C^\dagger\mathcal{R}(CD^{-1}B) = C^\dagger(Y) = \mathcal{R}(C^*)$, it follows that $\mathcal{R}(M)$ is closed, so there exists $M^\dagger \in \mathcal{B}(\mathcal{H})$. Since

$$M = \begin{bmatrix} W_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix},$$

and $R(M) = R(C^*)$, $N(M) = N(B^*)$, we conclude that W_1 is invertible and

$$M^\dagger = \begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}. \tag{4.23}$$

Now,

$$\begin{aligned}
& B^\dagger D(C^\dagger + (I - C^\dagger C)D^{-1}M^\dagger C^\dagger) \\
= \ & B^\dagger DC^\dagger + B^\dagger D(I - C^\dagger C)D^{-1}(C^\dagger CD^{-1}BB^\dagger)^\dagger C^\dagger \\
= \ & \begin{bmatrix} B_1^{-1}D_1C_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\
+ \ & \begin{bmatrix} B_1^{-1}D_1 & B_1^{-1}D_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix} \begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\
= \ & \begin{bmatrix} B_1^{-1}D_1C_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_1^{-1}D_2W_3W_1^{-1}C_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\
= \ & B_1^{-1}D_1C_1^{-1} + B_1^{-1}D_2W_3W_1^{-1}C_1^{-1}.
\end{aligned}$$

Using (10) and (4.12) we get

$$\begin{aligned}
& B^\dagger D(C^\dagger + (I - C^\dagger C)D^{-1}M^\dagger C^\dagger) \\
= \ & B_1^{-1}(D_1W_1 + D_2W_3)W_1^{-1}C_1^{-1} \\
= \ & B_1^{-1}W_1^{-1}C_1^{-1} \\
= \ & (CD^{-1}B)^{-1}.\square
\end{aligned}$$

Taking adjoints in Theorem 3.1, we immediately get the following expression for $(CD^{-1}B)^{-1}$:

Theorem 4.1.6 *Let $D \in \mathcal{B}(\mathcal{H})$ be invertible. Let $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ be such that $CD^{-1}B$ is invertible. Then*

$$(CD^{-1}B)^{-1} = (B^\dagger + B^\dagger M^\dagger D^{-1}(I - BB^\dagger))DC^\dagger,$$

where $M = C^\dagger CD^{-1}BB^\dagger$.

Remark 1. Notice that in Theorem 4.1.5, we have that $C^\dagger + (I - C^\dagger C)D^{-1}M^\dagger C^\dagger \in C\{1, 2, 3\}$ while in Theorem 4.1.6 $B^\dagger + B^\dagger M^\dagger D^{-1}(I - BB^\dagger) \in B\{1, 2, 4\}$.

Finally, we get a representation of $(CD^{-1}B)^{-1}$ in the form of $B^\dagger M^\dagger C^\dagger$.

Theorem 4.1.7 *Let $D \in \mathcal{B}(\mathcal{H})$ be invertible and let $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ be such that $CD^{-1}B$ is invertible. Then*

$$(CD^{-1}B)^{-1} = B^\dagger M^\dagger C^\dagger,$$

where $M = C^\dagger CD^{-1}BB^\dagger$.

Proof. Since $\mathcal{R}(C^\dagger CD^{-1}BB^\dagger) = \mathcal{R}(C^*)$ and $\mathcal{R}(C) = \mathcal{K}$, we have

$$\begin{aligned}
CD^{-1}B(B^\dagger M^\dagger C^\dagger) &= CD^{-1}BB^\dagger(C^\dagger CD^{-1}BB^\dagger)^\dagger C^\dagger \\
&= C(C^\dagger CD^{-1}BB^\dagger)(C^\dagger CD^{-1}BB^\dagger)^\dagger C^\dagger \\
&= CP_{R(C^\dagger CD^{-1}BB^\dagger)}C^\dagger \\
&= CP_{R(C^*)}C^\dagger \\
&= CC^\dagger CC^\dagger \\
&= CC^\dagger \\
&= I.
\end{aligned}$$

□

Bibliography

[1] W.N. Anderson, G.E. Trapp, *Shorted operators II* , SIAM J. Appl.Math. 28 (1975), 60—71.

[2] T. Ando, *Concavity of certain maps on positive definite matrices and applications to Hadamard products*, Linear Algebra Appl. 26 (1979), 203—241.

[3] T. Ando, *Generalized Schur complements*, Linear Algebra Appl. 27 (1979), 173—186.

[4] E. Arghiriade, *Sur les matrices qui sont permutables avec leur inverse généralisée*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 35 (1963), 244—251.

[5] F. V. Atkinson, *The normal solvability of linear equations in normed spaces* (Russian), Mat. Sb. (N.S.) 28(70) (1951), 3—14.

[6] F. V. Atkinson, *On relatively regular operators*, Acta Sci. Math. Szeged 15 (1953), 38—56.

[7] A. Ben-Israel, A. Charnes,*Contributions to the Theory of Generalized Inverses*, J. SIAM, 11 (1963), 667–699.

[8] A. Ben-Israel, T. N. E. Greville, *Generalized Inverse: Theory and Applications*, John Wiley, New York, 1974; 2nd Edition, Springer Verlag, New York, 2003.

[9] A. Bjerhamar, *A Generalized Matrix Algebra*, Trans. Roy. Inst. Tech. Stockholm 124 (1958), 32

[10] A. Bjerhammar, *Application of calculus of matrices to method of least squares with special reference to geodetic calculation*s, Trans. Roy. Inst. Tech. Stockholm (1951), no. 49, 86 pp. (2 plates).

[11] A. Bjerhammar, *Rectangular reciprocal matrices, with special reference to geodetic calculations*, Bull. Géodésique (1951), 188—220.

[12] E. Boasso, D.S. Cvetković-Ilić, R. Harte, *On weighted reverse order laws for the Moore-Penrose inverse and K-inverses*, Comm. Algebra, 40 (2012), 959–971.

[13] E. Bounitzky, *Sur la fonction de Green des équations differentielles linéaires ordinaires*, J. Math. Pures Appl. 5 (1909), no. 6, 65-–125.

[14] S.L. Campbell, C. D. Meyer, Jr., N. J. Rose, *Aplications of the Drazin inverse to linear system of differential equations*, SIAM J. appl. Math. 31 (1976), 411–425

[15] S.L. Campbell, C. D. Meyer, *Generaliyed inverses of linear transformations*, Pitman, London, (1979)

[16] D.J. Clements and H.K. Wimmer, *Monotonicity of the Optimal Cost in the Discrete-time Regulator Problem and Schur Complements*, Automatica, 37 (2001), 1779—1786.

[17] G. Corach, A. Maestripieri, D. Stojanoff, *Oblique projections and Schur complements*, Acta Sci. Math. (Szeged) 67 (2001), 439—459.

[18] D.S. Cvetković-Ilić, *Expression of the Drazin and MP-inverse of partitioned matrix and quotient identity of generalized Schur complement*, Appl. Math. Comput. 213(1) (2009), 18–24.

[19] D.S. Cvetković-Ilić, *New conditions for the reverse order laws for* $\{1,3\}$*- and* $\{1,4\}$*-generalized inverses*, Electron. J. Linear Algebra, 23 (2012), 231–242.

[20] D.S. Cvetković-Ilić, *Reverse order laws for* $\{1,3,4\}$*-generalized inverses in C*-algebras*, Appl. Math. Letters, 24(2) (2011), 210–213.

[21] D.S. Cvetković-Ilić, *Re-nnd solutions of the matrix equation AXB=C*, J. Aust. Math. Soc., 84 (2008), 63–72.

[22] D.S. Cvetković-Ilić, D.S. Djordjević, V. Rakočević, *Schur complements in C*-algebras*, Math. Nachr., 278 (2005), 808–814.

[23] D.S. Cvetković-Ilić, R. Harte, *Reverse order laws in C*-algebras*, Linear Algebra Appl. 434(5) (2011), 1388–1394.

[24] D.S. Cvetković-Ilić, R. Harte, *The spectral topology in rings*, Studia Mathematica, 200 (2010), 267–278.

[25] D.S. Cvetković-Ilić, X. Liu, H. Jin, *Representations of generalized inverses of partitioned matrix involving Schur complement* , Appl. Math. Comput. 219(18) (2013), 9615–9629.

[26] D.S. Cvetković-Ilić, J. Nikolov, *Reverse order laws for reflexive generalized inverse of operators*, Linear Multilinear Algebra 63 (6) (2015), 1167–1175.

[27] D.S. Cvetković-Ilić, J. Nikolov, *Reverse order laws for 1,2,3-generalized inverses*, Applied Mathematics and Computation, 234 (2014), 114–117.

[28] D.S. Cvetković-Ilić, V. Pavlović, *A comment on some recent results concerning the reverse order law for* $\{1,3,4\}$*-inverses*, App. Math. Comput., 217, (2010), 105–109.

[29] D.S. Cvetković-Ilić, B. Zheng, *Weighted generalized inverses of partitioned matrices in Banachiewicz-Schur form*, Appl. Math. Comput., 22(3) (2006), 175–184.

[30] A. Dajic, J.J. Koliha, *Equations $ax = c$ and $xb = d$ in rings and rings with involution with applications to Hilbert space operators*, Linear Algebras Appl., 429(7), (2008), 1779–1809.

[31] C.Y. Deng *Reverse order law for the group inverses*, J. Math. Anal. Appl., 382(2) (2011), 663–671.

[32] C. Deng, Y. Wei, *Representations for the Drazin inverse of* 2×2 *image block-operator matrix with singular Schur complement* , Linear Algebra Appl. 435(11) (2011), 2766–2783.

[33] N. Dinčić, D. S. Djordjević, *Hartwig's triple reverse order law revisited*, Linear Multilinear Algebra, 62 (7) (2014), 918–924.

[34] D.S. Djordjević, *Further results on the reverse order law for generalized inverses*, SIAM J. Matrix Anal. Appl. 29 (4)(2007), 1242–1246.

[35] D. Djordjević, N. Dinčić, *Reverse order law for the Moore–Penrose inverse*, J. Math. Anal. Appl., 361(1) (2010), 252–261.

[36] D.S. Djordjević, V. Rakočević, *Lectures on generalized inverses*, Faculty of Sciences and Mathematics, University of Nis, (2008)

[37] M.P. Drazin, *Pseudo-inverse in associative rings and semigroups*, Amer. Math. Monthly 65(1958), 506-–514.

[38] I. Fredholm, *Sur une classe d'équations fonctionnelles*, Acta Math. 27 (1903), 365—390.

[39] G.H. Golub, C.F. Van Loan, *Matrix Computations*, The John Hopkins University Press, Baltimore, MD, (1983)

[40] T.N.E. Greville, *Note on the generalized inverse of a matrix product*, SIAM Rev. 8 (1966), 518–521.

[41] C.W. Groetsch, *Generalized inverses of linear operators: representation and application*, Dekker, New York, (1977)

[42] J. Gross, *Explicit solutions to the matrix inverse problem AX=B*, Linear Algebra Appl. 289 (1999), 131–134.

[43] W. Guorong, W. Yimin, Q. Sanzheng, *Generalized Inverses: Theory and Computations*, Science Press, Beijing/New York (2004)

[44] R.E. Harte, *Invertibility and singularity*, Dekker (1988)

[45] R.E. Hartwig, *The reverse order law revisited*, Linear Algebra Appl. 76 (1986), 241–246.

[46] E. Haynsworth, *Determination of the inertia of a partitioned Hermitian matrix*, Linear Algebra Appl. 1 (1968), 73—81.

[47] W. A. Hurwitz, *On the pseudo-resolvent to the kernel of an integral equation*, Trans. Amer. Math. Soc. 13 (1912), 405—418.

[48] S. Izumino, *The product of operators with closed range and an extension of the reverse order law*, Tohoku Math. J. 34 (1982), 43–52.

[49] J.J. Koliha, D.S. Djordjević, D. Cvetković-Ilić, *Moore-Penrose inverse in rings with involution*, Linear Algebra Appl., 426 (2007), 371–381.

[50] J.J. Koliha, V. Rakočević, *Invertibility of the difference of idempotents*, Linear Multilinear Algebra, 51 (2003), 97–110.

[51] J.J. Koliha, V. Rakočević, it Differentiability of the g-Drazin inverse, Studia Math., 168 (2005), 193–201.

[52] J.J. Koliha, V. Rakočević, I. Starskraba, *The difference and sum of projectors*, Linear Algebra Appl., 388 (2004), 279–288.

[53] M.G. Krein, *The theory of self-adjoint extensions of semibounded Hermitian operators and its applications*, Math.Sb.(N.S.) 62 (1947), no.20, 431—495.

[54] X. Liu, J. Benítez, J. Zhong, *Some results on partial ordering and reverse order law of elements of image-algebras*, J. Math. Anal. Appl., 370(1) (2010), 295–301.

[55] X. Liu, S. Huang, D.S. Cvetković-Ilić, *Mixed-type reverse-order laws for* $\{1,3,4\}$-*generalized inverses over Hilbert spaces*, Appl. Math. Comp. 218(17) (2012), 8570–8577.

[56] J. Liu, Y. Huang, A. Liao, *Some inequalities for eigenvalues of Schur complements of Hermitian matrices*, Comput. Appl. Math. 196(2) (2006), 439–451.

[57] X. Liu, Shuxia Wu, D.S. Cvetković-Ilić, *New results on reverse order law for* $\{1,2,3\}$ *and* $\{1,2,4\}$*-inverses of bounded operators*, Mathematics of Computation, 82 (283), (2013), 1597–1607.

[58] Q. Liu, M. Wei, *Reverse order law for least squares g-inverses of multiple matrix products*, Linear Multilinear Algebra, 56 (5) (2008), 491–506.

[59] X. Liu, H. Yang, *An expression for the general common least-square solution to a pair of matrix equations with applications*, Comput. Math. Appl., 61 (2011), 3071–3078.

[60] D. Liu, H. Yang, *The reverse order law for* $\{1,3,4\}$- *inverse of the product of two matrices*, Appl. Math. Comput. 215 (12) (2010), 4293–4303.

[61] F.J. Murray, J. von Neumann, *On rings of operators*, Ann. of Math. 37 (1936), 116—229.

[62] G. Marsaglia, G.P.H. Styan, *Equalities and inequalities for ranks of matrices*, Linear Multilinear Algebra, 2 (1974), 269—292.

[63] M.Z. Nashed, *Inner, outer and generalized inverses in Banach and Hilbert spaces*, Numer. Funct. Anal. Optim. 9 (1987), 261—325.

[64] J. Nikolov, D.S. Cvetković-Ilić, *Reverse order laws for the weighted generalized inverses*, Appl. Math. Letters, 24 (2011), 2140–2145.

[65] J. Nikolov, *On the inverse of a special Schur complement*, Georgian Mathematical Journal, (accepted).

[66] J. Nikolov, D.S. Cvetković-Ilić, *Re-nnd generalized inverses*, Linear Algebra and its Applications, 439 (2013), 2999—3007.

[67] V. Pavlović, D.S. Cvetković-Ilić, *Applications of completions of operator matrices to reverse order law for* $\{1\}$*-inverses of operators on Hilbert spaces*, Linear Algebra Appl., accepted

[68] R. Penrose, *A generalized inverse for matrices*, Proc. Camb Phil. Soc., 51 (1955), 406–413.

[69] R. Penrose, *On best approximate solutions of linear matrix equations*, Proc. Camb Phil. Soc. 52 (1956), 17—19.

[70] A.R.D. Pierro, M. Wei, *Reverse order law for reflexive generalized inverses of products of matrices*, Linear Algebra Appl. 277 (1998), 299–311.

[71] D.V. Quellette, *Schur complements and statistics*, Linear Algebra Appl. 36 (1981), 187—295.

[72] J.N. Radenković, *Reverese order law for multiple operator product*, submitted

[73] V. Rakočević, *Funkcionalna analiza*, Naučna knjiga, Beograd, 1994.

[74] V. Rakočević, *On the norm of idempotent operators in a Hilbert space*, American Mathematical Monthly, 107 (2000), 748–750.

[75] V. Rakočević, Y. Wei, *A weighted Drazin inverse and applications*, Linear Algebra Appl. 350 (2002), 25—39.

[76] C.R. Rao, *A note on a generalized inverse of a matrix with applications to problems in mathematical statistics*, J. Roy. Statist. Sot. Sex B 24: 1. 52–158 (1962).

[77] C.R. Rao, S.K. Mitra, *Generalized Inverse of Matrices and Its Applications*, John Wiley, New York, (1971)

[78] T. Reid, *Generalized inverses of differential and integral operators*, In Boullion and Odell [121], 1—25.

[79] D.W. Robinson, *Gauss and generalized inverses*, Historia Mathematica 7 (1980), 118—125.

[80] C.A. Rohde, *Contributions to the theory, computation and application of generalized inverses*, Ph.d., University of North Carolina, Raleigh, N.C., May (1964)

[81] X. Sheng, G. Chen, *Some generalized inverses of partition matrix and quotient identity of generalized Schur complement*, Appl. Math. Comput. 196(1) (2008), 174–184.

[82] N. Shinozaki, M. Sibuya, *Further results on the reverse order law*, Linear Algebra Appl. 27 (1979) 9—16.

[83] N. Shinozaki, M. Sibuya, *The reverse order law* $(AB)^{-} = B^{-}A^{-}$, Linear Algebra Appl. 9 (1974), 29—40.

[84] C. L. Siegel, *Über die analytische Theorie der quadratischen Formen III*, Ann. of Math. 38 (1937), 212—291.

[85] W. Sun and Y. Wei, *Inverse Order Rule for Weighted Generalized Inverse*, SIAM. J. Matrix Anal. Appl. 19 (1996), 772–775.

[86] W. Sun, Y. Wei, *Inverse order rule for weighted generalized inverse*, SIAM. J. Matrix Anal. Appl., 19(3) (1998), 772–775.

[87] W. Sun, Y. Wei, *Triple reverse-order law for weighted generalized inverses*, Appl. Math. Comput., 125(2-3) (2002), 221–229

[88] W. Sun, Y. Yuan, *Optimization Theory and Methods*, Science Press, Beijing, (1996)

[89] Y. Tian, *Maximization and minimization of the rank and inertia of the Hermitian matrix expression* $A - BX - (BX)^{*}$ *with applications*, Linear Algebra Appl. 434, (2011), 2109–2139.

[90] H. Tian, *On the reverse order laws* $(AB)^{D} = B^{D}A^{D}$, J. Math. Research and Exposition 19 (1999), 355-–358.

[91] Y. Tian, *Reverse order laws for the generalized inverses of multiple matrix products*, Linear Algebra Appl., 211 (1994), 85–100.

[92] Y. Tian, *The Moore—Penrose inverse of a triple matrix product* (in Chinese), Math. Theory Practice 1 (1992), 64—70.

[93] Y.-Y. Tseng, *Generalized inverses iof unbounded operators between two unitary spaces*, Doklady Akad. Nauk SSSR (N.S.) 67 (1949), 431—434. i

[94] Y.-Y. Tseng, *Properties and classification of generalized inverses of closed operators*, Doklady Akad. Nauk SSSR (N.S.) 67 (1949), 607-–610.

[95] Y.-Y. Tseng, *Sur les solutions des équations opératrices fonctionnelles entre les espaces unitaires. Solutions extrémales. Solutions virtuelles*, C. R. Acad. Sci. Paris 228 (1949), 640—641.

[96] Y.-Y. Tseng, *The Characteristic Value Problem of Hermitian Functional Operatorsi in a Non–Hilbert Space*, Ph.d. in mathematics, University of Chicago, Chicago, (1933), (published by the University of Chicago Libraries, 1936).

[97] G. Wang, *The reverse order law for the Drazin inverses of multiple matrix products*, Linear Algebra Appl. 348 (2002), 265—272.

[98] G. Wang, Y. Wei, S. Qiao, *Generalized inverses: theory and computations*, Science Press, (2003)

[99] J. Wang, H. Zhang, G. Ji, *A generalized reverse order law for the products of two operators*, Journal of Shaanxi Normal University, 38(4)(2010), 13–17.

[100] M. Wei, *Equivalent conditions for generalized inverses product*, Linear Algebra Appl. 266 (1997) 347–363.

[101] M. Wei, *Reverse order laws for generalized inverses of multiple matrix products*, Linear Algebra Appl. 293 (1999) 273–288.

[102] M. Wei, W. Guo, *Reverse order laws for least-square g-inverses and minimum norm g-inverses of products of two matrices*, Linear Algebra Appl. 342 (2002), 117–132. i

[103] H.J. Werner, *G-inverse of matrix products*, Data Analysis and Statistical Inference, Eul-Verlag, Bergisch-Gladbach (1992), 531–546.

[104] H.J. Werner, *When is B^-A^- a generalized inverse of AB*, Linear Algebra Appl. 210 (1994), 255–263.

[105] L. Wu, *The Re-positive definite solution to the matrix inverse problem $AX = B$*, Linear Algebra Appl. 174 (1992), 145—151.

[106] L. Wu, B. Cain, *The Re-nonnegative definite solutions to the matrix inverse problem $AX = B$*, Linear Algebra Appl. 236 (1996), 137—146.

[107] Z. Xiong, Y. Qin, *A note on the reverse order law for reflexive generalized inverse of multiple matrix products*, Appl. Math. Comput., 219 (2013) 4255–4265.

[108] Y. Xiong, Y. Qin, *On the inverse of a special Schur complement*, Appl. Math. Comput. 218 (2012), 7679—7684.

[109] Z. Xiong, Y. Qin, *The common Re-nnd and Re-pd solutions to the matrix equations $AX = C$ and $XB = D$*, Appl. Math. Comput., 218 (2011), 3330–3337.

[110] Z. Xiong, Y. Qin, B. Zheng, *The least squares g-inverses for sum of matrices*, Linear Multilinear Algebra, 61 (4) (2013), 448–462.

[111] Z. Xiong, B. Zheng, *The reverse order law for* $\{1,2,3\}$*- and* $\{1,2,4\}$*- inverses of a two-matrix product*, Appl. Math. Letters 21 (2008), 649–655.

[112] B. Zheng, Z. Xiong, *On reverse order laws for the weighted generalized inverse*, Arab. J. Sci. Eng., 34 (2A) (2008), 195–203.

[113] B. Zheng, Z. Xiong, *The reverse order laws for* $\{1,2,3\}$*- and* $\{1,2,4\}$*-inverses of multiple matrix products*, Linear Multilinear Algebra, 58 (6) (2010), 765–782.

Biography

Jovana Nikolov Radenković was born on September 22nd, 1986, in Niš, Serbia. She completed Ćele-kula Elementary School in Niš, and Svetozar Marković Grammar School in a special class for talented mathematicians.

In the school year 2005/2006, she entered the Faculty of Sciences and Mathematics, University of Niš, at the Department of Mathematics and Informatics, and graduated in 2009 with a grade point average of 9.93/10. In 2008 she was honoured for the best student of Faculty of Science and Mathematics. In 2009/2010, she enrolled in PhD studies at the Department of Mathematics, the Faculty of Sciences and Mathematics, University of Niš, and passed all exams with a grade point average of 10/10. Since October 2012, Jovana has been working as a teaching assistant at the Faculty of Sciences and Mathematics in Niš in the Department of Mathematics. Since 2010 she participated in the project *Functional analysis, stochastic analysis and applications* supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia. She published five and has two accepted and one paper under review in international journals with IF.

Publications

1. P.S. Stanimirović, J. Nikolov, I.P. Stanimirović, *A generalization of Fibonacci and Lucas matrices*, Discrete Applied Mathematics, 156 (2008), 2606–2619. (**M23**)

2. J. Nikolov, D.S. Cvetković-Ilić, *Reverse order laws for weighted generalized inverses*, Applied Mathematics Letters, 24 (2011), 2140-2145. (**M21**)

3. J. Nikolov, D.S. Cvetković-Ilić, *Re-nnd generalized inverses*, Linear Algebra and its Applications, 439 (2013), 2999–3007. (**M22**)

4. D.S. Cvetković-Ilić, J. Nikolov, *Reverse order laws for* $\{1,2,3\}$*-generalized inverses*, Applied Mathematics and Computation, 234 (2014), 114-117. (**M21**)

5. D.S. Cvetković-Ilić, J. Nikolov, *Reverse order laws for reflexive generalized inverse of operators*, Linear and multilinear Algebra, 63 (2015), 1167-1175. (**M22**)

6. J. Nikolov Radenković, *On the inverse of a special Schur complement*, Georgian Mathematical Journal, (accepted). (**M23**)

7. J. Nikolov Radenković, *Some additive and multiplicative results for generalized inverses*, Filomat, (accepted). (**M21**)

8. J. Nikolov Radenković, *Reverse order law for multiple operator product*, (submitted).

Универзитет у Нишу

ИЗЈАВА О АУТОРСТВУ

Изјављујем да је докторска дисертација, под насловом

Псеудоинверзи и закон обрнутог редоследа за матрице и операторе

која је одбрањена на Природно-математичком факултету Универзитета у Нишу:

- резултат сопственог истраживачког рада;
- да ову дисертацију, ни у целини, нити у деловима, нисам пријављивао/ла на другим факултетима, нити универзитетима;
- да нисам повредио/ла ауторска права, нити злоупотребио/ла интелектуалну својину других лица.

Дозвољавам да се објаве моји лични подаци, који су у вези са ауторством и добијањем академског звања доктора наука, као што су име и презиме, година и место рођења и датум одбране рада, и то у каталогу Библиотеке, Дигиталном репозиторијуму Универзитета у Нишу, као и у публикацијама Универзитета у Нишу.

У Нишу, 10.06.2015.

Аутор дисертације: Јована Николов Раденковић

Потпис аутора дисертације:

Јована Николов Раденковић

Универзитет у Нишу

ИЗЈАВА О ИСТОВЕТНОСТИ ШТАМПАНОГ И ЕЛЕКТРОНСКОГ ОБЛИКА ДОКТОРСКЕ ДИСЕРТАЦИЈЕ

Име и презиме аутора: Јована Николов Раденковић

Наслов дисертације: *Псеудоинверзи и закон обрнутог редоследа за матрице и операторе*

Ментор: др Драгана Цветковић-Илић

Изјављујем да је штампани облик моје докторске дисертације истоветан електронском облику, који сам предао/ла за уношење у **Дигитални репозиторијум Универзитета у Нишу.**

У Нишу, 10.06.2015.

Потпис аутора дисертације:

Јована Николов Раденковић

Универзитет у Нишу

ИЗЈАВА О КОРИШЋЕЊУ

Овлашћујем Универзитетску библиотеку „Никола Тесла“ да, у Дигитални репозиторијум Универзитета у Нишу, унесе моју докторску дисертацију, под насловом:

Псеудоинверзи и закон обрнутог редоследа за матрице и операторе

Дисертацију са свим прилозима предао/ла сам у електронском облику, погодном за трајно архивирање.

Моју докторску дисертацију, унету у Дигитални репозиторијум Универзитета у Нишу, могу користити сви који поштују одредбе садржане у одабраном типу лиценце Креативне заједнице (Creative Commons), за коју сам се одлучио/ла.

1. Ауторство **(CC BY)**

2. Ауторство – некомерцијално **(CC BY-NC)**

3. Ауторство – некомерцијално – без прераде **(CC BY-NC-ND)**

4. Ауторство – некомерцијално – делити под истим условима **(CC BY-NC-SA)**

5. Ауторство – без прераде **(CC BY-ND)**

6. Ауторство – делити под истим условима **(CC BY-SA)**

У Нишу, 10.06.2015.

Аутор дисертације: Јована Николов Раденковић

Потпис аутора дисертације:

Јована Николов Раденковић