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Subnormal operators: A multivariable operator theory perspective

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ABSTRACT

This doctoral dissertation mainly explores the intricate domain of subnormal operators, shedding light on their diverse aspects and uncovering some insights within the domain of multivariable operator theory.

Firstly, the dissertation explores the relationship between subnormality and quasinormality of bounded operators. It investigates conditions under which an operator's subnormality and the quasinormality of its square implies quasinormality of an operator itself. Additionally, it demonstrates that the subnormal n-th roots of a quasinormal operator must also be quasinormal. The study provides sufficient conditions under which matricial and spherical quasinormality of operator pairs are equivalent to the matricial and spherical quasinormality of their n-th powers. It also addresses the converse of Fuglede Theorem, establishing when subnormal operators must be normal provided their product is normal.

The dissertation also introduces the concept of the spherical mean transform for operator pairs, extending the notion of the mean transform from one-dimensional cases to a multivariable operator setting. It explores various spectral properties of this transform, including the preservation of the Taylor spectrum, as well as some analytical properties. The study also establishes conditions under which the transform preserves p-hyponormality of two-variable weighted shifts.

Furthermore, in the context of subnormal operators and subnormal duals, the dissertation addresses the completion of upper-triangular operator matrices to normality. It introduces the concept of normal complements and provides characterizations and representation theorems for these pairs. The study delves into the joint spectral properties of normal complements, highlighting shared properties among coordinate operators in a pair of normal complements. It also establishes a connection between the theory of subnormal duals and Aluthge and Duggal transforms.

In addition, the dissertation delves into various classes of operators related to normal and subnormal operators, introducing novel concepts and addressing the solvability of specific operator equations along the way. It also examines several inequalities concerning the q-numerical radius of bounded operators and operator matrices, extending well-established equalities pertaining to the numerical radius.

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Hranislav Stanković,

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INTRODUCTION

The study of linear operators in functional analysis and operator theory has a rich history spanning over a century. These operators, often serving as mathematical models for a myriad of physical phenomena, find applications in diverse fields such as quantum mechanics, signal processing, and control theory. Among the various classes of linear operators, the class of normal operators stands out as one of the most fundamental due to its intrinsic connection with the Spectral Theorem. In this context, subnormal operators, which naturally generalize the class of normal operators, hold a significant place. They offer complex challenges and promising insights into the underlying mathematical structures.

This doctoral dissertation, titled ”Subnormal Operators: A Multivariable Operator Theory Perspective“, delves into the theory of subnormal operators from a multivariable operator theory standpoint. Subnormal operators, an extension of normal operators, have fascinated mathematicians for many years. However, their behavior, particularly when studied within the context of multivariable operator theory, remains an area of ongoing research with significant potential.

This dissertation takes a fresh approach, introducing new methods and viewpoints to unravel the complexities of subnormal operators. We aim to shed light on their intricacies, uncovering connections and insights that may have previously eluded us. The exploration is structured into seven chapters, each dedicated to specific aspects of the subject.

We commence our exploration by laying the groundwork in **Chapter 1** with a foundational overview of operator theory of subnormal operators, as well as operator theory in general. The primary objective is to equip our readers with the essential knowledge required to comprehend the results presented in this dissertation. Throughout this process, we strive to ensure that our presentation remains as self-contained as possible, minimizing the need for external references.

Chapter 2 is dedicated to exploring the interplay between subnormality and quasinormality of operators. Specifically, we investigate whether the subnormality of an operator and the quasinormality of its square implies the quasinormality of the operator itself. In Section 2.2, we extend our inquiry by establishing that the subnormal n-th roots of a quasinormal operator are also quasinormal. Additionally, we establish sufficient conditions under which the matricial and spherical quasinormality of operator pairs is equivalent to the corresponding properties observed in their respective n-th powers. In Section 2.3, our focus shifts to the conditions under which subnormal operators become quasinormal when their product is quasinormal. Furthermore, we provide sufficient conditions that dictate when quasinormal (or subnormal) operators must be normal if their product is normal. In essence, we discern the criteria for the converse of the Fuglede Theorem, establishing a connection with the multivariable operator theory concerning subnormal operators. The results covering this chapter are published in [163, 168].

Chapter 3 brings forth the concept of the spherical mean transform, which is introduced for operator pairs. This takes the mean transform of an operator and extends it to a more complex multivariable operator framework. In Section 3.2, we delve into an extensive analysis of the spectral properties associated with this transformation. One of the key aspects we explore is its ability to preserve the Taylor spectrum, one of the key concepts in the multivariable operator theory. The results ensure that in some particular cases, certain spectral characteristics remain consistent even after undergoing the spherical mean transformation. In addition to this, we unravel various analytical properties intrinsic to this transformation. Section 3.3 focuses on practical applications of the spherical mean transform. Here, we establish some sufficient conditions that guarantee the preservation of the p-hyponormality, a concept particularly relevant when dealing with 2-variable weighted shifts. Note that the obtained results in this chapter are also in [167].

Chapter 4 is dedicated to addressing the problem of completing upper-triangular operator matrices to normality. This issue holds fundamental significance within the theory of subnormal operators, particularly within the framework of subnormal duals. With the aim of answering the mentioned completion problem, we introduce the notion of normal complements. In Section 4.1, we delve into the characterizations of normal complements, unveiling their essential properties and providing representation theorems. Section 4.2 focuses on joint spectral properties of normal complements, and, as we shall see, many properties are shared among the coordinate operators in a pair of normal complements. Finally, in Section 4.3, we draw a bridge between the theory of subnormal duals and the Aluthge and Duggal transforms. These transformations have garnered substantial attention over the past few decades, making this connection a noteworthy contribution to the field. The results presented here are based on a joint paper [66].

Chapter 5 explores different classes of operators related to normal and subnormal operators. In a recent paper, A. Bachir, M. H. Mortad and N. A. Sayyaf [12] introduced generalized powers of linear operators. In other words, operators are not raised to numbers, but to other operators. They gave several properties as regards this notion. Within Section 5.1, we extend their results, delving deeper into the properties of this novel operator exponentiation. We also introduce the concept of generalized logarithms in Section 5.1.3. More precisely, for two positive and invertible operators, A and B, where $1 \notin \sigma(A)$, we define the logarithm of B with respect to base A, denoted as $\log_A B$. Our exploration encompasses a comprehensive investigation of its properties, further enriching the understanding of these mathematical constructs. In Section 5.2, we introduce a new class of operators on a complex Hilbert space $\mathcal{H}$ referred to as polynomially accretive operators. This concept extends the existing notions of accretive and n-real power positive operators. Our exploration into this new operator class reveals several intrinsic properties and generalizes established results for accretive operators. An intriguing finding emerges as we establish that every 2-normal and $(2k+1)$-real power positive operator, for a some $k \in \mathbb{N}$, must exhibit n-normality for all $n \geq 2$. Additionally, we provide sufficient conditions for the normality of T within the context of this implication. The concluding section of this chapter, Section 5.3, is devoted to the study of the solvability of a general system of operator equations: $A_i X B_i = C_i$ for $i = 1, 2$. Within this framework, we present necessary and sufficient conditions for the existence of solutions, encompassing Hermitian solutions and positive solutions. Furthermore, we derive the general forms of these solutions, paving the way for the exploration of $*$-order operator inequalities. Specifically, we scrutinize the solvability of $C \overset{*}{\leq} AXB$ and present the general form of solutions for $C \overset{*}{\leq} AX$ and $C \overset{*}{\leq} XB$. Most of the results on which this chapter is based are presented in [164, 165, 166].

Finally, in **Chapter 6**, we consider the q-numerical radius $\omega_q(\cdot)$ of operator matrices defined on a direct sum of Hilbert spaces and investigate the various inequalities involving these values. We also extend some well known equalities regarding the numerical radius that occurs when we plug in $q = 1$. Subsequently, we give explicit formulas for computing $\omega_q(\cdot)$ for some special cases of operator matrices and also establish some analytical properties of $\omega_q(\cdot)$ regarded as a function in q. In Section 6.4.2, we consider one-dimensional operators on a Hilbert space $\mathcal{H}$ and present a generalization of the well-known formula for the numerical radius of the rank-1 operator. We also prove the generalized Buzano inequality, as a corollary. The majority of the results in this chapter have already been disclosed in [75] and [170].

We finish the presentation with concluding remarks, summarizing our results and giving some final comments in **Chapter 7**.

Overall, we present a variety of new results, theorems, and illustrative examples to illuminate the subject. While this work may not claim to be exhaustive due to a vast theory of normal and subnormal operators, it aspires to provide a substantial contribution to the understanding of subnormal operators and subnormal tuples. We hope that the insights gained from this research will inspire further exploration in this intriguing area of mathematics and also lead to new discoveries.

CHAPTER 1

PRELIMINARIES

We commence by establishing the fundamental framework of operator theory. This initial step serves as a solid foundation for our forthcoming investigation into subnormal operators. Our objective here is to equip our readers with the essential knowledge required to grasp the findings presented in this dissertation. Throughout this process, we strive to make this exposition as self-contained as possible, ensuring that readers can follow along with the presented results.

1.1 OPERATORS ON HILBERT SPACES

In this section, we present classical results from operator theory, as well as the description of symbols that will be used throughout this dissertation.

We use the standard notation $\mathbb{C}$ to denote the complex plane, while $\mathbb{R}$ will denote the real axis. $\mathbb{N}$ will represent the set of all natural numbers, $\mathbb{Z}$ will denote integers, while $\mathbb{Z}_+$ will stand for $\mathbb{N} \cup \{0\}$. We shall also use $\mathbb{D}$ in order to signify the open unit complex disk around zero, i.e. $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

With $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$, ... we will denote the Hilbert spaces, which are always assumed to be complex. The inner product and norm on a Hilbert space will be denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. By a subspace $\mathcal{M}$ of $\mathcal{H}$ we always mean a linear subspace, which is not necessarily closed with respect to the topology generated by the inner product on $\mathcal{H}$. The closure of a subspace $\mathcal{M}$ we denote by $\overline{\mathcal{M}}$, while the subspace $\mathcal{M}$ is said to be dense if $\overline{\mathcal{M}} = \mathcal{H}$. The orthogonal complement of a subspace $\mathcal{M}$ will be denoted by $\mathcal{M}^\perp$, while $\mathcal{M} \oplus \mathcal{N}$ will represent the orthogonal sum of two subspaces $\mathcal{M}$ and $\mathcal{N}$.

If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, then we denote by $\mathfrak{B}(\mathcal{H}, \mathcal{K})$ the Banach space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. If $\mathcal{H} = \mathcal{K}$, then we simply write $\mathfrak{B}(\mathcal{H})$ instead of $\mathfrak{B}(\mathcal{H}, \mathcal{H})$. The norm of an operator T will be denoted by $\|T\|$. For a given operator $T \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$, we will denote by $\mathcal{R}(T)$ the range of T. $\mathcal{N}(T)$ will denote the null space (or kernel) of T. We say that operator $T \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$ is a closed-range operator (or has the closed range) if $\mathcal{R}(T)$ is a closed subspace of $\mathcal{K}$.

The adjoint of an operator $T \in \mathfrak{B}(\mathcal{H},\mathcal{K})$ will be denoted by $T^* \in \mathfrak{B}(\mathcal{K},\mathcal{H})$. With $\sigma(T)$ and $r(T)$ we denote the spectrum and the spectral radius of operator T, respectively. The numerical range of T is defined as the set

$$\mathcal{W}(T) = \{\langle Tx, x\rangle\colon\ x \in \mathcal{H},\ \|x\| = 1\},$$

while the numerical radius is taken to be

$$\omega(T) = \sup_{w\in\mathcal{W}(T)} |w|.$$

We say that operator $T \in \mathfrak{B}(\mathcal{H})$ is normal if $TT^* = T^*T$, i.e. if T commutes with T^*. The class of normal operators is very important (if not the most important!) in the operator theory due to the significance of the Spectral Theorem that holds for the operators in the mentioned class. Some of the basic examples of normal operators are unitary, Hermitian (self-adjoint), and positive operators. The mentioned terms will have the usual meaning: T is unitary if it is normal and invertible; Hermitian if $T = T^*$ and positive if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. The class of normal operators on a specific Hilbert space $\mathcal{H}$ will be denoted by $\mathcal{N}(\mathcal{H})$.

The set of positive operators represents a convex cone in $\mathfrak{B}(\mathcal{H})$, and the partial order on the set of Hermitian operators induced by this cone is called Löwner order, and will be denoted by $\leq$. Every positive operator T has a unique positive square root, i.e. there exists a unique positive operator S such that $T = S^2$. We denote S by $T^{1/2}$. Using the continuous functional calculus, we can also define an arbitrary positive power of T, i.e. for each $\alpha > 0$, the operator T^α makes sense. Since T^*T is positive for every $T \in \mathfrak{B}(\mathcal{H},\mathcal{K})$, the operator $(T^*T)^{1/2}$ is well defined and is called the modulus (or the absolute value) of T, denoted by $|T|$.

For any operator $T \in \mathfrak{B}(\mathcal{H})$, by $\mathrm{Comm}(T)$ we denote the commutant of T, i.e.

$$\mathrm{Comm}(T) = \{S \in \mathfrak{B}(\mathcal{H}) : TS = ST\}.$$

It is well known that for a positive operator $T \in \mathfrak{B}(\mathcal{H})$, we have that $\mathrm{Comm}(T) = \mathrm{Comm}(T^{1/2})$. Moreover, we also have the following:

Theorem 1.1.1. *If $n \in \mathbb{N}$, then the commutants of a positive operator and its n-th root coincide.*

Proof. The proof can be easily deduced from the Spectral Theorem (see, for example, [156, Theorem 12.23] or [178, Theorem 7.20]). ■

Operator $T \in \mathfrak{B}(\mathcal{H})$ is said to be a projection if $T^2 = T$, i.e. if T idempotent. T is an orthogonal projection if $T^2 = T = T^*$, i.e. if T is a Hermitian idempotent. The projection with the range $\mathcal{M}$ and the null space $\mathcal{N}$ will be denoted by $P_{\mathcal{M},\mathcal{N}}$, while $P_{\mathcal{M}}$ will denote the orthogonal projection with the range $\mathcal{M}$.

If $T : \mathcal{H} \to \mathcal{K}$ and $\mathcal{M} \subseteq \mathcal{H}$, the restriction of operator T to $\mathcal{M}$ will be denoted by $T\restriction_{\mathcal{M}}$. The corestriction T^{cr} of an operator T is defined as a map with domain $\mathcal{H}$, codomain $\mathcal{R}(A)$ and

$$A^{cr}x = Ax, \quad x \in \mathcal{H}.$$

A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be an invariant subspace for $T \in \mathfrak{B}(\mathcal{H})$ if $Tx \in \mathcal{M}$ for all $x \in \mathcal{M}$. A closed subspace $\mathcal{M}$ is a reducing subspace of $T \in \mathcal{B}(\mathcal{H})$ (or reduces $T \in \mathcal{B}(\mathcal{H})$) if it is invariant under both T and T^*, i.e. $T(\mathcal{M}) \subseteq \mathcal{M}$ and $T^*(\mathcal{M}) \subseteq \mathcal{M}$. The next simple, but useful observation, will be used in several proofs.

Lemma 1.1.2. *Let $A, P \in \mathfrak{B}(\mathcal{H})$ such that A is self-adjoint and P is an orthogonal projection. Then $\mathcal{R}(P)$ is invariant for A if and only if A and P commute.*

Proof. If $\mathcal{R}(P)$ is invariant for A, then, obviously, $PAP = AP$. By taking adjoints in the last equality, we have $PAP = PA$, and so $AP = PA$.

Conversely, if $AP = PA$, then $PAP = AP$, which implies that $\mathcal{R}(P)$ is invariant for A. ∎

Note that the previous lemma has a more general form in the view of the following theorem.

Theorem 1.1.3. *[90, p. 62] Let T be an operator on a Hilbert space $\mathcal{H}$ and $\mathcal{M}$ be a closed subspace of $\mathcal{H}$. Then the following conditions are mutually equivalent:*

(i) $\mathcal{M}$ reduces T;

(ii) $\mathcal{M}^{\perp}$ reduces T;

(iii) $TP_{\mathcal{M}} = P_{\mathcal{M}}T$.

Given a closed subspace $\mathcal{S}$, for any $T \in \mathfrak{B}(\mathcal{H})$, the operator matrix decomposition of T induced by $\mathcal{S}$ is given by

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \tag{1.1}$$

where $T_{11} = P_{\mathcal{S}}TP_{\mathcal{S}}\restriction_{\mathcal{S}}$, $T_{12} = P_{\mathcal{S}}T(I - P_{\mathcal{S}})\restriction_{\mathcal{S}^{\perp}}$, $T_{21} = (I - P_{\mathcal{S}})TP_{\mathcal{S}}\restriction_{\mathcal{S}}$ and $T_{22} = (I - P_{\mathcal{S}})T(I - P_{\mathcal{S}})\restriction_{\mathcal{S}^{\perp}}$. If $T_{12} = 0$ and $T_{21} = 0$, we will simply write $T = T_{11} \oplus T_{22}$.

For $T \in \mathfrak{B}(\mathcal{H})$ there exists a linear operator $T' : \mathcal{D}(T') \subseteq \mathcal{H} \mapsto \mathcal{H}$ such that $\mathcal{R}(T) \subseteq \mathcal{D}(T')$ and

$$TT'T = T.$$

Operator T' is called the inner inverse of T. In general, note that T' may not be bounded, i.e. $T' \notin \mathfrak{B}(\mathcal{H})$. Moreover, for $T \in \mathfrak{B}(\mathcal{H})$ there exists an inner inverse of T, T', such that $T' \in \mathfrak{B}(\mathcal{H})$ if and only if T has closed range [140]. In that case, operator T is called regular. Additionally, if T' also satisfies

$$T'TT' = T',$$

then T' is called a reflexive inverse of T. Furthermore, there exists a unique reflexive inverse X of operator T which satisfies the system of equations

$$XT = P_{\overline{R(T^*)}} \quad \text{and} \quad TX = P_{\overline{R(T)}} \restriction_{\mathcal{R}(T)\oplus\mathcal{R}(T)^{\perp}},$$

Such an operator is called the Moore-Penrose (generalized) inverse of T and will be denoted by $T^{\dagger}$. Equivalently, operator $T^{\dagger}$ satisfies the following system of equations:

$$TT^{\dagger}T = T, \quad T^{\dagger}TT^{\dagger} = T^{\dagger}, \quad (TT^{\dagger})^* = TT^{\dagger}, \quad (T^{\dagger}T)^* = T^{\dagger}T,$$

which are called the Penrose equations. Moore-Penrose inverse represents a major tool in solving many matrix and operator equations. The modern theory of generalized inverses can be traced back to the work of Bjerhammar in [22] and [23] when he pointed out that the Moore "reciprocal" [130] is exactly the least square solution of the equation $AXB = C$. After that Penrose in [144] and [145] extended Bjerhammar's result proving the following theorem:

Theorem 1.1.4. *[144] Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$ and $C \in \mathbb{C}^{m\times q}$. Then the matrix equation*

$$AXB = C$$

is consistent if and only if for some inner inverses A', B',

$$AA'CB'B = C,$$

in which case the general solution is

$$X = A'CB' + Y - A'AYBB',$$

for arbitrary $Y \in \mathbb{C}^{n\times p}$.

For more details, also see [61] and [68], and the references therein.

Remark 1.1.5. *The operator case when A and B are closed range operators, in its essence, is the same as a matrix case of the equation $AXB = C$, and therefore, Penrose's algebraic proof can be applied to the operator case, as well. Moreover, with slight modifications, the following result, due to Arias and Gonzalez [9], can be proved:*

Theorem 1.1.6. *[9] Let $A \in \mathfrak{B}(\mathcal{H},\mathcal{K})$, $B \in \mathfrak{B}(\mathcal{F},\mathcal{G})$ and $C \in \mathfrak{B}(\mathcal{F},\mathcal{K})$. If $\mathcal{R}(A)$, $\mathcal{R}(B)$ or $\mathcal{R}(C)$ is closed, then the following conditions are equivalent:*

(i) The equation $AXB = C$ is solvable;

(ii) $AA'CB'B = C$ *for every inner inverses,* A', B', *of* A *and* B, *respectively;*

(iii) $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ *and* $\mathcal{R}(C^*) \subseteq \mathcal{R}(B^*)$.

We also state the celebrated Douglas' Lemma, we can say freely, an irreplaceable tool when dealing with operator range inclusions.

Theorem 1.1.7 (Douglas' Lemma [79]). *Let* A *and* B *be bounded operators on Hilbert space* $\mathcal{H}$. *The following statements are equivalent:*

(i) $\mathcal{R}(A) \subseteq \mathcal{R}(B)$;

(ii) $AA^* \leq \lambda^2 BB^*$ *for some* $\lambda \geq 0$;

(iii) there exists a bounded operator C *on* $\mathcal{H}$ *such that* $A = BC$.

Moreover, if any of the previous conditions holds, then there exists a unique operator C *so that*

1. $\|C\|^2 = \inf\{\mu : AA^* \leq \mu BB^*\}$;
2. $\mathcal{N}(A) = \mathcal{N}(C)$;
3. $\mathcal{R}(C) \subseteq \overline{\mathcal{R}(B^*)}$.

An operator U on a Hilbert space $\mathcal{H}$ is said to be a partial isometry operator if there exists a closed subspace $\mathcal{M}$ such that

$$\|Ux\| = \|x\|$$

for any $x \in \mathcal{M}$, and $Ux = 0$ for any $x \in \mathcal{M}^\perp$, where $\mathcal{M}$ is said to be the initial space of U, and $\mathcal{N} = \mathcal{R}(U)$ is said to be the final space of U. The projections onto the initial space and the final space are said to be the initial projection and the final projection of U, respectively.

Theorem 1.1.8. *[90, p. 53] Let* U *be a partial isometry operator on Hilbert space* $\mathcal{H}$ *with the initial space* $\mathcal{M}$ *and final space* $\mathcal{N}$. *Then the following hold:*

(i) $UP_{\mathcal{M}} = U$ *and* $U^*U = P_{\mathcal{M}}$;

(ii) $\mathcal{N}$ *is a closed subspace of* $\mathcal{H}$;

(iii) U^* *is a partial isometry with the initial space* $\mathcal{N}$ *and the final space* $\mathcal{M}$, *that is,* $U^*P_{\mathcal{N}} = U^*$ *and* $UU^* = P_{\mathcal{N}}$.

For $T \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$, we say that $T = UP$ is the polar decomposition of operator T if P is positive, U is a partial isometry, and $\mathcal{N}(T) = \mathcal{N}(U) = \mathcal{N}(P)$. In that case, $P = |T|$.

Theorem 1.1.9. *[90, p. 59] Let $T = U|T|$ be the polar decomposition of an operator T on a Hilbert space $\mathcal{H}$. Then $T^* = U^*|T^*|$ is also the polar decomposition of an operator T^*.*

Theorem 1.1.10. *[90, p. 63] If $T = UP$ is the polar decomposition of an operator T, then U and P commute with A and A^*, where A denotes any operator which commutes with T and T^*.*

Also, for $T \in \mathfrak{B}(\mathcal{H})$, we can write

$$T = \mathrm{Re}\,(T) + i\,\mathrm{Im}\,(T),$$

where $\mathrm{Re}\,(T)$ and $\mathrm{Im}\,(T)$ are Hermitian. Such a decomposition is unique, and

$$\mathrm{Re}\,(T) = \frac{T + T^*}{2}, \quad \mathrm{Im}\,(T) = \frac{T - T^*}{2i}.$$

Operators $\mathrm{Re}\,(T)$ and $\mathrm{Im}\,(T)$ are called the real and imaginary part of T, respectively. The decomposition is called Cartesian decomposition of operator T.

Finally, we give several standard results concerning normal and positive operators.

Theorem 1.1.11. *[5] Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ and $T \in \mathfrak{B}(\mathcal{H})$ have the matrix operator decomposition induced by $\mathcal{S}$ and given by (1.1). Then, T is positive if and only if*

(i) $T_{11} \geq 0$;

(ii) $T_{21} = T_{12}^*$;

(iii) $\mathcal{R}(T_{12}) \subseteq \mathcal{R}(T_{11}^{1/2})$;

(iv) $T_{22} = \left((T_{11}^{1/2})^\dagger T_{12}\right)^* (T_{11}^{1/2})^\dagger T_{12} + F$, *where* $F \geq 0$.

Theorem 1.1.12 (Fuglede Theorem [88]). *Let T and N be bounded operators on a complex Hilbert space with N being normal. If $TN = NT$, then $TN^* = N^*T$.*

Theorem 1.1.13 (Fuglede-Putnam Theorem [149]). *Let $T \in \mathfrak{B}(\mathcal{H})$ and let M and N be two normal operators. Then*

$$TN = MT \iff TN^* = B^*T.$$

Corollary 1.1.14. *[88] If M and N are commuting normal operators, then MN is also normal.*

Proof. Let M, N in $\mathfrak{B}(\mathcal{H})$ be normal operators such that $MN = NM$. By direct computation, using Theorem 1.1.12,

$$\begin{aligned}(MN)(MN)^* &= MN(NM)^* = MNM^*N^* \\ &= MM^*NN^* = M^*MN^*N \\ &= M^*N^*MN = (NM)^*MN \\ &= (MN)^*(MN).\end{aligned}$$

Hence, MN is the normal operator. ■

For more information on Fuglede-Putnam theory, we refer a reader to [135]

Theorem 1.1.15. *[133, Corollary 5.1.36] If $A, B \in \mathfrak{B}(\mathcal{H})$ are two commuting and positive operators, then*

$$\sqrt[n]{AB} = \sqrt[n]{A}\sqrt[n]{B},$$

for all $n \in \mathbb{N}$.

Theorem 1.1.16 (Löwner-Heinz inequality [104, 124])**.** . *If $A, B \in \mathfrak{B}(\mathcal{H})$ are positive operators such that $B \leq A$ and $p \in [0, 1]$, then $B^p \leq A^p$.*

Remark 1.1.17. *In general, the previous theorem does not hold for $p > 1$ (see, for example, [134, page 55]). However, if A and B commute, and $p \in \mathbb{N}$, then $B \leq A$ implies $B^p \leq A^p$. Indeed, since A and B commute, we may write*

$$A^p - B^p = (A - B)(A^{p-1} + A^{p-2}B + \ldots + B^{p-1}).$$

Since A and B commute and $B \leq A$, we have that $A - B$ and $A^{p-1} + A^{p-2}B + \ldots + B^{p-1}$ are two commuting positive operators, and so $A^p - B^p$ is also positive. Thus, $B^p \leq A^p$, as desired.

1.2 Generalizations of normal operators

In operator theory, there are many generalizations of normal operators. One of the most important is the class of subnormal operators. Subnormal operators are bounded linear operators on a Hilbert space defined by naturally weakening the requirements for normal operators. The concept of subnormal operators was introduced by Paul R. Halmos [97] at the same time that he defined hyponormal operators, even larger class of operators. He was led to do so by a study of the properties of the unilateral shift, probably the most understood non-normal operator. In this section, we shall examine some basic properties of subnormal operators and see how the different generalization classes of normal operators are related to each other.

1.2.1 Subnormal operators

We begin with the definition of the subnormality of an operator.

Definition 1.2.1. An operator T on a Hilbert space $\mathcal{H}$ is *subnormal* if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator $N : \mathcal{K} \mapsto \mathcal{K}$ such that $N(\mathcal{H}) \subseteq \mathcal{H}$ and $Nx = Tx$ for every $x \in \mathcal{H}$.

In other words, an operator is subnormal if it has a normal extension, or equivalently, if there exists a Hilbert space $\mathcal{L}$ and a normal operator $N \in \mathcal{B}(\mathcal{H} \oplus \mathcal{L})$ such that

$$N = \begin{bmatrix} T & * \\ 0 & * \end{bmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{L} \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ \mathcal{L} \end{pmatrix}.$$

In [97], operators satisfying Definition 1.2.1 are called *completely subnormal*. The term *subnormal* as it is used here was first introduced in [98].

As mentioned earlier, the study of normal operators has been distinctly successful. The main reason may be the Spectral Theorem that holds for such operators. Also, it is natural to try to understand the structure of as many non-normal operators as possible. Since the concept of subnormality may be viewed as sufficiently close to normality, it is reasonable to expect that the theory of subnormal operators has the potential to follow a similar path. Indeed, many of the questions and conjectures regarding subnormal operators which are inspired by those concerning normal operators have been answered. For example, in [27] it was shown that every subnormal operator has a non-trivial invariant subspace. However, there are some essential differences between the two mentioned classes, which led to the theory of subnormal operators following its own path. Namely, the theory of normal operators heavily relies on measure theory and Spectral Theorem, while the theory of subnormal operators is based on analytic function theory.

In the literature, there are many characterizations of subnormal operators. See, for example, [97, 25, 83, 28].

Another elegant characterization of subnormality which signifies its closeness to the concept of normality in a topological sense is due to Bishop [21].

Theorem 1.2.1. *[21] If $T \in \mathfrak{B}(\mathcal{H})$, then the following statements are equivalent:*

(i) T is subnormal;

(ii) T is the SOT*-limit of a sequence of normal operators;*

(iii) T belongs to the SOT*-closure of the set of normal operators.*

If $T \in \mathfrak{B}(\mathcal{H})$ is subnormal operator and $N \in \mathfrak{B}(\mathcal{K})$ is normal, then obviously $S = T \oplus N$ is also subnormal. Sometimes, it is of interest to explore the ”non-normal part“ of S only. More precisely, we have the following:

Theorem 1.2.2. *[45, Proposition 2.1] If $T \in \mathfrak{B}(\mathcal{H})$, then there is a reducing subspace $\mathcal{H}_0$ for T such that*

$$T = \begin{bmatrix} T_n & 0 \\ 0 & T_p \end{bmatrix} : \begin{pmatrix} \mathcal{H}_0 \\ \mathcal{H}_0^{\perp} \end{pmatrix} \to \begin{pmatrix} \mathcal{H}_0 \\ \mathcal{H}_0^{\perp} \end{pmatrix},$$

where T_n is normal and T_p is pure operator.

Definition 1.2.2. An operator $T \in \mathfrak{B}(\mathcal{H})$ is *pure* if it has no nontrivial reducing subspace $\mathcal{M}$ such that $T|_{\mathcal{M}}$ is normal.

Note that T is pure if the subspace $\mathcal{H}_0$ in Theorem 1.2.2 is $\{0\}$. In the sequel, a reducing space $\mathcal{H}_0$ and its orthogonal complement we will denote by $\mathcal{H}_n(T)$ and $\mathcal{H}_p(T)$, respectively, while T_p and T_n we call respectively the *pure* and *normal part* of T. Decomposition $T = T_n \oplus T_p$ will be simply called the *pure-normal decomposition* of T.

The normal extension of a subnormal operator is never unique. Indeed, if N is a normal extension of T, and M is any normal operator, then $M \oplus N$ is also a normal extension of T. Thus, it makes sense to introduce the following definition.

Definition 1.2.3. If T is a subnormal operator acting on $\mathcal{H}$, and N is a normal extension of T acting on $\mathcal{K} \supseteq \mathcal{H}$, we say that N is a *minimal normal extension* of T if $\mathcal{K}$ has no proper subspace that reduces N and contains $\mathcal{H}$.

The next theorem shows that minimal normal extensions are unique. Consequently, we can legitimately speak of *the* minimal normal extension of a subnormal operator T.

Theorem 1.2.3. *[45, Corollary 2.7] If $T \in \mathfrak{B}(\mathcal{H})$ is a subnormal operator and N_1 and N_2 are minimal normal extensions of T, then N_1 and N_2 are unitarily equivalent.*

Let us emphasize that we say that operators $A \in \mathfrak{B}(\mathcal{H})$ and $B \in \mathfrak{B}(\mathcal{K})$ are unitarily equivalent if there exist a unitary transformation $U \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ ($U^*U = I_{\mathcal{H}}, UU^* = I_{\mathcal{K}}$) such that $A = U^*BU$. Another very useful result connects the concept of the minimal normal extension and the purity of an operator.

Theorem 1.2.4. *[45, Proposition 2.10] Let $T \in \mathfrak{B}(\mathcal{H})$ be subnormal and let*

$$N = \begin{bmatrix} T & A \\ 0 & B^* \end{bmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^{\perp} \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^{\perp} \end{pmatrix} \tag{1.2}$$

be a normal extension of T. The following statements are equivalent:

(i) T is pure;

(ii) N^* is the minimal normal extension of B;

(*iii*) *The smallest subspace of $\mathcal{H}$ that reduces T and contains $\mathcal{R}(A)$ is $\mathcal{H}$;*

(*iv*) *There is no nonzero projection P on $\mathcal{H}$ such that $PT = TP$ and $PA = 0$.*

Suppose that $T \in \mathcal{B}(\mathcal{H})$ is a subnormal operator and let $N \in \mathcal{B}(\mathcal{K})$ be its minimal normal extension given by (1.2). If T is a pure subnormal operator then S is unique up to unitary equivalence and is called the dual of T (see [44]). T is said to be self-dual if T is unitarily equivalent to its dual S. The dual of subnormal operator was also studied in [136] and [186]. We have the following characterizations regarding self-dual subnormal operators, which will be used in Chapter 4.

Theorem 1.2.5. *[136] Let T be a pure operator on a Hilbert space $\mathcal{H}$. Then T is a self-dual subnormal operator if and only if there exists a normal operator A on $\mathcal{H}$ such that*

$$[T^*, T] = AA^* \quad \text{and} \quad AT = T^*A.$$

Theorem 1.2.6. *[157] Let $T \in \mathfrak{B}(\mathcal{H})$ be a pure operator. Then, T is a self-dual subnormal operator if and only if there exists an operator $A \in \mathfrak{B}(\mathcal{H})$ such that the operator matrix $\begin{bmatrix} T & A \\ 0 & T^* \end{bmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$ is normal.*

1.2.2 Quasinormal operators

The class of quasinormal operators was introduced in [26].

Definition 1.2.4. An operator T on a Hilbert space $\mathcal{H}$ is *quasinormal* if it commutes with T^*T, i.e. $TT^*T = T^*T^2$.

Theorem 1.2.7. *[45, Proposition 1.6] If $T = U|T|$ is the polar decomposition of T, then T is quasinormal if and only if U and $|T|$ commute.*

Obviously, every normal operator is quasinormal, and the class of quasinormal operators is exactly the subset of $\mathfrak{B}(\mathcal{H})$ whose elements have commuting polar decompositions. Thus, the class of quasinormal operators is interesting on its own. Moreover, it also has many applications in the theory of subnormal operators as it forms a "bridge" between normality and subnormality. In other words, we have the following:

Theorem 1.2.8. *Every quasinormal operator is subnormal.*

Proof. Let $T \in \mathfrak{B}(\mathcal{H})$ be a quasinormal operator. It is straightforward to see that $T^*T - TT^*$ is positive. Let $A := (T^*T - TT^*)^{1/2}$, $B := T$, and consider the operator matrix

$$N = \begin{bmatrix} T & A \\ 0 & B^* \end{bmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{H} \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{H} \\ \mathcal{H} \end{pmatrix}.$$

It is easy to see that $N \in \mathfrak{B}(\mathcal{H} \oplus \mathcal{H})$ is a normal extension of operator T, and so T is subnormal. ∎

Theorem 1.2.9. *[136] Every pure quasinormal operator is a self-dual subnormal operator.*

The following lemma, which is due to Curto et al. [51], gives us the necessary and sufficient conditions for quasinormality (and normality) of a subnormal operator.

Lemma 1.2.10. *[51] Let $T \in \mathfrak{B}(\mathcal{H})$ be a subnormal operator with normal extension*

$$N = \begin{bmatrix} T & A \\ 0 & B^* \end{bmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{L} \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{H} \\ \mathcal{L} \end{pmatrix}.$$

Then T is quasinormal if and only if $A^*T = 0$, and normal if and only if $A = 0$.

Proof. First, we calculate

$$NN^*N = \begin{bmatrix} TT^*T + AA^*T & TT^*A + A(A^*A + BB^*) \\ B^*A^*T & B^*(A^*A + BB^*) \end{bmatrix}$$

and

$$N^*NN = \begin{bmatrix} T^*TT & T^*TA + T^*AB^* \\ A^*TT & A^*TA + (A^*A + BB^*)B^* \end{bmatrix}.$$

Since N is normal, and therefore quasinormal, we have that $NN^*N = N^*NN$. Hence, from the (1,1)-entry we get that

$$TT^*T + AA^*T = T^*TT.$$

It follows that T is quasinormal if and only if $AA^*T = 0$ which is equivalent with $\mathcal{R}(T) \subseteq \mathcal{N}(AA^*) = \mathcal{N}(A^*)$. The last statement is further equivalent with $A^*T = 0$, so we conclude that T is quasinormal if and only if $A^*T = 0$. ■

The following lemma, as we shall see, turned out to be much more useful for proving several results in this dissertation regarding quasinormal operators. The lemma first appeared in [30] (cf. [47]). We present it here in a slightly different form using a proof technique based on Lemma 1.1.2.

Lemma 1.2.11. *[45, Lemma 3.1] Let $T \in \mathfrak{B}(\mathcal{H})$ be a subnormal operator. If N is a normal extension for T, then T is quasinormal if and only if $\mathcal{H}$ is invariant for N^*N.*

Proof. Let N be a normal extension of T on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$ given by

$$N = \begin{bmatrix} T & A \\ 0 & B^* \end{bmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{pmatrix},$$

and let $P \in \mathfrak{B}(\mathcal{K})$ be the orthogonal projection onto $\mathcal{H}$. Note that $\mathcal{H}$ is invariant for N^*N if and only if $PN^*N = N^*NP$ (Lemma 1.1.2). A direct computation shows that

$$N^*NP = \begin{bmatrix} T^*T & 0 \\ A^*T & 0 \end{bmatrix} \quad \text{and} \quad PN^*N = \begin{bmatrix} T^*T & TA^* \\ 0 & 0 \end{bmatrix}.$$

Thus, $PN^*N = N^*NP$ if and only if $A^*T = 0$. The conclusion now follows from Lemma 1.2.10. ■

1.2.3 HYPONORMAL OPERATORS

As mentioned earlier, the concept of hyponormality was introduced in [97], while the term ”hyponormal“ first appeared in [19].

Definition 1.2.5. An operator T on a Hilbert space $\mathcal{H}$ is *hyponormal* if $TT^* \leq T^*T$.

The class of hyponormal operators is larger than the class of subnormal operators, as the next theorem shows.

Theorem 1.2.12. *Every subnormal operator is hyponormal.*

Proof. Let $T \in \mathfrak{B}(\mathcal{H})$ be a subnormal operator. There exists a Hilbert space $\mathcal{L}$ such that the operator matrix

$$N = \begin{bmatrix} T & A \\ 0 & B^* \end{bmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{L} \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{H} \\ \mathcal{L} \end{pmatrix}$$

is normal. From the $(1,1)$-entry of $NN^* = N^*N$, after performing matrix multiplication, we have that $TT^* + AA^* = T^*T$. It now immediately follows that $TT^* \leq T^*T$. Thus, T is hyponormal. ■

Directly from the definition, we have that an operator A on a Hilbert space $\mathcal{H}$ is hyponormal if and only if $\|Tx\| \geq \|T^*x\|$, for all $x \in \mathcal{H}$. Also, using Theorem 1.1.7, it follows that $\mathcal{R}(T) \subseteq \mathcal{R}(T^*)$ for any hyponormal operator $T \in \mathfrak{B}(\mathcal{H})$. If T^* is hyponormal, we say that T is *cohyponormal*. Operators that are either hyponormal or cohyponormal are called *seminormal*. The theory of seminormal operators is an extensive and highly developed area. More information about the subject can be found in [128], [43] and [182].

Many properties which hold for normal operators hold in analogous form for hyponormal operators, as the following results show.

Theorem 1.2.13. *[45, Proposition 4.4] Let $T \in \mathfrak{B}(\mathcal{H})$ be a hyponormal operator.*

(a) *If T is invertible, then T^{-1} is hyponormal.*

(b) *If $\lambda \in \mathbb{C}$, then $T - \lambda$ is hyponormal.*

(c) *If $\lambda \in \sigma_p(T)$ and $x \in \mathcal{H}$ such that $Tx = \lambda x$, then $T^*x = \overline{\lambda}x$.*

(d) *If x and g are eigenvectors corresponding to distinct eigenvalues of T, then $x \perp y$.*

Theorem 1.2.14. *[160] If T is hyponormal, then $\|T^n\| = \|T\|^n$, and consequently, $\|T\| = r(T)$.*

It is important to note that the theory of hyponormal operators (and hence subnormal operators) is strictly an infinite dimensional theory. More precisely, the class of hyponormal operators coincides with the class of normal operators, if the underlying Hilbert space is finite dimensional. Indeed, if $\mathcal{H}$ is finite dimensional and $T \in \mathfrak{B}(\mathcal{H})$ is hyponormal, then $T^*T - TT^* \geq 0$, while the trace of $T^*T - TT^*$ is 0. Thus, $T^*T = TT^*$, i.e. T is normal. The following theorem due to Putnam [152] shows that we actually have a much stronger result.

Theorem 1.2.15. *[152] If $T \in \mathfrak{B}(\mathcal{H})$ is hyponormal, then*

$$\|[T^*, T]\| \leq \frac{1}{\pi} \text{Area}(\sigma(T)).$$

1.2.4 p-HYPONORMAL OPERATORS AND ALUTHGE TRANSFORM

In this section, we briefly mention other generalizations of normal operators and the relations between them.

Definition 1.2.6. An operator T on a Hilbert space $\mathcal{H}$ is said to be *p-hyponormal* if $(TT^*)^p \leq (T^*T)^p$ for some $p \in (0, 1]$.

A p-hyponormal operator T is said to be *semi-hyponormal* if $p = \frac{1}{2}$, and clearly, T is hyponormal if $p = 1$. Using Theorem 1.1.16, we note that every hyponormal operator must be p-hyponormal for all $p \in (0, 1]$. More generally, if $0 < q \leq p \leq 1$ and $T \in \mathfrak{B}(\mathcal{H})$ is p-hyponormal, then it is also q-hyponormal. Thus, the class of p-hyponormal has been defined as an extension of hyponormal operators in [182], and it has been studied by many authors since then. See, for example, [1, 2, 181].

Combining the previous observations, we have the following chain of inclusions:

$$\text{normal} \Rightarrow \text{quasinormal} \Rightarrow \text{subnormal} \Rightarrow \text{hyponormal} \Rightarrow p\text{-hyponormal}.$$

In a close relationship to p-hyponormal operators are *Aluthge transform* and *Duggal tranform*. The Aluthge transform $\widetilde{T}$ of an operator $T \in \mathfrak{B}(\mathcal{H})$ with the polar decomposition $T = U|T|$ is defined as $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$, while the Duggal transform $\widehat{T}$ of T is given by $\widehat{T} = |T|U$. For more details on the Aluthge and Duggal transform, see, for instance, [1, 7, 39, 86, 110].

Aluthge transform $\widetilde{T}$ of an operator $T \in \mathfrak{B}(\mathcal{H})$ turned out to be quite an interesting and useful idea in the study of linear operators. For example, we have that $\sigma(\widetilde{T}) = \sigma(T)$. This follows from the fact that $\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}$ for any $A, B \in \mathfrak{B}(\mathcal{H})$. But maybe an even more remarkable and surprising fact is the following:

Theorem 1.2.16. *[1] Let $T = U|T|$ be p-hyponormal for some $0 < p \leq 1$ and U be unitary. Then*

(i) $\widetilde{T}$ *is* $(p+\frac{1}{2})$*-hyponormal if* $0 < p < \frac{1}{2}$.

(ii) $\widetilde{T}$ *is hyponormal if* $\frac{1}{2} \leq p < 1$.

Thus, the Aluthge transformation "sends" a p-hyponormal operator to a smaller class than the p-hyponormal class containing the operator originally. This is one of the main reasons for the applicability of the Aluthge transform to the various areas of operator theory.

1.3 Subnormal and quasinormal tuples

Let $n \in \mathbb{N}$. If $T_i \in \mathfrak{B}(\mathcal{H})$, $i = \overline{1,n}$, then $\mathbf{T} = (T_1, \ldots, T_n) \in \mathfrak{B}(\mathcal{H})^n$ will denote an n-tuple of operators acting on $\mathcal{H}$. By $\mathbf{T}^*$ we mean an operator n-tuple $\mathbf{T}^* = (T_1^*, \ldots, T_n^*) \in \mathfrak{B}(\mathcal{H})^n$. An operator n-tuple $\mathbf{T} = (T_1, \ldots, T_n) \in \mathfrak{B}(\mathcal{H})^n$ is said to be a *commuting* if $T_iT_j = T_jT_i$, for all $i, j \in \{1, \ldots, n\}$.

Many concepts and ideas from a single variable operator theory have been transferred to a multivariable operator setting. For example, a classical operator norm has a multivariable analogue in the forms of *joint operator norm* and *euclidean operator norm*. The joint operator norm of an n-tuple $\mathbf{T} = (T_1, \ldots, T_n)$ was introduced in [42] as

$$\|\mathbf{T}\| = \sup\left\{ \left(\sum_{k=1}^{n} \|T_k x\|^2 \right)^{1/2} : x \in \mathcal{H}, \|x\| = 1 \right\},$$

while the euclidean operator norm first appears in [143] and it is given by

$$\|\mathbf{T}\|^e = \left(\sum_{k=1}^{n} \|T_k\|^2 \right)^{1/2}.$$

The concepts of numerical range and numerical radius followed a similar path. Namely, the *joint numerical range* of an n-tuple $\mathbf{T} = (T_1, \ldots, T_n)$ is defined as

$$\mathcal{W}(\mathbf{T}) = \{(\langle T_1x, x\rangle, \ldots, \langle T_nx, x\rangle) : x \in \mathcal{H}, \|x\| = 1\},$$

and the *joint numerical radius* (also called *euclidean operator radius*) of $\mathbf{T}$ is given by

$$\omega(\mathbf{T}) = \sup\left\{ \left(\sum_{k=1}^{n} |\langle T_k x, x\rangle|^2 \right)^{1/2} : x \in \mathcal{H}, \|x\| = 1 \right\}.$$

The notion of the joint numerical range was first investigated by Halmos [99, Problem 166], while $\omega(\mathbf{T})$ was studied in [143]. For more information on these concepts, we refer a reader to [42, 70, 80, 112, 138, 139, 147, 162].

Unlike the single variable operator theory, the spectrum of an operator tuple has many definitions. See, for instance, [8], [71] and [171]. In this dissertation, we shall restrict ourselves to the Taylor invertibility for a pair of operators only. It is defined in the following way: let $\mathbf{T} = (T_1, T_2)$ be a commuting pair. Consider a Koszul complex $\mathcal{K}(\mathbf{T}, \mathcal{H})$ associated to $\mathbf{T}$ on $\mathcal{H}$:

$$\mathcal{K}(\mathbf{T}, \mathcal{H}) : \quad 0 \longrightarrow \mathcal{H} \xrightarrow{\mathbf{T}} \mathcal{H} \oplus \mathcal{H} \xrightarrow{(-T_2\ T_1)} \mathcal{H} \longrightarrow 0,$$

where $\mathbf{T} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$. Then, $\mathbf{T}$ is said to be *Taylor invertible* if its associated Koszul complex $\mathcal{K}(\mathbf{T}, \mathcal{H})$ is exact. We define the *Taylor spectrum* $\sigma_T(\mathbf{T})$ of $\mathbf{T}$ as follows:

$$\sigma_T(\mathbf{T}) = \left\{ (\lambda_1, \lambda_2) \in \mathbb{C}^2 : \mathcal{K}((T_1 - \lambda_1, T_2 - \lambda_2), \mathcal{H}) \text{ is not exact} \right\}.$$

For more information on the Taylor invertibility and Koszul complexes, we refer a reader to [102, 103, 115, 137, 171, 172].

For $S, T \in \mathfrak{B}(\mathcal{H})$ let $[S, T] = ST - TS$. Operator $[S, T]$ is called the *commutator* of operators S and T. If $S = T^*$, then $[T^*, T]$ is called the *self-commutator* of operator T. Analogously, if $\mathbf{T} = (T_1, \dots, T_n) \in \mathfrak{B}(\mathcal{H})^n$ is an n-tuple of operators, we denote by $[\mathbf{T}^*, \mathbf{T}]$ the self-commutator of $\mathbf{T}$ which is defined by

$$[\mathbf{T}^*, \mathbf{T}]_{i,j} = [T_j^*, T_i] = T_j^* T_i - T_i T_j^*,$$

for all $i, j \in \{1, \dots, n\}$. We say that an n-tuple $\mathbf{T} = (T_1, \dots, T_n)$ of operators on $\mathcal{H}$ is (jointly) *hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := \begin{bmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{bmatrix}$$

is positive on the direct sum of n copies of $\mathcal{H}$ (cf. [10, 48, 52]). The n-tuple $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal, and *subnormal* if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal n-tuple $\mathbf{N} = (N_1, \dots, N_n) \in \mathfrak{B}(\mathcal{K})^n$ such that $N_i(\mathcal{H}) \subseteq \mathcal{H}$ and $N_i x = T_i x$ for every $x \in \mathcal{H}$ and every $i \in \{1, \dots, n\}$. For $i, j, k \in \{1, 2, \dots, n\}$, $\mathbf{T}$ is called *matricially quasinormal* if each T_i commutes with each $T_j^* T_k$, $\mathbf{T}$ is (jointly) *quasinormal* if each T_i commutes with each $T_j^* T_j$, and *spherically quasinormal* if each T_i commutes with $\sum_{j=1}^n T_j^* T_j$. As shown in [11] and [92], we have

normal $\Rightarrow$ matricially quasinormal $\Rightarrow$ (jointly) quasinormal
$\Rightarrow$ spherically quasinormal $\Rightarrow$ subnormal $\Rightarrow$ (jointly) hyponormal

On the other hand, the results in [55] and [92] show that the inverse implications do not hold.

For $T_1, T_2 \in \mathfrak{B}(\mathcal{H})$, consider the pair $\mathbf{T} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$ as an operator from $\mathcal{H}$ into $\mathcal{H} \oplus \mathcal{H}$, that is,

$$\mathbf{T} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} : \mathcal{H} \to \begin{matrix} \mathcal{H} \\ \oplus \\ \mathcal{H} \end{matrix} .$$

We define *(canonical) spherical polar decomposition* of $\mathbf{T}$ (cf. [54], [55], [114]) as

$$\mathbf{T} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} P = \begin{pmatrix} V_1 P \\ V_2 P \end{pmatrix} = \mathbf{V}P,$$

where $P = (\mathbf{T}^*\mathbf{T})^{1/2} = \sqrt{T_1^* T_1 + T_2^* T_2}$ is a positive operator on $\mathcal{H}$, and

$$\mathbf{V} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} : \mathcal{H} \to \begin{matrix} \mathcal{H} \\ \oplus \\ \mathcal{H} \end{matrix} ,$$

is a spherical partial isometry from $\mathcal{H}$ into $\mathcal{H} \oplus \mathcal{H}$. Then, $\mathbf{V}^*\mathbf{V} = V_1^* V_1 + V_2^* V_2$ is the (orthogonal) projection onto the initial space of the partial isometry $\mathbf{V}$ which is

$$\mathcal{N}(\mathbf{T})^\perp = (\mathcal{N}(T_1) \cap \mathcal{N}(T_2))^\perp = \mathcal{N}(P)^\perp = (\mathcal{N}(V_1) \cap \mathcal{N}(V_2))^\perp.$$

With respect to the polar decomposition, spherically quasinormal pairs can be characterized as follows:

Theorem 1.3.1. *[56] Let $\mathbf{T} = (V_1 P, V_2 P)$ be the polar decomposition of $\mathbf{T}$. Then $\mathbf{T}$ is spherically quasinormal if and only if $V_i P = P V_i$, $i = 1, 2$.*

Finally, recall the class of 2-variable weighted shifts. Consider double-indexed non-negative bounded sequences $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in l^\infty(\mathbb{Z}_+^2)$, where $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}_+^2$ and let $l^2(\mathbb{Z}_+^2)$ be the Hilbert space of square-summable complex sequences indexed by $\mathbb{Z}_+^2$. We define the *2-variable weighted shift* $\mathbf{W}_{(\alpha,\beta)} = (T_1, T_2)$ by

$$T_1 e_{(k_1,k_2)} = \alpha_{(k_1,k_2)} e_{(k_1+1,k_2)}$$

and

$$T_2 e_{(k_1,k_2)} = \beta_{(k_1,k_2)} e_{(k_1,k_2+1)},$$

where $\{e_{(k,l)}\}_{k,l=0}^\infty$ is the canonical orthonormal basis in $l^2(\mathbb{Z}_+^2)$. For all $(k_1, k_2) \in \mathbb{Z}_+^2$, it is easy to see that

$$T_1 T_2 = T_2 T_1 \iff \beta_{(k_1+1,k_2)} \alpha_{(k_1,k_2)} = \alpha_{(k_1,k_2+1)} \beta_{(k_1,k_2)}.$$

For the basic properties of a 2-variable weighted shift $\mathbf{W}_{(\alpha,\beta)}$, we refer to [49] and [53].

CHAPTER 2

SUBNORMAL FACTORS OF QUASINORMAL OPERATORS

In this chapter, we address the question of whether the subnormality of an operator and the quasinormality of its square are sufficient for the quasinormality of the operator itself. Moreover, in Section 2.2, it will be shown that the subnormal n-th roots of a quasinormal operator must be quasinormal, as well. Additionally, some sufficient conditions are provided under which the matricial and spherical quasinormality of operator pairs are equivalent to the matricial and spherical quasinormality of their n-th powers. In Section 2.3, we deal with the problem of finding conditions under which subnormal operators must be quasinormal provided their product is quasinormal. Furthermore, sufficient conditions are given under which quasinormal (subnormal) operators must be normal provided their product is normal. In other words, sufficient conditions for the converse of the Fuglede Theorem have been found, making a connection with the multivariable operator theory of subnormal operators along the way.

2.1 SQUARE ROOT PROBLEM FOR QUASINORMAL OPERATORS

In a recent paper [51], R. E. Curto, S. H. Lee, and J. Yoon, partially motivated by the results of their previous articles [49] and [50], asked the following question:

Problem 2.1.1. *Let T be a subnormal operator, and assume that T^2 is quasinormal. Does it follow that T is quasinormal?*

With the additional assumption of left invertibility, they showed that a left invertible subnormal operator T whose square T^2 is quasinormal must be quasinormal (see Theorem 2.1.5 below). It remained an open question whether this is true in general without any assumption about the left invertibility until the paper [146] was published. Moreover, the authors proved an even stronger result:

Theorem 2.1.2. *[146] Let $T \in \mathfrak{B}(\mathcal{H})$ be a subnormal operator such that T^n is quasinormal for some $n \in \mathbb{N}$. Then T is quasinormal.*

The proof is based on the theory of operator monotone functions and Hansen's inequality. More precisely, Theorem 1.1.1, Theorem 1.1.16, and the following theorems were crucial for the proof.

Theorem 2.1.3. *[83] Let T be a bounded operator on $\mathcal{H}$. Then the following conditions are equivalent:*

(i) *T is quasinormal;*

(ii) *$(T^*)^n T^n = (T^*T)^n$, $n \in \mathbb{N}$;*

(iii) *there exists a (unique) spectral Borel measure E on $\mathbb{R}_+$ such that*

$$(T^*)^n T^n = \int_{\mathbb{R}_+} x^n \, E(dx), \quad n \in \mathbb{Z}_+.$$

Theorem 2.1.4 (Hansen inequality [101, 174])**.** *Let $A \in \mathfrak{B}(\mathcal{H})$ be a positive operator, $T \in \mathfrak{B}(\mathcal{H})$ be a contraction and $f : [0, \infty) \to \mathbb{R}$ be a continuous operator monotone function such that $f(0) \geq 0$. Then*

$$T^* f(A) T \leq f(T^* A T).$$

Moreover, if f is not an affine function and T is an orthogonal projection such that $T \neq I_{\mathcal{H}}$, then the equality holds if and only if $TA = AT$ and $f(0) = 0$.

In Section 2.2.1, using an elementary technique, we provide a much simpler proof of Theorem 2.1.2.

In the literature, similar properties to Problem 2.1.1 for other classes of operators are known. Namely, hyponormal n-th roots of normal operators are normal (see [160, Theorem 5]). The author used a technique based on the Spectral Theorem. Another more elementary proof can be found in [3]. Namely, the authors showed that for any p-hyponormal operator $T \in \mathfrak{B}(\mathcal{H})$ and any $n \in \mathbb{N}$,

$$((T^n)^* T^n)^{p/n} \geq (T^*T)^p \geq (TT^*)^p \geq (T^n (T^n)^*)^{p/n}.$$

If in addition, T^n is normal for some $n \in \mathbb{N}$, then we actually have that

$$((T^n)^* T^n)^{p/n} = (T^*T)^p = (TT^*)^p = (T^n (T^n)^*)^{p/n},$$

and so T must be normal. Another extension of [160, Theorem 5] can be found in [6]. However, if we replace the normality of an operator with some weaker assumption, the analogous conclusions may not hold. In turn, if T is hyponormal operator and T^n is subnormal then T doesn't have to be subnormal (see [161]).

Motivated by these type of problems, in this section we also consider problems of when matricial (spherical) quasinormality of $\mathbf{T}^{(n,n)} := (T_1^n, T_2^n)$ implies matricial (spherical) quasinormality of $\mathbf{T} = (T_1, T_2)$.

Let us briefly return to Problem 2.1.1. In order to answer the question asked in it, the authors in [51] first proved Lemma 1.2.10.

Although it may seem like a practical tool for determining whether some operator is quasinormal or not, this approach fails to give an answer to Problem 2.1.1 without imposing additional assumptions on T and becomes even more impractical if we replace the square of an operator with its arbitrary power.

Nevertheless, using the mentioned lemma, the mentioned authors proved the following result:

Theorem 2.1.5. *[51] Let $T \in \mathfrak{B}(\mathcal{H})$ be a subnormal operator and assume that T^2 is quasinormal. If T is bounded below, then T is quasinormal.*

Theorem 2.1.5 and Lemma 1.2.10 provided a foundation for proving multivariable analogues of these results. Namely, for a subnormal pair $\mathbf{T} = (T_1, T_2)$ with the normal extension $\mathbf{N} = (N_1, N_2)$, where

$$N_i = \begin{bmatrix} T_i & A_i \\ 0 & B_i^* \end{bmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{pmatrix},$$

the authors proved the following results:

Corollary 2.1.6. *[51] Let $\mathbf{T}$ be a subnormal and assume that T_i is bounded below and T_i^2 is quasinormal, $i = 1, 2$. Then $\mathbf{T}$ is spherically quasinormal.*

Theorem 2.1.7. *[51] Let $\mathbf{T}$ be subnormal, with normal extension $\mathbf{N}$. Then $\mathbf{T}$ is spherically quasinormal if and only if $A_1^* T_1 + A_2^* T_2 = 0$.*

Theorem 2.1.8. *[51] Let $\mathbf{T}$ be subnormal, with normal extension $\mathbf{N}$. Then $\mathbf{T}$ is (jointly) quasinormal if and only if $A_i^* T_j = 0, i, j = 1, 2$.*

Corollary 2.1.9. *[51] Let $\mathbf{T}$ be a subnormal pair with normal extension $\mathbf{N}$. Then $\mathbf{T}$ is matricially quasinormal if and only if $A_i A_j^* T_k = 0, i, j, k = 1, 2$.*

Here we use the opportunity to state that Theorem 2.1.8 is actually false. Namely, if $A_j^* T_k = 0$, $j, k = 1, 2$, then obviously, $A_i A_j^* T_k = 0, i, j, k = 1, 2$. If Theorem 2.1.8 is true, then this implies that every (jointly) quasinormal n-tuple must be matricially quasinormal. But as mentioned earlier, the results in [55] and [92] show that this is not the case. The correction and other ”unexpected“ implications of this mistake will be presented later in Section 2.2.2.

2.2 SUBNORMAL n-TH ROOTS OF QUASINORMAL OPERATORS

In this section, we give an answer to Problem 2.1.1 using an elementary technique. Among other things, we also show that we can relax a condition in the definition of matricially quasinormal n-tuples and we give a correction for Theorem 2.1.8.

2.2.1 ONE-DIMENSIONAL CASE

Proof of Theorem 2.1.2. Let $N \in \mathfrak{B}(\mathcal{K})$ be a normal extension for T, where $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$ and let $P \in \mathfrak{B}(\mathcal{K})$ be the orthogonal projection onto $\mathcal{H}$. Then, N^n is a normal extension for T^n and since $\mathcal{H}$ is invariant for $(N^n)^* N^n = (N^*N)^n$ (Lemma 1.2.11), it follows that P commutes with $(N^*N)^n$ (Lemma 1.1.2). Hence, P also commutes with N^*N, by Theorem 1.1.1. Therefore, $\mathcal{H}$ is invariant for N^*N and by applying Lemma 1.2.11 again, we conclude that T is quasinormal. ∎

The following corollary is a generalization of [51, Corollary 2.4].

Corollary 2.2.1. *Let $T \in \mathfrak{B}(\mathcal{H})$ be a subnormal operator such that T^n is pure quasinormal for some $n \in \mathbb{N}$. Then T is pure quasinormal.*

Proof. Quasinormality follows from Theorem 2.1.2. If T is not pure, then there is a non-zero reducing subspace $\mathcal{M}$ of $\mathcal{H}$ such that $P^{cr}_{\mathcal{M}} T \upharpoonright_{\mathcal{M}}$ is normal. Since $P^{cr}_{\mathcal{M}} T^n \upharpoonright_{\mathcal{M}} = (P^{cr}_{\mathcal{M}} T \upharpoonright_{\mathcal{M}})^n$ is also normal, T^n is not pure, which is a contradiction. Therefore, T must be pure. ∎

2.2.2 MULTIVARIABLE CASE

Now we can shift the focus to the multivariable case. Although we present our results for commuting pairs of operators, the reader will easily see that the same (or analogous) statements work well for commuting n-tuples of operators, when $n > 2$.

Theorem 2.1.2 allows us to remove the left invertibility assumption from Corollary 2.1.6. Moreover, we can prove an even stronger result:

Corollary 2.2.2. *Let $\mathbf{T} = (T_1, T_2)$ be a subnormal pair and assume that T_1^k and T_2^l are quasinormal for some $k, l \in \mathbb{N}$. Then $\mathbf{T}$ is spherically quasinormal.*

Proof. Since $T_i, i = 1, 2$ are subnormal and T_1^k and T_2^l are quasinormal, Theorem 2.1.2 implies that $T_i, i = 1, 2$ are quasinormal. Therefore, $\mathbf{T}$ is spherically quasinormal (see [51, Remark 2.6]). ∎

The following lemma can be considered as a multivariable analogue of Lemma 1.2.11 (see Remark 2.2.4 below).

Lemma 2.2.3. *Let* $\mathbf{T} = (T_1, T_2)$ *be a subnormal pair, with a normal extension* $\mathbf{N} = (N_1, N_2)$. *Then* $\mathbf{T}$ *is spherically quasinormal if and only if* $\mathcal{H}$ *is invariant for* $N_1^*N_1 + N_2^*N_2$.

Proof. Let $N_i, i = 1, 2,$ be the normal extensions of T_i on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$ given by

$$N_i = \begin{bmatrix} T_i & A_i \\ 0 & B_i^* \end{bmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{pmatrix},$$

and let $P \in \mathfrak{B}(\mathcal{K})$ be the orthogonal projection onto $\mathcal{H}$. Note that $\mathcal{H}$ is invariant for $N_1^*N_1 + N_2^*N_2$ if and only if $P(N_1^*N_1 + N_2^*N_2) = (N_1^*N_1 + N_2^*N_2)P$ (Lemma 1.1.2). By direct computation,

$$(N_1^*N_1 + N_2^*N_2)P = \begin{bmatrix} T_1^*T_1 + T_2^*T_2 & 0 \\ A_1^*T_1 + A_2^*T_2 & 0 \end{bmatrix}$$

and

$$P(N_1^*N_1 + N_2^*N_2) = \begin{bmatrix} T_1^*T_1 + T_2^*T_2 & T_1^*A_1 + T_2^*A_2 \\ 0 & 0 \end{bmatrix}.$$

Therefore, $P(N_1^*N_1 + N_2^*N_2) = (N_1^*N_1 + N_2^*N_2)P$ if and only if $A_1^*T_1 + A_2^*T_2 = 0$. Now it only remains to apply Theorem 2.1.7. ■

Remark 2.2.4. *If we treat* $\mathbf{N} = (N_1, N_2)$ *as a column vector, we may use the notation* $\mathbf{N}^*\mathbf{N} = N_1^*N_1 + N_2^*N_2$, *which gives us the following analogue of Lemma 1.2.11:*

Lemma 2.2.5. *Let* $\mathbf{T}$ *be a subnormal, with a normal extension* $\mathbf{N}$. *Then* $\mathbf{T}$ *is spherically quasinormal if and only if* $\mathcal{H}$ *is invariant for* $\mathbf{N}^*\mathbf{N}$.

As shown in [51, Example 3.6], there exists a spherically quasinormal 2-variable weighted shift $W_{(\alpha,\beta)}$ such that $W_{(\alpha,\beta)}^{(2,1)}$ is not spherically quasinormal. In other words, if $\mathbf{T} = (T_1, T_2)$ is a spherically quasinormal pair, then $\mathbf{T}^{(m,n)} = (T_1^m, T_2^n)$ may not be spherically quasinormal.

The following theorem gives a sufficient condition for the equivalence of spherical quasinormality of $\mathbf{T}^{(n,n)} = (T_1^n, T_2^n)$ and spherical quasinormality of $\mathbf{T} = (T_1, T_2)$.

Theorem 2.2.6. *Let* $\mathbf{T} = (T_1, T_2)$ *be a subnormal pair with the normal extension* $\mathbf{N} = (N_1, N_2)$ *such that* $N_1N_2 = 0$. *Then* $\mathbf{T}^{(n,n)} = (T_1^n, T_2^n)$ *is spherically quasinormal for some* $n \in \mathbb{N}$ *if and only if* $\mathbf{T}$ *is spherically quasinormal.*

Proof. Let $\mathbf{N} = (N_1, N_2) \in \mathfrak{B}(\mathcal{K})^2$ be a normal extension for $\mathbf{T}$, where $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$ and let $P \in \mathfrak{B}(\mathcal{K})$ be the orthogonal projection onto $\mathcal{H}$. Then, $\mathbf{N}^{(n,n)} = (N_1^n, N_2^n)$ is a normal extension for $\mathbf{T}^{(n,n)} = (T_1^n, T_2^n)$, and using the fact that $N_1N_2 = 0$ and Theorem 1.1.12, we have that

$$(N_1^*N_1 + N_2^*N_2)^n = (N_1^*N_1)^n + (N_2^*N_2)^n.$$

($\Rightarrow$:) Assume that $\mathbf{T}^{(n,n)}$ is spherically quasinormal. Then $\mathcal{H}$ is invariant for $(N_1^n)^*N_1^n + (N_2^n)^*N_2^n = (N_1^*N_1)^n + (N_2^*N_2)^n = (N_1^*N_1 + N_2^*N_2)^n$ (Lemma 2.2.3), and so P commutes with $(N_1^*N_1 + N_2^*N_2)^n$ (Lemma 1.1.2). It now follows that P also commutes with $N_1^*N_1 + N_2^*N_2$, by Theorem 1.1.1. Hence, $\mathcal{H}$ is invariant for $N_1^*N_1 + N_2^*N_2$. Lemma 2.2.3 now implies that $\mathbf{T}$ is spherically quasinormal.

($\Leftarrow$) : The converse can be proved in a similar manner. ∎

We now give another characterization of matricially quasinormal n-tuples and correct the mistake in [51]:

Lemma 2.2.7. *Let* $\mathbf{T} = (T_1, T_2)$ *be a subnormal pair, with a normal extension* $\mathbf{N} = (N_1, N_2)$*. Then* $\mathbf{T}$ *is matricially quasinormal if and only if* $A_i^*T_j = 0$, $i, j = 1, 2$.

Proof. Let $\mathbf{T} = (T_1, T_2)$ be a subnormal, with a normal extension $\mathbf{N} = (N_1, N_2)$. As shown in the proof of Theorem 2.1.8 (see [51, Theorem 2.9]),

$$T_iT_j^*T_k + A_iA_j^*T_k = T_j^*T_kT_i,$$

i.e. $[T_i, T_j^*T_k] = -A_iA_j^*T_k$.

If $\mathbf{T}$ is matricially quasinormal, then $A_iA_j^*T_k = 0$ for all $i, j, k = 1, 2$, and thus for $i = j$, we have $A_jA_j^*T_k = 0$. Since $\mathcal{N}(A_jA_j^*) = \mathcal{N}(A_j^*)$ it follows that $A_j^*T_k = 0$.

Now, assume that $A_j^*T_k = 0$ for all $j, k = 1, 2$. Then for all $i = 1, 2$, we have that $A_iA_j^*T_k = 0$, which means that $[T_i, T_j^*T_k] = 0$, $i, j, k = 1, 2$. By definition, $\mathbf{T}$ is matricially quasinormal. ∎

As a consequence of the previous result, we observe that we can relax a condition in the definition of matricial quasinormality:

Corollary 2.2.8. $\mathbf{T} = (T_1, T_2)$ *is matricially quasinormal if and only if* T_i *commutes with* $T_i^*T_j$, $i, j = 1, 2$.

Here is the correction of Theorem 2.1.8:

Corollary 2.2.9. *Let* $\mathbf{T} = (T_1, T_2)$ *be a subnormal pair, with a normal extension* $\mathbf{N} = (N_1, N_2)$*. Then* $\mathbf{T}$ *is (jointly) quasinormal if and only if* $A_iA_j^*T_j = 0$, $i, j = 1, 2$.

Proof. It follows from the proof of Lemma 2.2.7, by taking $j = k$. ∎

Inspired by Lemma 1.2.11, we give another analogous result in the multivariable case.

Lemma 2.2.10. *Let* $\mathbf{T} = (T_1, T_2)$ *be a subnormal pair, with a normal extension* $\mathbf{N} = (N_1, N_2)$*. Then* $\mathbf{T}$ *is matricially quasinormal if and only if* $\mathcal{H}$ *is invariant for* $N_i^*N_j$, $i, j = 1, 2$.

Proof. Let $N_i, i = 1,2$ be the normal extensions of T_i on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$ given by

$$N_i = \begin{bmatrix} T_i & A_i \\ 0 & B_i^* \end{bmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{pmatrix},$$

and let $P \in \mathfrak{B}(\mathcal{K})$ be the orthogonal projection onto $\mathcal{H}$. Note that $\mathcal{H}$ is invariant for $N_i^* N_j$ if and only if $PN_i^*N_jP = N_i^*N_jP$. Since

$$N_i^*N_jP = \begin{bmatrix} T_i^*T_j & 0 \\ A_i^*T_j & 0 \end{bmatrix} \quad \text{and} \quad PN_i^*N_jP = \begin{bmatrix} T_i^*T_j & 0 \\ 0 & 0 \end{bmatrix}.$$

if follows that $PN_i^*N_jP = N_i^*N_jP$ if and only if $A_i^*T_j = 0, i,j = 1,2$. By Lemma 2.2.7, this is further equivalent with matricial quasinormality of $\mathbf{T}$. ■

The following theorem gives sufficient conditions for the equivalence of matricial quasinormality of $\mathbf{T}^{(n,n)} = (T_1^n, T_2^n)$ and matricial quasinormality of $\mathbf{T} = (T_1, T_2)$:

Theorem 2.2.11. *Let $\mathbf{T} = (T_1, T_2)$ be a subnormal pair with the normal extension $\mathbf{N} = (N_1, N_2)$, such that $N_1^*N_2 \geq 0$. Then $\mathbf{T}^{(n,n)} = (T_1^n, T_2^n)$ is matricially quasinormal for some $n \in \mathbb{N}$ if and only if $\mathbf{T}$ is matricially quasinormal.*

Proof. Let $\mathbf{N} = (N_1, N_2) \in \mathfrak{B}(\mathcal{K})^2$ be a normal extension for $\mathbf{T}$, where $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$ and let $P \in \mathfrak{B}(\mathcal{K})$ be the orthogonal projection onto $\mathcal{H}$. Then, $\mathbf{N}^{(n,n)} = (N_1^n, N_2^n)$ is a normal extension for $\mathbf{T}^{(n,n)} = (T_1^n, T_2^n)$ and using the fact that N_1 and N_2 commute and Theorem 1.1.12, we have that

$$(N_i^*N_j)^n = (N_i^n)^*N_j^n.$$

Also, $N_1^*N_2 \geq 0$ implies $N_2^*N_1 = (N_1^*N_2)^* = N_1^*N_2 \geq 0$.

($\Rightarrow$:) Assume that $\mathbf{T}^{(n,n)}$ is matricially quasinormal and let $(i,j) \in \{1,2\} \times \{1,2\}$ be arbitrary. Then $\mathcal{H}$ is invariant for $(N_i^n)^*N_j^n = (N_i^*N_j)^n$ (Lemma 2.2.10), and so P commutes with $(N_i^*N_j)^n$ (Lemma 1.1.2). By assumption, $N_i^*N_j$ is positive, and thus P also commutes with $N_i^*N_j$ (Theorem 1.1.1). Hence, $\mathcal{H}$ is invariant for $N_i^*N_j$. Lemma 2.2.10 now implies that $\mathbf{T}$ is matricially quasinormal.

($\Leftarrow$) : The converse can be proved in a similar manner. ■

Remark 2.2.12. *We observe that Theorem 2.2.11 is a generalization of Theorem 2.1.2. More precisely, we get Theorem 2.1.2 as a corollary, by taking $T_1 = T_2 = T$ and $N_1 = N_2$ in Theorem 2.2.11.*

2.3 SUBNORMAL FACTORS OF NORMAL OPERATORS

We focus now on a more general approach to the Problem 2.1.1. More precisely, we regard the square as a product and move the problem from the one-variable instance to a multivariable setting. We treat the new (generalized) problem as a converse of Fuglede Theorem, and especially Corollary 1.1.14. The obtained versions, as we shall see, yield, in particular cases, the previously known results. The crucial step is the following observation:

We can reformulate Problem 2.1.1 as follows: *Let* $\mathbf{T} = (T, T)$ *be a subnormal pair and assume that* $T \cdot T$ *is quasinormal. Does it follow that* T *is quasinormal?*

This also gives us the motivation for the following problems:

Problem 2.3.1. *Let* $\mathbf{T} = (T_1, T_2)$ *be a subnormal pair such that* T_1T_2 *is quasinormal. Find sufficient conditions for* T_1 *and* T_2 *to be quasinormal.*

Problem 2.3.2. *Let* $\mathbf{T} = (T_1, T_2)$ *be a (jointly) quasinormal pair such that* T_1T_2 *is normal. Find sufficient conditions for* T_1 *and* T_2 *to be normal.*

Problem 2.3.3. *Let* $\mathbf{T} = (T_1, T_2)$ *be a subnormal pair such that* T_1T_2 *is normal. Find sufficient conditions for* T_1 *and* T_2 *to be normal.*

As we see, the Problem 2.3.3 can be treated as a converse of Corollary 1.1.14.

2.3.1 QUASINORMAL FACTORS OF NORMAL OPERATORS

The starting point in our discussion will be the following lemma:

Lemma 2.3.4. *Let* $\mathbf{T} = (T_1, T_2)$ *be a subnormal pair with the normal extension* $\mathbf{N} = (N_1, N_2)$ *such that* T_2 *is quasinormal and* T_1T_2 *is normal. If* T_1 *is left invertible, then* T_2 *is normal.*

Proof. Let

$$N_1 = \begin{bmatrix} T_1 & A_1 \\ 0 & B_1^* \end{bmatrix}, \quad N_2 = \begin{bmatrix} T_2 & A_2 \\ 0 & B_2^* \end{bmatrix}$$

be the normal extensions for T_1 and T_2, respectively, defined on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$. Since $N_1N_2 = N_2N_1$, by Corollary 1.1.14, N_1N_2 is normal. Thus,

$$N_1N_2 = \begin{bmatrix} T_1T_2 & T_1A_2 + A_1B_2^* \\ 0 & (B_2B_1)^* \end{bmatrix}$$

is a normal extension for T_1T_2. Operator T_1T_2 is normal, so by Lemma 1.2.10, we have that $T_1A_2 + A_1B_2^* = 0$, i.e. $T_1A_2 = -A_1B_2^*$. Since T_1 is left invertible, there exists $C_1 \in \mathfrak{B}(\mathcal{H})$ such that $A_2 = -C_1A_1B_2^*$. From here, $\mathcal{N}(B_2^*) \subseteq \mathcal{N}(A_2)$, and so $A_2|_{\mathcal{N}(B_2^*)} = 0$.

From $N_2^*N_2 = N_2N_2^*$ it follows that $A_2^*T_2 = B_2^*A_2^*$. Since T_2 is quasinormal, using Lemma 1.2.10 again, we have that $A_2^*T_2 = 0$, i.e. $A_2B_2 = 0$. Thus, $A_2|_{\mathcal{R}(B_2)} = 0$, and by continuity, $A_2|_{\overline{\mathcal{R}(B_2)}} = 0$. Since $\mathcal{L} = \mathcal{N}(B_2^*) \oplus \overline{\mathcal{R}(B_2)}$, it follows that $A_2 = 0$. By Lemma 1.2.10, we obtain that T_2 is normal. ■

Lemma 2.3.5. *Let* $\mathbf{T} = (T_1, T_2)$ *be a subnormal pair with the normal extension* $\mathbf{N} = (N_1, N_2)$ *such that* T_2 *is quasinormal and* T_1T_2 *is normal. If* $\mathcal{R}(T_1) = \overline{\mathcal{R}(T_1)} \subseteq \overline{\mathcal{R}(T_2^*)}$ *and* $\mathcal{N}(T_1) = \mathcal{N}(T_2)$*, then* T_2 *is normal.*

Proof. Since $\mathcal{R}(T_1) = \overline{\mathcal{R}(T_1)} \subseteq \overline{\mathcal{R}(T_2^*)}$ and $\mathcal{N}(T_1) = \mathcal{N}(T_2)$ we have that the operators T_1 and T_2 have representations

$$T_1 = \begin{bmatrix} T_1^1 & 0 \\ 0 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} T_2^1 & 0 \\ 0 & 0 \end{bmatrix},$$

respectively, with respect to $\mathcal{H} = \mathcal{N}(T_2)^\perp \oplus \mathcal{N}(T_2)$ decomposition. It follows that

$$N_1 = \begin{bmatrix} T_1^1 & 0 & A_1^1 \\ 0 & 0 & A_1^2 \\ 0 & 0 & B_1^* \end{bmatrix} : \begin{pmatrix} \mathcal{N}(T_2)^\perp \\ \mathcal{N}(T_2) \\ \mathcal{H}^\perp \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{N}(T_2)^\perp \\ \mathcal{N}(T_2) \\ \mathcal{H}^\perp \end{pmatrix}$$

is a normal extension for T_1^1 and

$$N_2 = \begin{bmatrix} T_2^1 & 0 & A_2^1 \\ 0 & 0 & A_2^2 \\ 0 & 0 & B_2^* \end{bmatrix} : \begin{pmatrix} \mathcal{N}(T_2)^\perp \\ \mathcal{N}(T_2) \\ \mathcal{H}^\perp \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{N}(T_2)^\perp \\ \mathcal{N}(T_2) \\ \mathcal{H}^\perp \end{pmatrix}$$

is a normal extension for T_2^1. Since $N_1N_2 = N_2N_1$, operator pair $\mathbf{T}^1 = (T_1^1, T_2^1)$ is subnormal. From quasinormality of T_2 we have that T_2^1 is quasinormal, and since T_1T_2 is normal, it follows that $T_1^1T_2^1$ is also normal.

Obviously, $\mathcal{R}(T_1^1) = \mathcal{R}(T_1)$, and so $\mathcal{R}(T_1^1)$ is closed. Now let $x \in \mathcal{N}(T_1^1) \subseteq \mathcal{N}(T_2)^\perp$. Then $P_{\mathcal{N}(T_2)^\perp}T_1x = 0$. From here and the fact that $\mathcal{R}(T_1) \subseteq \mathcal{N}(T_2)^\perp$, we have that $T_1x = 0$, i.e. $x \in \mathcal{N}(T_1) = \mathcal{N}(T_2)$. It must be $x = 0$, and so $\mathcal{N}(T_1^1) = \{0\}$. Therefore, T_1^1 is left invertible.

If we apply Lemma 2.3.4 to the operator pair $\mathbf{T}^1 = (T_1^1, T_2^1) \in \mathfrak{B}(\mathcal{N}(T_2)^\perp)^2$, we conclude that T_2^1 is normal. Now it directly follows that T_2 is also normal. ■

Corollary 2.3.6. *Let* $\mathbf{T} = (T_1, T_2)$ *be a (jointly) quasinormal pair such that* T_1T_2 *is normal. If* $\mathcal{R}(T_1) = \mathcal{R}(T_2) = \overline{\mathcal{R}(T_2)}$ *and* $\mathcal{N}(T_1) = \mathcal{N}(T_2)$*, then* $\mathbf{T}$ *is normal.*

Proof. Since T_1 and T_2 are hyponormal, we have $\mathcal{R}(T_i) \subseteq \mathcal{R}(T_i^*)$, $i = 1, 2$. Thus, if $\overline{\mathcal{R}(T_1)} = \mathcal{R}(T_2) = \overline{\mathcal{R}(T_2)}$, then $\mathcal{R}(T_1) = \overline{\mathcal{R}(T_1)} \subseteq \overline{\mathcal{R}(T_2^*)}$ and $\mathcal{R}(T_2) = \overline{\mathcal{R}(T_2)} \subseteq \overline{\mathcal{R}(T_1^*)}$. The conclusion now follows directly from Lemma 2.3.5. ■

Combining the previously obtained results, we arrive at the following theorem:

Theorem 2.3.7. *Let $\mathbf{T} = (T_1, T_2)$ be a (jointly) quasinormal pair such that T_1T_2 is normal. Then $\mathbf{T}$ is normal if one of the following conditions holds:*

(i) *T_1 or T_2 is right invertible;*

(ii) *T_1 and T_2 are left invertible;*

(iii) *$\mathcal{R}(T_i) = \overline{\mathcal{R}(T_i)} \subseteq \overline{\mathcal{R}(T_j^*)}$ for $i \neq j$, and $\mathcal{N}(T_1) = \mathcal{N}(T_2)$;*

(iv) *$\mathcal{R}(T_1) = \mathcal{R}(T_2) = \overline{\mathcal{R}(T_2)}$ and $\mathcal{N}(T_1) = \mathcal{N}(T_2)$.*

Proof. (i) Without loss of generality, assume that T_1 is right invertible. Then T_1^* is left invertible and $\mathcal{N}(T_1) \subseteq \mathcal{N}(T_1^*) = \{0\}$, as T_1 is hyponormal. Thus T_1 is invertible. From the quasinormality of T_1, it now follows that T_1 is normal. Operator T_2 is normal by Lemma 2.3.4. Thus, $\mathbf{T}$ is normal.

The rest of the proof follows directly from Lemma 2.3.4, Lemma 2.3.5 and Corollary 2.3.6. ■

Remark 2.3.8. *In Corollary 2.3.6 and Theorem 2.3.7 it is enough to assume that T_1 and T_2 are quasinormal instead of (joint) quasinormality of $\mathbf{T} = (T_1, T_2)$. We will show in the sequel that we can actually remove the quasinormality condition on one (or both) of the coordinate operators.*

Remark 2.3.9. *Although condition (iv) of Theorem 2.3.7 actually implies condition (iii) of the same theorem (as shown in the proof of Corollary 2.3.6), we listed it due to its elegant form.*

2.3.2 Subnormal factors of quasinormal operators and converse of Fuglede theorem

The previous section concludes our consideration of Problem 2.3.2. We shift our focus now to the ”implied quasinormality problem“ and the converse of Fuglede Theorem, i.e. we deal with Problem 2.3.1 and Problem 2.3.3.

Lemma 2.3.10. *Let $\mathbf{T} = (T_1, T_2)$ be a subnormal pair with the normal extension $\mathbf{N} = (N_1, N_2)$ such that T_1T_2 is quasinormal. Then T_2 is quasinormal if one of the following conditions holds:*

(i) $\mathrm{Comm}(|N_1N_2|) \subseteq \mathrm{Comm}(|N_2|)$*;*

(ii) *T_1 is quasinormal and right invertible;*

(iii) *T_1 is quasinormal and N_1 is injective.*

Proof. (i) Let $\mathbf{N} = (N_1, N_2) \in \mathfrak{B}(\mathcal{K})^2$ be the normal extension for $\mathbf{T} = (T_1, T_2)$ where $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$, and let $P \in \mathfrak{B}(\mathcal{K})$ be the orthogonal projection onto $\mathcal{H}$. Since $N_1N_2 = N_2N_1$, by Corollary 1.1.14, N_1N_2 is a normal extension for T_1T_2. Also, T_1T_2 is quasinormal, and so we have that $\mathcal{H}$ is invariant for $(N_1N_2)^*(N_1N_2)$. By Lemma 1.2.11, P commutes with $(N_1N_2)^*(N_1N_2) = |N_1N_2|^2$. Using Theorem 1.1.1, we have that P commutes with $N_2^*N_2$, and so $\mathcal{H}$ is invariant for $N_2^*N_2$. Therefore, T_2 is quasinormal, by Lemma 1.2.11.

(ii) As in the proof of Theorem 2.3.7, we have that T_1 is invertible normal operator. Using the fact that T_1T_2 is quasinormal and T_1 and T_2 commute, Theorem 1.1.12 implies that

$$T_1T_1^*T_1T_2T_2^*T_2 = T_1^*T_1T_1T_2^*T_2T_2.$$

Multiplying from the left-hand side by $(T_1T_1^*T_1)^{-1}$, it immediately follows that T_2 is quasinormal.

(iii) As shown in part (i), we have that P commutes with $(N_1N_2)^*(N_1N_2) = N_1^*N_1N_2^*N_2$, i.e. $PN_1^*N_1N_2^*N_2 = N_1^*N_1N_2^*N_2P$. By assumption, T_1 is quasinormal, and so P commutes with $N_1^*N_1$ (Lemma 1.2.11). Hence, $N_1^*N_1PN_2^*N_2 = N_1^*N_1N_2^*N_2P$. Since $\mathcal{N}(N_1^*N_1) = \mathcal{N}(N_1) = \{0\}$, it follows that $PN_2^*N_2 = N_2^*N_2P$. The quasinormality of T_2 is now guaranteed by Lemma 1.2.11. ■

Theorem 2.1.2 now follows as a simple corollary:

Proof of 2.1.2. Let N be a normal extension for T and let $T_1 = T^{n-1}$ and $T_2 = T$. Then $\mathbf{T} = (T_1, T_2)$ is a subnormal pair with the normal extension $\mathbf{N} = (N_1, N_2) = (N^{n-1}, N)$. Note that $(N_1N_2)^*(N_1N_2) = (N^*N)^n$ and so the first condition of Lemma 2.3.10 is satisfied, by Theorem 1.1.1. Thus, $T_2 = T$ is quasinormal. ■

Using Lemma 2.3.10 and the same technique as in the proof of Lemma 2.3.5, we can prove the next lemma:

Lemma 2.3.11. *Let $\mathbf{T} = (T_1, T_2)$ be a subnormal pair with the normal extension $\mathbf{N} = (N_1, N_2)$ such that T_1 and T_1T_2 are quasinormal. If $\mathcal{R}(T_1) = \overline{\mathcal{R}(T_2^*)}$ and $\mathcal{N}(T_2) \subseteq \mathcal{N}(T_1)$, then T_2 is quasinormal.*

Proof. Since $\mathcal{R}(T_1) = \overline{\mathcal{R}(T_2^*)}$ and $\mathcal{N}(T_2) \subseteq \mathcal{N}(T_1)$ we have that operators T_1 and T_2 have representations

$$T_1 = \begin{bmatrix} T_1^1 & 0 \\ 0 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} T_2^1 & 0 \\ 0 & 0 \end{bmatrix},$$

respectively, with respect to $\mathcal{H} = \mathcal{N}(T_2)^\perp \oplus \mathcal{N}(T_2)$ decomposition. It follows that

$$N_1 = \begin{bmatrix} T_1^1 & 0 & A_1^1 \\ 0 & 0 & A_1^2 \\ 0 & 0 & B_1^* \end{bmatrix} : \begin{pmatrix} \mathcal{N}(T_2)^\perp \\ \mathcal{N}(T_2) \\ \mathcal{H}^\perp \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{N}(T_2)^\perp \\ \mathcal{N}(T_2) \\ \mathcal{H}^\perp \end{pmatrix}$$

is a normal extension for T_1^1 and

$$N_2 = \begin{bmatrix} T_2^1 & 0 & A_2^1 \\ 0 & 0 & A_2^2 \\ 0 & 0 & B_2^* \end{bmatrix} : \begin{pmatrix} \mathcal{N}(T_2)^{\perp} \\ \mathcal{N}(T_2) \\ \mathcal{H}^{\perp} \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{N}(T_2)^{\perp} \\ \mathcal{N}(T_2) \\ \mathcal{H}^{\perp} \end{pmatrix}$$

is a normal extension for T_2^1. Since $N_1N_2 = N_2N_1$, operator pair $\mathbf{T}^1 = (T_1^1, T_2^1)$ is subnormal. From quasinormality of T_1 we have that T_1^1 is quasinormal, and since T_1T_2 is quasinormal, it follows that $T_1^1T_2^1$ is quasinormal.

Obviously, $\mathcal{R}(T_1^1) = \mathcal{R}(T_1) = \overline{\mathcal{R}(T_2^*)} = \mathcal{N}(T_2)^{\perp}$, and so T_1^1 is onto. In other words, T_1^1 is right invertible.

We conclude that operator pair $\mathbf{T}^1 = (T_1^1, T_2^1) \in \mathfrak{B}(\mathcal{N}(T_2)^{\perp})^2$ satisfies condition (ii) of Lemma 2.3.10, and so T_2^1 is quasinormal. Now it directly follows that T_2 is also quasinormal. ∎

In order to prove our next result, similar in spirit to Lemma 2.3.10, but also of independent interest, we need the following theorem:

Theorem 2.3.12. *[82] Let A and B be operators with $\sigma(A) \cap \sigma(B) = \emptyset$. Then every operator that commutes with $A + B$ and with AB also commutes with A and B.*

Theorem 2.3.13. *Let $\mathbf{T} = (T_1, T_2)$ be a spherically quasinormal pair with a normal extension $\mathbf{N} = (N_1, N_2)$ such that $\sigma(|N_1|) \cap \sigma(|N_2|) = \emptyset$. If T_1T_2 is quasinormal, then $\mathbf{T}$ is (jointly) quasinormal.*

Proof. Let N_i, $i = 1, 2$, be the normal extensions of T_i on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^{\perp}$ given by

$$N_i = \begin{bmatrix} T_i & A_i \\ 0 & B_i^* \end{bmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^{\perp} \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^{\perp} \end{pmatrix},$$

and let $P \in \mathfrak{B}(\mathcal{K})$ be the orthogonal projection onto $\mathcal{H}$. As in the proof of Lemma 2.3.10, we can show that quasinormality of T_1T_2 implies that P commutes with $N_1^*N_1N_2^*N_2$. Since $\mathbf{T}$ is spherically quasinormal, by Lemma 2.2.3, we have that $\mathcal{H}$ is invariant for $N_1^*N_1 + N_2^*N_2$. Therefore, P commutes with $N_1^*N_1 + N_2^*N_2$, as well.

By assumption, $\sigma(|N_1|) \cap \sigma(|N_2|) = \emptyset$, and so by the Spectral Mapping Theorem, $\sigma(N_1^*N_1) \cap \sigma(N_2^*N_2) = \emptyset$. We conclude that P commutes with $N_1^*N_1$ and $N_2^*N_2$ (Theorem 2.3.12). Hence, $\mathcal{H}$ is invariant for $N_1^*N_1$ and $N_2^*N_2$. By Lemma 1.2.11, T_1 and T_2 are quasinormal. Since T_1 commutes with $T_1^*T_1$ and $T_1^*T_1 + T_2^*T_2$, it also commutes with $T_2^*T_2$. Similarly, T_2 commutes with $T_1^*T_1$. Therefore, $\mathbf{T}$ is (jointly) quasinormal. ∎

Finally, we arrive at the main result of this section:

Theorem 2.3.14 (Converse of Fuglede Theorem)**.** *Let $\mathbf{T} = (T_1, T_2)$ be a subnormal pair with the normal extension $\mathbf{N} = (N_1, N_2)$ such that T_1T_2 is normal. Then $\mathbf{T}$ is normal if one of the following conditions holds:*

(i) *T_1 or T_2 is a right invertible quasinormal operator;*

(ii) *T_1 is quasinormal and N_1 and T_2 are left invertible, or T_2 is quasinormal and T_1 and N_2 are left invertible;*

(iii) *T_1 or T_2 is quasinormal, $\mathcal{R}(T_i) = \overline{\mathcal{R}(T_j^*)}$ for $i \neq j$, and $\mathcal{N}(T_1) = \mathcal{N}(T_2)$.*

(iv) $\text{Comm}(|N_1 N_2|) \subseteq \text{Comm}(|N_1|) \cap \text{Comm}(|N_2|)$ *and any condition $(i) - (iv)$ of Theorem 2.3.7 holds;*

(v) $\mathbf{T}$ *is spherically quasinormal, $\sigma(|N_1|) \cap \sigma(|N_2|) = \emptyset$ and any condition $(i) - (iv)$ of Theorem 2.3.7 holds.*

Proof. (i) Without loss of generality, assume that T_1 is right invertible quasinormal operator. By Lemma 2.3.10, it follows that T_2 is quasinormal. Thus, condition (i) of Theorem 2.3.7 is satisfied, and so $\mathbf{T}$ is normal.

(ii) Without loss of generality, assume that T_1 is quasinormal and N_1 and T_2 are left invertible. By Lemma 2.3.10, it follows that T_2 is quasinormal. Also, the left invertibility of N_1 implies the left invertibility of T_1. This means that condition (ii) of Theorem 2.3.7 holds. Therefore, $\mathbf{T}$ must be normal.

(iii) Again, we may assume that T_1 is quasinormal. By Lemma 2.3.11, we have that T_2 is quasinormal. The condition (iii) of Theorem 2.3.7 is obviously satisfied in this case, and hence, $\mathbf{T}$ is normal.

(iv) Condition

$$\text{Comm}(|N_1 N_2|) \subseteq \text{Comm}(|N_1|) \cap \text{Comm}(|N_2|)$$

implies that both T_1 and T_2 are quasinormal. Any condition of Theorem 2.3.7 is now sufficient for the normality of $\mathbf{T}$.

(v) Conditions $\mathbf{T}$ is spherically quasinormal and $\sigma(|N_1|) \cap \sigma(|N_2|) = \emptyset$ implies that $\mathbf{T}$ is (jointly) quasinormal. As in the previous case, any condition of Theorem 2.3.7 now implies that $\mathbf{T}$ is normal. ∎

CHAPTER 3

SPHERICAL MEAN TRANSFORM OF OPERATOR PAIRS

This chapter introduces the concept of the spherical mean transform for commuting operator pairs. This extension allows us to broaden the definition of the mean transform, which was originally defined in one-dimensional settings, to the domain of multivariable operator theory. Our primary objective is to explore various spectral properties of this transformation, including its ability to preserve the Taylor spectrum and some analytical characteristics, as well. Furthermore, we establish specific conditions under which the transform maintains the property of being "p-hyponormal" for two-variable weighted shifts.

3.1 MOTIVATION AND PRELIMINARIES

Let $T = U|T|$ be the polar decomposition of an operator $T \in \mathfrak{B}(\mathcal{H})$. In Section 1.2.4, we gave the definitions of the Aluthge and Duggal transforms of the operator T. Recently, the authors in [119] introduced yet another transform of an operator. The *mean transform* of operator T, denoted by $\mathcal{M}(T)$, is

$$\mathcal{M}(T) = \frac{1}{2}(U|T| + |T|U) = \frac{1}{2}(T + \widehat{T}).$$

In recent years, besides Aluthge and Duggal transforms, the mean transform has also attracted considerable attention (see, for example, [14, 31, 32, 33, 109, 142, 185, 187]). In the view of the practical use, one of the major advantages of the mean transform is the following: it may be really hard to find the Aluthge transform of the given operator because it involves finding the square root of a positive operator, while the mean transform involves the sums of two operators, and so it is easier to get the mean transforms if we know the polar decompositions of the operators.

Now let $\mathbf{T} = (V_1P, V_2P)$ be the (canonical) spherical polar decomposition of an operator pair $\mathbf{T}$ (see Section 1.3). In an analogous way to the one-dimensional case, we obtain the *spherical Aluthge transform* $\widetilde{\mathbf{T}}$ as $\widetilde{\mathbf{T}} = (\sqrt{P}V_1\sqrt{P}, \sqrt{P}V_2\sqrt{P})$ and the *spherical Duggal transform* $\widehat{\mathbf{T}}$ as $\widehat{\mathbf{T}} = (PV_1, PV_2)$ (cf. [16, 54, 55, 84, 114]). Naturally, we extend the notion of the mean transform to a multivariable setting, as well.

Definition 3.1.1. Let $\mathbf{T} = (T_1, T_2) = (V_1P, V_2P)$ be the canonical spherical polar decomposition of $\mathbf{T}$. The *spherical mean transform* of $\mathbf{T}$ is defined as

$$\mathcal{M}(\mathbf{T}) = (\mathcal{M}_1(\mathbf{T}), \mathcal{M}_2(\mathbf{T})) = \frac{1}{2}(V_1P + PV_1, V_2P + PV_2).$$

The notion can be easily generalized to any n-tuple of operators.

Remark 3.1.1. *Observe that $\mathcal{M}_1(\mathbf{T})$ and $\mathcal{M}_2(\mathbf{T})$ in the previous definition are not the mean transforms of T_1 and T_2, respectively, as $T_i = V_iP$, $i = 1, 2$, are not the standard polar decompositions of bounded operators on $\mathcal{H}$.*

In a recent paper (see [113]), the authors introduced the notion of *spherical p-hyponormality* in the following way: we say that a commuting pair $\mathbf{T} = (T_1, T_2)$ of operators on $\mathcal{H}$ is *spherically p-hyponormal* $(0 < p \leq 1)$, if

$$(T_1^*T_1 + T_2^*T_2)^p \geq (T_1T_1^* + T_2T_2^*)^p.$$

They also showed the following theorem:

Theorem 3.1.2. *[113] Consider a 2-variable weighted shift $\mathbf{W}_{(\alpha,\beta)} = (T_1, T_2)$. Then, for $0 < p \leq 1$, we have that $\mathbf{W}_{(\alpha,\beta)}$ is spherically p-hyponormal if and only if*

$$\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)} \geq \alpha^2_{(k_1-1,k_2)} + \beta^2_{(k_1,k_2-1)}, \quad \text{for all } k_1, k_2 \geq 0, \tag{3.1}$$

where $\alpha_{(-1,0)} = \beta_{(0,-1)} = 0$.

Throughout the chapter, for brevity, we will use the following notation: for an operator pair $\mathbf{T} = (T_1, T_2)$, and $A, B \in \mathfrak{B}(\mathcal{H})$, $A\mathbf{T}B$ means

$$A\mathbf{T}B = (AT_1B, AT_2B).$$

Also, for two operator pairs $\mathbf{A} = (A_1, A_2)$ and $\mathbf{B} = (B_1, B_2)$, we write

$$\mathbf{AB} = (A_1B_1, A_2B_2).$$

Finally, $\mathbf{0}$ and $\mathbf{I}$ stand for operator pairs $(0, 0)$ and (I, I), respectively.

The chapter is organized as follows. In Section 3.2, we give some properties of the spherical mean transform, which may present the basis for further study on the topic. In Section 3.3, we describe how 2-variable weighted shifts behave under the transform. More specifically, we focus on the p-hyponormality of 2-variable weighted shifts and their spherical mean transforms.

3.2 General properties

We start with the following simple observation.

Theorem 3.2.1. *Let* $\mathbf{T} = (T_1, T_2)$ *be a commuting pair of operators on* $\mathcal{H}$. *The following conditions are equivalent:*

(i) $\mathbf{T}$ *is spherically quasinormal;*

(ii) $\mathcal{M}(\mathbf{T}) = \mathbf{T}$.

Proof. Observe that

$$\mathcal{M}(\mathbf{T}) = \mathbf{T} \Leftrightarrow \frac{1}{2}(V_i P + P V_i) = V_i P, \ i = 1, 2.$$

The last equality translates to $V_i P = P V_i$, $i = 1, 2$, which is by Theorem 1.3.1 further equivalent with the spherical quasinormality of $\mathbf{T}$. ■

The next theorem states that the kernel of an operator pair $\mathbf{T}$ is preserved under the spherical mean transform.

Theorem 3.2.2. *Let* $\mathbf{T} = (T_1, T_2)$ *be a pair of operators on* $\mathcal{H}$. *Then*

$$\ker(\mathcal{M}(\mathbf{T})) = \ker(\mathbf{T}).$$

Proof. Let $\mathbf{T} = (V_1 P, V_2 P)$ be the canonical spherical polar decomposition of $\mathbf{T}$ and assume that $x \in \ker(\mathbf{T})$. Since $\ker(\mathbf{T}) = \ker(P) = \ker(\mathbf{V})$, we have that $Px = V_i x = 0$, $i = 1, 2$. Thus,

$$\mathcal{M}_i(\mathbf{T})x = \frac{1}{2}(V_i P x + P V_i x) = 0, \quad i = 1, 2.$$

Thus, $x \in \ker(\mathcal{M}_1(\mathbf{T})) \cap \ker(\mathcal{M}_2(\mathbf{T})) = \ker(\mathcal{M}(\mathbf{T}))$, and so $\ker(\mathbf{T}) \subseteq \ker(\mathcal{M}(\mathbf{T}))$.

Conversely, assume now that $x \in \ker(\mathcal{M}(\mathbf{T}))$. Then

$$(V_1 P + P V_1)x = (V_2 P + P V_2)x = 0.$$

From here,

$$V_1^* V_1 P x + V_1^* P V_1 x = 0$$

and

$$V_2^* V_2 P x + V_2^* P V_2 x = 0.$$

By adding the previous two equations, and using the fact that $V_1^* V_1 + V_2^* V_2 = I$ on $\overline{\mathcal{R}(P)}$, we obtain

$$Px + V_1^* P V_1 x + V_2^* P V_2 x = 0,$$

i.e.

$$\langle Px, x \rangle + \langle P V_1 x, V_1 x \rangle + \langle P V_2 x, V_2 x \rangle = 0.$$

Since operator P is positive, it follows that $x \in \ker(P) = \ker(\mathbf{T})$. This implies that $\ker(\mathcal{M}(\mathbf{T})) \subseteq \ker(\mathbf{T})$. ■

Corollary 3.2.3. *Let* $\mathbf{T} = (T_1, T_2)$ *be a pair of operators on* $\mathcal{H}$*. The following conditions are equivalent:*

(i) $\mathbf{T} = \mathbf{0}$*;*

(ii) $\mathcal{M}(\mathbf{T}) = \mathbf{0}$.

Similar in spirit to the previous corollary, the next theorem deals with the operator **I** (instead of **0**).

Theorem 3.2.4. *Let* $\mathbf{T} = (T_1, T_2)$ *be a pair of operators on* $\mathcal{H}$ *with the spherical polar decomposition* $\mathbf{T} = (V_1P, V_2P)$*. The following conditions are equivalent:*

(i) $\mathbf{T} = \mathbf{I}$*;*

(ii) $\mathcal{M}(\mathbf{T}) = \mathbf{I}$ *and* $\operatorname{Re}(V_1^*V_2) = \frac{1}{2}I$*.*

Proof. $(i) \Rightarrow (ii)$: If $\mathbf{T} = \mathbf{I}$, then, obviously, $P = \sqrt{2}I$ and $V_1 = V_2 = \frac{1}{\sqrt{2}}I$. It immediately follows that (ii) holds.

$(ii) \Rightarrow (i)$: Assume now that $\mathcal{M}(\mathbf{T}) = \mathbf{I}$ and $\operatorname{Re}(V_1^*V_2) = \frac{1}{2}I$. We have that $2I = V_iP + PV_i$, $i = 1, 2$, and so

$$\begin{aligned} V_1^* &= V_1^*I = \frac{1}{2}(V_1^*V_1P + V_1^*PV_1), \\ V_2^* &= V_2^*I = \frac{1}{2}(V_2^*V_2P + V_2^*PV_2). \end{aligned}$$

Therefore,

$$\begin{aligned} V_1^* + V_2^* &= \frac{1}{2}(V_1^*V_1 + V_2^*V_2)P + \frac{1}{2}(V_1^*PV_1 + V_2^*PV_2) \\ &= \frac{1}{2}P + \frac{1}{2}(V_1^*PV_1 + V_2^*PV_2). \end{aligned}$$

It follows that $V_1^* + V_2^*$ is positive. Observe that $\ker(\mathcal{M}(\mathbf{T})) = \{0\}$, and thus $\ker(\mathbf{T}) = \{0\}$, by Theorem 3.2.2. Since $V_1^*V_1 + V_2^*V_2$ is an orthogonal projection onto $(\ker(\mathbf{T}))^\perp = \{0\}^\perp = \mathcal{H}$, we have that $V_1^*V_1 + V_2^*V_2 = I$. Using the fact that $\operatorname{Re}(V_1^*V_2) = \frac{1}{2}I$, i.e. $V_1^*V_2 + V_2^*V_1 = I$, it follows that

$$\begin{aligned} (V_1 + V_2)^2 &= (V_1 + V_2)^*(V_1 + V_2) \\ &= V_1^*V_1 + V_1^*V_2 + V_2^*V_1 + V_2^*V_2 \\ &= 2I. \end{aligned}$$

From the uniqueness of a positive square root, we have that $V_1 + V_2 = \sqrt{2}I$. Again, from $\mathcal{M}(\mathbf{T}) = \mathbf{I}$, it follows that

$$2I = V_i P + PV_i, \quad i = 1, 2,$$

and so,

$$4I = (V_1 + V_2)P + P(V_1 + V_2) = 2\sqrt{2}P.$$

Therefore, $P = \sqrt{2}I$. Now, $I = \frac{1}{2}(V_i P + PV_i)$, $i = 1, 2$, implies $V_1 = V_2 = \frac{1}{\sqrt{2}}I$. Finally, we conclude that $\mathbf{T} = \mathbf{I}$. ■

Theorem 3.2.5. *Let $\mathbf{T} = (T_1, T_2)$ be a pair of operators on $\mathcal{H}$ and $U \in \mathfrak{B}(\mathcal{H})$ be a unitary operator. Then*

$$\mathcal{M}(U\mathbf{T}U^*) = U\mathcal{M}(\mathbf{T})U^*.$$

Proof. Let $\mathbf{S} = U\mathbf{T}U^* = (UT_1U^*, UT_2U^*)$. Consider the spherical polar decompositions $\mathbf{T} = \mathbf{V}P_T$ and $\mathbf{S} = \mathbf{W}P_S$. Let us show first that

$$P_S = UP_TU^*.$$

Indeed,

$$\begin{aligned} P_S^2 &= UT_1^*U^*UT_1U^* + UT_2^*U^*UT_2U^* \\ &= U(T_1^*T_1 + T_2^*T_2)U^* = UP_T^2U^* = (UP_TU^*)^2 \end{aligned}$$

From the uniqueness of a positive square root, the conclusion follows.

Now, let us show that $\mathbf{W} = U\mathbf{V}U^*$. We have

$$S_i = UT_iU^* = UV_iP_TU^* = UV_iU^*UP_TU^* = \tilde{V}_iP_S, \quad i = 1, 2,$$

where $\tilde{V}_i = UV_iU^*$, i.e. $\tilde{\mathbf{V}} = (\tilde{V}_1, \tilde{V}_2) = U\mathbf{V}U^*$. Note that

$$\tilde{V}_1^*\tilde{V}_1 + \tilde{V}_2^*\tilde{V}_2 = U(V_1^*V_1 + V_2^*V_2)U^*.$$

Let $\mathcal{S} = U((\ker(V_1) \cap \ker(V_2))^{\perp})$, and let $y \in \mathcal{S}$ be arbitrary. Then, $y = Ux$, for some $x \in (\ker(V_1) \cap \ker(V_2))^{\perp}$, and so

$$\begin{aligned} (\tilde{V}_1^*\tilde{V}_1 + \tilde{V}_2^*\tilde{V}_2)y &= U(V_1^*V_1 + V_2^*V_2)U^*Ux \\ &= U(V_1^*V_1 + V_2^*V_2)x = Ux = y. \end{aligned}$$

Thus, we conclude that $(\tilde{V}_1, \tilde{V}_2)$ is a spherical partial isometry such that $\tilde{V}_1^*\tilde{V}_1 + \tilde{V}_2^*\tilde{V}_2 = I$ on $U((\ker(V_1) \cap \ker(V_2))^{\perp})$. Also, for $x \in \mathcal{H}$,

$$\begin{aligned} x \in \ker(\mathbf{S}) &\iff x \in \ker(UT_1U^*) \cap \ker(UT_2U^*) \\ &\iff UT_1U^*x = UT_2U^*x = 0 \\ &\iff T_1U^*x = T_2U^*x = 0 \\ &\iff U^*x \in \ker(T_1) \cap \ker(T_2) \\ &\iff U^*x \in\in \ker(V_1) \cap \ker(V_2) \\ &\iff UV_1U^*x = UV_2U^*x = 0 \\ &\iff x \in \ker(\tilde{\mathbf{V}}). \end{aligned}$$

Therefore, $\ker(\mathbf{S}) = \ker(\tilde{\mathbf{V}})$, and so it must be $\mathbf{W} = \tilde{\mathbf{V}} = (UV_1U^*, UV_2U^*)$. This implies that

$$\begin{aligned}\mathcal{M}_i(\mathbf{S}) &= \frac{1}{2}(W_iP_S + P_SW_i)\\ &= \frac{1}{2}(UV_iU^*UP_TU^* + UP_TU^*UV_iU^*)\\ &= \frac{1}{2}(U(V_iP_T + P_TV_i)U^*)\\ &= U\mathcal{M}_i(\mathbf{T})U^*,\end{aligned}$$

for $i = 1, 2$, which yields the wanted result. ■

By the previous theorem, we see that the spherical mean transform ”behaves nicely“ with respect to the unitary equivalence.

In general, it is not true that $\sigma_T(\mathbf{T}) = \sigma_T(\mathcal{M}(\mathbf{T}))$ (see Example 3.3.8 below). However, in the case when $\mathbf{T}$ is a commuting spherical partial isometry, we get the affirmative answer. In order to prove our claim, we need the following definition and the theorem.

Definition 3.2.1. [15] Let $\mathbf{A} = (A_1, \ldots, A_n)$ and $\mathbf{B} = (B_1, \ldots, B_n)$ be the be two n-tuples of operators on $\mathcal{H}$. We say that $\mathbf{A}$ and $\mathbf{B}$ *criss-cross commute* (or that $\mathbf{A}$ *criss-cross commutes* with $\mathbf{B}$) if $A_iB_jA_k = A_kB_jA_i$ and $B_iA_jB_k = B_kA_jB_i$, for all $i, j, k = 1, \ldots, n$.

Theorem 3.2.6. *(cf. [17, 18]) Let $\mathbf{A}$ criss-cross commute with $\mathbf{B}$ on $\mathcal{H}$, and assume that $\mathbf{AB}$ is commuting. Then*

$$\sigma_T(\mathbf{BA}) \setminus \{\mathbf{0}\} = \sigma_T(\mathbf{AB}) \setminus \{\mathbf{0}\}.$$

Theorem 3.2.7. *Let $\mathbf{V} = (V_1, V_2)$ be a spherical partial isometry. Then*

$$\mathcal{M}(\mathbf{V}) = \frac{1}{2}\left((I + P)V_1, (I + P)V_2\right),$$

where $P = \sqrt{V_1^*V_1 + V_2^*V_2}$.
Moreover,

$$\sigma_T(\mathbf{V}) = \sigma_T(\mathcal{M}(\mathbf{V})).$$

Proof. The spherical polar decomposition of $\mathbf{V}$ is given by $V_i = V_iP$, $i = 1, 2$, where $P = \sqrt{V_1^*V_1 + V_2^*V_2}$. For $i \in \{1, 2\}$, we have

$$\mathcal{M}_i(\mathbf{V}) = \frac{1}{2}(V_iP + PV_i) = \frac{1}{2}(V_i + PV_i) = \frac{1}{2}(I + P)V_i,$$

which proves the first part of the claim.

Using the fact that $\mathbf{V} = (V_1, V_2)$ is a commuting pair, we will show that $\mathbf{V} = (V_1, V_2)$ criss-cross commutes with $\mathbf{P} = (I+P, I+P)$ and that $\mathbf{VP}$ is commuting. Since $V_iP = V_i$, $i = 1, 2$, we have that

$$\begin{aligned} V_1(I+P)V_2 &= (V_1 + V_1P)V_2 = 2V_1V_2 \\ &= 2V_2V_1 = (V_2 + V_2P)V_1 = V_2(I+P)V_1. \end{aligned}$$

We conclude that $\mathbf{V}$ criss-cross commute with $\mathbf{P}$ as the other condition in cross-commutativity trivially holds. $\mathbf{VP}$ is also a commuting pair since

$$V_1(I+P)V_2(I+P) = V_2(I+P)V_1(I+P).$$

All conditions of Theorem 3.2.6 are therefore satisfied. Thus,

$$\begin{aligned} \sigma_T(\mathcal{M}(\mathbf{V})) \setminus \{\mathbf{0}\} &= \frac{1}{2}\sigma_T((I+P)V_1, (I+P)V_2) \setminus \{\mathbf{0}\} \\ &= \frac{1}{2}\sigma_T(\mathbf{PV}) \setminus \{\mathbf{0}\} = \frac{1}{2}\sigma_T(\mathbf{VP}) \setminus \{\mathbf{0}\} \\ &= \frac{1}{2}\sigma_T(V_1(I+P), V_2(I+P)) \setminus \{\mathbf{0}\} \\ &= \frac{1}{2}\sigma_T(2V_1, 2V_2) \setminus \{\mathbf{0}\} \\ &= \sigma_T(\mathbf{V}) \setminus \{\mathbf{0}\}. \end{aligned}$$

Furthermore, assume that $\mathbf{V}$ is Taylor invertible. Then $V_1^*V_1 + V_2^*V_2$ is invertible orthogonal projection, and so $V_1^*V_1 + V_2^*V_2 = I$, i.e. $P = I$. Thus, $\mathcal{M}(\mathbf{V}) = \mathbf{V}$ and so $\mathcal{M}(\mathbf{V})$ is Taylor invertible.
Now assume that $\mathcal{M}(\mathbf{V})$ is Taylor invertible. In particular, we have that $\ker(\mathcal{M}(\mathbf{V})) = \{0\}$. By Theorem 3.2.2, it follows that $\ker(\mathbf{V}) = \{0\}$. Again, using the fact that $V_1^*V_1 + V_2^*V_2$ is an orthogonal projection onto $(\ker(\mathbf{V}))^{\perp}$, it follows that $V_1^*V_1 + V_2^*V_2 = I$, and so, $P = I$. This implies that $\mathcal{M}(\mathbf{V}) = \mathbf{V}$, and so $\mathbf{V}$ is Taylor invertible. This completes the proof. ■

We shift our focus now to the topological properties of the spherical mean transform. In order to prove our next result, we need the following theorem.

Theorem 3.2.8. *[55] The spherical Aluthge transform* $(T_1, T_2) \mapsto \widetilde{(T_1, T_2)}$ *is* $(\|\cdot\|, \|\cdot\|)$*-continuous on* $\mathfrak{B}(\mathcal{H})$*.*

Theorem 3.2.9. *Let* $\mathbf{T} = (T_1, T_2)$ *be a commuting pair of operators with* $\ker(\mathbf{T}) = \{0\}$*. The spherical mean transform* $(T_1, T_2) \mapsto (\mathcal{M}_1(\mathbf{T}), \mathcal{M}_2(\mathbf{T}))$ *is* $(\|\cdot\|, \text{SOT})$*-continuous.*

Proof. Let $\mathbf{T}_0 = (T_0^1, T_0^2) = (V_0^1 P_0, V_0^2 P_0) \in (\mathfrak{B}(\mathcal{H}))^2$ be an arbitrary commuting pair such that $\ker(\mathbf{T}_0) = \{0\}$ and assume that the sequence $\mathbf{T}_n = (T_n^1, T_n^2) = (V_n^1 P_n, V_n^2 P_n)$ converges in norm to $\mathbf{T}_0$. It follows that T_n^i converges in norm to T_0^i, $i = 1, 2$, and so, by continuity, $T_n^{1*} T_n^1 + T_n^{2*} T_n^2$ converges in norm to $T_0^{1*} T_0^1 + T_0^{2*} T_0^2$. In other words,

$$\|P_n - P_0\| \to 0 \quad \text{and} \quad \|\sqrt{P_n} - \sqrt{P_0}\| \to 0, \quad n \to \infty. \tag{3.2}$$

Let $i \in \{1, 2\}$ be arbitrary. First, note that

$$\begin{aligned} \|P_n V_n^i \sqrt{P_n} - P_n V_n^i \sqrt{P_0}\| &\leq \|P_n V_n^i\| \|\sqrt{P_n} - \sqrt{P_0}\| \\ &\leq \|P_n\| \|\sqrt{P_n} - \sqrt{P_0}\| \to 0, \end{aligned}$$

as $\|P_n\|$ is a bounded sequence and (3.2) holds. Furthermore, by Theorem 3.2.8, it follows that

$$\begin{aligned} \|P_n V_n^i \sqrt{P_n} - \sqrt{P_n}\sqrt{P_0} V_0^i \sqrt{P_0}\| &= \|\sqrt{P_n}\sqrt{P_n} V_n^i \sqrt{P_n} - \sqrt{P_n}\sqrt{P_0} V_0^i \sqrt{P_0}\| \\ &\leq \|\sqrt{P_n}\| \|\sqrt{P_n} V_n^i \sqrt{P_n} - \sqrt{P_0} V_0^i \sqrt{P_0}\| \\ &= \|\sqrt{P_n}\| \|\widehat{T_n^i} - \widehat{T_0^i}\| \\ &\leq \|\sqrt{P_n}\| \|\widehat{\mathbf{T}}_n - \widehat{\mathbf{T}}_0\| \to 0. \end{aligned}$$

Again, using (3.2),

$$\begin{aligned} \|\sqrt{P_n}\sqrt{P_0} V_0^i \sqrt{P_0} - P_0 V_0^i \sqrt{P_0}\| &= \|\sqrt{P_n}\sqrt{P_0} V_0^i \sqrt{P_0} - \sqrt{P_0}\sqrt{P_0} V_0^i \sqrt{P_0}\| \\ &\leq \|\sqrt{P_0} V_0^i \sqrt{P_0}\| \|\sqrt{P_n} - \sqrt{P_0}\| \to 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \|P_n V_n^i \sqrt{P_0} - P_0 V_0^i \sqrt{P_0}\| &\leq \|P_n V_n^i \sqrt{P_0} - P_n V_n^i \sqrt{P_n}\| \\ &\quad + \|P_n V_n^i \sqrt{P_n} - \sqrt{P_n}\sqrt{P_0} V_0^i \sqrt{P_0}\| \\ &\quad + \|\sqrt{P_n}\sqrt{P_0} V_0^i \sqrt{P_0} - P_0 V_0^i \sqrt{P_0}\| \to 0. \end{aligned}$$

Hence, for each $x \in \mathcal{R}(\sqrt{P_0})$, we have

$$\|P_n V_n^i x - P_0 V_0^i x\| \to 0. \tag{3.3}$$

By the continuity argument, the previous statement holds for each x in $\overline{\mathcal{R}(\sqrt{P_0})} = (\ker(\sqrt{P_0}))^{\perp}$. But $\ker(\sqrt{P_0}) = \ker(P_0) = \ker(\mathbf{T}_0) = \{0\}$, and so $\overline{\mathcal{R}(\sqrt{P_0})} = \mathcal{H}$. Thus, (3.3) is true for all $x \in \mathcal{H}$, which completes the proof. ∎

3.3 SPHERICAL MEAN TRANSFORM OF 2-VARIABLE WEIGHTED SHIFTS

We start this section with the derivation of the general formula of the spherical mean transform of an arbitrary 2-variable weighted shift.

Theorem 3.3.1. *Let $\mathbf{W}_{(\alpha,\beta)} = (T_1, T_2)$ be a 2-variable weighted shift. Then $\mathcal{M}(\mathbf{W}_{(\alpha,\beta)}) = (\mathcal{M}_1(\mathbf{W}_{(\alpha,\beta)}), \mathcal{M}_2(\mathbf{W}_{(\alpha,\beta)}))$ is given by*

$$\mathcal{M}_1(\mathbf{W}_{(\alpha,\beta)})e_{\mathbf{k}} = \gamma_{\mathbf{k}} \frac{\sqrt{\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2} + \sqrt{\alpha_{\mathbf{k}+\varepsilon_1}^2 + \beta_{\mathbf{k}+\varepsilon_1}^2}}{2} e_{\mathbf{k}+\varepsilon_1},$$
$$\mathcal{M}_2(\mathbf{W}_{(\alpha,\beta)})e_{\mathbf{k}} = \delta_{\mathbf{k}} \frac{\sqrt{\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2} + \sqrt{\alpha_{\mathbf{k}+\varepsilon_2}^2 + \beta_{\mathbf{k}+\varepsilon_2}^2}}{2} e_{\mathbf{k}+\varepsilon_2},$$

for all $\mathbf{k} \in \mathbb{Z}_+^2$, where $\varepsilon_1 = (1,0)$, $\varepsilon_2 = (0,1)$,

$$\gamma_{\mathbf{k}} = \begin{cases} \dfrac{\alpha_{\mathbf{k}}}{\sqrt{\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2}}, & \text{if } \alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2 \neq 0, \\ 0, & \text{if } \alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2 = 0, \end{cases} \tag{3.4}$$

and

$$\delta_{\mathbf{k}} = \begin{cases} \dfrac{\beta_{\mathbf{k}}}{\sqrt{\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2}}, & \text{if } \alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2 \neq 0, \\ 0, & \text{if } \alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2 = 0. \end{cases} \tag{3.5}$$

Proof. Let $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}_+^2$ be arbitrary. First, observe that

$$(T_1^* T_1 + T_2^* T_2)e_{(k_1,k_2)} = (\alpha_{(k_1,k_2)}^2 + \beta_{(k_1,k_2)}^2)e_{(k_1,k_2)}.$$

Therefore,

$$Pe_{(k_1,k_2)} = \sqrt{\alpha_{(k_1,k_2)}^2 + \beta_{(k_1,k_2)}^2}\, e_{(k_1,k_2)}.$$

It is now easy to see that

$$V_1 e_{(k_1,k_2)} = \gamma_{(k_1,k_2)} e_{(k_1+1,k_2)} \quad \text{and} \quad V_2 e_{(k_1,k_2)} = \delta_{(k_1,k_2)} e_{(k_1,k_2+1)},$$

where $\gamma_{(k_1,k_2)}$ and $\delta_{(k_1,k_2)}$ are given by (3.4) and (3.5), respectively.

We have that

$$\begin{aligned} V_1 P e_{(k_1,k_2)} &= \sqrt{\alpha_{(k_1,k_2)}^2 + \beta_{(k_1,k_2)}^2}\, V_1 e_{(k_1,k_2)} \\ &= \gamma_{(k_1,k_2)} \sqrt{\alpha_{(k_1,k_2)}^2 + \beta_{(k_1,k_2)}^2}\, e_{(k_1+1,k_2)}, \end{aligned}$$

while

$$\begin{aligned} PV_1 e_{(k_1,k_2)} &= \gamma_{(k_1,k_2)} P e_{(k_1+1,k_2)} \\ &= \gamma_{(k_1,k_2)} \sqrt{\alpha^2_{(k_1+1,k_2)} + \beta^2_{(k_1+1,k_2)}}\, e_{(k_1+1,k_2)}. \end{aligned}$$

Similarly,

$$\begin{aligned} V_2 P e_{(k_1,k_2)} &= \sqrt{\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)}}\, V_2 e_{(k_1,k_2)} \\ &= \delta_{(k_1,k_2)} \sqrt{\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)}}\, e_{(k_1,k_2+1)}, \end{aligned}$$

and

$$\begin{aligned} PV_2 e_{(k_1,k_2)} &= \delta_{(k_1,k_2)} P e_{(k_1,k_2+1)} \\ &= \delta_{(k_1,k_2)} \sqrt{\alpha^2_{(k_1,k_2+1)} + \beta^2_{(k_1,k_2+1)}}\, e_{(k_1,k_2+1)}. \end{aligned}$$

Using the notation $\mathbf{k} = (k_1, k_2)$, $\varepsilon_1 = (1, 0)$, $\varepsilon_2 = (0, 1)$, we immediately obtain that

$$\begin{aligned} \mathcal{M}_1(\mathbf{W}_{(\alpha,\beta)}) e_{\mathbf{k}} &= \frac{1}{2}(V_1 P + PV_1) e_{\mathbf{k}} \\ &= \gamma_{\mathbf{k}} \frac{\sqrt{\alpha^2_{\mathbf{k}} + \beta^2_{\mathbf{k}}} + \sqrt{\alpha^2_{\mathbf{k}+\varepsilon_1} + \beta^2_{\mathbf{k}+\varepsilon_1}}}{2} e_{\mathbf{k}+\varepsilon_1}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}_2(\mathbf{W}_{(\alpha,\beta)}) e_{\mathbf{k}} &= \frac{1}{2}(V_2 P + PV_2) e_{\mathbf{k}} \\ &= \delta_{\mathbf{k}} \frac{\sqrt{\alpha^2_{\mathbf{k}} + \beta^2_{\mathbf{k}}} + \sqrt{\alpha^2_{\mathbf{k}+\varepsilon_2} + \beta^2_{\mathbf{k}+\varepsilon_2}}}{2} e_{\mathbf{k}+\varepsilon_2}. \end{aligned}$$

This completes the proof. ■

Remark 3.3.2. *Let $\mathbf{W}_{(\alpha,\beta)} = (T_1, T_2)$ be a 2-variable weighted shift. From the previous theorem, we have that $\mathcal{M}(\mathbf{W}_{(\alpha,\beta)}) = (\mathcal{M}_1(\mathbf{W}_{(\alpha,\beta)}), \mathcal{M}_2(\mathbf{W}_{(\alpha,\beta)}))$ is a 2-variable weighted shift, as well.*

Corollary 3.3.3. *Let $\mathbf{W}_{(\alpha,\beta)} = (T_1, T_2)$ be a 2-variable weighted shift such that $\alpha^2_{\mathbf{k}} + \beta^2_{\mathbf{k}} \neq$*

0 *for each* $\mathbf{k} \in \mathbb{Z}_+^2$*. Then,* $\mathcal{M}(\mathbf{W}_{(\alpha,\beta)}) = (\mathcal{M}_1(\mathbf{W}_{(\alpha,\beta)}), \mathcal{M}_2(\mathbf{W}_{(\alpha,\beta)}))$ *is given by*

$$\mathcal{M}_1(\mathbf{W}_{(\alpha,\beta)})e_{\mathbf{k}} = \frac{\alpha_{\mathbf{k}}}{2}\left(1 + \sqrt{\frac{\alpha^2_{\mathbf{k}+\varepsilon_1} + \beta^2_{\mathbf{k}+\varepsilon_1}}{\alpha^2_{\mathbf{k}} + \beta^2_{\mathbf{k}}}}\right) e_{\mathbf{k}+\varepsilon_1},$$

$$\mathcal{M}_2(\mathbf{W}_{(\alpha,\beta)})e_{\mathbf{k}} = \frac{\beta_{\mathbf{k}}}{2}\left(1 + \sqrt{\frac{\alpha^2_{\mathbf{k}+\varepsilon_2} + \beta^2_{\mathbf{k}+\varepsilon_2}}{\alpha^2_{\mathbf{k}} + \beta^2_{\mathbf{k}}}}\right) e_{\mathbf{k}+\varepsilon_2},$$

for all $\mathbf{k} \in \mathbb{Z}_+^2$*, where* $\varepsilon_1 = (1,0)$*,* $\varepsilon_2 = (0,1)$*,*

In the sequel, for the sake of simplicity, we will always assume that $\alpha^2_{\mathbf{k}} + \beta^2_{\mathbf{k}} \neq 0$ for each $\mathbf{k} \in \mathbb{Z}_+^2$.

First observe that directly from Theorem 3.1.2, we have the following result.

Corollary 3.3.4. *Let* $\mathbf{W}_{(\alpha,\beta)} = (T_1, T_2)$ *be a 2-variable weighted shift. Then,* $\mathcal{M}(\mathbf{W}_{(\alpha,\beta)}) = (\mathcal{M}_1(\mathbf{W}_{(\alpha,\beta)}), \mathcal{M}_2(\mathbf{W}_{(\alpha,\beta)}))$ *is spherically p-hyponormal (for* $0 < p \leq 1$*) if and only if*

$$m(\alpha)^2_{\mathbf{k}} + m(\beta)^2_{\mathbf{k}} \geq m(\alpha)^2_{\mathbf{k}-\varepsilon_1} + m(\beta)^2_{\mathbf{k}-\varepsilon_2}, \tag{3.6}$$

for all $\mathbf{k} \in \mathbb{Z}_+^2$*, where* $\varepsilon_1 = (1,0)$*,* $\varepsilon_2 = (0,1)$*,*

$$m(\alpha)_{\mathbf{k}} = \alpha_{\mathbf{k}}\left(1 + \sqrt{\frac{\alpha^2_{\mathbf{k}+\varepsilon_1} + \beta^2_{\mathbf{k}+\varepsilon_1}}{\alpha^2_{\mathbf{k}} + \beta^2_{\mathbf{k}}}}\right),$$

$$m(\beta)_{\mathbf{k}} = \beta_{\mathbf{k}}\left(1 + \sqrt{\frac{\alpha^2_{\mathbf{k}+\varepsilon_2} + \beta^2_{\mathbf{k}+\varepsilon_2}}{\alpha^2_{\mathbf{k}} + \beta^2_{\mathbf{k}}}}\right),$$

and $m(\alpha)_{(-1,0)} = m(\beta)_{(0,-1)} = 0$*.*

In general, spherical (Aluthge, Duggal, mean) transforms do not preserve hyponormality of operator pairs. In the following theorem, we give some sufficient conditions for the map $\mathbf{W}_{(\alpha,\beta)} \mapsto \mathcal{M}(\mathbf{W}_{(\alpha,\beta)})$ to be hyponormality preserving.

Theorem 3.3.5. *Let* $\mathbf{W}_{(\alpha,\beta)} = (T_1, T_2)$ *be a 2-variable weighted shift. Assume that the following conditions hold:*

(i) $\mathbf{W}_{(\alpha,\beta)}$ *is hyponormal;*

(ii) $\alpha_{(k_1+1,k_2)} = \alpha_{(k_1,k_2+1)}$ *and* $\beta_{(k_1+1,k_2)} = \beta_{(k_1,k_2+1)}$ *for all* $k_1, k_2 \geq 0$*;*

(iii) for all $k_1, k_2 \geq 0$,

$$\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)} \leq \sqrt{\left(\alpha^2_{(k_1-1,k_2)} + \beta^2_{(k_1-1,k_2)}\right)\left(\alpha^2_{(k_1+1,k_2)} + \beta^2_{(k_1+1,k_2)}\right)}.$$

Then $\mathcal{M}(\mathbf{W}_{(\alpha,\beta)})$ *is hyponormal.*

Proof. Let $(k_1, k_2) \in \mathbb{Z}^2_+$ be arbitrary and let $\{\theta^{(k_2)}_k\}_{k\in\mathbb{Z}_+}$ be the sequence defined as $\theta^{(k_2)}_k = \dfrac{\alpha^2_{(k+1,k_2)} + \beta^2_{(k+1,k_2)}}{\alpha^2_{(k,k_2)} + \beta^2_{(k,k_2)}}$, $k \in \mathbb{Z}_+$. Using condition (ii), we have that

$$m(\alpha)^2_{(k_1,k_2)} + m(\beta)^2_{(k_1,k_2)} = \left(\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)}\right)\left(1 + \sqrt{\theta^{(k_2)}_{k_1}}\right)^2,$$

and

$$m(\alpha)^2_{(k_1-1,k_2)} + m(\beta)^2_{(k_1,k_2-1)} = \left(\alpha^2_{(k_1-1,k_2)} + \beta^2_{(k_1,k_2-1)}\right)\left(1 + \sqrt{\theta^{(k_2)}_{k_1-1}}\right)^2.$$

It is easy to see that condition (iii) implies that the sequence $\{\theta^{(k_2)}_k\}_{k\in\mathbb{Z}_+}$ is non-decreasing. Hence, $\theta^{(k_2)}_{k_1} \geq \theta^{(k_2)}_{k_1-1}$, and so

$$\left(1 + \sqrt{\theta^{(k_2)}_{k_1}}\right)^2 \geq \left(1 + \sqrt{\theta^{(k_2)}_{k_1-1}}\right)^2.$$

Using condition (i) and (3.1), we have that

$$\begin{aligned} m(\alpha)^2_{(k_1,k_2)} + m(\beta)^2_{(k_1,k_2)} &= \left(\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)}\right)\left(1 + \sqrt{\theta^{(k_2)}_{k_1}}\right)^2 \\ &\geq \left(\alpha^2_{(k_1-1,k_2)} + \beta^2_{(k_1,k_2-1)}\right)\left(1 + \sqrt{\theta^{(k_2)}_{k_1-1}}\right)^2 \\ &= m(\alpha)^2_{(k_1-1,k_2)} + m(\beta)^2_{(k_1,k_2-1)}. \end{aligned}$$

Thus, inequality (3.6) is satisfied. Using Corollary 3.3.4, we finally conclude that $\mathcal{M}(\mathbf{W}_{(\alpha,\beta)})$ is hyponormal. ■

Remark 3.3.6. *The analogue of the previous theorem where hyponormality is replaced with p-hyponormality, for any* $0 < p \leq 1$, *holds as well.*

Recall the following definition:

Definition 3.3.1. A sequence $\{\sigma_k\}_{k\in\mathbb{Z}_+}$ of real numbers is said to be a *Stieltjes moment sequence* if there exists a positive Borel measure μ on the closed half-line $[0,+\infty)$ such that

$$\sigma_k = \int_0^{+\infty} t^k \, d\mu(t), \quad k \in \mathbb{Z}_+.$$

The measure μ is called a *representing measure* of $\{\sigma_k\}_{k\in\mathbb{Z}_+}$.

The reader is referred to [20] for comprehensive information regarding the Stieltjes moment sequence and the Stieltjes moment problem.

Note that for all $k \in \mathbb{N}$, by applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\sigma_k^2 &= \left(\int_0^{+\infty} t^k \, d\mu(t)\right)^2 \\ &= \left(\int_0^{+\infty} t^{\frac{k-1}{2}} t^{\frac{k+1}{2}} \, d\mu(t)\right)^2 \\ &\leq \int_0^{+\infty} t^{k-1} \, d\mu(t) \cdot \int_0^{+\infty} t^{k+1} \, d\mu(t) \\ &= \sigma_{k-1}\sigma_{k+1},\end{aligned}$$

and thus, $|\sigma_k| \leq \sqrt{\sigma_{k-1}\sigma_{k+1}}$. Therefore, we obtain the following corollary of Theorem 3.3.5.

Corollary 3.3.7. *Let $\mathbf{W}_{(\alpha,\beta)} = (T_1, T_2)$ be a 2-variable weighted shift. Assume that the following conditions hold:*

(i) $\mathbf{W}_{(\alpha,\beta)}$ *is hyponormal;*

(ii) $\alpha_{(k_1+1,k_2)} = \alpha_{(k_1,k_2+1)}$ *and* $\beta_{(k_1+1,k_2)} = \beta_{(k_1,k_2+1)}$ *for all* $k_1, k_2 \geq 0$*;*

(iii) For all $k_2 \geq 0$*, the sequence* $\{\sigma_k^{(k_2)}\}_{k\in\mathbb{Z}_+}$ *given by*

$$\sigma_k^{(k_2)} = \alpha_{(k,k_2)}^2 + \beta_{(k,k_2)}^2, \quad k \in \mathbb{Z}_+,$$

is a Stieltjes moment sequence.

Then $\mathcal{M}(\mathbf{W}_{(\alpha,\beta)})$ *is hyponormal.*

Proof. The proof follows immediately from Theorem 3.3.5 and the discussion preceding the corollary. ∎

We finish this section by providing an example of an operator pair $\mathbf{T}$ such that $\sigma_T(\mathbf{T}) \neq \sigma_T(\mathcal{M}(\mathbf{T}))$. Moreover, we will show that $\mathcal{M}(\mathbf{T})$ is Taylor invertible, even though $\mathbf{T}$ is not.

Example 3.3.8. *Let* $\{e_{(k_1,k_2)}\}_{(k_1,k_2)\in\mathbb{Z}^2}$ *be the canonical basis of* $l^2(\mathbb{Z}^2)$ *and for* $n \in \mathbb{Z}$ *let*

$$\theta_n = \begin{cases} 1, & \text{if } n \text{ is even,} \\ 1/n^2, & \text{if } n \text{ is odd.} \end{cases}$$

Let $\mathbf{W}_{(\alpha,\beta)} = (T_1, T_2)$ *be the bilateral 2-variable weighted shift defined as follows:*

$$T_1 e_{(k_1,k_2)} = \alpha_{(k_1,k_2)} e_{(k_1+1,k_2)}, \quad \text{and} \quad T_2 e_{(k_1,k_2)} = \alpha_{(k_1,k_2)} e_{(k_1,k_2+1)},$$

where $\alpha_{(k_1,k_2)} = \theta_{k_1+k_2}$, $(k_1,k_2) \in \mathbb{Z}^2$.
We have that

$$T_1 e_{(k_1,k_1+1)} = \theta_{2k_1+1} e_{(k_1+1,k_1)} = \frac{1}{(2k_1+1)^2} e_{(k_1+1,k_1)}$$

and, similarly,

$$T_2 e_{(k_1,k_1+1)} = \frac{1}{(2k_1+1)^2} e_{(k_1,k_1+2)}.$$

Therefore, $T_1 e_{(k_1,k_1+1)} \to 0$ *and* $T_2 e_{(k_1,k_1+1)} \to 0$ *as* $k_1 \to \infty$*. Therefore,* $\mathbf{W}_{(\alpha,\beta)} = (T_1, T_2)$ *is not Taylor invertible (as it is not bounded below).*

On the other hand, $\mathcal{M}(\mathbf{W}_{(\alpha,\beta)}) = (\mathcal{M}_1(\mathbf{W}_{(\alpha,\beta)}), \mathcal{M}_2(\mathbf{W}_{(\alpha,\beta)}))$ *is also bilateral 2-variable weighted shift (Remark 3.3.2), and by Corollary 3.3.3, we have*

$$\mathcal{M}_1(\mathbf{W}_{(\alpha,\beta)}) e_{(k_1,k_2)} = \frac{\theta_{k_1+k_2} + \theta_{k_1+k_2+1}}{2} e_{(k_1+1,k_2)},$$

and

$$\mathcal{M}_2(\mathbf{W}_{(\alpha,\beta)}) e_{(k_1,k_2)} = \frac{\theta_{k_1+k_2} + \theta_{k_1+k_2+1}}{2} e_{(k_1,k_2+1)}.$$

Since for each $(k_1,k_2) \in \mathbb{Z}^2$*, we have*

$$1 \geq \frac{\theta_{k_1+k_2} + \theta_{k_1+k_2+1}}{2} \geq \frac{\max\{\theta_{k_1+k_2}, \theta_{k_1+k_2+1}\}}{2} = \frac{1}{2},$$

it follows that $\mathcal{M}(\mathbf{W}_{(\alpha,\beta)})$ *is Taylor invertible.* ■

CHAPTER 4

SUBNORMAL DUALS AND COMPLETION TO NORMALITY

Motivated by the definitions of subnormal duals (see Section 1.2.1), and with the main goal to consider the completion of upper triangular 2×2 operator matrix (with known diagonal blocks) to a normal operator, we introduce the following definition:

Definition 4.0.1. Let $A \in \mathfrak{B}(\mathcal{H})$ and $B \in \mathfrak{B}(\mathcal{K})$. We say that operators A and B are *normal complements* if there exists $C \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$ such that the operator matrix

$$M_C = \begin{bmatrix} A & C \\ 0 & B^* \end{bmatrix} \tag{$\star$}$$

is normal.

Also, let

$$\mathfrak{N}(A, B) = \{C \in \mathfrak{B}(\mathcal{K}, \mathcal{H}) : M_C \text{ given by } (\star) \text{ is normal}\}.$$

Clearly, operators A and B are normal complements if and only if there exists $C \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$ such that

$$A^*A - AA^* = CC^* \tag{4.1}$$

$$B^*B - BB^* = C^*C \tag{4.2}$$

$$A^*C = CB. \tag{4.3}$$

It is important to observe the difference between duals and normal complements, since if A and B are normal complements by Definition 4.0.1, then it does not follow that B is the dual of A, by definition introduced in [44], as A may not even be pure. As already mentioned, the definition 4.0.1 is introduced with the aim of answering the question on the completion of operator matrices to normality, and thus we do not impose any additional restrictions on operators A and B.

4.1 DIFFERENT CHARACTERIZATIONS OF NORMAL COMPLEMENTS

In the next two theorems, we will give some characterizations for operators A, B to be normal complements in terms of some blocks of these operators. First, recall that an operator $T \in \mathfrak{B}(\mathcal{H})$ is called *posinormal* if there exists a positive operator $Q \in \mathfrak{B}(\mathcal{H})$ such that $TT^* = T^*QT$. That every hyponormal operator is posinormal follows directly by Theorem 1.1.7 and the fact that T is posinormal if and only if $\mathcal{R}(T) \subseteq \mathcal{R}(T^*)$ (see [155, Theorem 2.1]).

Theorem 4.1.1. *Let $A \in \mathfrak{B}(\mathcal{H})$, $B \in \mathfrak{B}(\mathcal{K})$. Let $A_1 = P^{cr}_{\mathcal{N}(A)^\perp} A|_{\mathcal{N}(A)^\perp}$ and $B_1 = P^{cr}_{\mathcal{N}(B)^\perp} B|_{\mathcal{N}(B)^\perp}$. The following conditions are equivalent:*

(i) Operators A and B are normal complements;

(ii) Operators A and B are posinormal, and A_1 and B_1 are normal complements.

Moreover,

$$\mathfrak{N}(A, B) = \{C_1 \oplus 0 : C_1 \in \mathfrak{N}(A_1, B_1)\}.$$

Proof. $(i) \Rightarrow (ii)$: Assume that A and B are normal complements and let $C \in \mathfrak{N}(A, B)$. Then A and B are subnormal, and therefore posinormal. Thus, they have the following representations:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{pmatrix} \mathcal{N}(A)^\perp \\ \mathcal{N}(A) \end{pmatrix} \to \begin{pmatrix} \mathcal{N}(A)^\perp \\ \mathcal{N}(A) \end{pmatrix}, \tag{4.4}$$

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{pmatrix} \mathcal{N}(B)^\perp \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathcal{N}(B)^\perp \\ \mathcal{N}(B) \end{pmatrix}. \tag{4.5}$$

Let $C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} : \begin{pmatrix} \mathcal{N}(B)^\perp \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathcal{N}(A)^\perp \\ \mathcal{N}(A) \end{pmatrix}$. By (4.1), we have that $C_3 = 0$ and $C_4 = 0$, which by (4.2) gives that $C_2 = 0$. Therefore $C = C_1 \oplus 0$, where by (4.1)-(4.3) we have that $C_1 \in \mathfrak{N}(A_1, B_1)$, i.e. A_1 and B_1 are normal complements.

$(ii) \Rightarrow (i)$: The converse is obvious. ■

Theorem 4.1.2. *Let $A \in \mathfrak{B}(\mathcal{H})$, $B \in \mathfrak{B}(\mathcal{K})$ be given with pure parts A_p and B_p, respectively. The following conditions are equivalent:*

(i) Operators A and B are normal complements;

(ii) Operators A_p and B_p are normal complements.

Moreover,

$$\mathfrak{N}(A, B) = \{C_1 \oplus 0 : C_1 \in \mathfrak{N}(A_p, B_p)\}.$$

Proof. $(i) \Rightarrow (ii)$: Assume that operators A and B are normal complements and let $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ be such that M_C is normal. Then A and B are subnormal, and using Theorem 1.2.2, we have an appropriate decompositions of $\mathcal{H}$ and $\mathcal{K}$ such that

$$A = \begin{bmatrix} A_p & 0 \\ 0 & A_n \end{bmatrix} : \begin{pmatrix} \mathcal{H}_p(A) \\ \mathcal{H}_n(A) \end{pmatrix} \to \begin{pmatrix} \mathcal{H}_p(A) \\ \mathcal{H}_n(A) \end{pmatrix}, \tag{4.6}$$

$$B = \begin{bmatrix} B_p & 0 \\ 0 & B_n \end{bmatrix} : \begin{pmatrix} \mathcal{K}_p(B) \\ \mathcal{K}_n(B) \end{pmatrix} \to \begin{pmatrix} \mathcal{K}_p(B) \\ \mathcal{K}_n(B) \end{pmatrix} \tag{4.7}$$

Let $C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} : \begin{pmatrix} \mathcal{K}_p(B) \\ \mathcal{K}_n(B) \end{pmatrix} \to \begin{pmatrix} H_p(A) \\ H_n(A) \end{pmatrix}$. By (4.1) and normality of $A_n \in \mathfrak{B}(\mathcal{H}_n(A))$ we have that $C_3 = 0$ and $C_4 = 0$, which further by (4.2) and normality of B_n implies that $C_2 = 0$. Now (4.3) implies $A_p^* C_1 = C_1 B_p$. Therefore, $C = C_1 \oplus 0$ and $C_1 \in \mathfrak{N}(A_p, B_p)$. Thus, A_p and B_p are normal complements.

$(ii) \Rightarrow (i)$: The converse is obvious. ■

The next theorem presents necessary and sufficient conditions for the completion of operator matrix M_C given by ($\star$) to a normal operator, in terms of the existence of a partial isometry with prescribed initial and final spaces. Let us remark that the assumption that $A \in \mathfrak{B}(\mathcal{H})$ and $B \in \mathfrak{B}(\mathcal{K})$ are hyponormal operators is natural since it is the necessary condition for A and B to be normal complements.

Theorem 4.1.3. *Let $A \in \mathfrak{B}(\mathcal{H})$ and $B \in \mathfrak{B}(\mathcal{K})$ be hyponormal operators. The following conditions are equivalent:*

(i) *Operators A and B are normal complements;*

(ii) *There exists a partial isometry $U \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$ with the initial space $\mathcal{M} \supseteq \overline{\mathcal{R}([B^*, B])}$ and final space $\mathcal{N} \supseteq \overline{\mathcal{R}([A^*, A])}$ such that*

$$[A^*, A]U = U[B^*, B], \tag{4.8}$$

$$A^*[A^*, A]^{1/2}U = U[B^*, B]^{1/2}B. \tag{4.9}$$

Moreover,

$$\mathfrak{N}(A, B) = \{U[B^*, B]^{1/2} : \; U \text{ is a partial isometry from part } (ii)\}.$$

Proof. $(i) \Rightarrow (ii)$: Let $C \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$ be such that M_C is normal and let $C = U|C|$ be its polar decomposition. By (4.1), we have $|C^*| = [A^*, A]^{1/2}$, while (4.2) gives $|C| = [B^*, B]^{1/2}$. Then $C = U[B^*, B]^{1/2}$, and since $C^* = U^*|C^*|$ is a polar decomposition for C^* (Theorem 1.1.9), it follows that

$$[A^*, A]^{1/2}U = U[B^*, B]^{1/2},$$

which further implies that

$$[A^*,A]U = [A^*,A]^{1/2}U[B^*,B]^{1/2} = U[B^*,B].$$

Finally, from (4.3) we get

$$A^*[A^*,A]^{1/2}U = A^*C = CB = U[B^*,B]^{1/2}B.$$

It is easy to check that $\overline{\mathcal{R}([B^*,B])}$ and $\overline{\mathcal{R}([A^*,A])}$ are respectively, initial and final space of U.

$(ii) \Rightarrow (i)$: Assume that there exists a partial isometry $U \in \mathfrak{B}(\mathcal{K},\mathcal{H})$ with the mentioned properties and let $C := U[B^*,B]^{1/2}$. We will show that the operator matrix M_C is normal.

Note that (4.8) implies that $C = [A^*,A]^{1/2}U$. Indeed, let

$$V = \begin{bmatrix} 0 & 0 \\ U & 0 \end{bmatrix} \quad \text{and } P = \begin{bmatrix} [B^*,B] & 0 \\ 0 & [A^*,A] \end{bmatrix}.$$

Then, $PV = VP$ so by $P^{1/2}V = VP^{1/2}$, we get $C = U[B^*,B]^{1/2} = [A^*,A]^{1/2}U$.

By (4.9), it follows that $A^*C = CB$. Using the fact that $\overline{R(T)} = \overline{R(T^{1/2})}$ for any positive operator T, we obtain

$$C^*C = [B^*,B]^{1/2}U^*U[B^*,B]^{1/2} = B^*B - BB^*.$$

Furthermore,

$$CC^* = [A^*,A]^{1/2}UU^*[A^*,A]^{1/2} = A^*A - AA^*.$$

Thus, equalities (4.1)-(4.3) are satisfied, and thus A and B are normal complements. ■

Now, let us consider the following notation introduced in [106]: For two positive operators $A \in \mathfrak{B}(\mathcal{H})$ and $B \in \mathfrak{B}(\mathcal{K})$,

$$\mathfrak{C}(A,B) = \{A^{1/2}UB^{1/2} : U \in \mathfrak{B}(\mathcal{K},\mathcal{H}),\ \|U\| \leq 1\}.$$

In the next theorem, we show that for normal complements $A, B \in \mathfrak{B}(\mathcal{H})$ an operator $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ for which M_C is a normal operator can be represented for any $\lambda \in [0,1]$ in the form $C = [A^*,A]^{\frac{\lambda}{2}}U_\lambda[B^*,B]^{\frac{1-\lambda}{2}}$, where $U_\lambda \in \mathfrak{B}(\mathcal{K},\mathcal{H})$ is a contraction.

Theorem 4.1.4. *Let $A, B \in \mathfrak{B}(\mathcal{H})$ be normal complements. Then*

$$\mathfrak{N}(A,B) \subseteq \bigcap_{\lambda\in[0,1]} \mathfrak{C}([A^*,A]^{\lambda},[B^*,B]^{1-\lambda}). \tag{4.10}$$

Proof. Let $C \in \mathfrak{N}(A,B)$ and let $\lambda \in [0,1]$ be arbitrary. Now, the fact that for an arbitrary operator $A \in \mathfrak{B}(\mathcal{H})$ and all $x, y \in \mathcal{H}$ (see [111, Theorem 3]),

$$|\langle Ax, y\rangle| \leq \langle |A|^{2\lambda}x, x\rangle\langle |A^*|^{2(1-\lambda)}y, y\rangle, \quad 0 \leq \lambda \leq 1,$$

together with (4.1) and (4.2), gives that

$$|\langle Cx, y\rangle| \leq \langle [B^*, B]^{\lambda}x, x\rangle\langle [A^*, A]^{1-\lambda}y, y\rangle.$$

Now, by Lemma 1 from [117] we have that

$$\begin{bmatrix} [B^*, B]^{\lambda} & C^* \\ C & [A^*, A]^{1-\lambda} \end{bmatrix} \geq 0,$$

which implies the existence of a contraction $S \in \mathfrak{B}(\mathcal{H})$ such that

$$C^* = [B^*, B]^{\frac{\lambda}{2}} S [A^*, A]^{\frac{1-\lambda}{2}},$$

i.e. $C = [A^*, A]^{\frac{1-\lambda}{2}} S^* [B^*, B]^{\frac{\lambda}{2}}$ (see [85, p. 547]). This implies that

$$C \in \mathfrak{C}([A^*, A]^{1-\lambda}, [B^*, B]^{\lambda}).$$

Since $C \in \mathfrak{N}(A,B)$ and $\lambda \in [0,1]$ were arbitrary, inclusion in (4.10) follows immediately. ■

In the case when $A \in \mathfrak{B}(\mathcal{H})$ and $B \in \mathfrak{B}(\mathcal{K})$ are normal complements and one of them is quasinormal we have that both are quasinormal. Furthermore, in that case, pure parts of A and B are unitarily equivalent. The converse also holds which will be shown in the next theorem.

Theorem 4.1.5. *Let $A \in \mathfrak{B}(\mathcal{H})$ and $B \in \mathfrak{B}(\mathcal{K})$ be such that one of them is quasinormal. The following conditions are equivalent:*

(i) A and B are normal complements;

(ii) Pure parts of A and B are unitarily equivalent.

Proof. $(i) \Rightarrow (ii)$: Assume that A and B are normal complements and let C be such that M_C is normal. Observe that if one of A and B is quasinormal, then both of them are quasinormal. Indeed, using (4.1) (or from Lemma 1.2.10), we have that A is quasinormal if and only if $A^*C = 0$, i.e. $CB = 0$ which is by (4.2) equivalent with the fact that B is quasinormal. Thus, the assumption of the theorem implies that A and B are quasinormal. Let $A = A_p \oplus A_n$ and $B = B_p \oplus B_n$ be the pure-normal decompositions of operators A and B, respectively. Since A and B are quasinormal, it follows that A_p and B_p are pure quasinormal operators. Also, by Theorem 4.1.2, it follows that A_p and B_p are normal complements. In other words, there exists $D \in \mathfrak{B}(\mathcal{K}_p(B), \mathcal{H}_p(A))$ such that the operator matrix

$$N_D = \begin{bmatrix} A_p & D \\ 0 & B_p^* \end{bmatrix}$$

is normal. Since B_p is pure, by Theorem 1.2.4, we have that N_D is the minimal normal extension for A_p. Also, A_p is a pure quasinormal operator, and so it is a self-dual, by Theorem 1.2.9. Therefore, it is unitarily equivalent to its dual, i.e. A_p and B_p are unitarily equivalent.

$(ii) \Rightarrow (i)$: Now assume that the pure parts of A and B are unitarily equivalent. Then, there exists a unitary transformation $U \in \mathcal{B}(\mathcal{H}_p(A), \mathcal{K}_p(B))$ such that $A_p = U^* B_p U$. Quasinormality of one of the operators A and B implies the quasinormality of its pure part which by $A_p = U^* B_p U$ implies that pure parts of both operators are quasinormal. Thus, A_p is a pure quasinormal operator, and since any pure quasinormal operator is a self-dual subnormal operator, there exists $C \in \mathcal{B}(\mathcal{H}_p(A))$ such that

$$N_C = \begin{bmatrix} A_p & C \\ 0 & A_p^* \end{bmatrix}$$

is normal. Now let $V = \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix}$. Using the fact that $A_p = U^* B_p U$, we have

$$\begin{aligned} V N_C' V^* &= \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} A_p & C \\ 0 & A_p^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & U^* \end{bmatrix} \\ &= \begin{bmatrix} A_p & CU^* \\ 0 & UA_pU^* \end{bmatrix} = \begin{bmatrix} A_p & CU^* \\ 0 & B_p^* \end{bmatrix}. \end{aligned}$$

Since N_C is normal and V is unitary, it follows that $V N_C' V^*$ is normal, and thus A_p and B_p are normal complements. Theorem 4.1.2 now yields that A and B are normal complements. This completes the proof. ∎

The previous theorem can be treated as a generalization of the next simple observation.

Corollary 4.1.6. *Let $A \in \mathfrak{B}(\mathcal{H})$ be normal and $B \in \mathfrak{B}(\mathcal{K})$ be subnormal operator. The following conditions are equivalent:*

(i) A and B are normal complements;

(ii) B is normal.

Proof. The proof follows immediately by noting that the normality of A implies that the pure parts of A and B are unitarily equivalent if and only if B is normal. ∎

4.2 Joint spectral properties of normal complements

It is interesting to remark that in the case when $A \in \mathfrak{B}(\mathcal{H})$ and $B \in \mathfrak{B}(\mathcal{K})$ are given operators the problem of completion of the operator matrix M_C to Fredholm operator has one very interesting property. Namely, the existence of an operator $C \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$ such that M_C is a Fredholm operator is equivalent to the existence of such invertible $C \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$ (see [64]). The same property holds for some other completions (see [59, 60, 190]). As we will see in the next result, this is not the case for the completion of M_C to a normal operator. In other words, if $A \in \mathfrak{B}(\mathcal{H})$ and $B \in \mathfrak{B}(\mathcal{K})$ are normal complements, then $C \in \mathfrak{N}(A, B)$ can not be invertible.

Theorem 4.2.1. *Let $A, B \in \mathfrak{B}(\mathcal{H})$ be normal complements. Then $0 \in \sigma(C)$ for all $C \in \mathfrak{N}(A, B)$.*

Proof. Let $C \in \mathfrak{N}(A, B)$ be arbitrary. Assume to the contrary, that $0 \notin \sigma(C)$, i.e. C is invertible. By (4.3), we have that $C^*A = B^*C^*$, i.e. A and B^* are similar. Since A and B^* are subnormal, by [89, Corollary 1] it follows that both are normal and unitarily equivalent. Then (4.1) implies that $C = 0$, which is a contradiction. Thus, $0 \in \sigma(C)$. ■

Furthermore, in the next theorem, we will prove that for any subnormal operator $A \in \mathcal{B}(\mathcal{H})$ the spectrum of its self-commutator contains 0, i.e. self-commutator is not invertible.

Theorem 4.2.2. *Let $A \in \mathfrak{B}(\mathcal{H})$ be a subnormal (or hyponormal) operator. Then $0 \in \sigma([A^*, A])$.*

Proof. Since A is subnormal it follows that A is hyponormal, i.e. $A^*A - AA^* \geq 0$. If we suppose that $0 \notin \sigma([A^*, A])$, then $P = A^*A - AA^* > 0$ and we have that

$$\|A^*A\| = \sup_{\|x\|=1} (A^*Ax, x) = \sup_{\|x\|=1} ((AA^*x, x) + (Px, x)) > \|AA^*\|,$$

which is not true (since $\|A^*A\| = \|AA^*\|$). Thus $0 \in \sigma([A^*, A])$. ■

From the above theorem, we can conclude that for $C \in \mathfrak{N}(A, B)$ we have that $0 \in \sigma_l(C) \cap \sigma_r(C)$, i.e. we have the following corollary:

Corollary 4.2.3. *Let $A, B \in \mathfrak{B}(\mathcal{H})$ be normal complements. Then $0 \in \sigma([A^*, A]) \cap \sigma([B^*, B])$ and $\sigma([A^*, A]) = \sigma([B^*, B])$.*

In general, when we consider different properties of M_C, we can realize the similarity of A and B concerning some properties. Having in mind certain results on completions of the upper triangular operator matrix to invertibility and Fredholmness, we can reach the following conclusions:

Theorem 4.2.4. *Let $A \in \mathfrak{B}(\mathcal{H})$ and $B \in \mathfrak{B}(\mathcal{K})$ be given. Then the following hold:*

(i) *If the operator matrix M_C given by $(\star)$ is invertible for some $C \in \mathfrak{N}(A, B)$, then both A and B are left invertible. Furthermore, if M_C is invertible, then the invertibility of one of the operators A and B implies the invertibility of the other one.*

(ii) *If the operator matrix M_C given by $(\star)$ is Fredholm for some $C \in \mathfrak{N}(A, B)$, then A and B are left semi-Fredholm. Furthermore, if M_C is Fredholm, then the Fredholmness of one of the operators A and B implies the Fredholmness of the other one.*

Proof. (i) This follows from the well-known result that concerns the completion of an upper triangular operator matrix to an invertible operator (see [81, 100]). So, there exists $C \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$ such that the operator matrix M_C is invertible if and only if A and B are left invertible and $\dim \mathcal{N}(B^*) = \dim \mathcal{R}(A)^\perp$.

Part (ii) follows by Theorem 3.8 from [64]. ■

In the next theorems, we will consider special cases when $A \in \mathfrak{B}(\mathcal{H})$ and $B \in \mathfrak{B}(\mathcal{K})$ are normal complements or when $A \in \mathfrak{B}(\mathcal{H})$ and $B \in \mathfrak{B}(\mathcal{K})$ are normal complements such that one of them is quasinormal. First, we will see that if $A \in \mathfrak{B}(\mathcal{H})$ and $B \in \mathfrak{B}(\mathcal{K})$ are normal complements and if for some $C \in \mathfrak{N}(A, B)$ we have that M_C is injective, then both A and B are injective (that is not valid in the general case). Also, we will show that instead of implications that we have in items $(i) - (ii)$ of Theorem 4.2.4, we will get equivalences when $A \in \mathfrak{B}(\mathcal{H})$ and $B \in \mathfrak{B}(\mathcal{K})$ are normal complements such that one of them is quasinormal.

Theorem 4.2.5. *Let $A \in \mathfrak{B}(\mathcal{H})$ and $B \in \mathfrak{B}(\mathcal{K})$ be normal complements. If the operator matrix M_C is injective for some $C \in \mathfrak{N}(A, B)$, then both A and B are injective.*

Proof. Assume that M_C is injective for some $C \in \mathfrak{N}(A, B)$. Then, it is easy to see that A is injective and $\mathcal{N}(C) \cap \mathcal{N}(B^*) = \{0\}$. Since $\mathcal{N}(C) = \mathcal{N}(C^*C) = \mathcal{N}([B^*, B])$, and $\mathcal{N}([B^*, B]) \cap \mathcal{N}(B^*) = \mathcal{N}(B) \cap \mathcal{N}(B^*)$, we can conclude that $\mathcal{N}(B) \cap \mathcal{N}(B^*) = \{0\}$. Since operator B is subnormal, we have $\mathcal{N}(B) \subseteq \mathcal{N}(B^*)$. Thus $\mathcal{N}(B) = \{0\}$, i.e. B is injective. ■

The case when one of $A \in \mathfrak{B}(\mathcal{H})$ and $B \in \mathfrak{B}(\mathcal{K})$ which are normal complements is quasinormal we consider in the next theorem:

Theorem 4.2.6. *Let $A \in \mathfrak{B}(\mathcal{H})$ and $B \in \mathfrak{B}(\mathcal{K})$ be normal complements such that one of them is quasinormal and let $C \in \mathfrak{N}(A, B)$. Then the following hold:*

(i) *M_C is invertible if and only if both A and B are left invertible;*

(ii) *M_C is Fredholm if and only if both A and B are left semi-Fredholm operators;*

(iii) *M_C is regular if and only if both A and B are regular operators.*

Proof. Let us suppose that A is quasinormal. Then $A^*C = 0$, which together with (4.1) − (4.3) implies that

$$M_C^* M_C = \begin{bmatrix} A^*A & 0 \\ 0 & B^*B \end{bmatrix}. \tag{4.11}$$

(i) Using the fact that a normal operator is invertible if and only if it is left invertible, we get by (4.11) that M_C is invertible if and only if A and B are left invertible.

(ii) Using (4.11) and normality of M_C, it is clear that M_C is Fredholm if and only if A and B are left semi Fredholm operators.

(iii) Since the regularity of operator X is equivalent with the regularity of X^*X, it is clear from (4.11) that M_C is regular if and only if A and B are regular. ■

By Theorem 4.2.6 we can conclude that if $A \in \mathfrak{B}(\mathcal{H})$ and $B \in \mathfrak{B}(\mathcal{K})$ are normal complements such that one of them is quasinormal, then the operator matrix M_C is invertible (Fredholm or regular) for some $C \in \mathfrak{N}(A, B)$ if and only if it is invertible (Fredholm or regular) for all $C \in \mathfrak{N}(A, B)$.

Concerning regularity of M_C in the case when $C \in \mathfrak{N}(A, B)$ we have the following result:

Theorem 4.2.7. *Let $A \in \mathfrak{B}(\mathcal{H})$ and $B \in \mathfrak{B}(\mathcal{K})$ be normal complements such that one of them is quasinormal and let $C \in \mathfrak{N}(A, B)$. If A, B and C are all regular, then $(A^*)^\dagger$ and $(B^*)^\dagger$ are normal complements and $(C^*)^\dagger \in \mathfrak{N}((A^*)^\dagger, (B^*)^\dagger)$.*

Proof. First, observe that the Moore-Penrose inverse of a regular normal operator is also normal. The proof now follows by the fact that $A^*C = CB = 0$, since in that case it is easy to check that

$$M_C^\dagger = \begin{bmatrix} A^\dagger & 0 \\ C^\dagger & (B^*)^\dagger \end{bmatrix}.$$

■

Definition 4.2.1. An operator $A \in \mathfrak{B}(\mathcal{H})$ is called a *self-complemented* subnormal operator if $\mathfrak{N}(A, A)$ is a non-empty set.

In the sequel, we shall write $\mathfrak{N}(A, A) = \mathfrak{N}(A)$. Following (4.1) − (4.3), we have that A is a self-complemented subnormal operator if and only if there exists a normal operator $C \in \mathfrak{B}(\mathcal{H})$ such that

$$[A^*, A] = CC^* \quad \text{and} \quad A^*C = CA. \tag{4.12}$$

In that case, $\mathfrak{N}(A)$ consists of normal operators only.

Obviously, every quasinormal operator A is self-complemented subnormal (take $C = (A^*A - AA^*)^{1/2}$) and A is self-complemented if and only if the pure part of A is self-dual.

The next theorem gives that any linear combination of $C \in \mathfrak{N}(A)$ and its adjoint is non invertible operator.

Theorem 4.2.8. *Let $A \in \mathfrak{B}(\mathcal{H})$ be a self-complemented subnormal operator. Then*

$$0 \in \sigma(\lambda C + \mu C^*)$$

for any $C \in \mathfrak{N}(A)$ and any $\lambda, \mu \in \mathbb{C}$.

Proof. Assume that $A \in \mathfrak{B}(\mathcal{H})$ is a self-complemented subnormal operator and let $C \in \mathfrak{N}(A)$ and $\lambda, \mu \in \mathbb{C}$ be arbitrary. Observe that $A^*C = CA$ implies that $A^*C^* = C^*A$. Therefore,

$$A^*(\lambda C + \mu C^*) = (\lambda C + \mu C^*)A.$$

We can now finish the proof in an analogous way as in Theorem 4.2.1. ■

As a corollary, we have that real and imaginary parts of $C \in \mathfrak{N}(A)$ are not invertible.

Corollary 4.2.9. *Let $A \in \mathfrak{B}(\mathcal{H})$ be a self-complemented subnormal operator and let $C \in \mathfrak{N}(A)$. Then* $\text{Re}\,(C)$ *and* $\text{Im}\,(C)$ *are not invertible.*

If A is self-complemented and $C \in \mathfrak{N}(A)$, Theorem 4.2.1 yields that $0 \in \sigma(C)$. The following results give some sufficient conditions for $\sigma(C) = \{0\}$ which is equivalent to the normality of A.

Theorem 4.2.10. *Let $A \in \mathfrak{B}(\mathcal{H})$ be a self-complemented subnormal operator. If there exists $C \in \mathfrak{N}(A)$ such that A and C^n commute for some $n \in \mathbb{N}$, then A is normal.*

Proof. Let $A \in \mathfrak{B}(\mathcal{H})$ be a self-complemented subnormal operator and assume that there exists $C \in \mathfrak{N}(A)$ and $n \in \mathbb{N}$ such that $AC^n = C^nA$. Since C is normal we have that C^n is normal, and so by Theorem 1.1.12, it follows that $A(C^n)^* = (C^n)^*A$. Therefore,

$$A(CC^*)^n = C^nA(C^n)^* = (CC^*)^nA.$$

Theorem 1.1.1 now implies that A commutes with $CC^* = [A^*, A]$. The normality of A now directly follows from [150, Corollary 1]. ■

Theorem 4.2.11. *Let $A \in \mathfrak{B}(\mathcal{H})$ be a self-complemented subnormal operator. If $C \in \mathfrak{N}(A)$ satisfies any of the following conditions:*

(*i*) $AC = CA$,

(*ii*) $A^*C = AC$,

(*iii*) $\mathcal{R}(C) \perp \mathcal{R}(\text{Im}\,(A))$,

(*iv*) $\text{Re}\,(C)$ *and* $\text{Re}\,(C^2)$ *commute with* A,

then A is normal.

Proof. If C satisfies any of the conditions $(i)-(iii)$, then the proof follows immediately from Theorem 4.2.10 and (4.12).

Now assume that C satisfies condition (iv). Since A commutes with $\mathrm{Re}\,(C)$, we have that

$$A(\mathrm{Re}\,(C))^2 = (\mathrm{Re}\,(C))^2 A.$$

Using the normality of C, it follows that

$$A(C^2 + 2CC^* + (C^*)^2) = (C^2 + 2CC^* + (C^*)^2)A,$$

i.e. $A(\mathrm{Re}\,(C^2) + CC^*) = (\mathrm{Re}\,(C^2) + CC^*)A$. Now, $A\mathrm{Re}\,(C^2) = \mathrm{Re}\,(C^2)A$ implies that A commutes with $CC^* = [A^*, A]$, and so A is normal. ∎

Corollary 4.2.12. *Let $A \in \mathfrak{B}(\mathcal{H})$ be a self-dual subnormal operator. Then A does not commute with any $C \in \mathfrak{N}(A)$.*

4.3 Self-duality of Aluthge and Duggal Transforms

This section commences with two particular results that provide sufficient conditions that Aluthge and Duggal transforms of pure hyponormal (semi-hyponormal) operator $A \in \mathfrak{B}(\mathcal{H})$ are self-dual subnormal operators.

Let us first observe that if $A \in \mathfrak{B}(\mathcal{H})$ is a pure hyponormal operator with a dense range, then A is injective and in the polar decomposition of $A = U|A|$, we have that U is a unitary operator. Indeed, by $A^*A \geq AA^*$ we have that $\mathcal{N}(A) \subseteq \mathcal{N}(A^*) = \mathcal{R}(A)^{\perp} = \{0\}$. Also, by $\mathcal{N}(U) = \mathcal{N}(A)$, $\mathcal{R}(U) = \overline{\mathcal{R}(A)}$, $UU^* = P_{\mathcal{R}(U)}$ and $U^*U = P_{\mathcal{N}(U)^{\perp}}$, it follows that U is a unitary operator. Moreover, using the polar decomposition of $A = U|A|$ and unitarity of U, we have that hyponormality of A is equivalent with

$$|A|^2 \geq U|A|^2U^*,$$

i.e.

$$U^*|A|^2U \geq |A|^2.$$

Now, we are ready to prove our first result.

Theorem 4.3.1. *Let $A \in \mathfrak{B}(\mathcal{H})$ be a pure hyponormal operator of dense range with the polar decomposition $A = U|A|$ and let $P = (U^*|A|^2U - |A|^2)^{1/2}$. If $P\widehat{A}$ is a self-adjoint operator, then $\widehat{A}$ is a self-dual subnormal operator.*

Proof. First observe that $[(\widehat{A})^*, \widehat{A}] = U^*|A|^2U - |A|^2 = P^2$ and that self-adjointness of $P\widehat{A}$ implies that $P\widehat{A} = (\widehat{A})^*P$. By Theorem 1.2.5, these two facts will be sufficient for $\widehat{A}$ to be a self-dual subnormal operator if we show that $\widehat{A}$ is pure. Suppose that

$\mathcal{L}$ is a non-zero reducing subspace for $\widehat{A}$ on which $\widehat{A}$ is normal. Then $|A|U(\mathcal{L}) \subseteq \mathcal{L}$, $U^*|A|(\mathcal{L}) \subseteq \mathcal{L}$ and

$$U^*|A|^2Ux = |A|^2x, \text{ for all } x \in \mathcal{L}. \tag{4.13}$$

These imply that

$$|A|^2(\mathcal{L}) = U^*|A|(|A|U(\mathcal{L})) \subseteq \mathcal{L},$$

so $\mathcal{L}$ is a reducing subspace for $|A|^2$, and thus it is a reducing subspace for $|A|$, as well. Since $\mathcal{N}(|A|) = \mathcal{N}(A) = \{0\}$, it follows that $|A|$ has dense range and since $\mathcal{L}$ is a reducing space for $|A|$ it follows that $\overline{\mathcal{R}(|A|^2|_{\mathcal{L}})} = \overline{\mathcal{R}(|A||_{\mathcal{L}})} = \mathcal{L}$. Now by $U^*|A|(\mathcal{L}) \subseteq \mathcal{L}$ and $\overline{\mathcal{R}(|A||_{\mathcal{L}})} = \mathcal{L}$, we get that $\mathcal{L}$ is an invariant space for U^*. Using (4.13), we have that $U|A|^2x = |A|^2Ux$, for any $x \in \mathcal{L}$ and from $\overline{\mathcal{R}(|A|^2|_{\mathcal{L}})} = \mathcal{L}$ it follows that

$$U(\mathcal{L}) \subseteq \overline{|A|^2U(\mathcal{L})} \subseteq \overline{|A|(\mathcal{L})} = \mathcal{L}.$$

Hence, $\mathcal{L}$ is invariant for U. Since it is also invariant for U^*, it is reducing for U. Thus, $\mathcal{L}$ is a reducing subspace for both $|A|$ and U, and so it is a reducing subspace for A. Also, by (4.13) and the fact that for any $x \in \mathcal{L}$ we have that $U^*x \in \mathcal{L}$, it follows that

$$U^*|A|^2UU^*x = |A|^2U^*x \quad \text{for all } x \in \mathcal{L},$$

i.e.

$$|A|^2x = U|A|^2U^*x \quad \text{for all } \ x \in \mathcal{L},$$

which is equivalent to $AA^*x = A^*Ax$, for any $x \in \mathcal{L}$. Hence, A is normal on $\mathcal{L}$, which is a contradiction. We conclude that $\widehat{A}$ is pure, which completes the proof. ∎

In order to prove the next result that considers the case when the Aluthge transform of a pure semi-hyponormal operator is a self-dual subnormal operator, we need the following auxiliary result:

Theorem 4.3.2. *[38, Lemma 4] Let $T = U|T|$ be a pure p-hyponormal operator with dense range. Then the Aluthge transformation $\widetilde{T}$ is pure $(p+\frac{1}{2})$-hyponormal.*

Theorem 4.3.3. *Let $A \in \mathfrak{B}(\mathcal{H})$ be a pure semi-hyponormal operator of dense range with the polar decomposition $A = U|A|$ and let $P := (U^*|A|U - U|A|U^*)^{1/2}$. If $[A^*, \widehat{A}]$ and $P\widehat{A}|A|^{1/2}$ are self-adjoint operators, then $\widetilde{A}$ is a self-dual subnormal operator.*

Proof. Since A is semi-hyponormal and has a dense range, we have that U is unitary. By the properties of the polar decomposition we know that $A = U|A|$ implies that $A = |A^*|U$ and $|A| = U^*A$ (Theorem 1.1.9). Hence $|A| = U^*|A^*|U$. Since A is semi-hyponormal, we have that

$$|A| = U^*|A^*|U \leq U^*|A|U,$$

i.e. $U|A|U^* \leq |A|$ because U is unitary. Thus, $U^*|A|U - U|A|U^*$ is positive, and so P is well-defined.

Also, we have that

$$[(\widetilde{A})^*, \widetilde{A}] = |A|^{1/2}P^2|A|^{1/2} = Q^*Q,$$

where $Q = P|A|^{1/2}$ is a positive operator. Positivity of Q follows by the fact that $[A^*, \widehat{A}]$ is self-adjoint if and only if P^2 commute with $|A|$ that is equivalent with the commutativity of P and $|A|^{1/2}$. Also, using that $P\widehat{A}|A|^{1/2}$ is a self-adjoint operator and that P and $|A|^{1/2}$ commute, we get that $Q\widetilde{A} = (\widetilde{A})^*Q$. Finally, using Theorem 4.3.2 and Theorem 1.2.5 we can reach the conclusion that $\widetilde{A}$ is a self-dual subnormal operator. ∎

Definition 4.3.1 (cf. [119]). Let $T = U|T|$ be the polar decomposition of $T \in \mathfrak{B}(\mathcal{H})$. Operator T is the δ-class operator if $U^2|T| = |T|U^2$.

Motivated by the above definition and Theorem 4.3.3, we have the following corollary:

Corollary 4.3.4. *Let $A \in \mathfrak{B}(\mathcal{H})$ be a pure semi-hyponormal δ-class operator of dense range which polar decomposition is given by $A = U|A|$. Then $\widetilde{A}$ is a self-dual subnormal operator.*

Proof. Let $P = (U^*|A|U - U|A|U^*)^{1/2}$. Then $[A^*, \widehat{A}] = |A|P^2$. Now the proof follows immediately by Theorem 4.3.3 and a simple observation that A is A δ-class operator if and only if $P = 0$. ∎

Theorem 4.3.5. *Let $A \in \mathfrak{B}(\mathcal{H})$ be given. If the polar decomposition of A is $A = U|A|$ where U is unitary, then the following conditions are equivalent:*

(i) A is normal;

(ii) A is a δ-class operator and $\mathcal{N}(\mathcal{H}) \cap \mathfrak{N}(A, \widehat{A}) \neq \emptyset$.

Proof. Notice that normality of A is equivalent with the fact that $|A|$ and U commute, i.e. $\widehat{A} = A$.

$(i) \Rightarrow (ii)$: Let us suppose that A is normal. By commutativity of $|A|$ and U we have that A is a δ-class operator. Also, $0 \in \mathcal{N}(\mathcal{H}) \cap \mathfrak{N}(A, A) = \mathcal{N}(\mathcal{H}) \cap \mathfrak{N}(A, \widehat{A})$, so the second condition holds.

$(ii) \Rightarrow (i)$: Assume that A is a δ-class operator and that there exists $C \in \mathcal{N}(\mathcal{H}) \cap \mathfrak{N}(A, \widehat{A})$. Then, $[A^*, A] = [(\widehat{A})^*, \widehat{A}]$, i.e.

$$|A|^2 - U|A|^2U^* = U^*|A|^2U - |A|^2. \tag{4.14}$$

Since A is the δ-class operator, we have that $|A|U^2 = U^2|A|$ which implies

$$U^*|A|^2U = U|A|^2U^*. \tag{4.15}$$

Now by (4.14) and (4.15) we have that

$$|A|^2 = U|A|^2U^*,$$

i.e. $|A|^2U = U|A|^2$. Thus, A is normal. ■

The next result is an auxiliary one which gives us that an arbitrary reducing space for $A \in \mathfrak{B}(\mathcal{H})$ is also a reducing space for both transformations, Aluthge and Duggal.

Lemma 4.3.6. *Let $A \in \mathfrak{B}(\mathcal{H})$ be given operator. Then any reducing subspace for A is also a reducing subspace for $\widehat{A}$ and $\widetilde{A}$.*

Proof. The fact that $\mathcal{M}$ is a reducing subspace for A is equivalent with the fact that A commutes with $P_{\mathcal{M}}$ (Theorem 1.1.3). If the polar decomposition of A is $A = U|A|$, then we have that U and $|A|$ commute with $P_{\mathcal{M}}$ (see [90, p. 63]). Evidently, $\widehat{A}$ and $\widetilde{A}$ commute with $P_{\mathcal{M}}$, i.e. $\mathcal{M}$ is a reducing subspace for $\widehat{A}$ and $\widetilde{A}$. ■

Let $A \in \mathcal{B}(\mathcal{H})$ be a self-dual operator with the polar decomposition $A = U|A|$ and let $C \in \mathfrak{N}(A)$. If we assume that A has a dense range, then U is unitary. Operator $C \in \mathfrak{N}(A)$ is normal, it can be represented as $C = V'|C|$ for some unitary V' (see [90, p. 66]). The next theorem, however, shows that, in some cases, for any nontrivial common reducing subspace for A and C we have that the restrictions of U and V must be different.

Theorem 4.3.7. *Let $A \in \mathfrak{B}(\mathcal{H})$ be a self-dual δ-class operator with dense range with the polar decomposition $A = U|A|$ and let $C \in \mathfrak{N}(A)$. If $C = V|C|$ is a decomposition of C where V is unitary, and $\mathcal{M}$ is a nontrivial common reducing subspace for both A and C, then $U|_{\mathcal{M}} \neq V|_{\mathcal{M}}$.*

Proof. Assume to the contrary, that there exists a nontrivial closed subspace $\mathcal{M}$ which reduces both A and C such that $U|_{\mathcal{M}} = V|_{\mathcal{M}}$. By Lemma 4.3.6, $\mathcal{M}$ also reduces $\widehat{A}$. That $\mathcal{M}$ is a reducing subspace for $|C|$ and V follows from [90, p. 66] and Theorem 1.1.3. Thus, $\mathcal{M}$ is reducing subspace for $A, C, |A|, U, |C|, V$ and $\widehat{A}$.

Let us accept the following notations: if $\mathcal{M}$ is reducing subspace for an operator X then we can represent X as 2×2 operator matrix with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$ and by X_1 and X_2 we denote its $(1,1)$ and $(2,2)$ blocks, respectively.

Since A has a dense range, we have that A and $\widehat{A}$ are unitarily equivalent. Thus, A and $\widehat{A}$ are normal complements and it is straightforward to check that $U^*C \in \mathfrak{N}(\widehat{A}, A)$. Also, it is easy to see that A_1 and $\widehat{A}_1$ are normal complements and

$$|C_1| = V_1^*C_1 = U_1^*C_1 \in \mathfrak{N}(\widehat{A}_1, A_1).$$

Therefore, $\mathcal{N}(\mathcal{H}) \cap \mathfrak{N}(\widehat{A}_1, A_1)$ is non-empty. Since A_1 is the δ-class operator, Theorem 4.3.5 implies that A_1 must be normal, which is in contradiction with A being pure. Thus, it must be $U|_{\mathcal{M}} \neq V|_{\mathcal{M}}$, which completes the proof. ■

Remark 4.3.8. *Let us remark that the previous theorem also holds in the case when* $C = V|C|$ *is the "ordinary" polar decomposition of* C.

CHAPTER 5

A NOTE ON SOME RELATED CLASSES OF LINEAR OPERATORS

This chapter explores different classes of operators related to normal and subnormal operators. In Section 5.1, we further extend some results regarding generalized powers of linear operators and also introduce the concept of generalized logarithms. In Section 5.2, we define a new class of operators called polynomially accretive operators, and thereby extend the notion of accretive and n-real power positive operators. We give several properties of the newly introduced class and generalize some results for accretive operators. The final section of this chapter, i.e. Section 5.3, focuses on the study of solvability of the general system of operator equations $A_iXB_i = C_i$, $i = 1, 2$. Here we present some necessary and sufficient conditions for the existence of the solutions (Hermitian solutions, positive solutions) and also obtain the general forms of Hermitian solutions and positive solutions to the system above. We also study the solvability of the $*$-order operator inequality $C \overset{*}{\leq} AXB$.

5.1 POSITIVE OPERATORS AND GENERALIZED POWERS

In a recent paper, A. Bachir, M. H. Mortad and N. A. Sayyaf [12] introduced generalized powers of linear operators, allowing operators to be raised not only to numbers but to other operators, as well. They gave several properties as regards this notion. The aim of this section is to further extend their results and also answer the question regarding the monotonicity of the map $T \mapsto A^T$. We also introduce the concept of generalized logarithms. More precisely, for two positive and invertible operators A and B such that $1 \notin \sigma(A)$, we define the logarithm of B to base A, denoted by $\log_A B$, and investigate some of its properties.

5.1.1 Previous results

For any $A \in \mathfrak{B}(\mathcal{H})$, it is well known that the series $\sum_{n=0}^{\infty} \frac{A^n}{n!}$ converges in $\mathfrak{B}(\mathcal{H})$. This allows us to define e^A for any $A \in \mathfrak{B}(\mathcal{H})$, without using all the theory of the functional calculus.

If A is self-adjoint, then it is easy to see that e^A is self-adjoint, as well. The opposite, however, may not be true. The simplest example is $A = 2\pi i I$. For more information on the topic, we refer a reader to [118], [151], and [158].

As regards the logarithms of bounded operators, there are several possible ways to define them. For example, we may use a series

$$\log A := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (A - I)^n.$$

The previous series converges in $\mathfrak{B}(\mathcal{H})$ whenever $A \in \mathfrak{B}(\mathcal{H})$ and $\|A - I\| < 1$. Another way is to say that if operators $A, B \in \mathfrak{B}(\mathcal{H})$ satisfy $e^A = B$, then A is defined to be a logarithm of B. Since $e^A = e^{A+2\pi i I}$, the logarithm defined this way is not a single valued. The third way is to use the continuous functional calculus, as follows.

Let $A \in \mathfrak{B}(\mathcal{H})$ be positive and invertible, and so $\sigma(A) \subset (0, \infty)$. Hence the function log is well-defined on $\sigma(A)$. Therefore, it also makes sense to define $\log A$. We call it the logarithm of A. It is also clear why $\log A$ is self-adjoint. The following example shows the analogy with an ordinary exponential and logarithm of real numbers.

Example 5.1.1. *Let $A \in \mathfrak{B}(\mathcal{H})$ be a self-adjoint operator. Then*

$$\log e^A = A.$$

Moreover, if A is positive and invertible, then

$$e^{\log A} = A.$$

■

It is easy to see that $\log A$ is a unique self-adjoint solution to the operator equation $e^T = A$. Indeed, by the previous example, $e^{\log A} = A$, and if T_1 and T_2 are self-adjoint operators such that $e^{T_1} = A = e^{T_2}$, then

$$T_1 = \log(e^{T_1}) = \log(e^{T_2}) = T_2.$$

In [12], the authors introduced the concept of generalized powers of operators as follows.

Definition 5.1.1. Let $A, B \in \mathfrak{B}(\mathcal{H})$ such that A is positive and invertible. Define

$$A^B := e^{B \log A}.$$

Then B is called the *generalized power* of A.

The preceding definition does generalize the usual definition of (ordinary) powers of operators as, for example, if $B = nI$, where $n \in \mathbb{N}$, then

$$A^{nI} = e^{nI \log A} = e^{n \log A} = e^{\log A^n} = A^n.$$

The authors also proved somewhat expected identities as regards generalized powers.

Theorem 5.1.2. *[12] Let $A, B, C \in \mathfrak{B}(\mathcal{H})$ such that A is positive and invertible. Then*

(i) $I^B = I$, $A^0 = I$ *and* $A^I = A$;

(ii) $(AB)^C = A^C B^C$ *whenever* A, B *and* C *pairwise commute, and* B *is positive and invertible;*

(iii) $(A^B)^C = A^{CB}$;

(iv) $A^B A^C = A^{B+C}$ *whenever* A, B *and* C *pairwise commute.*

Note that in part (iii) of the previous theorem, the self-adjointness of B and commutativity condition $AB = BA$ should also be added. Otherwise, A^B may not even be positive (see Example 5.1.4 below).

In [12, Proposition 2.2], the self-adjointness of B is also missing. Here we state the correct version.

Proposition 5.1.3. *[12] Let $A \in \mathfrak{B}(\mathcal{H})$ be positive and invertible. Then A^B is positive and invertible for any self-adjoint $B \in \mathfrak{B}(\mathcal{H})$ such that $AB = BA$. Moreover,*

$$\log A^B = B \log A.$$

The proof remains the same. Namely, the authors showed that if $AB = BA$ then $B \log A = \log(A)B$, from which they concluded that $B \log A$ is self-adjoint. But this is only possible if B is self-adjoint. The following example illustrates the necessity of self-adjointness of B.

Example 5.1.4. *Let $A = eI$ and $B = i\pi I$. Obviously, $AB = BA$, but A^B is not positive since $A^B = (eI)^{i\pi I} = e^{i\pi} I = -I$.* ■

There is also a connection with Theorem 1.1.16. Namely, the authors in [12] proved an analogue of Remark 1.1.17.

Theorem 5.1.5. *[12] Let $A, B \in \mathfrak{B}(\mathcal{H})$ be both positive such that $A \leq B$ and A is invertible. Then*

$$A^T \leq B^T$$

for any positive $T \in \mathfrak{B}(\mathcal{H})$ provided T, A and B pairwise commute.

After that, the authors asked whether the previous result holds if the commutativity of A and B is dropped. The answer is negative, as the following example shows:

Example 5.1.6. *Let*

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Obviously, both A and B are positive and invertible since $\det(A) = 1$ *and* $\det(B) = 2$. *Also, $A \leq B$ as*

$$B - A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \geq 0,$$

Since $T = 2I$, we have

$$B^T - A^T = B^2 - A^2 = \begin{bmatrix} 10 & 4 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}.$$

But $\begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix} \not\geq 0$, *as*

$$\begin{vmatrix} 5 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0.$$

Thus, it is not true that $A^T \leq B^T$.

Frequently in the sequel, we will tacitly use Spectral Theorem for self-adjoint operators. For more information on the topic, we refer a reader to [46], [72] and [133].

5.1.2 Generalized powers of operators

In this section, we present some new results regarding generalized powers of operators.

Theorem 5.1.7. *Let $A \in \mathfrak{B}(\mathcal{H})$ be a positive and invertible operator such that $1 \notin \sigma(A)$ and let $B, C \in \mathfrak{B}(\mathcal{H})$ be self-adjoint operators such that both B and C commute with A. Then the following conditions are equivalent:*

(i) $BC = CB$;

(ii) $A^B A^C = A^C A^B$.

Proof. $(i) \Rightarrow (ii)$: Assume that $BC = CB$. Since B and C commute with A it follows that they also commute with $\log A$, and therefore $B \log A$ commutes with $C \log A$. Hence, $e^{B \log A}$ commutes with $e^{C \log A}$, i.e. $A^B A^C = A^C A^B$.

$(ii) \Rightarrow (i)$: Assume that $A^B A^C = A^C A^B$, i.e.

$$e^{B \log A} e^{C \log A} = e^{C \log A} e^{B \log A}.$$

Since $BA = AB$, it follows that $B \log A = \log(A) B$. Both B and $\log A$ are self-adjoint, and so $B \log A$ is self-adjoint, as well. It now follows that $e^{B \log A}$ is positive, and thus

$$\log(e^{B \log A}) e^{C \log A} = e^{C \log A} \log(e^{B \log A}).$$

Therefore,

$$B \log A e^{C \log A} = e^{C \log A} B \log A.$$

Since C is also self-adjoint, using a similar argument, we conclude that

$$B \log(A) C \log A = C \log(A) B \log A,$$

i.e.

$$BC \log^2 A = CB \log^2 A.$$

By the Spectral Mapping Theorem, $1 \notin \sigma(A)$ implies that $0 \notin \log(\sigma(A)) = \sigma(\log A)$. Thus, $BC = CB$. ■ ■

As an immediate corollary, we obtain the following well-known result:

Corollary 5.1.8. *[179] Let $A, B \in \mathfrak{B}(\mathcal{H})$ be self-adjoint. Then*

$$AB = BA \quad \Leftrightarrow \quad e^A e^B = e^B e^A.$$

Proof. Obviously, eI is a positive operator such that $0, 1 \notin \sigma(eI)$ and both A and B commute with eI. The conclusion now follows directly from Theorem 5.1.7. ■

Theorem 5.1.7 does not hold in general if the condition $1 \notin \sigma(A)$ is dropped, as the following example shows.

Example 5.1.9. *Let*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Obviously, B and C are self-adjoint operators which commute with A, and

$$A^B A^C = A^C A^B,$$

as $A^B = A^C = I$. But

$$BC - CB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and so $BC \neq CB$. ■

In Theorem 5.1.2, it was shown that if A is positive and invertible, and B and C are such that A, B and C pairwise commute, we have that

$$A^B A^C = A^{B+C}.$$

The following two theorems provide the generalizations of this result.

Theorem 5.1.10. *Let $A \in \mathfrak{B}(\mathcal{H})$ be a positive and invertible operator and let $B, C \in \mathfrak{B}(\mathcal{H})$ be such that B and C commute with both A and $[B, C]$. Then*

$$A^B A^C = A^{\frac{1}{2}[B,C]\log A} A^{B+C}.$$

Proof. Note that $AB = BA$ implies that B commutes with $\log A$, and by the same reasoning, C also commutes with $\log A$. Thus,

$$\begin{aligned} B\log(A)[B\log A, C\log A] &= \log(A)B(BC - CB)\log^2 A \\ &= \log(A)(BC - CB)B\log^2 A \\ &= [B\log A, C\log A]B\log(A). \end{aligned}$$

We get that $B\log A$ commutes with $[B\log A, C\log A]$, and similarly, $C\log A$ commutes with $[B\log A, C\log A]$, as well. Using the definition of generalized powers and the Baker-Campbell-Hausdorff formula (see [96, Theorem 5.1.]), we obtain

$$\begin{aligned} A^B A^C &= e^{B\log A} e^{C\log A} \\ &= e^{\frac{1}{2}[B\log A, C\log A]} e^{B\log A + C\log A} \\ &= e^{\frac{1}{2}[B,C]\log^2 A} e^{(B+C)\log A} \\ &= A^{\frac{1}{2}[B,C]\log A} A^{B+C}. \end{aligned}$$

This completes the proof. ■

In the following theorem, we further remove the commutativity conditions.

Theorem 5.1.11. *Let $A \in \mathfrak{B}(\mathcal{H})$ be a positive and invertible operator and let $B, C \in \mathfrak{B}(\mathcal{H})$. Then*

$$A^{B+C} = \lim_{n\to\infty} \left(A^{B/n} A^{C/n}\right)^n.$$

Proof. Using the Lie–Trotter product formula (see [173]), we have

$$\begin{aligned} A^{B+C} &= e^{(B+C)\log A} \\ &= e^{B\log A + C\log A} \\ &= \lim_{n\to\infty} \left(e^{(B\log A)/n} e^{(C\log A)/n}\right)^n \\ &= \lim_{n\to\infty} \left(e^{B/n\log A} e^{C/n\log A}\right)^n \\ &= \lim_{n\to\infty} \left(A^{B/n} A^{C/n}\right)^n. \end{aligned}$$

■

Theorem 5.1.12. *Let $A \in \mathfrak{B}(\mathcal{H})$ be positive and invertible and let $B, C \in \mathfrak{B}(\mathcal{H})$. If there exists an invertible operator $S \in \mathfrak{B}(\mathcal{H})$ which commutes with A and such that $C = SBS^{-1}$, then*

$$A^C = SA^BS^{-1}.$$

In other words, A^B is similar to A^C.

Proof. Since S commutes with A, we have that S^{-1} commutes with $\log A$. Using the properties of the exponential function, we have

$$\begin{aligned} A^C &= A^{SBS^{-1}} = e^{SBS^{-1}\log A} \\ &= e^{S(B\log A)S^{-1}} = Se^{B\log A}S^{-1} \\ &= SA^BS^{-1}. \end{aligned}$$

■

We finish this section by giving several results regarding the monotonicity of the function $T \mapsto A^T$.

Theorem 5.1.13. *Let $A \in \mathfrak{B}(\mathcal{H})$ be positive and invertible and let $S, T \in \mathfrak{B}(\mathcal{H})$ be self-adjoint such that $A^T \le A^S$ and both S and T commute with A.*

(i) *If $\sigma(A) \subset (1, \infty)$, then $T \le S$.*

(ii) *If $\sigma(A) \subset (0, 1)$, then $T \ge S$.*

Proof. Let us prove part (i) first. Assume that $A^T \le A^S$, i.e. $e^{T\log A} \le e^{S\log A}$. Since T and S commute with $\log A$, we have that $e^{T\log A}$ and $e^{S\log A}$ are positive and invertible. As a consequence of the classical Heinz inequality (Theorem 1.1.16), we have that

$$\log e^{T\log A} \le \log e^{S\log A},$$

i.e.

$$T\log A \le S\log A.$$

Since $\sigma(A) \subset (1, \infty)$ it follows that $\sigma(\log A) = \log \sigma(A) \subset (0, \infty)$. Thus, $\log A$ is positive. Hence, the inequality $T\log A \le S\log A$ becomes

$$(\log A)^{1/2}T(\log A)^{1/2} \le (\log A)^{1/2}S(\log A)^{1/2}.$$

Multiplying from both sides by $(\log A)^{-1/2}$, the last inequality yields $T \le S$.

In order to prove (ii), note that $\sigma(A^{-1}) \subset (1, \infty)$. Also, using Theorem 5.1.2, part (iii), we have that

$$(A^{-1})^T = A^{T(-I)} = A^{(-I)T} = (A^T)^{-1}.$$

Similarly $(A^{-1})^S = (A^S)^{-1}$. Now,

$$0 \le A^T \le A^S \quad \Rightarrow \quad 0 \le (A^S)^{-1} \le (A^T)^{-1} \quad \Rightarrow \quad 0 \le (A^{-1})^S \le (A^{-1})^T.$$

By applying part (i) to the operators A^{-1}, T and S, we conclude that $T \ge S$. ■

The converse does not hold in general, as there are self-adjoint matrices A and B such that $A \geq B$ but $e^A \geq e^B$ (see [134] for a counterexample).

In the following theorem, we are going to use the fact that if A and B are commuting self-adjoint operators and $A \leq B$, then $e^A \leq e^B$.

Theorem 5.1.14. *Let $A \in \mathfrak{B}(\mathcal{H})$ be positive and invertible and let $S, T \in \mathfrak{B}(\mathcal{H})$ be self-adjoint such that A, S and T pairwise commute.*

(i) If $\sigma(A) \subseteq (1, \infty)$, then
$$T \leq S \quad \Leftrightarrow \quad A^T \leq A^S.$$

(ii) If $\sigma(A) \subseteq (0, 1)$, then
$$T \leq S \quad \Leftrightarrow \quad A^T \geq A^S.$$

Proof. In both cases, we only need to show the $\Rightarrow$ direction.

(i) Assume that $T \leq S$. Since $\log A$ is positive and commutes with S and T, it follows that
$$(\log A)^{1/2} T (\log A)^{1/2} \leq (\log A)^{1/2} S (\log A)^{1/2},$$
i.e.
$$T \log A \leq S \log A.$$

Now, $T \log A$ commutes with $S \log A$, and so $e^{T \log A} \leq e^{S \log A}$. In other words, $A^T \leq A^S$.

(ii) In this case, $-\log A$ is positive, and so
$$-T \log A \leq -S \log A,$$
i.e. $T \log A \geq S \log A$. Therefore, $e^{T \log A} \geq e^{S \log A}$. Hence, $A^T \geq A^S$. ∎

5.1.3 GENERALIZED LOGARITHMS OF OPERATORS

Here, we introduce the concept of generalized logarithms of operators.

Definition 5.1.2. Let $A, B \in \mathfrak{B}(\mathcal{H})$ be positive and invertible operators such that $1 \notin \sigma(A)$. The *logarithm* of B to base A, denoted by $\log_A B$, is defined as
$$\log_A B = \log(B)(\log A)^{-1}.$$

Note that $\log_A B$ given by the previous definition is well-defined since $1 \notin \sigma(A)$, and so $0 \notin \log(\sigma(A)) = \sigma(\log A)$.

In general, $\log_A B$ may not be a self-adjoint operator, as the following example illustrates.

Example 5.1.15. *Let*

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$\log A = \begin{bmatrix} \log 2 & 0 \\ 0 & \log 3 \end{bmatrix} \quad \textit{and} \quad \log B = \begin{bmatrix} 0 & -\pi/2 \\ \pi/2 & 0 \end{bmatrix},$$

and so

$$\begin{aligned} \log_A B &= \log(B)(\log A)^{-1} \\ &= \begin{bmatrix} 0 & -\pi/2 \\ \pi/2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/\log 2 & 0 \\ 0 & 1/\log 3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\pi/\log 9 \\ \pi/\log 4 & 0 \end{bmatrix}, \end{aligned}$$

which is not a self-adjoint matrix. ■

The following theorem gives a characterization of $\log_A B$ as a solution to the operator equation $A^T = B$.

Theorem 5.1.16. *Let $A, B \in \mathfrak{B}(\mathcal{H})$ be positive and invertible operators such that $1 \notin \sigma(A)$ and let $T \in \mathfrak{B}(\mathcal{H})$ be an arbitrary operator. The following conditions are equivalent:*

(i) $T = \log_A B$ *and* $AB = BA$*;*

(ii) T *is self-adjoint,* $A^T = B$ *and* $AT = TA$*.*

Proof. $(i) \Rightarrow (ii)$: Assume that $T = \log_A B$ and $AB = BA$. Since

$$A^{\log_A B} = A^{\log(B)(\log A)^{-1}} = e^{\log(B)(\log A)^{-1}\log A} = e^{\log B} = B,$$

we have that $A^T = B$. Using the fact that A commutes with B, we have that A and $\log A$ both commute with $\log B$. Therefore,

$$\log(B)(\log A)^{-1} = (\log A)^{-1} \log B.$$

This implies the self-adjointness of T and also

$$AT = A\log(B)(\log A)^{-1} = \log(B)(\log A)^{-1}A = TA.$$

$(ii) \Rightarrow (i)$: Now assume that T is self-adjoint, $A^T = B$ and $AT = TA$. As in part (i), we can show that $A^{\log_A B} = B$. Thus, $A^{\log_A B} = A^T$, and so

$$e^{\log_A(B)\log A} = e^{T\log A},$$

i.e.

$$e^{\log B} = e^{T \log A}.$$

From the self-adjointness of T and the fact that A and T commute, we have that $T \log A$ is also a self-adjoint operator. Therefore,

$$e^{\log B} = e^{T \log A} \quad \Rightarrow \quad \log e^{\log B} = \log e^{T \log A},$$

i.e. $\log B = T \log A$. By multiplying from the right-hand side by $(\log A)^{-1}$, we get that $T = \log(B)(\log A)^{-1} = \log_A B$.

It remains to show that $AB = BA$. From

$$A \log(B)(\log A)^{-1} = \log(B)(\log A)^{-1} A = \log(B) A (\log A)^{-1},$$

by multiplying from the right-hand side by $\log A$, we have that A commutes with $\log B$ and so it also commutes with $e^{\log B} = B$. This completes the proof. ∎

Remark 5.1.17. *From Theorem 5.1.16, we conclude that $\log_A B$ is a unique self-adjoint solution to the operator equation $A^T = B$ which commutes with A, provided A and B commute.*

Remark 5.1.18. *Under the commutativity assumption of A and B, we also have*

$$\log_A A^B = \log(A^B)(\log A)^{-1} = B \log(A)(\log A)^{-1} = B.$$

Thus, in case when A and B commute, we have that

$$\log_A A^B = B \quad \text{and} \quad A^{\log_A B} = B.$$

Here we state some elementary properties of generalized logarithm.

Theorem 5.1.19. *Let $A, B, C \in \mathfrak{B}(\mathcal{H})$ such that A is positive and invertible. Then*

(i) $\log_A I = 0$ and $\log_A A = I$;

(ii) if B and C are commuting positive and invertible operators, then

$$\log_A(BC) = \log_A B + \log_A C.$$

(iii) $\log_A(B^{-1}) = -\log_A(B)$.

Proof. (i) This is obvious from the definition of logarithm.

(ii) Using [12, Proposition 1.1.], we have

$$\begin{aligned}\log_A(BC) &= \log(BC)(\log A)^{-1} \\ &= (\log B + \log C)(\log A)^{-1} \\ &= \log(B)(\log A)^{-1} + \log(C)(\log A)^{-1} \\ &= \log_A B + \log_A C.\end{aligned}$$

(iii) This follows from the previous two parts. ∎

We also obtain another expected property of the generalized logarithm.

Theorem 5.1.20. *Let $A, B \in \mathfrak{B}(\mathcal{H})$ be positive and $C, D \in \mathfrak{B}(\mathcal{H})$ be self-adjoint such that A, B and C are invertible and $1 \notin \sigma(A)$. If A commutes with C and B commutes with D, then*

$$\log_{A^C}(B^D) = D\log_A(B)C^{-1}.$$

Proof. Since C commutes with A and both C and $\log A$ are invertible (as $1 \notin \sigma(A)$), it follows that $C\log A$ is self-adjoint invertible operator. Thus, $0 \notin \sigma(C\log A)$, and therefore $A^C = e^{C\log A}$ is positive invertible operator such that $1 \notin \sigma(A^C)$, by the Spectral Mapping Theorem. Thus, $X \mapsto \log_{A^C} X$ is a well-defined map.

Since A commutes with C and B commutes with D, by Proposition 5.1.3, we have that

$$\log(A^C) = C\log A \quad \text{and} \quad \log(B^D) = D\log B.$$

Finally,

$$\begin{aligned}
\log_{A^C}(B^D) &= \log(B^D)(\log A^C)^{-1} \\
&= D\log(B)(C\log A)^{-1} \\
&= D\log(B)(\log A)^{-1}C^{-1} \\
&= D\log_A(B)C^{-1}.
\end{aligned}$$

This completes the proof. ■

The following theorem is similar in spirit to Theorem 5.1.7.

Theorem 5.1.21. *Let $A, B, C \in \mathfrak{B}(\mathcal{H})$ be positive and invertible operator such that $1 \notin \sigma(A)$ and B and C both commute with A. Then the following conditions are equivalent:*

(i) $BC = CB$;

(ii) $\log_A(B)\log_A C = \log_A(C)\log_A(B)$.

Proof. $(i) \Rightarrow (ii)$: Assume that $BC = CB$. Since B and C commute with A, it follows that $\log B$ and $\log C$ commute with $\log A$, and therefore $\log(B)(\log A)^{-1}$ commutes with $\log(C)(\log A)^{-1}$. Hence, $\log_A(B)\log_A C = \log_A(C)\log_A(B)$.

$(ii) \Rightarrow (i)$: Assume that $\log_A(B)\log_A C = \log_A(C)\log_A(B)$., i.e.

$$\log(B)(\log A)^{-1}\log(C)(\log A)^{-1} = \log(C)(\log A)^{-1}\log(B)(\log A)^{-1}.$$

Since, B and C commute with A, it follows that $\log(B)\log C = \log(C)\log B$. Furthermore,

$$e^{\log B}\log C = \log(C)e^{\log B},$$

i.e.

$$B\log C = \log(C)B.$$

From here,

$$Be^{\log C} = e^{\log C}B,$$

and so, $BC = CB$. ■

The following theorems regard the monotonicity of a function $T \mapsto \log_A T$.

Theorem 5.1.22. *Let $A, S, T \in \mathfrak{B}(\mathcal{H})$ be positive and invertible operators such that both S and T commute with A and $T \leq S$.*

(i) If $\sigma(A) \subset (1, \infty)$, then $\log_A T \leq \log_A S$.

(ii) If $\sigma(A) \subset (0, 1)$, then $\log_A T \geq \log_A S$.

Proof. Let us prove part (i) first. Assume that $T \leq S$. Then $\log T \leq \log S$. Since S and T commute with A, it follows that $\log T$ and $\log S$ commute with $(\log A)^{-1}$, and thus

$$\log(T)(\log A)^{-1} \leq \log(S)(\log A)^{-1}.$$

Hence, $\log_A T \leq \log_A S$.

Now assume that $\sigma(A) \subset (0, 1)$. Then $\sigma(A^{-1}) \subset (1, \infty)$. By applying part (i) to the operators A^{-1}, T and S, we have that

$$\log_{A^{-1}} T \leq \log_{A^{-1}} S.$$

Also, using Theorem 5.1.20, we have that

$$-\log_A T = \log_{A^{-1}} T \leq \log_{A^{-1}} S = -\log_A S.$$

Therefore, $\log_A T \geq \log_A S$. ■

Theorem 5.1.23. *Let $A, S, T \in \mathfrak{B}(\mathcal{H})$ be positive and invertible operators such that A, S and T pairwise commute.*

(i) If $\sigma(A) \subseteq (1, \infty)$, then

$$T \leq S \quad \Leftrightarrow \quad \log_A T \leq \log_A S.$$

(ii) If $\sigma(A) \subseteq (0, 1)$, then

$$T \leq S \quad \Leftrightarrow \quad \log_A T \geq \log_A S.$$

Proof. In both cases, we only need to show the $\Leftarrow$ direction.

(i) Assume that $\log_A T \leq \log_A S$, i.e. $\log(T)(\log A)^{-1} \leq \log(S)(\log A)^{-1}$. Since S and T commute with $\log A$, it follows that $\log T \leq \log S$. From the commutativity of $\log T$ and $\log S$, it now follows that $e^{\log T} \leq e^{\log S}$. Thus $S \leq T$.

(ii) Now assume that $\sigma(A) \subseteq (0, 1)$ and $\log_A T \geq \log_A S$, i.e.

$$\log(T)(\log A)^{-1} \geq \log(S)\log(A)^{-1}.$$

From here,

$$\log(T)(-\log A)^{-1} \leq \log(S)(-\log A)^{-1}.$$

Since $-\log(A)$ is positive and commutes with $\log T$ and $\log S$, we have that $\log T \leq \log S$. Again, since T and S commute, it follows that $e^{\log T} \leq e^{\log S}$, i.e. $T \leq S$. ■

5.2 POLYNOMIALLY ACCRETIVE OPERATORS

We introduce a new class of operators called *polynomially accretive operators*, with an aim to extend the notion of accretive and n-real power positive operators. We give several properties of the newly introduced class and generalize some results for accretive operators. We also prove that every 2-normal and $(2k+1)$-real power positive operator, for some $k \in \mathbb{N}$, must be n-normal for all $n \geq 2$. Finally, we give sufficient conditions for the normality of T in the previous implication.

5.2.1 MOTIVATION

Besides several classes mentioned in Chapter 1 which generalize the class of normal operators, there are many more. In [108], the author introduced another generalization of normal operators, called n-power normal operators. Namely, the operator T is n-power normal operator for some $n \in \mathbb{N}$ if T^n commutes with T^*, i.e. $T^nT^* = T^*T^n$. For more information on n-power normal operators, see [4], [40] and [41].

More recently, in [77], the authors further generalized the notion of n-power normal operator to the class of polynomially normal operators. An operator T is said to be polynomially normal if there exists a non-trivial polynomial p such that $p(T)$ is normal. We also have to mention that the idea of considering this class of operators is not new, and can be traced back to the work of Kittaneh [116].

The class of accretive operators recently followed a similar path as the class of normal operators. Recall that the class of accretive operators is a subset of $\mathfrak{B}(\mathcal{H})$ consisting of all operators that have the positive real part. In other words, operator T is accretive if and only if $\mathrm{Re}\,(T) \geq 0$. Throughout the literature, accretive operators are also known as real positive operators in the case of general Hilbert spaces, and *Re-nnd* (Re-nonnegative definite) matrices, in a finite-dimensional case (cf. [24, 58, 175, 180, 189]).

In [94], the authors introduced and studied the operator T satisfying $T^2 \geq -T^{*2}$, and in [13], the author further generalized the notion of accretive operators by introducing the n-real power positive operator. Namely, for $n \in \mathbb{N}$, an operator T is said to be n-real power positive operator if

$$T^n + T^{*n} \geq 0,$$

or, equivalently, $\mathrm{Re}\,(T^n) \geq 0$. The author in [13] also gave several properties regarding this notion. Inspired by these results, as well as the development and the path taken in generalizing the class of normal operators, it is natural to extend the notion of n-real power positive operators to an even wider class related to polynomials.

Through the rest of this chapter, $\mathbb{C}[z]$ will denote the set of all non-trivial complex polynomials in one variable. Note that if $p \in \mathbb{C}[z]$, then $\overline{p} \in \mathbb{C}[z]$, as well, where $\overline{p}(z) = \overline{p(\overline{z})}$, $z \in \mathbb{C}$.

Definition 5.2.1. Let $T \in \mathfrak{B}(\mathcal{H})$ and $p \in \mathbb{C}[z]$. If T satisfies the inequality

$$p(T) + \overline{p}(T^*) \geq 0, \tag{5.1}$$

then T is called p-accretive operator.

Operator $T \in \mathfrak{B}(\mathcal{H})$ is polynomially accretive, if T is q-accretive for some polynomial $q \in \mathbb{C}[z]$.

Remark 5.2.1. *Note that if $T \in \mathfrak{B}(\mathcal{H})$ and $p(t) = t^n$, for some $n \in \mathbb{N}$, then T is a n-real power positive operator. Also, if T is p-accretive for $p(t) = t$, then T is accretive. Thus, the set of all polynomially accretive operators contains all accretive and all n-real power positive operators.*

Remark 5.2.2. *In the sequel, real positive and n-power real positive operators will be called accretive and n-accretive operators, respectively. Also, n-power normal operators will be simply called n-normal operators.*

5.2.2 General properties

We start with the following elementary observation.

Theorem 5.2.3. *Let $T \in \mathfrak{B}(\mathcal{H})$ and $p \in \mathbb{C}[z]$. The following conditions are equivalent:*

- (i) *T is p-accretive;*
- (ii) *$p(T)$ is accretive.*
- (iii) $\operatorname{Re}\langle p(T)x, x\rangle \geq 0$, *for all $x \in \mathcal{H}$;*
- (iv) *T^* is $\overline{p}$-accretive.*

Proof. $(i) \Leftrightarrow (ii)$: Obvious.

$(i) \Leftrightarrow (iii)$: We have that

$$\begin{aligned}
p(T) + \overline{p}(T^*) \geq 0 &\iff \langle (p(T) + \overline{p}(T^*))x, x\rangle \geq 0, \quad \text{for all } x \in \mathcal{H}, \\
&\iff \langle p(T)x, x\rangle + \langle \overline{p}(T^*)x, x\rangle \geq 0, \quad \text{for all } x \in \mathcal{H}, \\
&\iff \langle p(T)x, x\rangle + \langle x, p(T)x\rangle \geq 0, \quad \text{for all } x \in \mathcal{H}, \\
&\iff \langle p(T)x, x\rangle + \overline{\langle p(T)x, x\rangle} \geq 0, \quad \text{for all } x \in \mathcal{H}, \\
&\iff \operatorname{Re}\langle p(T)x, x\rangle \geq 0, \quad \text{for all } x \in \mathcal{H}.
\end{aligned}$$

$(i) \Leftrightarrow (iv)$: This follows directly from the definition.

∎

Theorem 5.2.4. *Let $p \in \mathbb{C}[z]$ and $T \in \mathfrak{B}(\mathcal{H})$ be p-accretive. Then*

- (i) *If zeroes of p do not belong to $\sigma(T)$, then $p(T)^{-1}$ is accretive.*

(ii) *If S is unitarily equivalent to T, then S is p-accretive.*

(iii) *If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$ which reduces T, then $P^{cr}_{\mathcal{M}}T|_{\mathcal{M}}$ is p-accretive.*

Proof. (i) Assume that the zeroes of p do not belong to $\sigma(T)$. Then, by the Spectral Mapping Theorem, we have that $p(T)$ is invertible. Since $p(T)$ is accretive, for all $x \in \mathcal{H}$, we have that

$$0 \le \mathrm{Re}\,\langle p(T)p(T)^{-1}x, p(T)^{-1}x\rangle = \mathrm{Re}\,\langle x, p(T)^{-1}x\rangle = \mathrm{Re}\,\langle p(T)^{-1}x, x\rangle.$$

Thus, $p(T)^{-1}$ is accretive.

(ii) By assumption, S is unitarily equivalent to T, and so there exists a unitary operator $U \in \mathfrak{B}(\mathcal{H})$ such that $S = U^*TU$. Then $S^* = S^*T^*S$, and it is easy to see that $p(S) = U^*p(T)U$ and $\overline{p}(S^*) = U^*\overline{p}(T^*)U$. From (5.1) it now follows that

$$p(S) + \overline{p}(S^*) = U^*p(T)U + U^*\overline{p}(T^*)U = U^*(p(T) + \overline{p}(T^*))U \ge 0,$$

and so S is p-accretive.

(iii) If $\mathcal{M}$ is a closed reducing subspace for T, then T can be represented as

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} : \begin{pmatrix} \mathcal{M} \\ \mathcal{M}^\perp \end{pmatrix} \to \begin{pmatrix} \mathcal{M} \\ \mathcal{M}^\perp \end{pmatrix}.$$

From here,

$$T^* = \begin{bmatrix} T_1^* & 0 \\ 0 & T_2^* \end{bmatrix}, \quad p(T) = \begin{bmatrix} p(T_1) & 0 \\ 0 & p(T_2) \end{bmatrix} \text{ and } \overline{p}(T) = \begin{bmatrix} \overline{p}(T_1^*) & 0 \\ 0 & \overline{p}(T_2^*) \end{bmatrix}.$$

Using the fact that T is p-accretive, for any $x \in \mathcal{H}$, we have

$$0 \le \mathrm{Re}\,\langle \begin{bmatrix} p(T_1) & 0 \\ 0 & p(T_2) \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix} \rangle = \mathrm{Re}\,\langle p(T_1)x, x\rangle.$$

Thus, $P^{cr}_{\mathcal{M}}T|_{\mathcal{M}} = T_1$ is p-accretive. ■

Theorem 5.2.5. *Let $p \in \mathbb{C}[z]$ and $T \in \mathfrak{B}(\mathcal{H})$. If $T = T_1 \oplus T_2$, then T is p-accretive if and only if T_1 and T_2 are p-accretive.*

Proof. The "*if*" part follows from part (iii) of the previous theorem.

Now assume that T_1 and T_2 are p-accretive and let $[x\ y]^\top \in \mathcal{H} \oplus \mathcal{H}$ be arbitrary. Then,

$$\begin{aligned} \mathrm{Re}\,\langle \begin{bmatrix} p(T_1) & 0 \\ 0 & p(T_2) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \rangle &= \mathrm{Re}\,\langle \begin{bmatrix} p(T_1)x \\ p(T_2)y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \rangle \\ &= \mathrm{Re}\,(\langle p(T_1)x, x\rangle + \langle p(T_2)y, y\rangle) \\ &= \mathrm{Re}\,\langle p(T_1)x, x\rangle + \mathrm{Re}\,\langle p(T_2)y, y\rangle \\ &\ge 0. \end{aligned}$$

Thus, T is p-accretive. ■

Theorem 5.2.6. *Let $T \in \mathfrak{B}(\mathcal{H})$. If T is k-accretive for all $1 \le k \le n$, then T is p-accretive for any polynomial p of a degree n with nonnegative coefficients.*

Proof. Let $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ be an n-th degree polynomial with nonnegative coefficients. Then,

$$\begin{aligned}\mathrm{Re}\,\langle p(T)x, x\rangle &= \mathrm{Re}\,\langle \sum_{k=0}^{n} a_k T^k x, x\rangle \\ &= \mathrm{Re}\,\sum_{k=0}^{n} a_k \langle T^k x, x\rangle \\ &= \sum_{k=0}^{n} a_k \mathrm{Re}\,\langle T^k x, x\rangle \\ &= a_0\|x\|^2 + \sum_{k=1}^{n} a_k \mathrm{Re}\,\langle T^k x, x\rangle \\ &\ge 0.\end{aligned}$$

Theorem 5.2.3 now yields the wanted result. ∎

Theorem 5.2.7. *Let $T \in \mathfrak{B}(\mathcal{H})$ and $q, r \in \mathbb{C}[z]$. Consider $F = q(T) + \overline{r}(T^*)$ and $G = q(T) - \overline{r}(T^*)$ and let $p(z) = q(z)r(z)$, $z \in \mathbb{C}$. The following conditions are equivalent:*

(i) T is p-accretive;

(ii) $GG^* \le FF^*$.

Proof. By direct computation, we have

$$\begin{aligned}FF^* - GG^* &= (q(T) + \overline{r}(T^*))(\overline{q}(T^*) + r(T)) \\ &\quad - (q(T) - \overline{r}(T^*))(\overline{q}(T^*) - r(T)) \\ &= q(T)\overline{q}(T^*) + q(T)r(T) + \overline{r}(T^*)\overline{q}(T^*) + \overline{r}(T^*)r(T) \\ &\quad - (q(T)\overline{q}(T^*) - q(T)r(T) - \overline{r}(T^*)\overline{q}(T^*) + \overline{r}(T^*)r(T)) \\ &= 2(q(T)r(T) + \overline{r}(T^*)\overline{q}(T^*)) \\ &= 2(p(T) + \overline{p}(T^*)).\end{aligned}$$

Therefore,

$$T \text{ is } p\text{-accretive} \iff p(T) + \overline{p}(T^*) \ge 0 \iff FF^* - GG^* \ge 0,$$

from where the conclusion follows. ∎

Corollary 5.2.8. *Let $T \in \mathfrak{B}(\mathcal{H})$ and $q, r \in \mathbb{C}[z]$. Consider $F = q(T) + \overline{r}(T^*)$ and $G = q(T) - \overline{r}(T^*)$ and let $p(z) = q(z)r(z)$, $z \in \mathbb{C}$. If T is p-accretive, then $\mathcal{R}(G) \subseteq \mathcal{R}(F)$.*

Proof. The proof follows from the previous theorem and Theorem 1.1.7. ∎

5.2.3 The structure of p-accretive operators

Now we aim to give some representations of the structure of polynomially accretive operators. The starting point in our discussion will be the following representation theorem of 2-normal operators proved by Radjavi and Rosenthal in [153]. We present it here in a slightly different form.

Theorem 5.2.9. *[153] Let $T \in \mathfrak{B}(\mathcal{H})$. Operator T is 2-normal if and only if*

$$T = \begin{bmatrix} A & 0 & 0 \\ 0 & B & C \\ 0 & 0 & -B \end{bmatrix}, \tag{5.2}$$

where A, B are normal, $C \geq 0$, C is one-to-one and $BC = CB$. Moreover, B can be chosen so that $\sigma(B)$ lies in the closed upper half-plane and the Hermitian part of B is non-negative.

In the case when polynomial p has only even powers, the characterization of polynomially accretive operators is rather simple.

Theorem 5.2.10. *Let $T \in \mathfrak{B}(\mathcal{H})$ be a 2-normal operator with the matrix representation given by (5.2) and let $p \in \mathbb{C}[z]$ be a polynomial with even powers only. Then T is p-accretive if and only if A and B are p-accretive.*

Proof. First note that, since B and C commute, we have that

$$T^2 = \begin{bmatrix} A^2 & 0 & 0 \\ 0 & B^2 & 0 \\ 0 & 0 & B^2 \end{bmatrix}.$$

Since polynomial p has even powers only, we have that $p(z) = q(z^2)$ for some polynomial $q \in \mathbb{C}[z]$. Therefore,

$$p(T) = q(T^2) = \begin{bmatrix} q(A^2) & 0 & 0 \\ 0 & q(B^2) & 0 \\ 0 & 0 & q(B^2) \end{bmatrix} = \begin{bmatrix} p(A) & 0 & 0 \\ 0 & p(B) & 0 \\ 0 & 0 & p(B) \end{bmatrix}.$$

The conclusion now follows by combining Theorem 5.2.3 and Theorem 5.2.5. ■

Lemma 5.2.11. *Let $T \in \mathfrak{B}(\mathcal{H})$ be a 2-normal operator with the matrix representation given by (5.2) and let $p \in \mathbb{C}[z]$ be a polynomial of a degree at least 3 and with exactly one odd power $k \geq 3$. If $\operatorname{Re}(p(B))$ and $\operatorname{Re}(p(-B))$ have closed ranges, then T is p-accretive if and only if the following conditions hold:*

(i) A is p-accretive;

(ii) $|B| \leq \mu(\operatorname{Re}(p(B)))^{\frac{1}{k-1}}$, for some $\mu > 0$.

(iii) $|B|(\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}} \leq \nu(\mathrm{Re}\,(p(-B)))^{\frac{1}{k-1}}$ *for some* $\nu > 0$.

Proof. Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, $n \geq 3$. Using representation (5.2), we have that

$$p(T) = \begin{bmatrix} p(A) & 0 & 0 \\ 0 & p(B) & D \\ 0 & 0 & p(-B) \end{bmatrix},$$

for some D. Since $k \geq 3$ is the only odd integer such that $a_k \neq 0$, we have that $D = a_k B^{k-1} C$. Therefore,

$$p(T) + \overline{p}(T^*) = \begin{bmatrix} p(A) + \overline{p}(A^*) & 0 & 0 \\ 0 & p(B) + \overline{p}(B^*) & a_k B^{k-1} C \\ 0 & \overline{a_k}(B^{k-1}C)^* & p(-B) + \overline{p}(-B^*) \end{bmatrix},$$

i.e.

$$\mathrm{Re}\,(p(T)) = \begin{bmatrix} \mathrm{Re}\,(p(A)) & 0 & 0 \\ 0 & \mathrm{Re}\,(p(B)) & \frac{a_k}{2} B^{k-1} C \\ 0 & (\frac{a_k}{2} B^{k-1} C)^* & \mathrm{Re}\,(p(-B)) \end{bmatrix}.$$

Thus, we have that T is p-accretive if and only if the following two conditions hold:

(i') $\mathrm{Re}\,(p(A)) \geq 0$;

(ii') $\begin{bmatrix} \mathrm{Re}\,(p(B)) & \frac{a_k}{2} B^{k-1} C \\ (\frac{a_k}{2} B^{k-1} C)^* & \mathrm{Re}\,(p(-B)) \end{bmatrix} \geq 0.$

Obviously, conditions (i) and (i') are equivalent. By Theorem 1.1.11, condition (ii') is equivalent to the conjunction of the following three conditions:

(i'') $\mathrm{Re}\,(p(B)) \geq 0$;

(ii'') $\mathcal{R}(B^{k-1}C) \subseteq \mathcal{R}((\mathrm{Re}\,(p(B)))^{1/2})$

(iii'') $\mathrm{Re}\,(p(-B)) \geq \frac{|a_k|^2}{4} F^* F$, where $F = \left((\mathrm{Re}\,(p(B)))^{1/2}\right)^{\dagger} B^{k-1} C$.

First, we focus on condition (ii'). Let us show that $(ii'') \Rightarrow (ii)$. Note that since C is one-to-one, we have that $\mathcal{R}(C)$ is dense in $\mathcal{H}$. Thus, $B^{k-1}(\mathcal{R}(C)) \subseteq \mathcal{R}((\mathrm{Re}\,(p(B)))^{1/2})$ now implies that

$$\mathcal{R}(B^{k-1}) \subseteq \overline{\mathcal{R}((\mathrm{Re}\,(p(B)))^{1/2})}.$$

By assumption, a positive operator $\mathrm{Re}\,(p(B))$ has closed range, and thus,

$$\mathcal{R}(\mathrm{Re}\,(p(B))) = \mathcal{R}((\mathrm{Re}\,(p(B)))^{1/2}).$$

Therefore,

$$\mathcal{R}(B^{k-1}) \subseteq \mathcal{R}(\mathrm{Re}\,(p(B))).$$

By Theorem 1.1.7, there exists $\mu' > 0$ such that

$$B^{k-1}(B^*)^{k-1} \leq \mu' \,(\mathrm{Re}\,(p(B)))^2$$

Using the fact that B is normal, it follows that

$$|B|^{2(k-1)} = (B^*B)^{k-1} \leq \mu' \,(\mathrm{Re}\,(p(B)))^2 .$$

Since the function $f(x) = x^{\frac{1}{2(k-1)}}$ is operator monotone (Theorem 1.1.16), we have

$$|B| \leq (\mu')^{\frac{1}{2(k-1)}} (\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}}.$$

By taking $\mu = (\mu')^{\frac{1}{2(k-1)}}$, condition (ii) now follows. The reverse implication can be proved in a similar manner by noting that normality of B and Theorem 1.1.1 imply that $|B|$ and $(\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}}$ commute, and thus the function $g(x) = x^{2(k-1)}$ preserves monotonicity, by Remark 1.1.17. Thus, $(ii) \Leftrightarrow (ii'')$.

Let us now show that $(iii) \Leftrightarrow (iii'')$. Assume that (iii'') holds. Since B is normal, C is positive and $BC = CB$, we have that both B and C commute with a positive operator $\mathrm{Re}\,(p(B))$. By Theorem 1.1.1, they also commute with $\mathrm{Re}\,(p(B)))^{1/2}$, which further implies, using the Spectral Theorem for normal operators, that they commute with $\left((\mathrm{Re}\,(p(B)))^{1/2}\right)^{\dagger}$, as well. Thus, by Theorem 1.1.14, we have that operator $F = \left((\mathrm{Re}\,(p(B)))^{1/2}\right)^{\dagger} B^{k-1}C$ is normal. Hence, $F^*F = FF^*$, and so

$$\mathrm{Re}\,(p(-B)) \geq \frac{|a_k|^2}{4} FF^*.$$

By Theorem 1.1.7 and using the closedness of range of $\mathrm{Re}\,(p(-B))$, we conclude that

$$\mathcal{R}(F) \subseteq \mathcal{R}((\mathrm{Re}\,(p(-B))^{1/2})) = \mathcal{R}(\mathrm{Re}\,(p(-B))),$$

i.e.

$$B^{k-1}\mathcal{R}\left(\left((\mathrm{Re}\,(p(B)))^{1/2}\right)^{\dagger} C\right) \subseteq \mathcal{R}(\mathrm{Re}\,(p(-B))). \tag{5.3}$$

Observe that

$$\begin{aligned}
\overline{\mathcal{R}\left(\left((\mathrm{Re}\,(p(B)))^{1/2}\right)^{\dagger} C\right)} &= \overline{\left((\mathrm{Re}\,(p(B)))^{1/2}\right)^{\dagger}\overline{\mathcal{R}\,(C)}} \\
&= \overline{\mathcal{R}\left(\left((\mathrm{Re}\,(p(B)))^{1/2}\right)^{\dagger}\right)} \\
&= \overline{\mathcal{R}\left(\left((\mathrm{Re}\,(p(B)))^{1/2}\right)^{*}\right)} \\
&= \overline{\mathcal{R}\left((\mathrm{Re}\,(p(B)))^{1/2}\right)} \\
&= \mathcal{R}\,(\mathrm{Re}\,(p(B)))\,.
\end{aligned}$$

Combining this with (5.3), and again using the fact that $\mathrm{Re}\,(p(-B))$ has closed range, we have

$$\mathcal{R}(B^{k-1}\mathrm{Re}\,(p(B))) \subseteq \mathcal{R}(\mathrm{Re}\,(p(-B))).$$

Theorem 1.1.7 now implies that there exists $\nu > 0$ such that

$$|B|^{2(k-1)}\mathrm{Re}\,(p(B))^2 \leq \nu'\mathrm{Re}\,(p(-B))^2.$$

Using the fact that $|B|$ and $\mathrm{Re}\,(p(B))$ commute and Theorem 1.1.15, monotonicity of $f(x) = x^{\frac{1}{2(k-1)}}$ now implies

$$|B|(\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}} \leq \nu(\mathrm{Re}\,(p(-B)))^{\frac{1}{k-1}},$$

where $\nu = (\nu')^{\frac{1}{2(k-1)}}$. Therefore, (iii) holds.
Using the similar arguments and comments as in part $(ii) \Rightarrow (ii'')$, we can show that $(iii) \Rightarrow (iii'')$. This completes the proof. ■

Remark 5.2.12. *If T and $p \in \mathbb{C}[z]$ are as in Lemma 5.2.11, we can see that p-accretivity of T implies that the operators A, B and $-B$ are also p-accretive.*

Theorem 5.2.13. *Let $T \in \mathfrak{B}(\mathcal{H})$ and $p \in \mathbb{C}[z]$ be as in Lemma 5.2.11. If operator T is p-accretive then A is p-accretive and there exists $\lambda > 0$ such that*

$$\frac{1}{\lambda}|B|^2 \leq |B|\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}} \leq \lambda(\mathrm{Re}\,(p(-B)))^{\frac{1}{k-1}}. \tag{5.4}$$

Moreover, if B is left-invertible, then the reverse implication holds, as well.

Proof. ($\Longrightarrow$:) Assume that T is p-accretive. By Lemma 5.2.11, we have that A is p-accretive and there exists $\mu, \nu > 0$ such that

$$|B| \leq \mu(\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}} \tag{5.5}$$

and

$$|B|(\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}} \leq \nu(\mathrm{Re}\,(p(-B)))^{\frac{1}{k-1}}.$$

Let $\lambda = \max\{\mu, \nu\}$. Then the second inequality in (5.4) is obviously satisfied. Now using the fact that B is normal, (5.5) yields

$$\begin{aligned}|B|\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}} &= |B|^{\frac{1}{2}}\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}}|B|^{\frac{1}{2}}\\ &\geq \frac{1}{\mu}|B|^{\frac{1}{2}}|B||B|^{\frac{1}{2}} \geq \frac{1}{\lambda}|B|^2.\end{aligned}$$

Hence, (5.4) holds.

($\Longleftarrow$:) To prove the reverse inequality, it is enough to show that condition (ii) in Lemma 5.2.11 is satisfied. Observe that $|B|$ is invertible since B is left-invertible, by assumption. Therefore, using the normality of B, the inequality

$$\frac{1}{\lambda}|B|^2 \leq |B|\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}}$$

implies that

$$|B|^{-\frac{1}{2}}|B|^2|B|^{-\frac{1}{2}} \leq \lambda|B|^{-\frac{1}{2}}\left(|B|\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}}\right)|B|^{-\frac{1}{2}},$$

i.e.

$$|B| \leq \lambda(\mathrm{Re}\,(p(B)))^{\frac{1}{k-1}}.$$

Thus, T is p-accretive and the proof is completed. ■

Theorem 5.2.14. *Let $T \in \mathfrak{B}(\mathcal{H})$ and let $k \in \mathbb{N}$. The following conditions are equivalent:*

(i) *T is 2-normal and $(2k+1)$-accretive;*

(ii) *$T = T_1 \oplus T_2$, where T_1 is normal and $(2k+1)$-accretive, and T_2 is nilpotent of index 2.*

Proof. $(i) \Longrightarrow (ii)$. Assume that (i) holds. Since T is 2-normal, it is given by (5.2). By analysing the proof of Lemma 5.2.11, we have that $(2k+1)$-accretivity of T implies that A is $(2k+1)$-accretive, and also the following conditions hold:

(i'') $B^{2k+1} + (B^*)^{2k+1} \geq 0$;

(ii'') $\mathcal{R}(B^{2k}C) \subseteq \mathcal{R}((B^{2k+1} + (B^*)^{2k+1})^{1/2})$;

(iii'') $(-B)^{2k+1} + (-B^*)^{2k+1} \geq 0$.

Condition (iii'') is equivalent with the fact that

$$-\left(B^{2k+1} + (B^*)^{2k+1}\right) = (-B)^{2k+1} + (-B^*)^{2k+1} \geq 0.$$

This, together with (i''), implies that $B^{2k+1} + (B^*)^{2k+1}$ must be equal to the zero operator. Therefore,

$$\mathcal{R}((B^{2k+1} + (B^*)^{2k+1})^{1/2}) = \{0\}.$$

Condition (ii'') yields that $\mathcal{R}(B^{2k}C) \subseteq \{0\}$, and thus $CB^{2k} = 0$. But C is one-to-one, and so $B^{2k} = 0$. The only nilpotent normal operator is the zero operator, and hence, $B = 0$. Let

$$T_1 = A \text{ and } T_2 = \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix}.$$

Then, $T = T_1 \oplus T_2$, where T_1 is $(2k+1)$-accretive and T_2 is nilpotent of index 2, as required.

$(ii) \Longrightarrow (i)$ Now assume that (ii) holds. Then $T^2 = T_1^2 \oplus 0$ implies that T^2 is normal. Similarly, $T^{2k+1} = T_1^{2k+1} \oplus 0$ yields the $(2k+1)$-accretivity of T. ■

Corollary 5.2.15. *Let $T \in \mathfrak{B}(\mathcal{H})$. If T is 2-normal and $(2k+1)$-accretive for some $k \in \mathbb{N}$, then T is n-normal for all $n \geq 2$.*

Proof. It follows immediately from the representation of T given in Theorem 5.2.14. ■

Remark 5.2.16. *In general, under the conditions of Corollary 5.2.15, we cannot conclude that the operator T is normal. To see this, it is enough to take any non-normal operator T such that $T^2 = 0$.*

In the following proposition, motivated by [4, Lemma 2.28], we give a necessary condition for the normality of T.

Corollary 5.2.17. *Let $T \in \mathfrak{B}(\mathcal{H})$ be such that T is 2-normal and $(2k+1)$-accretive for some $k \in \mathbb{N}$. If $\mathcal{R}([T^*, T]) \subseteq \mathcal{N}(T^l)^\perp$, for some $l \geq 2$, then T is normal.*

Proof. Since T is 2-normal and $(2k+1)$-accretive, we have that T is n-normal for all $n \geq 2$, by Corollary 5.2.15. Specially, T is l-normal and $(l+1)$-normal. Thus,

$$T^l T T^* = T^{l+1} T^* = T^* T^{l+1} = T^* T^l T = T^l T^* T,$$

i.e. $T^l(TT^* - T^*T) = 0$. Thus, $\mathcal{R}([T^*, T]) \subseteq \mathcal{N}(T^l) \cap \mathcal{N}(T^l)^\perp = \{0\}$, from where it follows that $TT^* = T^*T$, i.e. T is normal. ■

Corollary 5.2.18. *Let $T \in \mathfrak{B}(\mathcal{H})$. If T is injective, 2-normal and $(2k+1)$-accretive for some $k \in \mathbb{N}$, then T is normal.*

The following corollaries are matrix analogues of the previous results, presented in the language of matrix theory.

Corollary 5.2.19. *Let A be a $n \times n$ complex matrix and let $k \in \mathbb{N}$. The following conditions are equivalent:*

(i) A^2 is normal and A^{2k+1} is Re-nnd.

(ii) $A = A_1 \oplus A_2$, where A_1 is normal, A_1^{2k+1} is Re-nnd, and A_2 is nilpotent of index 2.

Corollary 5.2.20. *Let A be a $n \times n$ complex matrix. If A^2 is normal and A^{2k+1} is Re-nnd for some $k \in \mathbb{N}$, then A^n is normal for all $n \geq 2$.*

Corollary 5.2.21. *Let A be a $n \times n$ complex non-singular matrix. If A^2 is normal and A^{2k+1} is Re-nnd for some $k \in \mathbb{N}$, then A is normal.*

5.3 Positive and self-adjoint solutions to the system of operator equations

Operator equations is one of the branches of operator theory that has many applications in other fields of mathematics and science. For example, the most notable applications can be found in control theory, information theory, linear system theory, and other areas (cf. [9]). In 1966, Douglas [79] established the celebrated "Douglas Range Inclusion Theorem", in which he gave some equivalent conditions for the existence of a solution to the operator equation $AX = B$ (see Theorem 1.1.7). Later, many mathematicians considered some related problems of the solvability of some operator equations derived from $AX = B$ (see [57, 58, 62, 65, 76, 78, 177, 183, 184, 125]). For example, Dajić and Koliha [69] considered the existence of hermitian solutions and positive solutions to the system of operator equations $AX = C, XB = D$ and also gave their concrete forms. Based on that, Arias and Gonzalez [9] studied the existence and expression of positive solutions to operator equation $AXB = C$ with arbitrary operators A, B and C. The authors in [67] further generalized the results of [9]. For the C^*-algebra setting see [63]. Recently, Vosough and Moslehian [176] considered the system of operator equations $BXA = B = AXB$ and characterized representations of the solutions to the system, while Zhang and Ji [188] extended their results to the system $AXB = C = BXA$. In this section, we further extend the problem, and consider the relevant problems of the solutions to the system of operator equations $A_iXB_i = C_i$, $i = 1, 2$, as well as solving the operator inequality $C \overset{*}{\leq} AXB$, where $A \overset{*}{\leq} B$ means

$$AA^* = BA^* \quad \text{and} \quad A^*A = A^*B.$$

The following simple, but useful results, will often be used throughout this section:

Lemma 5.3.1. *[9] Let $T \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$ and $C \in \mathfrak{B}(\mathcal{G}, \mathcal{K})$ such that $\mathcal{R}(C) \subseteq \mathcal{R}(T)$. Then $T^\dagger C \in \mathfrak{B}(\mathcal{G}, \mathcal{H})$.*

Theorem 5.3.2. *[90, p. 55] Let $\mathcal{M}$ be a dense subspace of a Hilbert space $\mathcal{H}$, and let T be a linear operator from $\mathcal{M}$ to a Hilbert space $\mathcal{K}$. If T is bounded, then there exists a unique $\overline{T}$ which is the extension of T from $\mathcal{H}$ to $\mathcal{K}$, that is $\overline{T}x = Tx$ for all $x \in \mathcal{M}$ and $\|\overline{T}\| = \|T\|$.*

The operator $\overline{T}$ is called the *continuous linear extension* of T. In the sequel, if T is (possibly) not defined on a whole space $\mathcal{H}$, with $\mathcal{D}(T)$ we shall denote the domain of T.

5.3.1 Solvability of $A_iXB_i = C_i$, $i = 1, 2$

We start by giving some conditions for the existence of solutions, Hermitian solutions, and positive solutions to the system of operator equations $A_iXB_i = C_i$, $i = 1, 2$.

Theorem 5.3.3. *Let $A_i \in \mathfrak{B}(\mathcal{H},\mathcal{K})$, $B_i \in \mathfrak{B}(\mathcal{F},\mathcal{G})$ and $C_i \in \mathfrak{B}(\mathcal{F},\mathcal{K})$, $i = 1,2$. Assume that $\mathcal{D}(B_1^\dagger) = \mathcal{D}(B_2^\dagger)$ and $\mathcal{R}(C_iB_i^\dagger) \subseteq \mathcal{D}(A_i^\dagger)$, $i = 1,2$. If $A_1^\dagger C_1 B_1^\dagger = A_2^\dagger C_2 B_2^\dagger$, then the following statements are equivalent:*

(i) The system of operator equations

$$A_1XB_1 = C_1,$$
$$A_2XB_2 = C_2$$

is solvable;

(ii) $\mathcal{R}(C_i) \subseteq \mathcal{R}(A_i)$ and $\mathcal{R}((A_i^\dagger C_i)^*) \subseteq \mathcal{R}(B_i^*)$, $i = 1,2$.

Proof. $(i) \Rightarrow (ii)$ Assume that there exist $\widetilde{X}$ such that $A_i\widetilde{X}B_i = C_i$, $i = 1,2$. Then, obviously $\mathcal{R}(C_i) \subseteq \mathcal{R}(A_i)$ and so $A_i^\dagger C_i \in \mathfrak{B}(\mathcal{F},\mathcal{H})$. Moreover, as $A_i^\dagger A_i\widetilde{X}B_i = A_i^\dagger C_i$, we have that $\mathcal{R}((A_i^\dagger C_i)^*) \subseteq \mathcal{R}(B_i^*)$, $i = 1,2$.

$(ii) \Rightarrow (i)$ Now assume that (ii) holds. Since $\mathcal{R}(C_i) \subseteq \mathcal{R}(A_i)$, it follows that $C_i = A_iA_i^\dagger C_i$. Also, from $\mathcal{R}((A_i^\dagger C_i)^*) \subseteq \mathcal{R}(B_i^*) \subseteq \mathcal{R}(B_i^\dagger B_i)$, we get that $A_i^\dagger C_i = A_i^\dagger C_i B_i^\dagger B_i$. Thus,

$$C_i = A_iA_i^\dagger C_i = A_iA_i^\dagger C_iB_i^\dagger B_i.$$

Let $\widetilde{X} := A_1^\dagger C_1 B_1^\dagger = A_2^\dagger C_2 B_2^\dagger$. Then

$$A_1\widetilde{X}B_1 = A_1A_1^\dagger C_1B_1^\dagger B_1 = C_1$$
$$A_2\widetilde{X}B_2 = A_2A_2^\dagger C_2B_2^\dagger B_2 = C_2.$$

We will show that $\widetilde{X}$ is bounded. From $\mathcal{R}(C_1) \subseteq \mathcal{R}(A_1)$ we have that $A_1^\dagger C_1 \in \mathfrak{B}(\mathcal{F},\mathcal{H})$. Since $\mathcal{R}((A_1^\dagger C_1)^*) \subseteq \mathcal{R}(B_1^*)$, by Theorem 1.1.7, there exists $\widetilde{Z} \in \mathfrak{B}(\mathcal{G},\mathcal{H})$ such that $(A_1^\dagger C_1)^* = B_1^*\widetilde{Z}^*$, i.e.

$$A_1^\dagger C_1 = \widetilde{Z}B_1.$$

We get that $\widetilde{X} = A_1^\dagger C_1 B_1^\dagger = \widetilde{Z}B_1B_1^\dagger \in \mathfrak{B}(\mathcal{D}(B_1^\dagger),\mathcal{H})$. Let $X_0 \in \mathfrak{B}(\mathcal{G},\mathcal{H})$ be a continuous linear extension of $\widetilde{X}$. Since $\mathcal{R}(B_1) \subseteq \mathcal{D}(B_1^\dagger) = \mathcal{D}(\widetilde{X})$, we have that $X_0B_1 = \widetilde{X}B_1$. Similarly, $X_0B_2 = \widetilde{X}B_2$. It is obvious now that $X_0 \in \mathfrak{B}(\mathcal{G},\mathcal{H})$ is a solution to the system $A_iXB_i = C_i$, $i = 1,2$. ■

In the next theorem, we establish a relationship between solutions to the system $A_iXB_i = C_i$, $i = 1,2$ and solutions to the system $XB_1 = A_1^\dagger C_1, A_2X = \overline{C_2B_2^\dagger}$ which shall be useful to give the general hermitian solutions and positive solutions to the system $A_iXB_i = C_i$, $i = 1,2$.

Theorem 5.3.4. *Let $A_i \in \mathfrak{B}(\mathcal{H},\mathcal{K})$, $B_i \in \mathfrak{B}(\mathcal{F},\mathcal{G})$ and $C_i \in \mathfrak{B}(\mathcal{F},\mathcal{K})$, $i = 1,2$. If $\mathcal{R}(B_1) \subseteq \mathcal{R}(B_2)$ and $\mathcal{R}(A_2^*) \subseteq \overline{\mathcal{R}(A_1^*)}$, then the following statements are equivalent:*

(i) The system of operator equations

$$A_1XB_1 = C_1,$$
$$A_2XB_2 = C_2$$

is solvable;

(ii) $\mathcal{R}(C_1) \subseteq \mathcal{R}(A_1)$*,* $\mathcal{R}(C_2^*) \subseteq \mathcal{R}(B_2^*)$ *and the system of operator equations*

$$XB_1 = A_1^\dagger C_1,$$
$$A_2X = \overline{C_2B_2^\dagger}$$

is solvable.

Moreover, if one of the previous conditions holds, then every solution of $XB_1 = A_1^\dagger C_1$*,* $A_2X = \overline{C_2B_2^\dagger}$ *is also a solution of* $A_iXB_i = C_i$*,* $i = 1,2$*. Also, for* $X \in \mathfrak{B}(\mathcal{G},\mathcal{H})$ *such that* $A_iXB_i = C_i$*,* $i = 1,2$*, we have that*

$$\overline{P_{\overline{\mathcal{R}(A_1^*)}}XP_{\overline{\mathcal{R}(B_2)}}\restriction_{\mathcal{D}(B_2^\dagger)}}$$

is a solution of $XB_1 = A_1^\dagger C_1$*,* $A_2X = \overline{C_2B_2^\dagger}$*.*

Proof. $(i) \Rightarrow (ii)$ Assume that there exist $\widetilde{X}$ such that $A_i\widetilde{X}B_i = C_i$, $i = 1,2$. Then, obviously $\mathcal{R}(C_1) \subseteq \mathcal{R}(A_1)$ and $\mathcal{R}(C_2^*) \subseteq \mathcal{R}(B_2^*)$. Also, from $\mathcal{R}(B_1) \subseteq \mathcal{R}(B_2)$ and $\mathcal{R}(A_2^*) \subseteq \overline{\mathcal{R}(A_1^*)}$, we may conclude that $B_1 = B_2B_2^\dagger B_1$ and $A_2 = A_2A_1^\dagger A_1$. Let $\widetilde{Y} := A_1^\dagger A_1\widetilde{X}B_2B_2^\dagger \in \mathfrak{B}(\mathcal{D}(B_2^\dagger),\mathcal{H})$. Then

$$\widetilde{Y}B_1 = A_1^\dagger A_1\widetilde{X}B_2B_2^\dagger B_1 = A_1^\dagger A_1\widetilde{X}B_1 = A_1^\dagger C_1$$
$$A_2\widetilde{Y} = A_2A_1^\dagger A_1\widetilde{X}B_2B_2^\dagger = A_2\widetilde{X}B_2B_2^\dagger = C_2B_2^\dagger.$$

Let $Y_0 \in \mathfrak{B}(\mathcal{G},\mathcal{H})$ be a continuous linear extension of $\widetilde{Y}$. Since $\mathcal{R}(B_1) \subseteq \mathcal{R}(B_2) \subseteq \mathcal{D}(B_2^\dagger) = \mathcal{D}(\widetilde{Y})$, we have that $Y_0B_1 = \widetilde{Y}B_1$, and thus

$$Y_0B_1 = A_1^\dagger C_1.$$

Also, since $\mathcal{R}(C_2^*) \subseteq \mathcal{R}(B_2^*)$, by Theorem 1.1.7, there exists $\widetilde{Z} \in \mathfrak{B}(\mathcal{G},\mathcal{H})$ such that $C_2^* = B_2^*\widetilde{Z}^*$, i.e.

$$C_2 = \widetilde{Z}B_2.$$

We get that $C_2B_2^\dagger = \widetilde{Z}B_2B_2^\dagger \in \mathfrak{B}(\mathcal{D}(B_2^\dagger),\mathcal{H})$. Thus, it allows a continuous linear extension on $\mathcal{G}$. Let $x \in \mathcal{G}$ be arbitrary and let $(x_n)_n \in \mathcal{D}(B_2^\dagger)$ such that $x_n \to x$, $n \to \infty$. Then

$$A_2Y_0x = A_2(\lim_{n\to\infty} \widetilde{Y}x_n) = \lim_{n\to\infty} A_2\widetilde{Y}x_n$$
$$= \lim_{n\to\infty} C_2B_2^\dagger x_n = \overline{C_2B_2^\dagger}x.$$

Thus, $Y_0 \in \mathfrak{B}(\mathcal{G}, \mathcal{H})$ is a solution to the system $XB_1 = A_1^\dagger C_1$, $A_2X = \overline{C_2B_2^\dagger}$.

$(ii) \Rightarrow (i)$ Now assume that (ii) holds. Since $\mathcal{R}(C_1) \subseteq \mathcal{R}(A_1)$ and $\mathcal{R}(C_2^*) \subseteq \mathcal{R}(B_2^*)$, we have that $C_1 = A_1A_1^\dagger C_1$ and $C_2 = C_2B_2^\dagger B_2$. Now let $\widetilde{X} \in \mathfrak{B}(\mathcal{G}, \mathcal{H})$ such that $\widetilde{X}B_1 = A_1^\dagger C_1$ and $A_2\widetilde{X} = \overline{C_2B_2^\dagger}$. Since $\mathcal{R}(B_2) \subseteq \mathcal{D}(B_2^\dagger) = \mathcal{D}(C_2B_2^\dagger)$, it follows that

$$A_1\widetilde{X}B_1 = A_1A_1^\dagger C_1 = C_1$$
$$A_2\widetilde{X}B_2 = \overline{C_2B_2^\dagger}B_2 = C_2B_2^\dagger B_2 = C_2.$$

Thus, the system $A_iXB_i = C_i$, $i = 1, 2$ is solvable. ■

The next theorem gives conditions for the existence of a Hermitian solution of the system $A_iXB_i = C_i$, $i = 1, 2$.

Theorem 5.3.5. *Let $A_i, B_i, C_i \in \mathfrak{B}(\mathcal{H})$, $i = 1, 2$ such that $C_1 \overset{*}{\leq} A_2 \overset{*}{\leq} A_1$, $C_2 \overset{*}{\leq} B_1 \overset{*}{\leq} B_2$ and $A_1^\dagger A_1 \restriction_{\mathcal{D}(B_2^\dagger)} = B_2B_2^\dagger$. If the system of operator equations $A_iXB_i = C_i$, $i = 1, 2$ is solvable, then the following statements are equivalent:*

(i) The system of operator equations

$$A_1XB_1 = C_1,$$
$$A_2XB_2 = C_2$$

has a hermitian solution;

(ii) The system of operator equations

$$XB_1 = A_1^\dagger C_1,$$
$$A_2X = \overline{C_2B_2^\dagger}$$

has a hermitian solution.

Proof. $(i) \Rightarrow (ii)$ Assume that there exist a hermitian operator $\widetilde{X} \in \mathfrak{B}(\mathcal{H})$ such that $A_i\widetilde{X}B_i = C_i$, $i = 1, 2$. It follows from the proof of Theorem 5.3.4 that

$$Y_0 := \overline{A_1^\dagger A_1\widetilde{X}B_2B_2^\dagger} \in \mathfrak{B}(\mathcal{H})$$

is a solution of the system $XB_1 = A_1^\dagger C_1$, $A_2X = \overline{C_2B_2^\dagger}$. Let $x \in \mathcal{H}$ be arbitrary and let $(x_n)_n \in \mathcal{D}(B_2^\dagger)$ such that $x_n \to x$, $n \to \infty$. As $A_1^\dagger A_1 = B_2B_2^\dagger$ on $\mathcal{D}(B_2^\dagger)$, we have

$$\begin{aligned}
\langle Y_0x, x\rangle &= \lim_{n\to\infty}\langle Y_0x_n, x_n\rangle = \lim_{n\to\infty}\langle A_1^\dagger A_1\widetilde{X}B_2B_2^\dagger x_n, x_n\rangle \\
&= \lim_{n\to\infty}\langle A_1^\dagger A_1\widetilde{X}A_1^\dagger A_1x_n, x_n\rangle = \lim_{n\to\infty}\langle x_n, A_1^\dagger A_1\widetilde{X}^*A_1^\dagger A_1x_n\rangle \\
&= \lim_{n\to\infty}\langle x_n, A_1^\dagger A_1\widetilde{X}B_2B_2^\dagger x_n\rangle = \lim_{n\to\infty}\langle x_n, Y_0x_n\rangle \\
&= \langle x, Y_0x\rangle.
\end{aligned}$$

Thus, $Y_0 \in \mathfrak{B}(\mathcal{H})$ is a hermitian solution of the system $XB_1 = A_1^\dagger C_1$, $A_2X = \overline{C_2B_2^\dagger}$.

$(ii) \Rightarrow (i)$ Now let $\widetilde{Y} \in \mathfrak{B}(\mathcal{H})$ be a hermitian solution of the system $XB_1 = A_1^\dagger C_1$, $A_2X = \overline{C_2B_2^\dagger}$. It follows that $\widetilde{X} = \widetilde{Y}$ is a hermitian solution of the system $A_iXB_i = C_i$, $i = 1, 2$ by Theorem 5.3.4. ■

The next theorem, similar in spirit to Theorem 5.3.5, gives conditions for the existence of a positive solution of the system $A_iXB_i = C_i$, $i = 1, 2$.

Theorem 5.3.6. *Let $A_i, B_i, C_i \in \mathfrak{B}(\mathcal{H})$, $i = 1, 2$ such that $C_1 \overset{*}{\leq} A_2 \overset{*}{\leq} A_1$, $C_2 \overset{*}{\leq} B_1 \overset{*}{\leq} B_2$ and $A_1^\dagger A_1 \restriction_{\mathcal{D}(B_2^\dagger)} = B_2B_2^\dagger$. If the system of operator equations $A_iXB_i = C_i$, $i = 1, 2$ is solvable, then the following statements are equivalent:*

(i) The system of operator equations

$$A_1XB_1 = C_1,$$
$$A_2XB_2 = C_2$$

has a positive solution;

(ii) The system of operator equations

$$XB_1 = A_1^\dagger C_1,$$
$$A_2X = \overline{C_2B_2^\dagger}$$

has a positive solution.

Proof. $(i) \Rightarrow (ii)$ Assume that there exist a positive operator $\widetilde{X} \in \mathfrak{B}(\mathcal{H})$ such that $A_i\widetilde{X}B_i = C_i$, $i = 1, 2$. It follows from the proof of Theorem 5.3.4 that

$$Y_0 := \overline{A_1^\dagger A_1 \widetilde{X} B_2 B_2^\dagger} \in \mathfrak{B}(\mathcal{H})$$

is a solution of the system $XB_1 = A_1^\dagger C_1$, $A_2X = \overline{C_2B_2^\dagger}$. Let $x \in \mathcal{H}$ be arbitrary and let $(x_n)_n \in \mathcal{D}(B_2^\dagger)$ such that $x_n \to x$, $n \to \infty$. As $A_1^\dagger A_1 = B_2B_2^\dagger$ on $\mathcal{D}(B_2^\dagger)$, we have

$$\begin{aligned}\langle Y_0x, x\rangle &= \lim_{n\to\infty} \langle Y_0x_n, x_n\rangle = \lim_{n\to\infty} \langle A_1^\dagger A_1 \widetilde{X} B_2B_2^\dagger x_n, x_n\rangle \\ &= \lim_{n\to\infty} \langle \widetilde{X} A_1^\dagger A_1 x_n, A_1^\dagger A_1 x_n\rangle \geq 0.\end{aligned}$$

Thus, $Y_0 \in \mathfrak{B}(\mathcal{H})$ is a positive solution of the system $XB_1 = A_1^\dagger C_1$, $A_2X = \overline{C_2B_2^\dagger}$.

$(ii) \Rightarrow (i)$ Now let $\widetilde{Y} \in \mathfrak{B}(\mathcal{H})$ be a positive solution of the system $XB_1 = A_1^\dagger C_1$, $A_2X = \overline{C_2B_2^\dagger}$. It follows that $\widetilde{X} = \widetilde{Y}$ is a positive solution of the system $A_iXB_i = C_i$, $i = 1, 2$ by Theorem 5.3.4. ■

Now we are ready to establish the general form of the Hermitian solution of the system $A_iXB_i = C_i$, $i = 1,2$.

Theorem 5.3.7. *Let $A_i, B_i, C_i \in \mathfrak{B}(\mathcal{H})$, $i = 1,2$ such that $C_1 \overset{*}{\leq} A_2 \overset{*}{\leq} A_1$, $C_2 \overset{*}{\leq} B_1 \overset{*}{\leq} B_2$ and*

$$A_1^\dagger A_1 \restriction_{\mathcal{D}(A_1^\dagger)} = A_1A_1^\dagger = B_2^\dagger B_2 \restriction_{\mathcal{D}(B_2^\dagger)} = B_2B_2^\dagger. \tag{5.6}$$

If the system of operator equations $A_iXB_i = C_i$, $i = 1,2$ has a hermitian solution, then the general Hermitian solution has the matrix representation

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} \tag{5.7}$$

in terms of $\mathcal{H} = \overline{\mathcal{R}(A_1^*)} \oplus \mathcal{N}(A_1)$, where X_{22} is a hermitian and $X_{11} = P_{\overline{\mathcal{R}(A_1^*)}}\widetilde{Y} \restriction_{\overline{\mathcal{R}(B_2)}}$ satisfying that $\widetilde{Y}$ is a hermitian solution of $XB_1 = A_1^\dagger C_1$, $A_2X = \overline{C_2B_2^\dagger}$.

Proof. Suppose that $X \in \mathfrak{B}(\mathcal{H})$ has the matrix decomposition (5.7). Since $A_2 \leq^* A_1$, $B_1 \leq^* B_2$ and (5.6) holds, we have that A_i and B_i, $i = 1,2$, have the matrix representations

$$A_i = \begin{bmatrix} A_{i1} & 0 \\ 0 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} B_{i1} & 0 \\ 0 & 0 \end{bmatrix}, \quad i = 1,2,$$

with respect to $\mathcal{H} = \overline{\mathcal{R}(A_1^*)} \oplus \mathcal{N}(A_1)$. Therefore,

$$A_iXB_i = A_ix_{11}B_i = A_iP_{\overline{\mathcal{R}(A_1^*)}}\widetilde{Y} \restriction_{\overline{\mathcal{R}(B_2)}} B_i, \quad i = 1,2.$$

Since $\mathcal{R}(B_1) \subseteq \mathcal{R}(B_2)$ and $\mathcal{R}(C_1) \subseteq \mathcal{R}(A_1)$, we have

$$\begin{aligned} A_1XB_1 &= A_1P_{\overline{\mathcal{R}(A_1^*)}}\widetilde{Y} \restriction_{\overline{\mathcal{R}(B_2)}} B_1 = A_1A_1^\dagger A_1\widetilde{Y}B_1 \\ &= A_1\widetilde{Y}B_1 = A_1A_1^\dagger C_1 = C_1. \end{aligned}$$

From $\mathcal{R}(A_2^*) \subseteq \mathcal{R}(A_1^*)$ and $\mathcal{R}(C_2^*) \subseteq \mathcal{R}(B_2^*)$, we get

$$\begin{aligned} A_2XB_2 &= A_2P_{\overline{\mathcal{R}(A_1^*)}}\widetilde{Y} \restriction_{\overline{\mathcal{R}(B_2)}} B_2 = A_2A_1^\dagger A_1\widetilde{Y}B_2 \\ &= A_2\widetilde{Y}B_2 = \overline{C_2B_2^\dagger}B_2 = C_2B_2^\dagger B_2 = C_2. \end{aligned}$$

Also, for an arbitrary $z \in \overline{\mathcal{R}(A_1^*)} = \overline{\mathcal{R}(B_2)}$,

$$\begin{aligned} \langle X_{11}z, z\rangle &= \langle P_{\overline{\mathcal{R}(A_1^*)}}\widetilde{Y}z, z\rangle = \langle \widetilde{Y}z, P_{\overline{\mathcal{R}(A_1^*)}}z\rangle \\ &= \langle z, \widetilde{Y}^*z\rangle = \langle P_{\overline{\mathcal{R}(A_1^*)}}z, \widetilde{Y}z\rangle \\ &= \langle z, P_{\overline{\mathcal{R}(A_1^*)}}\widetilde{Y}z\rangle = \langle z, X_{11}z\rangle, \end{aligned}$$

so $X_{11}^* = X_{11}$. Hence, X is a hermitian solution to the system $A_i X B_i = C_i$, $i = 1,2$.

On the contrary, assume that $X \in \mathfrak{B}(\mathcal{H})$ is a hermitian solution to the system $A_i X B_i = C_i$, $i = 1,2$. Set

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

in terms of $\mathcal{H} = \overline{\mathcal{R}(A_1^*)} \oplus \mathcal{N}(A_1)$. From Theorem 5.3.5, it follows that

$$\widetilde{Y} := \overline{P_{\overline{\mathcal{R}(A_1^*)}} X P_{\overline{\mathcal{R}(B_2)}} \restriction_{\mathcal{D}(B_2^\dagger)}} \in \mathfrak{B}(\mathcal{H})$$

is a solution of the system $XB_1 = A_1^\dagger C_1$, $A_2 X = \overline{C_2 B_2^\dagger}$. Now,

$$\begin{aligned} P_{\overline{\mathcal{R}(A_1^*)}} \widetilde{Y} \restriction_{\mathcal{R}(B_2)} &= P_{\overline{\mathcal{R}(A_1^*)}} \overline{P_{\overline{\mathcal{R}(A_1^*)}} X P_{\overline{\mathcal{R}(B_2)}} \restriction_{\mathcal{D}(B_2^\dagger)}} \restriction_{\mathcal{R}(B_2)} \\ &= P_{\overline{\mathcal{R}(A_1^*)}} P_{\overline{\mathcal{R}(A_1^*)}} X P_{\overline{\mathcal{R}(B_2)}} \restriction_{\mathcal{R}(B_2)} \\ &= P_{\overline{\mathcal{R}(A_1^*)}} X \restriction_{\mathcal{R}(B_2)}, \end{aligned}$$

and therefore,

$$X_{11} = P_{\overline{\mathcal{R}(A_1^*)}} X \restriction_{\overline{\mathcal{R}(B_2)}} = P_{\overline{\mathcal{R}(A_1^*)}} \widetilde{Y} \restriction_{\overline{\mathcal{R}(B_2)}} .$$

The fact that X is hermitian implies $X_{21} = X_{12}^*$ and X_{22} is hermitian. Hence, X has the form of (5.7). ■

Corollary 5.3.8. *Let $A_i, B_i, C_i \in \mathfrak{B}(\mathcal{H})$, $i = 1,2$ such that $\mathcal{R}(A_1)$ and $\mathcal{R}(B_2)$ are closed, $C_1 \overset{*}{\leq} A_2 \overset{*}{\leq} A_1$, $C_2 \overset{*}{\leq} B_1 \overset{*}{\leq} B_2$ and*

$$A_1^\dagger A_1 = A_1 A_1^\dagger = B_2^\dagger B_2 = B_2 B_2^\dagger.$$

If the system of operator equations $A_i X B_i = C_i$, $i = 1,2$ has a hermitian solution, then the general Hermitian solution has the matrix representation

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}$$

in terms of $\mathcal{H} = \mathcal{R}(A_1^*) \oplus \mathcal{N}(A_1)$, where X_{22} is a hermitian and $X_{11} = P_{\mathcal{R}(A_1^*)} \widetilde{Y} \restriction_{\mathcal{R}(B_2)}$ satisfying that $\widetilde{Y}$ is a hermitian solution of $XB_1 = A_1^\dagger C_1$, $A_2 X = C_2 B_2^\dagger$.

In the following, we obtain the representation of positive solutions to the system of equations $A_i X B_i = C_i$, $i = 1,2$.

Theorem 5.3.9. *Let $A_i, B_i, C_i \in \mathfrak{B}(\mathcal{H})$, $i = 1,2$ such that $C_1 \overset{*}{\leq} A_2 \overset{*}{\leq} A_1$, $C_2 \overset{*}{\leq} B_1 \overset{*}{\leq} B_2$ and*

$$A_1^\dagger A_1 \restriction_{\mathcal{D}(A_1^\dagger)} = A_1 A_1^\dagger = B_2^\dagger B_2 \restriction_{\mathcal{D}(B_2^\dagger)} = B_2 B_2^\dagger. \tag{5.8}$$

If the system of operator equations $A_iXB_i = C_i$, $i = 1,2$ has a positive solution, then the general positive solution has the matrix representation

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & ((X_{11}^{\frac{1}{2}})^\dagger X_{12})^*(X_{11}^{\frac{1}{2}})^\dagger X_{12} + F \end{bmatrix} \tag{5.9}$$

in terms of $\mathcal{H} = \overline{\mathcal{R}(A_1^*)} \oplus \mathcal{N}(A_1)$, where F is positive, $\mathcal{R}(X_{12}) \subseteq \mathcal{R}(X_{11}^{\frac{1}{2}})$ and $X_{11} = P_{\overline{\mathcal{R}(A_1^*)}}\widetilde{Y} \restriction_{\overline{\mathcal{R}(B_2)}}$ satisfying that $\widetilde{Y}$ is a positive solution of $XB_1 = A_1^\dagger C_1$, $A_2X = \overline{C_2B_2^\dagger}$.

Proof. Suppose that $X \in \mathfrak{B}(\mathcal{H})$ has the matrix decomposition (5.9). Since $A_2 \leq^* A_1$, $B_1 \leq^* B_2$ and (5.8) hold, we have that A_i and B_i, $i = 1,2$, have the matrix representations

$$A_i = \begin{bmatrix} A_{i1} & 0 \\ 0 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} B_{i1} & 0 \\ 0 & 0 \end{bmatrix}, \quad i = 1,2,$$

with respect to $\mathcal{H} = \overline{\mathcal{R}(A_1^*)} \oplus \mathcal{N}(A_1)$. Therefore,

$$A_iXB_i = A_ix_{11}B_i = A_iP_{\overline{\mathcal{R}(A_1^*)}}\widetilde{Y} \restriction_{\overline{\mathcal{R}(B_2)}} B_i, \quad i = 1,2.$$

Since $\mathcal{R}(B_1) \subseteq \mathcal{R}(B_2)$ and $\mathcal{R}(C_1) \subseteq \mathcal{R}(A_1)$, we have

$$\begin{aligned} A_1XB_1 &= A_1P_{\overline{\mathcal{R}(A_1^*)}}\widetilde{Y} \restriction_{\overline{\mathcal{R}(B_2)}} B_1 = A_1A_1^\dagger A_1\widetilde{Y}B_1 \\ &= A_1\widetilde{Y}B_1 = A_1A_1^\dagger C_1 = C_1. \end{aligned}$$

From $\mathcal{R}(A_2^*) \subseteq \mathcal{R}(A_1^*)$ and $\mathcal{R}(C_2^*) \subseteq \mathcal{R}(B_2^*)$, we get

$$\begin{aligned} A_2XB_2 &= A_2P_{\overline{\mathcal{R}(A_1^*)}}\widetilde{Y} \restriction_{\overline{\mathcal{R}(B_2)}} B_2 = A_2A_1^\dagger A_1\widetilde{Y}B_2 \\ &= A_2\widetilde{Y}B_2 = \overline{C_2B_2^\dagger}B_2 = C_2B_2^\dagger B_2 = C_2. \end{aligned}$$

Also, for an arbitrary $z \in \overline{\mathcal{R}(A_1^*)} = \overline{\mathcal{R}(B_2)}$,

$$\begin{aligned} \langle X_{11}z, z\rangle &= \langle P_{\overline{\mathcal{R}(A_1^*)}}\widetilde{Y}z, z\rangle = \langle \widetilde{Y}z, P_{\overline{\mathcal{R}(A_1^*)}}z\rangle \\ &= \langle \widetilde{Y}P_{\overline{\mathcal{R}(A_1^*)}}z, P_{\overline{\mathcal{R}(A_1^*)}}z\rangle \geq 0, \end{aligned}$$

so $X_{11} \geq 0$. Hence, X is a positive solution to the system $A_iXB_i = C_i$, $i = 1,2$.

On the contrary, assume that $X \in \mathfrak{B}(\mathcal{H})$ is a positive solution to the system $A_iXB_i = C_i$, $i = 1,2$. Set

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

in terms of $\mathcal{H} = \overline{\mathcal{R}(A_1^*)} \oplus \mathcal{N}(A_1)$. From Theorem 5.3.6, it follows that

$$\widetilde{Y} := \overline{P_{\overline{\mathcal{R}(A_1^*)}}XP_{\overline{\mathcal{R}(B_2)}} \restriction_{\mathcal{D}(B_2^\dagger)}} \in \mathfrak{B}(\mathcal{H})$$

is a solution of the system $XB_1 = A_1^\dagger C_1$, $A_2 X = \overline{C_2 B_2^\dagger}$. Now,

$$\begin{aligned} P_{\overline{\mathcal{R}(A_1^*)}} \widetilde{Y} \restriction_{\mathcal{R}(B_2)} &= P_{\overline{\mathcal{R}(A_1^*)}} \overline{P_{\overline{\mathcal{R}(A_1^*)}} X P_{\overline{\mathcal{R}(B_2)}} \restriction_{\mathcal{D}(B_2^\dagger)}} \restriction_{\mathcal{R}(B_2)} \\ &= P_{\overline{\mathcal{R}(A_1^*)}} P_{\overline{\mathcal{R}(A_1^*)}} X P_{\overline{\mathcal{R}(B_2)}} \restriction_{\mathcal{R}(B_2)} \\ &= P_{\overline{\mathcal{R}(A_1^*)}} X \restriction_{\mathcal{R}(B_2)}, \end{aligned}$$

and therefore,

$$X_{11} = P_{\overline{\mathcal{R}(A_1^*)}} X \restriction_{\overline{\mathcal{R}(B_2)}} = P_{\overline{\mathcal{R}(A_1^*)}} \widetilde{Y} \restriction_{\overline{\mathcal{R}(B_2)}} .$$

The fact that X is positive implies that $X_{21} = X_{12}^*$, $\mathcal{R}(X_{12}) \subseteq \mathcal{R}(X_{11}^{\frac{1}{2}})$ and

$$X_{22} = ((X_{11}^{\frac{1}{2}})^\dagger X_{12})^* (X_{11}^{\frac{1}{2}})^\dagger X_{12} + F,$$

where F is positive (see Theorem 1.1.11). Therefore X has the form of (5.9). ■

Corollary 5.3.10. *Let $A_i, B_i, C_i \in \mathfrak{B}(\mathcal{H})$, $i = 1, 2$ such that $\mathcal{R}(A_1)$ and $\mathcal{R}(B_2)$ are closed, $C_1 \overset{*}{\leq} A_2 \overset{*}{\leq} A_1$, $C_2 \overset{*}{\leq} B_1 \overset{*}{\leq} B_2$ and*

$$A_1^\dagger A_1 = A_1 A_1^\dagger = B_2^\dagger B_2 = B_2 B_2^\dagger.$$

If the system of operator equations $A_i X B_i = C_i$, $i = 1, 2$ has a positive solution, then the general positive solution has the matrix representation

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & ((X_{11}^{\frac{1}{2}})^\dagger X_{12})^* (X_{11}^{\frac{1}{2}})^\dagger X_{12} + F \end{bmatrix}$$

in terms of $\mathcal{H} = \mathcal{R}(A_1^*) \oplus \mathcal{N}(A_1)$, where F is positive, $\mathcal{R}(X_{12}) \subseteq \mathcal{R}(X_{11}^{\frac{1}{2}})$ and $X_{11} = P_{\mathcal{R}(A_1^*)} \widetilde{Y} \restriction_{\mathcal{R}(B_2)}$ satisfying that $\widetilde{Y}$ is a positive solution of $XB_1 = A_1^\dagger C_1$, $A_2 X = C_2 B_2^\dagger$.

5.3.2 SOLVABILITY OF $C \overset{*}{\leq} AXB$

The next theorem provides necessary and sufficient conditions for the solvability of operator inequality $C \overset{*}{\leq} AXB$.

Theorem 5.3.11. *Let $A, B, C \in \mathfrak{B}(\mathcal{H})$. The following statements are equivalent:*

(i) Operator inequality $C \overset{*}{\leq} AXB$ is solvable;

(ii) The system of operator equations

$$\begin{aligned} AXBC^* &= CC^*, \\ C^*AXB &= C^*C \end{aligned}$$

is solvable.

(iii) $\mathcal{R}(CC^*) \subseteq \mathcal{R}(A)$, $\mathcal{R}(C^*C) \subseteq \mathcal{R}(B^*)$ *and the system of operator equations*

$$XBC^* = A^\dagger CC^*,$$
$$C^*AX = \overline{C^*CB^\dagger}$$

is solvable.

Proof. $(i) \Leftrightarrow (ii)$ This follows immediately from the definition of $*$-order.

$(ii) \Leftrightarrow (iii)$ Observe that $\mathcal{R}(BC^*) \subseteq \overline{\mathcal{R}(B)}$ and $\mathcal{R}((C^*A)^*) = \mathcal{R}(A^*C) \subseteq \overline{R(A^*)}$. The proof now follows from Theorem 5.3.4. ∎

Corollary 5.3.12. *Let $A, B, C \in \mathfrak{B}(\mathcal{H})$ such that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed or $\mathcal{R}(C)$ is closed. Then the following statements are equivalent:*

(i) Operator inequality $C \overset{*}{\leq} AXB$ is solvable;

(ii) $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subseteq \mathcal{R}(B^*)$;

(iii) Operator equation $C = AXB$ is solvable.

Proof. $(i) \Rightarrow (ii)$ Assume that there exists $X \in \mathfrak{B}(\mathcal{H})$ such that $C \overset{*}{\leq} AXB$. Then $\mathcal{R}(CC^*) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(C^*C) \subseteq \mathcal{R}(B^*)$, by Theorem 5.3.11. If $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed, we have that

$$\mathcal{R}(C) \subseteq \overline{\mathcal{R}(CC^*)} \subseteq \overline{\mathcal{R}(A)} = \mathcal{R}(A),$$
$$\mathcal{R}(C^*) \subseteq \overline{\mathcal{R}(C^*C)} \subseteq \overline{\mathcal{R}(B^*)} = \mathcal{R}(B^*).$$

If $\mathcal{R}(C)$ is closed, then

$$\mathcal{R}(C) = \mathcal{R}(CC^*) \subseteq \mathcal{R}(A),$$
$$\mathcal{R}(C^*) = \mathcal{R}(C^*C) \subseteq \mathcal{R}(B^*).$$

In both cases, $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subseteq \mathcal{R}(B^*)$.

$(ii) \Rightarrow (iii)$ This follows from Theorem 1.1.6.

$(iii) \Rightarrow (i)$ Obvious. ∎

We now consider a special case when $B = I$, i.e. we consider the equation $C \overset{*}{\leq} AX$. Using Theorem 5.3.11 and Corollary 5.3.12, we get the following corollaries:

Corollary 5.3.13. *Let $A, C \in \mathfrak{B}(\mathcal{H})$. The following statements are equivalent:*

(i) Operator inequality $C \overset{*}{\leq} AX$ is solvable;

(ii) *The system of operator equations*

$$AXC^* = CC^*,$$
$$C^*AX = C^*C$$

is solvable.

(iii) $\mathcal{R}(CC^*) \subseteq \mathcal{R}(A)$ *and the system of operator equations*

$$XC^* = A^{\dagger}CC^*,$$
$$C^*AX = C^*C$$

is solvable.

Corollary 5.3.14. *Let* $A, C \in \mathfrak{B}(\mathcal{H})$ *such that* $\mathcal{R}(A)$ *or* $\mathcal{R}(C)$ *is closed. Then the following statements are equivalent:*

(i) *Operator inequality* $C \overset{*}{\leq} AX$ *is solvable;*

(ii) $\mathcal{R}(C) \subseteq \mathcal{R}(A)$*;*

(iii) *Operator equation* $C = AX$ *is solvable.*

We now give a general form of the solution of $C \overset{*}{\leq} AX$.

Theorem 5.3.15. *Let* $A, C \in \mathfrak{B}(\mathcal{H})$ *such that* $\mathcal{R}(C) = \overline{\mathcal{R}(C)} \subseteq \mathcal{R}(A)$*. Then, the general solution of the inequality* $C \overset{*}{\leq} AX$ *is*

$$X = A^{\dagger}C + P_{\mathcal{N}(C^*A)}Z - P_{\overline{\mathcal{R}(A^*)}\cap\mathcal{N}(C^*A)}ZP_{\mathcal{R}(C^*)},$$

where $Z \in \mathfrak{B}(\mathcal{H})$ *is arbitrary.*

Proof. Assume that $\mathcal{R}(C) = \overline{\mathcal{R}(C)} \subseteq \mathcal{R}(A)$ holds. By the previous corollary, the inequality $C \overset{*}{\leq} AX$ is solvable. Also, the general solution of the equation $AXC^* = CC^*$ is

$$\begin{aligned} X &= A^{\dagger}CC^*C^{*\dagger} + Y - A^{\dagger}AYC^*C^{*\dagger} \\ &= A^{\dagger}CC^{\dagger}C + Y - A^{\dagger}AYC^{\dagger}C \\ &= A^{\dagger}C + Y - A^{\dagger}AYC^{\dagger}C. \end{aligned}$$

where $Y \in \mathfrak{B}(\mathcal{H})$ is arbitrary. Note that $A^{\dagger}C : \mathcal{H} \mapsto \mathcal{H}$ is bounded since $\mathcal{R}(C) \subseteq \mathcal{R}(A)$. If X satisfies the equation $C^*AX = C^*C$, then

$$\begin{aligned} & C^*A(A^{\dagger}C + Y - A^{\dagger}AYC^{\dagger}C) = C^*C \\ \Rightarrow\ & C^*AA^{\dagger}C + C^*AY - C^*AA^{\dagger}AYC^{\dagger}C = C^*C \\ \Rightarrow\ & C^*C + C^*AY - C^*AYC^{\dagger}C = C^*C \\ \Rightarrow\ & C^*AY(I - C^{\dagger}C) = 0. \end{aligned}$$

Thus, Y is a solution of the equation $C^*AX(I - C^\dagger C) = 0$. Since

$$(I - C^\dagger C)(I - C^\dagger C)^\dagger = I - C^\dagger C,$$

it follows that

$$\begin{aligned} Y &= Z - (C^*A)^\dagger C^*AZ(I - C^\dagger C)(I - C^\dagger C)^\dagger \\ &= Z - (C^*A)^\dagger C^*AZ(I - C^\dagger C), \end{aligned}$$

where $Z \in \mathfrak{B}(\mathcal{H})$ is arbitrary. Replacing Y in the formula for X, we get

$$\begin{aligned} X &= A^\dagger C + Z - (C^*A)^\dagger C^*AZ(I - C^\dagger C) \\ &\quad - A^\dagger A(Z - (C^*A)^\dagger C^*AZ(I - C^\dagger C))C^\dagger C \\ &= A^\dagger C + Z - (C^*A)^\dagger C^*AZ(I - C^\dagger C) - A^\dagger AZC^\dagger C \\ &= A^\dagger C + (I - (C^*A)^\dagger C^*A)Z - (A^\dagger A - (C^*A)^\dagger C^*A)ZC^\dagger C \\ &= A^\dagger C + P_{\mathcal{N}(C^*A)}Z - (P_{\overline{\mathcal{R}(A^*)}} - P_{\overline{\mathcal{R}(A^*C)}})ZP_{\mathcal{R}(C^*)}. \end{aligned}$$

Since $\overline{\mathcal{R}(A^*C)} \subseteq \overline{\mathcal{R}(A^*)}$, it follows that

$$P_{\overline{\mathcal{R}(A^*)}} - P_{\overline{\mathcal{R}(A^*C)}} = P_{\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(A^*C)}^\perp} = P_{\overline{\mathcal{R}(A^*)} \cap \mathcal{N}(C^*A)}.$$

Finally, we conclude that the general solution of $C \overset{*}{\leq} AX$ is

$$X = A^\dagger C + P_{\mathcal{N}(C^*A)}Z - P_{\overline{\mathcal{R}(A^*)} \cap \mathcal{N}(C^*A)}ZP_{\mathcal{R}(C^*)},$$

where $Z \in \mathfrak{B}(\mathcal{H})$ is arbitrary. ■

Using the fact that

$$B \overset{*}{\leq} A \Leftrightarrow \begin{cases} BB^* = AB^* \\ B^*B = B^*A \end{cases} \Leftrightarrow \begin{cases} BB^* = BA^* \\ B^*B = A^*B \end{cases} \Leftrightarrow B^* \overset{*}{\leq} A^*,$$

we also have the following corollaries for the inequality $C \overset{*}{\leq} XB$:

Corollary 5.3.16. *Let $B, C \in \mathfrak{B}(\mathcal{H})$. The following statements are equivalent:*

(i) Operator inequality $C \overset{*}{\leq} XB$ is solvable;

(ii) The system of operator equations

$$\begin{aligned} B^*XC &= C^*C, \\ CB^*X &= CC^* \end{aligned}$$

is solvable.

(iii) $\mathcal{R}(C^*C) \subseteq \mathcal{R}(B^*)$ *and the system of operator equations*

$$XC = B^{*\dagger}C^*C,$$
$$CB^*X = CC^*$$

is solvable.

Corollary 5.3.17. *Let* $B, C \in \mathfrak{B}(\mathcal{H})$ *such that* $\mathcal{R}(B)$ *or* $\mathcal{R}(C)$ *is closed. Then the following statements are equivalent:*

(i) *Operator inequality* $C \overset{*}{\leq} XB$ *is solvable;*

(ii) $\mathcal{R}(C^*) \subseteq \mathcal{R}(B^*)$*;*

(iii) *Operator equation* $C = XB$ *is solvable.*

Corollary 5.3.18. *Let* $B, C \in \mathfrak{B}(\mathcal{H})$ *such that* $\mathcal{R}(C^*) = \overline{\mathcal{R}(C^*)} \subseteq \mathcal{R}(B^*)$*. Then, the general solution of the inequality* $C \overset{*}{\leq} XB$ *is*

$$X = \overline{CB^\dagger} + ZP_{\mathcal{N}(CB^*)} - P_{\mathcal{R}(C)}ZP_{\overline{\mathcal{R}(B)}\cap\mathcal{N}(CB^*)},$$

where $Z \in \mathfrak{B}(\mathcal{H})$ *is arbitrary.*

CHAPTER 6

A NOTE ON q-NUMERICAL RADIUS OF LINEAR OPERATORS

In this chapter, our primary focus is on exploring the concept of the q-numerical radius, denoted as $\omega_q(\cdot)$, for operators on Hilbert spaces. We delve into the investigation of various inequalities associated with these values, extending along the way the well-known results regarding the numerical radius that occurs when we plug in $q = 1$. Additionally, we provide explicit formulas for calculating $\omega_q(\cdot)$ in specific cases of operator matrices, as well as in the case of the rank one operators. Finally, we explore various analytical properties of $\omega_q(\cdot)$ when it is treated as a function of the parameter q.

6.1 MOTIVATION

As already mentioned in Section 1.1, the numerical range of some $A \in \mathfrak{B}(\mathcal{H})$ is defined as the set

$$\mathcal{W}(A) = \{\langle Ax, x\rangle\colon\ x \in \mathcal{H},\ \|x\| = 1\},$$

while the numerical radius is given by

$$\omega(A) = \sup_{w\in\mathcal{W}(A)} |w|.$$

Similarly, the q-numerical range for $q \in \overline{\mathbb{D}}$ is defined via

$$\mathcal{W}_q(A) = \{\langle Ax, y\rangle\colon\ x, y \in \mathcal{H},\ \|x\| = \|y\| = 1,\ \langle x, y\rangle = q\} \tag{6.1}$$

while the q-numerical radius represents the value

$$\omega_q(A) = \sup_{w\in\mathcal{W}_q(A)} |w|. \tag{6.2}$$

Here, it is worth pointing out that the q-numerical radius is merely a generalization of the numerical radius, given the fact that $\omega_q(A) = \omega(A)$ for $|q| = 1$. This observation follows from the fact that the equality would have to hold in the Cauchy-Schwarz inequality $|q| = |\langle x, y\rangle| \leq \|x\|\|y\| = 1$, provided $|q| = 1$. From here, it is clear that $y = \lambda x$ would need to be true for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$, hence $|\langle Ax, y\rangle| = |\langle Ax, x\rangle|$.

Obviously, if $\dim(\mathcal{H}) = 1$, then $\mathcal{W}_q(A)$ is nonempty if and only if $|q| = 1$, while if $\dim(\mathcal{H}) \geq 2$, it is easy to see that $\mathcal{W}_q(A)$ is always nonempty. Thus, in the sequel, we shall restrict ourselves to the Hilbert spaces of dimension at least two. The set $\mathcal{W}_q(A)$ was originally introduced in [127] for operators on finite-dimensional unitary spaces. The motivation for (6.1) is natural and comes from the problem of constrained optimization of bilinear functionals. More details about this set can be found in [91, Chapter 8]. We also refer a reader to some recent contributions to the subject. See, for example, [34, 35, 36, 37, 93, 107, 123, 121, 148, 154].

In a recent paper by Moghaddam et al. [129], the q-numerical radius $\omega_q(A)$ was considered for arbitrary operators $A \in \mathfrak{B}(\mathcal{H})$ and values of $q \in (0, 1)$, and the following two inequalities were derived.

Theorem 6.1.1. *[129, Theorem 2.1] Let $A \in \mathfrak{B}(\mathcal{H})$ and $q \in (0, 1)$. Then*

$$\frac{q}{2(2-q^2)}\|A\| \leq \omega_q(A) \leq \|A\|, \tag{6.3}$$

and for any normal operator A,

$$\frac{q}{2-q^2}\|A\| \leq \omega_q(A) \leq \|A\|.$$

Theorem 6.1.2. *[129, Eq. (23)] Let $A \in \mathfrak{B}(\mathcal{H})$ and $q \in (0, 1)$. Then*

$$\frac{q}{2-q^2}\omega(A) \leq \omega_q(A) \leq \frac{q}{1-\sqrt{1-q^2}}\omega(A). \tag{6.4}$$

An inequality similar to (6.4) holds in the scenario when we are dealing with a finite-dimensional Hilbert space $\mathcal{H}$, as demonstrated by Li et al. [122].

Theorem 6.1.3. *[122, Theorem 2.5] Suppose $q_1, q_2 \in \mathbb{C}$ satisfy $0 < |q_2| < |q_1| < 1$. Then for a finite-dimensional Hilbert space $\mathcal{H}$ and any $A \in \mathfrak{B}(\mathcal{H})$, we have*

$$q_2\, \mathcal{W}_{q_1}(A) \subseteq q_1\, \mathcal{W}_{q_2}(A).$$

Motivated by the aforementioned results, our central goal is to investigate the various properties of the q-numerical radius with the purpose of obtaining stronger results. As an immediate improvement over the inequalities stated in Theorems 6.1.1 and 6.1.2, we present the following theorem.

Theorem 6.1.4. *Let $A \in \mathfrak{B}(\mathcal{H})$ and $q \in \overline{\mathbb{D}}$. We then have*

$$|q|\,\omega(A) \le \omega_q(A), \tag{6.5}$$

as well as

$$\frac{|q|}{2}\|A\| \le \omega_q(A) \le \|A\|. \tag{6.6}$$

Also, if the operator A is normal, then

$$|q| \cdot \|A\| \le \omega_q(A) \le \|A\|. \tag{6.7}$$

Besides that, our second main result represents an inequality which generalizes the well-known equality (see, for example, [91, Proposition 5.1, (i)])

$$\omega\left(\bigoplus_{n=1}^{+\infty} A_n\right) = \sup_{n\in\mathbb{N}} \omega(A_n) \tag{6.8}$$

regarding the numerical radius of an operator $\bigoplus_{n=1}^{+\infty} A_n$, where $A_n \in \mathcal{B}(\mathcal{H}_n)$ for each $n \in \mathbb{N}$, and $(\mathcal{H}_n)_{n\in\mathbb{N}}$ is a given sequence of Hilbert spaces. We disclose the aforementioned result in the form of the next theorem.

Theorem 6.1.5. *Let $(\mathcal{H}_n)_{n\in\mathcal{N}}$ be a sequence of Hilbert spaces and let $A_n \in \mathcal{B}(\mathcal{H}_n)$ for all the $n \in \mathbb{N}$. If $q \in \overline{\mathbb{D}} \setminus \{0\}$, then*

$$\sup_{n\in\mathbb{N}} \omega_q(A_n) \le \omega_q\left(\bigoplus_{n=1}^{+\infty} A_n\right) \le \frac{|q| + 2\sqrt{1-|q|^2}}{|q|} \sup_{n\in\mathbb{N}} \omega_q(A_n). \tag{6.9}$$

Bearing everything in mind, the structure of the chapter shall be organized as follows. In Section 6.1.1 we disclose certain well-known theoretical results regarding the numerical and q-numerical radius. Afterward, we use Section 6.2 in order to give an elementary proof of the fact that $(\mathfrak{B}(\mathcal{H}), \omega_q)$ is a Banach space for each $q \in \overline{\mathbb{D}} \setminus \{0\}$. In this section, we also demonstrate the validity of Theorem 6.1.4 along the way. Subsequently, we regard ω_q as a function in $q \in \overline{\mathbb{D}}$ in Section 6.3, and establish some of its analytical properties, including its continuity. In Section 6.4 we offer a full proof of Theorem 6.1.5 along with some additional results regarding the q-numerical radius of special cases of operator matrices. We also give an explicit formula for the q-radius of rank one operators, as well as a generalization of the Buzano inequality (see [29, 87]).

6.1.1 PRELIMINARIES

This brief section will serve to disclose some theoretical results regarding the numerical and q-numerical radius which will be used throughout the remainder of the chapter. It is well known (see, for example, [91, 95]) that

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\| \quad \text{for each } A \in \mathfrak{B}(\mathcal{H}). \tag{6.10}$$

The left-hand side inequality in (6.10) is a direct consequence of the polarization identity and the parallelogram law. Besides that, we will heavily rely on the following lemma which elaborates on the properties of the numerical radius of some operator matrices.

Lemma 6.1.6. *[105, Lemma 2.1] Let $A, B \in \mathfrak{B}(\mathcal{H})$ and $\theta \in \mathbb{R}$. Then*

(i) $\omega\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right) = \max\{\omega(X), \omega(Y)\}$;

(ii) $\omega\left(\begin{bmatrix} 0 & X \\ e^{i\theta}Y & 0 \end{bmatrix}\right) = \omega\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right)$;

(iii) $\omega\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) = \omega\left(\begin{bmatrix} 0 & Y \\ X & 0 \end{bmatrix}\right)$;

(iv) $\omega\left(\begin{bmatrix} X & Y \\ Y & X \end{bmatrix}\right) = \max\{\omega(X+Y), \omega(X-Y)\}$.

In particular, $\omega\left(\begin{bmatrix} 0 & Y \\ Y & 0 \end{bmatrix}\right) = \omega(Y)$.

In the rest of the chapter, we will apply Lemma 6.1.6 in order to obtain certain generalized results regarding the q-numerical radius of corresponding operator matrices. While dealing with the q-numerical radius, one of its most important properties used will be its radial symmetry. This observation is quite simple to prove, alongside the following statements given in the next lemma which represents a direct consequence of the results obtained in [91, Proposition 3.1, Chapter 8], as well as (6.2).

Lemma 6.1.7. *For $A, B \in \mathfrak{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$, $q \in \overline{\mathbb{D}}$ we have following properties:*

(i) $\omega_q(\lambda A) = |\lambda|\omega_q(A)$;

(ii) $\omega_q(A+B) \leq \omega_q(A) + \omega_q(B)$;

(iii) $\omega_q(U^*AU) = \omega_q(A)$, *where $U \in \mathfrak{B}(\mathcal{H})$ is a unitary operator;*

(iv) $\omega_{\lambda q}(A) = \omega_q(A)$ *for all $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.*

6.2 Norm properties

By implementing (6.3), it is not difficult to conclude that the function $\omega_q\colon \mathfrak{B}(\mathcal{H}) \mapsto \mathbb{R}$ is a norm on the space $\mathfrak{B}(\mathcal{H})$, for each $q \in \overline{\mathbb{D}} \setminus \{0\}$. This result is actually well-known and can be proved in many ways. For example, if the Hilbert space $\mathcal{H}$ is finite-dimensional, then the corresponding proof can be carried out by using the strategy from [122, Theorem 3.1].

In this section, our goal will be to prove Theorem 6.1.4 and then provide an elementary proof of the fact that $(\mathfrak{B}(\mathcal{H}), \omega_q)$ is a Banach space for each $q \in \overline{\mathbb{D}} \setminus \{0\}$. We begin by interpreting equations (6.1) and (6.2) in order to derive an alternative formula for defining the q-numerical range and q-numerical radius of an operator. The corresponding result is disclosed in the following elementary lemma.

Lemma 6.2.1. *For any $A \in \mathcal{B}(\mathcal{H})$ and $q \in \overline{\mathbb{D}}$, we have*

$$\mathcal{W}_q(A) = \left\{ q\langle Ay, y\rangle + \sqrt{1-|q|^2}\langle At, y\rangle : \ \|y\| = \|t\| = 1,\ \langle t, y\rangle = 0 \right\}. \tag{6.11}$$

Also, we have

$$\omega_q(A) = \sup\left\{ |q| \cdot |\langle Ay, y\rangle| + \sqrt{1-|q|^2} \cdot |\langle At, y\rangle| \ : \ \|y\| = \|t\| = 1,\ \langle t, y\rangle = 0 \right\}. \tag{6.12}$$

Proof. Let $x, y \in \mathcal{H}$ be unit vectors such that $\langle x, y\rangle = q$. From $\mathcal{H} = \operatorname{lin}\{y\} \oplus \{y\}^\perp$, we have that there exist some $\lambda \in \mathbb{C}$ and $v \in \{y\}^\perp$ such that $x = \lambda y + v$. This implies $q = \langle x, y\rangle = \lambda$, hence we obtain $x = qy + v$. From $1 = \|x\|^2 = |q|^2 + \|v\|^2$, we get $\|v\| = \sqrt{1-|q|^2}$, which means that

$$x = qy + \sqrt{1-|q|^2}\, t \quad \text{for some unit vector } t \in \{y\}^\perp. \tag{6.13}$$

It is obvious that the converse is also true, i.e. if (6.13) holds, then $\langle x, y\rangle = q$. The noted observation directly implies (6.11) from (6.1).

It is clear that $\omega_q(A)$ is finite. By using (6.2), we see that

$$\omega_q(A) = \sup\left\{ |q\langle Ay, y\rangle + \sqrt{1-|q|^2}\langle At, y\rangle| \colon \ \|y\| = \|t\| = 1,\ \langle t, y\rangle = 0 \right\}.$$

If we denote

$$s = \sup\left\{ |q| \cdot |\langle Ay, y\rangle| + \sqrt{1-|q|^2} \cdot |\langle At, y\rangle| \ : \ \|y\| = \|t\| = 1,\ \langle t, y\rangle = 0 \right\},$$

it becomes trivial to see that s is finite and that $\omega_q(A) \le s$. Thus, in order to complete the proof regarding (6.12), it is sufficient to demonstrate that $\omega_q(A) \ge s$.

Let $\varepsilon > 0$ be an arbitrary constant. Then, there certainly exist two vectors $y, t \in \mathcal{H}$ such that $\|y\| = \|t\| = 1$, $\langle y, t \rangle = 0$ and

$$|q| \cdot |\langle Ay, y \rangle| + \sqrt{1 - |q|^2} \cdot |\langle At, y \rangle| > s - \varepsilon,$$

which promptly gives

$$|q\langle Ay, y \rangle| + \left| \sqrt{1 - |q|^2} \langle At, y \rangle \right| > s - \varepsilon.$$

Now, it is easy to see that there exists a $\theta \in \mathbb{R}$ such that $t_1 = e^{i\theta}\, t \in \mathcal{H}$ satisfies

$$|q\langle Ay, y \rangle| + \left| \sqrt{1 - |q|^2} \langle At_1, y \rangle \right| = \left| q\langle Ay, y \rangle + \sqrt{1 - |q|^2} \langle At_1, y \rangle \right|.$$

From here, we quickly obtain that

$$\omega_q(A) \geq \left| q\langle Ay, y \rangle + \sqrt{1 - |q|^2} \langle At_1, y \rangle \right| > s - \varepsilon.$$

Since $\varepsilon > 0$ can be arbitrarily chosen, we see that $\omega_q(A) \geq s$. ∎

By relying on Lemma 6.2.1, we are now able to give the complete proof of Theorem 6.1.4.

Proof of Theorem 6.1.4. By implementing (6.12), we immediately see that the inequality $|q| \cdot |\langle Ay, y \rangle| \leq \omega_q(A)$ holds for each unit vector $y \in \mathcal{H}$. Inequality (6.5) follows directly from here. Furthermore, it is obvious that $\omega_q(A) \leq \|A\|$. By combining (6.5) and (6.10), it becomes straightforward to obtain (6.6). Also, if the operator A is normal, then $\omega(A) = \|A\|$ necessarily holds (see, for example, [95, Theorem 1.4-2.]), which directly leads us to 6.7 by simply applying (6.5). ∎

We are now in a position to implement Theorem 6.1.4 in order to provide a quick and concise elementary proof regarding the norm properties of ω_q on $\mathfrak{B}(\mathcal{H})$. We demonstrate the said result in the following theorem.

Theorem 6.2.2. *If $\mathcal{H}$ is a Hilbert space and $q \in \overline{\mathbb{D}} \setminus \{0\}$, then $(\mathfrak{B}(\mathcal{H}), \omega_q)$ is a Banach space and the norm ω_q is equivalent to the standard operator norm $\|\cdot\|$ on $\mathfrak{B}(\mathcal{H})$.*

Proof. By implementing (6.6), it is not difficult to establish that $\omega_q \colon \mathfrak{B}(\mathcal{H}) \mapsto \mathbb{R}$ is a norm on $\mathfrak{B}(\mathcal{H})$, hence $(\mathfrak{B}(\mathcal{H}), \omega_q)$ is a normed space. From the same inequality, we deduce that the norm ω_q must be equivalent to the standard operator norm $\|\cdot\|$ on $\mathfrak{B}(\mathcal{H})$. For this reason, $(\mathfrak{B}(\mathcal{H}), \omega_q)$ must be a Banach space. ∎

Remark 6.2.3. *Theorem 6.2.2 could also be proved in a different manner, by using the convexity argument. Namely, let $y \in \mathcal{H}$ be an arbitrary unit vector and let $z \in \{y\}^{\perp}$ be some unit vector. Moreover, let $t_1 = z$, $t_2 = -z$ and*

$$a_1 = q\langle Ay, y\rangle + \sqrt{1-|q|^2}\langle At_1, y\rangle \quad \text{and} \quad a_2 = q\langle Ay, y\rangle + \sqrt{1-|q|^2}\langle At_2, y\rangle.$$

Since $\langle t_1, y\rangle = \langle t_2, y\rangle = 0$, (6.11) tells us that $a_1, a_2 \in \mathcal{W}_q(A)$. Due to the fact that $\mathcal{W}_q(A)$ is a convex subset in $\mathbb{C}$ (see, for example, [91, Theorem 3.2]), we have that

$$q\langle Ay, y\rangle = \frac{1}{2}a_1 + \frac{1}{2}a_2 \in \mathcal{W}_q(A),$$

which promptly gives (6.5).

Here, we note that Theorem 6.1.4 can also be applied to extend some other well-known claims. For example, it is possible to obtain a generalization of an earlier result given in [91, Proposition 5.1 (g)] by relying on a similar proof strategy.

Corollary 6.2.4. *Let $A \in \mathfrak{B}(\mathcal{H})$ and $q \in \overline{\mathbb{D}} \setminus \{0\}$. Then*

$$\lim_{n\to\infty} \sqrt[n]{\omega_q(A^n)} = r(A). \tag{6.14}$$

Proof. By applying (6.6), we obtain the inequalities

$$\sqrt[n]{\frac{|q|}{2}} \cdot \sqrt[n]{\|A^n\|} \leq \sqrt[n]{\omega_q(A^n)} \leq \sqrt[n]{\|A^n\|} \quad \text{for all the } n \in \mathbb{N}.$$

Now, it is sufficient to use Gelfand's spectral radius formula and the fact that $q \neq 0$ in order to reach $\lim_{n\to\infty} \sqrt[n]{\omega_q(A^n)} = r(A)$. ■

Remark 6.2.5. *Let $A \in \mathfrak{B}(\mathcal{H})$. It is easy to show that*

$$\omega(A) = \sup_{\theta\in\mathbb{R}} \|\mathrm{Re}\,(e^{i\theta}A)\|.$$

Indeed, for an arbitrary $x \in \mathcal{H}$, we have that $|\langle Ax, x\rangle| = \sup_{\theta\in\mathbb{R}} \mathrm{Re}\,(e^{i\theta}\langle Ax, x\rangle)$. Thus,

$$\sup_{\theta\in\mathbb{R}} \|\mathrm{Re}\,(e^{i\theta}A)\| = \sup_{\theta\in\mathbb{R}} \omega(\mathrm{Re}\,(e^{i\theta}A)) = \omega(A).$$

However, if $q \in \mathbb{D} \setminus \{0\}$, then

$$\omega_q(A) = \sup_{\theta\in\mathbb{R}} \|\mathrm{Re}\,(e^{i\theta}A)\|$$

may not be true. For example, it is enough to take $A = I$, where $I\colon \mathcal{H} \to \mathcal{H}$ is the identity operator on $\mathcal{H}$. Moreover,

$$\omega(I) \neq \sup_{\theta\in\mathbb{R}} N(\mathrm{Re}\,(e^{i\theta}I)),$$

where N is an arbitrary C^*-norm on $\mathfrak{B}(\mathcal{H})$. This is due to the fact that

$$\sup_{\theta\in\mathbb{R}} N(\mathrm{Re}\,(e^{i\theta}I)) = N(I) = 1,$$

while $\omega_q(I) = |q|$.

It is well known that the subalgebra of all invertible operators $\mathfrak{B}(\mathcal{H})^{-1}$ in $\mathfrak{B}(\mathcal{H})$ is open with respect to the topology generated by the operator norm. By Theorem 6.2.2, it is also open with respect to the topology generated by $\omega_q(\cdot)$, where $q \in \overline{\mathbb{D}} \setminus \{0\}$. The following theorem provides an estimate for the radius of a ball centered at some invertible operator and contained in $\mathfrak{B}(\mathcal{H})^{-1}$.

Theorem 6.2.6. *Let $A, B \in \mathfrak{B}(\mathcal{H})$ and let $q \in \overline{\mathbb{D}} \setminus \{0\}$. If A is invertible and*

$$\omega_q(A - B) < \frac{|q|^2}{4\omega_q(A^{-1})}, \tag{6.15}$$

then B is also invertible.

Proof. Assume that $B \in \mathfrak{B}(\mathcal{H})$ satisfies (6.15). Using (6.6), it follows that

$$\begin{aligned} |q|^2 &> 4\omega_q(A^{-1}) \cdot \omega_q(A - B) \\ &\geq 4 \cdot \frac{|q|}{2}\|A^{-1}\| \cdot \frac{|q|}{2}\|A - B\| \\ &\geq |q|^2\|A^{-1}(A - B)\| = |q|^2\|I - A^{-1}B\| \implies \|I - A^{-1}B\| < 1. \end{aligned}$$

Thus, $A^{-1}B$ is invertible, and so the operator B is also invertible. ∎

Another well-known result regarding the ordinary numerical radius is the power inequality. More precisely, for $A \in \mathfrak{B}(\mathcal{H})$ and $n \in \mathbb{N}$, we have that $\omega(A^n) \leq \omega^n(A)$ (see, for example, [91, Theorem 3.1]). An analogous power inequality holds in the case of the q-numerical radius as well, as the following two theorems show.

Theorem 6.2.7. *Let $A \in \mathfrak{B}(\mathcal{H})$ and $q \in \overline{\mathbb{D}} \setminus \{0\}$. Then $\omega_q(A) \leq |q|$ if and only if $\omega_q(A^n) \leq |q|$ for all $n \in \mathbb{N}$.*

Proof. We shall present the sketch of the proof as it essentially uses the same ideas as the ones used in the proof of [95, Theorem 2.1-1]. First, we have that $\omega_q(A) \leq |q|$ if and only if $\mathrm{Re}\langle (I - zA)x, y\rangle \geq 0$ for all $z \in \mathbb{D}$ and all $x, y \in \mathcal{H}$ such that $\|x\| = \|y\| = 1$, $\langle x, y\rangle = q$. Also, from $\omega_q(A) \leq |q|$ and (6.5), it follows that $r(A) \leq \omega(A) \leq 1$.

Now, for an arbitrary $z \in \mathbb{D}$ we have that the operator $I - zA$ is invertible since $r(zA) < 1$. Also, for all $z \in \mathbb{D}$ and all $x, y \in \mathcal{H}$ such that $\|x\| = \|y\| = 1$, $\langle x, y\rangle = q$, the inequality $\mathrm{Re}\langle (I - zA)x, y\rangle \geq 0$ implies that $\mathrm{Re}\langle (I - zA)^{-1}x, y\rangle \geq 0$. Finally, let $n \in \mathbb{N}$ be arbitrary and let ε denote the n-th root of unity. Using the identity

$$(I - z^n A^n)^{-1} = \frac{1}{n}\left[(I - zA)^{-1} + (I - \varepsilon zA)^{-1} + \cdots + \left(I - \varepsilon^{n-1} zA\right)^{-1}\right], \quad z \in \mathbb{D},$$

we have that $\mathrm{Re}\langle (I - z^n A^n)x, y\rangle \geq 0$ for all $z \in \mathbb{D}$, and all $x, y \in \mathcal{H}$ such that $\|x\| = \|y\| = 1$, $\langle x, y\rangle = q$, which is equivalent with the fact that $\omega_q(A^n) \leq |q|$. ■

Theorem 6.2.8. *Let $A \in \mathfrak{B}(\mathcal{H})$ and $q \in \overline{\mathbb{D}}$. Then*

$$|q|^{n-1}\omega_q(A^n) \leq \omega_q^n(A) \text{ for all } n \in \mathbb{N}.$$

Proof. If $A = 0$ or $q = 0$, the theorem trivially holds. Thus, we may assume that $A \neq 0$ and $q \in \overline{\mathbb{D}} \setminus \{0\}$. By Theorem 6.2.2, we have that $\omega_q(A) \neq 0$ and so $\omega_q\left(\frac{q}{\omega_q(A)}A\right) = |q|$. For an arbitrary $n \in \mathbb{N}$, Theorem 6.2.7 now yields $\omega_q\left(\left(\frac{q}{\omega_q(A)}A\right)^n\right) \leq |q|$, which promptly gives the inequality $|q|^{n-1}\omega_q(A^n) \leq \omega_q^n(A)$. ■

As a direct consequence of Theorem 6.2.7 and (6.6), we have the following result.

Corollary 6.2.9. *Let $A \in \mathfrak{B}(\mathcal{H})$ and $q \in \overline{\mathbb{D}} \setminus \{0\}$. If $\omega_q(A) \leq |q|$, then*

$$\|A^n\| \leq 2 \text{ for all } n \in \mathbb{N}.$$

We finish the section by pointing out that, although ω_q is a norm on $\mathfrak{B}(\mathcal{H})$ for any $q \in \overline{\mathbb{D}} \setminus \{0\}$, the same cannot be said about ω_0. This observation is not difficult to prove and the corresponding result is given within the following short lemma.

Lemma 6.2.10. *The function ω_0 is not a norm on $\mathfrak{B}(\mathcal{H})$ for any Hilbert space $\mathcal{H}$. The set $P_0 = \{A \in \mathfrak{B}(\mathcal{H}) : \omega_0(A) = 0\}$ is a closed subspace in $(\mathfrak{B}(\mathcal{H}), \|\cdot\|)$.*

Proof. In order to show the first statement disclosed in the lemma, it suffices to consider the identity operator $I\colon \mathcal{H} \to \mathcal{H}$. It is straightforward to obtain $\omega_0(I) = 0$ directly from (6.2), which clearly indicates that ω_0 is not a norm.

We shall now prove the rest of the lemma. Let $A, B \in P_0$ and $\alpha, \beta \in \mathbb{C}$. By implementing *(i)* and *(ii)* from Lemma 6.1.7, we get

$$\omega_0(\alpha A + \beta B) \leq |\alpha| \cdot \omega_0(A) + |\beta| \cdot \omega_0(B) = 0,$$

which leads us to $\alpha A + \beta B \in P_0$. If $(A_n)_{n \in \mathcal{N}}$ is a sequence in P_0 and $A \in \mathfrak{B}(\mathcal{H})$ is such that $\lim_{n\to\infty} \|A_n - A\| = 0$, then from *(ii)* of Lemma 6.1.7 and (6.6), we have

$$\omega_0(A) \leq \omega_0(A - A_n) + \omega_0(A_n) \leq \|A - A_n\| + 0 = \|A - A_n\|$$

for all the $n \in \mathcal{N}$. This implies $\omega_0(A) = 0$. ■

Example 6.2.11. *We may also point out that* (6.14) *is not necessarily true when* $q = 0$*. As in the proof of Lemma* 6.2.10*, the identity operator* $I \in \mathfrak{B}(\mathcal{H})$ *satisfies* $\omega_0(I^n) = \omega_0(I) = 0$ *for all the* $n \in \mathcal{N}$*. This means means that* $\lim_{n\to\infty} \sqrt[n]{\omega_0(I^n)} = 0$*. However, on the other hand,* $r(I) = 1$*.* ■

6.3 Analytical properties

In this section, we will regard $\omega_q(\cdot)$ as a function in $q \in \overline{\mathbb{D}}$, and our primary intention will be to inspect some of its analytical properties. To begin, we notice that the aforementioned function is necessarily continuous whenever $A \in \mathfrak{B}(\mathcal{H})$ for some finite-dimensional Hilbert space $\mathcal{H}$, as shown by [122, Theorem 2.9]. We shall promptly prove that $\omega_q(\cdot)$ is a continuous function in $q \in \overline{\mathbb{D}}$ for any Hilbert space $\mathcal{H}$.

Theorem 6.3.1. *Let* $A \in \mathcal{B}(\mathcal{H})$ *for some Hilbert space* $\mathcal{H}$*. The function* $f_A \colon \overline{\mathbb{D}} \to \mathbb{R}$ *given by*

$$f_A(q) = \omega_q(A) \quad \textit{for all the } q \in \overline{\mathbb{D}},$$

is continuous on $\overline{\mathbb{D}}$*. Also, we have*

$$\max_{q\in\overline{\mathbb{D}}} f_A(q) = \|A\|.$$

Proof. We shall use $g_A(q, y, t)$ to denote

$$g_A(q, y, t) = |q| \cdot |\langle Ay, y\rangle| + \sqrt{1 - |q|^2} \cdot |\langle At, y\rangle|, \quad q \in \mathbb{C},\ y, t \in \mathcal{H}.$$

By applying (6.12), it becomes clear that

$$f_A(q) = \sup_{\substack{\|y\|=\|t\|=1 \\ \langle y,t\rangle=0}} g_A(q, y, t) \quad \text{for all } q \in \overline{\mathbb{D}}.$$

Let $q_0 \in \overline{\mathbb{D}}$ be a fixed complex number. For any $q \in \overline{\mathbb{D}}$ and all unit vectors $y, t \in \mathcal{H}$ satisfying $\langle y, t\rangle = 0$, it is not difficult to establish

$$\begin{aligned}
&|g_A(q, y, t) - g_A(q_0, y, t)| = \\
&= \left| (|q| - |q_0|) \cdot |\langle Ay, y\rangle| + \left(\sqrt{1 - |q|^2} - \sqrt{1 - |q_0|^2} \right) \cdot |\langle At, y\rangle| \right| \\
&\le |q - q_0| \cdot \|A\| \|y\|^2 + \left| \sqrt{1 - |q|^2} - \sqrt{1 - |q_0|^2} \right| \cdot \|A\| \|t\| \|y\| \\
&= \left(|q - q_0| + \left| \sqrt{1 - |q|^2} - \sqrt{1 - |q_0|^2} \right| \right) \cdot \|A\|.
\end{aligned}$$

From here, we quickly conclude that

$$\begin{aligned}|f_A(q) - f_A(q_0)| &= \Big| \sup_{\substack{\|y\|=\|t\|=1 \\ \langle y,t\rangle=0}} g_A(q,y,t) - \sup_{\substack{\|y\|=\|t\|=1 \\ \langle y,t\rangle=0}} g_A(q_0,y,t)\Big| \\ &\leq \sup_{\substack{\|y\|=\|t\|=1 \\ \langle y,t\rangle=0}} |g_A(q,y,t) - g_A(q_0,y,t)| \\ &\leq \left(|q-q_0| + \left|\sqrt{1-|q|^2} - \sqrt{1-|q_0|^2}\right|\right) \cdot \|A\|.\end{aligned}$$

Given the fact that

$$\lim_{\substack{q \to q_0 \\ q \in \overline{\mathbb{D}}}} \left(|q-q_0| + \left|\sqrt{1-|q|^2} - \sqrt{1-|q_0|^2}\right|\right) = 0,$$

it is clear that f_A must be continuous on the set $\overline{\mathbb{D}}$. Since the set $\overline{\mathbb{D}}$ is compact, it is obvious that $\max_{q \in \overline{\mathbb{D}}} f_A(q)$ certainly exists. Furthermore, we have that

$$\begin{aligned}\|A\| &= \sup_{\|x\|=\|y\|=1} |\langle Ax, y\rangle| = \sup_{q \in \overline{\mathbb{D}}} \sup_{\substack{\|x\|=\|y\|=1 \\ \langle x,y\rangle=q}} |\langle Ax, y\rangle| \\ &= \sup_{q \in \overline{\mathbb{D}}} f_A(q) = \max_{q \in \overline{\mathbb{D}}} f_A(q) = \max_{q \in \overline{\mathbb{D}}} \omega_q(A),\end{aligned}$$

which completes the proof. ∎

Theorem 6.3.1 tells us that the function f_A must be continuous on $\overline{\mathbb{D}}$. We shall finish the section by inspecting the complex differentiability of f_A on $\mathbb{D}$.

Theorem 6.3.2. *Let $A \in \mathcal{B}(\mathcal{H})$. We have:*

(i) If $a \in \mathbb{D}$, then the derivative $f_A'(a)$ either does not exist or $f'(a) = 0$.

(ii) If f_A is differentiable on $\mathbb{D}$, then $f_A(q) = \|A\|$ for all the $q \in \overline{\mathbb{D}}$.

Proof. (i) Let $a \in \mathbb{D} \setminus \{0\}$ be a given point and assume that $f_A'(a)$ exists. By implementing (iv) from Lemma 6.1.7, it is not difficult to conclude that $f_A'(a) = 0$. On the other hand, if $a = 0$ and if the complex derivative $f_A'(0)$ exists, then

$$\lim_{\substack{t \to 0 \\ t \in [-1,1]\setminus\{0\}}} \frac{f_A(t) - f_A(0)}{t - 0}$$

must also exist. The function

$$[-1,1] \setminus \{0\} \ni t \mapsto \frac{f_A(t) - f_A(0)}{t - 0}$$

is clearly odd, which implies $f_A'(0) = 0$.

(ii) From (6.6) we have that $\omega_q(A) \leq \|A\|$. Thus, it is sufficient to show the converse. Let $\varepsilon > 0$ be an arbitrary constant. It is clear that there must exist unit vectors $x, y \in \mathcal{H}$ such that $|\langle Ax, y\rangle| > \|A\| - \varepsilon$. If we denote $\langle x, y\rangle = s$, it is then evident that $s \in \overline{\mathbb{D}}$ and $w_s(A) \geq \|A\| - \varepsilon$. Since the function f_A is differentiable on $\mathbb{D}$, it is surely constant on $\mathbb{D}$. Due to the fact that f_A is continuous, it is easy to see that it must also be constant on $\overline{\mathbb{D}}$. Now, we obtain $\omega_q(A) = w_s(A) \geq \|A\| - \varepsilon$ for any $q \in \overline{\mathbb{D}}$. Given the fact that $\varepsilon > 0$ was arbitrarily chosen, it promptly follows that $\omega_q(A) \geq \|A\|$. ■

Example 6.3.3. *If $A = 0$, then, obviously, $f_A'(q)$ exists for all the $q \in \mathbb{D}$. However, the converse is not true. In other words, if there exists a derivative $f_A'(q)$ for all the $q \in \mathbb{D}$, then it does not follow that $A = 0$. For example, we can choose $\mathcal{H} = \mathbb{C}^2$ and $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. First of all, we notice that $\|A\| = \max\{|1|, |-1|\} = 1$. We shall now prove that $f_A(q) = 1 = \|A\|$ holds for all the $q \in \overline{\mathbb{D}}$. By virtue of (iv) from Lemma 6.1.7, it is sufficient to take into consideration the values $q \in [0, 1]$. Let $x = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} \in \mathbb{C}^2$ and $y = \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} \in \mathbb{C}^2$ for $(\alpha, \beta) = \left(\frac{1}{2}\arccos q, -\frac{1}{2}\arccos q\right)$. In that case, we clearly have $\|x\| = \|y\| = 1$. Also, $\langle x, y\rangle = \cos(\alpha - \beta) = \cos 2\alpha = q$, while $\langle Ax, y\rangle = \cos(\alpha + \beta) = 1$. Hence, $\omega_q(A) \geq 1$, which means that $\omega_q(A) = 1$.* ■

6.4 q-NUMERICAL RADIUS INEQUALITIES

We now aim to investigate some properties of the q-numerical radius of operator matrices, as well as the rank one operators. By virtue of property *(iv)* from Lemma 6.1.7, it is enough to restrict ourselves to the case $q \in [0, 1]$.

6.4.1 INEQUALITIES INVOLVING OPERATOR MATRICES

We will start by giving a full proof of Theorem 6.1.5.

Proof of Theorem 6.1.5. In order to prove the left-hand side inequality in (6.9), it is enough to note that the operator $\bigoplus_{n=1}^{+\infty} A_n$ is a dilation of A_k for each $k \in \mathbb{N}$, which means that $\omega_q(A_k) \leq \omega_q\left(\bigoplus_{n=1}^{+\infty} A_n\right)$ must hold for any $k \in \mathbb{N}$. Thus,

$$\sup_{n \in \mathbb{N}} \omega_q(A_n) \leq \omega_q\left(\bigoplus_{n=1}^{+\infty} A_n\right).$$

We shall now demonstrate the right-hand side inequality in (6.9). Take $s \in \mathcal{W}_q\left(\bigoplus_{n=1}^{+\infty} A_n\right)$ to be an arbitrary value from the q-numerical range of $\bigoplus_{n=1}^{+\infty} A_n$. By applying Lemma 6.2.1, we see that for each $n \in \mathbb{N}$, there exist vectors $t_n, y_n \in \mathcal{H}_n$ such that

$$\sum_{n=1}^{+\infty} \|t_n\|^2 = \sum_{n=1}^{+\infty} \|y_n\|^2 = 1, \qquad \sum_{n=1}^{+\infty} \langle t_n, y_n \rangle = \left\langle \bigoplus_{n=1}^{+\infty} t_n, \bigoplus_{n=1}^{+\infty} y_n \right\rangle = 0,$$

and

$$\begin{aligned} s &= q\langle \left(\bigoplus_{n=1}^{+\infty} A_n \right) \bigoplus_{n=1}^{+\infty} y_n, \bigoplus_{n=1}^{+\infty} y_n \rangle + \sqrt{1-q^2}\langle \left(\bigoplus_{n=1}^{+\infty} A_n \right) \bigoplus_{n=1}^{+\infty} t_n, \bigoplus_{n=1}^{+\infty} y_n \rangle \\ &= q \sum_{n=1}^{+\infty} \langle A_n y_n, y_n \rangle + \sqrt{1-q^2} \sum_{n=1}^{+\infty} \langle A_n t_n, y_n \rangle. \end{aligned} \tag{6.16}$$

For the first sum stated in (6.16), we promptly obtain

$$\begin{aligned} \left| \sum_{n=1}^{+\infty} \langle A_n y_n, y_n \rangle \right| &\leq \sum_{n=1}^{+\infty} |\langle A_n y_n, y_n \rangle| = \sum_{\substack{n=1 \\ \|y_n\|>0}}^{+\infty} |\langle A_n y_n, y_n \rangle| \\ &= \sum_{\substack{n=1 \\ \|y_n\|>0}}^{+\infty} \|y_n\|^2 \left|\langle A_n \left(\frac{y_n}{\|y_n\|} \right), \frac{y_n}{\|y_n\|} \rangle\right| \\ &\leq \sum_{\substack{n=1 \\ \|y_n\|>0}}^{+\infty} \|y_n\|^2 \omega(A_n) \leq \sup_{n\in\mathbb{N}} \omega(A_n) \sum_{\substack{n=1 \\ \|y_n\|>0}}^{+\infty} \|y_n\|^2 = \sup_{n\in\mathbb{N}} \omega(A_n). \end{aligned}$$

As for the second sum given in (6.16), it is possible to use the polarization identity in order to quickly obtain

$$|\langle A_n t_n, y_n \rangle| \leq \omega(A_n) \left(\|t_n\|^2 + \|y_n\|^2 \right) \quad \text{for all } n \in \mathbb{N}.$$

Hence, it is not difficult to get

$$\begin{aligned} \left| \sum_{n=1}^{+\infty} \langle A_n t_n, y_n \rangle \right| &\leq \sum_{n=1}^{+\infty} |\langle A_n t_n, y_n \rangle| \\ &\leq \sum_{n=1}^{+\infty} \omega(A_n) \left(\|t_n\|^2 + \|y_n\|^2 \right) \\ &\leq \sup_{n\in\mathbb{N}} \omega(A_n) \left(\sum_{n=1}^{+\infty} \|t_n\|^2 + \sum_{n=1}^{+\infty} \|y_n\|^2 \right) = 2 \sup_{n\in\mathbb{N}} \omega(A_n). \end{aligned}$$

Therefore, we have

$$\begin{aligned} |s| &\leq q \left| \sum_{n=1}^{+\infty} \langle A_n y_n, y_n \rangle \right| + \sqrt{1-q^2} \left| \sum_{n=1}^{+\infty} \langle A_n t_n, y_n \rangle \right| \\ &\leq q \sup_{n\in\mathbb{N}} \omega(A_n) + 2\sqrt{1-q^2} \sup_{n\in\mathbb{N}} \omega(A_n) = \left(q + 2\sqrt{1-q^2} \right) \sup_{n\in\mathbb{N}} \omega(A_n). \end{aligned}$$

Since the value $s \in \mathcal{W}_q\left(\bigoplus_{n=1}^{+\infty} A_n\right)$ was chosen arbitrarily, it promptly follows that

$$\omega_q\left(\bigoplus_{n=1}^{+\infty} A_n\right) \leq \left(q+2\sqrt{1-q^2}\right) \sup_{n\in\mathbb{N}} \omega(A_n). \tag{6.17}$$

By combining (6.17) with (6.5), we reach

$$\omega_q\left(\bigoplus_{n=1}^{+\infty} A_n\right) \leq \frac{q+2\sqrt{1-q^2}}{q} \sup_{n\in\mathbb{N}} \omega_q(A_n),$$

thus completing the proof. ■

By applying Theorem 6.1.5 on the simple scenario when we have just two operators $A, B \in \mathfrak{B}(\mathcal{H})$, it is straightforward to obtain the following corollary.

Corollary 6.4.1. *Let $A, B \in \mathfrak{B}(\mathcal{H})$ and $q \in (0,1]$. Then*

$$\max\{\omega_q(A), \omega_q(B)\} \leq \omega_q\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \leq \frac{q+2\sqrt{1-q^2}}{q} \max\{\omega_q(A), \omega_q(B)\}. \tag{6.18}$$

By plugging in $q = 1$ in Theorem 6.1.5 and Corollary 6.4.1, we obtain (6.8) and the equality from part (i) of Lemma 6.1.6, respectively. Furthermore, the statements (ii) and (iii) from Lemma 6.1.6 hold for the q-radius as well. We demonstrate this fact in the following quick lemma.

Lemma 6.4.2. *Let $A, B \in \mathfrak{B}(\mathcal{H})$, $q \in \overline{\mathbb{D}}$ and $\theta \in \mathbb{R}$. Then*

(i) $\omega_q\left(\begin{bmatrix} 0 & A \\ e^{i\theta}B & 0 \end{bmatrix}\right) = \omega_q\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)$;

(ii) $\omega_q\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \omega_q\left(\begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix}\right)$;

(iii) $\omega_q\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = \omega_q\left(\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}\right)$.

Proof. The proof is basically the same as the one for the ordinary numerical radius by using the property (iii) from Lemma 6.1.7. Namely, the result of the given lemma stated in (i) follows by applying the equality $\omega_q(U^*TU) = \omega_q(T)$, where $T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$ represents the starting operator, while $U = \begin{bmatrix} I & 0 \\ 0 & e^{i\theta/2}I \end{bmatrix}$ is the unitary transformation operator. The remaining two statements of the lemma can be obtained by applying the same equality on the unitary operator $U = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. ■

We now provide another quick lemma that represents a generalization of the statement (iv) from Lemma 6.1.6.

Lemma 6.4.3. *Let $A, B \in \mathfrak{B}(\mathcal{H})$ and $q \in (0,1]$. Then*

$$\begin{aligned}\max\{\omega_q(A+B), \omega_q(A-B)\} &\leq \omega_q\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) \\ &\leq \frac{q+2\sqrt{1-q^2}}{q} \max\{\omega_q(A+B), \omega_q(A-B)\}.\end{aligned} \tag{6.19}$$

In particular, we have the following inequalities

$$\omega_q(B) \leq \omega_q\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}\right) \leq \frac{q+2\sqrt{1-q^2}}{q}\omega_q(B).$$

Proof. Let $U = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} I & I \\ -I & I \end{smallmatrix}\right]$. It is then clear that U is a unitary operator on $\mathcal{H} \oplus \mathcal{H}$ and it is easy to see that

$$U \begin{bmatrix} A & B \\ B & A \end{bmatrix} U^* = \begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix}.$$

Thus, we obtain

$$\omega_q\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) = \omega_q\left(\begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix}\right).$$

The inequality (6.19) follows immediately by applying Corollary 6.4.1. ∎

It is worth pointing out that Lemma 6.4.3 can be applied in order to give a generalization of a well-known result (see, for example, [105, Theorem 2.4])

$$\frac{\max\{\omega(A+B), \omega(A-B)\}}{2} \leq \omega\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \frac{\omega(A+B)+\omega(A-B)}{2}$$

regarding the numerical radius of operator matrices. We disclose the corresponding inequality in the form of the following theorem.

Theorem 6.4.4. *Let $A, B \in \mathfrak{B}(\mathcal{H})$ and $q \in (0,1]$. Then*

$$\begin{aligned}\frac{\max\{\omega_q(A+B), \omega_q(A-B)\}}{2} &\leq \omega_q\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \\ &\leq \frac{q+2\sqrt{1-q^2}}{2q}\left(\omega_q(A+B)+\omega_q(A-B)\right).\end{aligned} \tag{6.20}$$

Proof. By applying Lemma 6.4.3 and the statement (iii) from Lemma 6.4.2, we get

$$\begin{aligned}\omega_q(A+B) &\leq \omega_q\left(\begin{bmatrix} 0 & A+B \\ A+B & 0 \end{bmatrix}\right) \\ &= \omega_q\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} + \begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}\right) \\ &\leq \omega_q\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) + \omega_q\left(\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}\right) = 2\omega_q\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right).\end{aligned}$$

Thus, we have

$$\frac{\omega_q(A+B)}{2} \leq \omega_q\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right). \tag{6.21}$$

Now, by using the statement (i) from Lemma 6.4.2 and by applying (6.21) where we take $-B$ instead of B, we further obtain

$$\frac{\omega_q(A-B)}{2} \leq \omega_q\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right).$$

The left-hand side inequality from (6.20) follows from here.

In order to prove the right-hand side inequality given in (6.20), we observe that for the unitary operator $U = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} I & -I \\ I & I \end{smallmatrix}\right]$, we get

$$U^* \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} U = \frac{1}{2}\begin{bmatrix} A+B & A-B \\ -(A-B) & -(A+B) \end{bmatrix}. \tag{6.22}$$

By implementing Corollary 6.4.1, (6.22), the statement (i) from Lemma 6.4.2 and Lemma 6.4.3, we conclude that

$$\begin{aligned}
\omega_q\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &= \omega_q\left(U^* \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} U\right) \\
&= \frac{1}{2}\omega_q\left(\begin{bmatrix} A+B & A-B \\ -(A-B) & -(A+B) \end{bmatrix}\right) \\
&\leq \frac{1}{2}\omega_q\left(\begin{bmatrix} A+B & 0 \\ 0 & -(A+B) \end{bmatrix}\right) + \frac{1}{2}\omega_q\left(\begin{bmatrix} 0 & A-B \\ -(A-B) & 0 \end{bmatrix}\right) \\
&\leq \frac{1}{2}\frac{q+2\sqrt{1-q^2}}{q}\omega_q(A+B) + \frac{1}{2}\frac{q+2\sqrt{1-q^2}}{q}\omega_q(A-B) \\
&= \frac{q+2\sqrt{1-q^2}}{2q}\left(\omega_q(A+B) + \omega_q(A-B)\right),
\end{aligned}$$

thus completing the proof. ■

By implementing Theorem 6.4.4, it now becomes possible to easily derive an inequality regarding the operator matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, where the operators $A, B, C, D \in \mathfrak{B}(\mathcal{H})$ are entirely arbitrary. We disclose the said result in the next theorem.

Theorem 6.4.5. *Let* $A, B, C, D \in \mathfrak{B}(\mathcal{H})$ *and* $q \in (0, 1]$. *Then*

$$\omega_q\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq \frac{q+2\sqrt{1-q^2}}{q}\left(\max\{\omega_q(A), \omega_q(D)\} + \frac{\omega_q(B+C)+\omega_q(B-C)}{2}\right).$$

Proof. The desired inequality follows directly from the statement (ii) given in Lemma 6.1.7, Corollary 6.4.1 and Theorem 6.4.4. ■

By using the statement (i) from Lemma 6.1.6, it is evident that $\omega(A) = w\left(\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]\right)$ for any operator $A \in \mathfrak{B}(\mathcal{H})$. The given equality brings up the problem of examining the relationship between $\omega_q(A)$ and $\omega_q\left(\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]\right)$ for a given operator $A \in \mathfrak{B}(\mathcal{H})$. We shall soon find out that these numbers do not have to be equal even in the simplest cases, such as $A = I$. In the remainder of the section, we shall deal with such questions for certain special cases of operator matrices.

Theorem 6.4.6. *Let $q \in [0, 1]$. Then*

$$\omega_q\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right) = \frac{1+q}{2}.$$

Proof. For any vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathcal{H} \oplus \mathcal{H}$, we have

$$\left\langle \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle = \langle x_1, y_1 \rangle.$$

Thus, we need to compute

$$\sup\left\{ |\langle x_1, y_1\rangle| : \left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\| = 1,\ \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle = q \right\}.$$

Consider some two vectors $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a' \\ b' \end{bmatrix} \in \mathcal{H} \oplus \mathcal{H}$ such that $a = a'$, $b = -b'$, as well as

$$\|a\| = \|a'\| = \sqrt{\frac{1+q}{2}} \quad \text{and} \quad \|b\| = \|b'\| = \sqrt{\frac{1-q}{2}}.$$

It is not difficult to see that both of these vectors are unit vectors. Also, it is straightforward to check that $\langle a, a'\rangle + \langle b, b'\rangle = q$ and $\langle a, a'\rangle = \frac{1+q}{2}$. For this reason, we conclude that $\omega_q\left(\left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]\right) \geq \frac{1+q}{2}$.

In order to finalize the proof, it is sufficient to demonstrate that there do not exist two vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathcal{H} \oplus \mathcal{H}$ such that

$$\left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\| = 1,\ \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle = q \quad \text{and} \quad |\langle x_1, y_1\rangle| > \frac{1+q}{2}.$$

If such vectors were to exist, we would have that $|\langle x_1, y_1\rangle - q| > \frac{1-q}{2}$ since the triangle inequality would yield

$$|\langle x_1, y_1\rangle - q| \geq |\langle x_1, y_1\rangle| - q > \frac{1+q}{2} - q = \frac{1-q}{2}.$$

Moreover, by using the Cauchy-Schwarz inequality, we obtain

$$\|x_1\|\|y_1\| > \frac{1+q}{2} \quad \text{and} \quad \|x_2\|\|y_2\| > \frac{1-q}{2}.$$

From here, it follows that $\|x_1\|\|y_1\| + \|x_2\|\|y_2\| > 1$. However, this is not possible given the fact that

$$\|x_1\|\|y_1\| + \|x_2\|\|y_2\| \leq \frac{\|x_1\|^2 + \|y_1\|^2}{2} + \frac{\|x_2\|^2 + \|y_2\|^2}{2} = 1.$$

We have obtained a contradiction, which completes the proof. ■

It is trivial to derive the following two direct corollaries of Theorem 6.4.6.

Corollary 6.4.7. *Let $q \in [0,1]$. Then*

$$\omega_q(I) = \omega_q\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right) \quad \Longleftrightarrow \quad q = 1.$$

Corollary 6.4.8. *Let $q \in [0,1]$. Then*

$$\omega_q\left(\begin{bmatrix} I & I \\ I & I \end{bmatrix}\right) = \omega_q\left(\begin{bmatrix} I & -I \\ -I & I \end{bmatrix}\right) = 1 + q.$$

Proof. Let $U \in \mathfrak{B}(\mathcal{H})$ be the unitary operator from the proof of Lemma 6.4.3. By applying Theorem 6.4.6 and Lemma 6.4.2, we quickly obtain

$$\omega_q\left(\begin{bmatrix} I & I \\ I & I \end{bmatrix}\right) = \omega_q\left(\begin{bmatrix} I+I & 0 \\ 0 & I-I \end{bmatrix}\right) = 2\omega_q\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right) = 1 + q,$$

as well as

$$\left(\begin{bmatrix} I & -I \\ -I & I \end{bmatrix}\right) = \omega_q\left(\begin{bmatrix} I-I & 0 \\ 0 & I-(-I) \end{bmatrix}\right) = 2\omega_q\left(\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\right) = 1 + q,$$

thus completing the proof. ■

We finish the section by providing an explicit formula for computing the q-numerical radius of a special type of operator matrix.

Theorem 6.4.9. *Let $q \in [0,1]$. Then*

$$\omega_q\left(\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}\right) = \frac{1 + \sqrt{1-q^2}}{2}.$$

Proof. It is easy to establish that the desired value represents the supremum of $|\langle x_2, y_1\rangle|$ under the conditions

$$\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = q \quad \text{and} \quad \|x_1\|^2 + \|x_2\|^2 = 1 = \|y_1\|^2 + \|y_2\|^2,$$

where $x_1, x_2, y_1, y_2 \in \mathcal{H}$. Let $\|x_1\| = \cos\alpha$, $\|x_2\| = \sin\alpha$, $\|y_1\| = \cos\beta$ and $\|y_2\| = \sin\beta$ for some $\alpha, \beta \in [0, \frac{\pi}{2}]$. By using the Cauchy–Schwarz inequality, we have

$$|\langle x_2, y_1\rangle| \leq \|x_2\|\|y_1\| = \sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha-\beta) + \frac{1}{2}\sin(\alpha+\beta).$$

By additionally using the triangle inequality, we also obtain

$$q = |q| = |\langle x_1, y_1\rangle + \langle x_2, y_2\rangle| \leq \|x_1\|\|y_1\| + \|x_2\|\|y_2\| = \cos(\alpha-\beta),$$

which promptly leads to $\sin(\alpha-\beta) \leq \sqrt{1-q^2}$. From here, we immediately reach

$$|\langle x_2, y_1\rangle| \leq \frac{\sqrt{1-q^2}}{2} + \frac{1}{2} = \frac{1+\sqrt{1-q^2}}{2}. \tag{6.23}$$

which means that $\omega_q\left(\left[\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right]\right) \leq \frac{1+\sqrt{1-q^2}}{2}$.

In order to show that $\omega_q\left(\left[\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right]\right) = \frac{1+\sqrt{1-q^2}}{2}$, it is sufficient to find concrete feasible vectors $x_1, x_2, y_1, y_2 \in \mathcal{H}$ for which $|\langle x_2, y_1\rangle| = q$. This can be achieved by simply showing that the equality in (6.23) can be achieved for certain vectors. It is not difficult to see that we may choose the value $\alpha, \beta \in [0, \frac{\pi}{2}]$ so that $\alpha - \beta = \arccos q$ and $\alpha + \beta = \frac{\pi}{2}$. By choosing the vectors $x_1, x_2, y_1, y_2 \in \mathcal{H}$ so that

$$x_1 = \cos\alpha\, e,\ x_2 = \sin\alpha\, e,\ y_1 = \cos\beta\, e,\ y_2 = \sin\beta\, e,$$

for some arbitrary unit vector $e \in \mathcal{H}$, all the applied inequalities will quickly become equalities, hence $|\langle x_2, y_1\rangle| = \frac{1+\sqrt{1-q^2}}{2}$. ■

6.4.2 q-NUMERICAL RADIUS OF RANK 1 OPERATORS

If $a, b \in \mathcal{H}$, we can represent the one-dimensional operator $A \in \mathfrak{B}(\mathcal{H})$ as follows:

$$Ax = \langle x, a\rangle b, \quad x \in \mathcal{H}.$$

We shall symbolize this operator as $a \otimes b$. For an operator defined in this manner, we observe the following properties:

- The adjoint of $a \otimes b$, denoted as $(a \otimes b)^*$, is equal to $b \otimes a$;
- The operator norm of $a \otimes b$ is equal to the product of the norms of a and b, i.e. $\|a \otimes b\| = \|a\|\|b\|$;

- The q-numerical radius of $a \otimes b$ is the same as that of $b \otimes a$ for all q in the interval [0,1].

In [29], the author obtained an extension of Cauchy-Schwarz inequality in the following way: if a, b, x are vectors in a Hilbert space $\mathcal{H}$, then

$$|\langle a, x\rangle\langle x, b\rangle| \leq \frac{\|a\|\|b\| + |\langle a, b\rangle|}{2}\|x\|^2. \tag{6.24}$$

This is known as *Buzano inequality*. For a simple proof, see [87]. As a direct consequence, the authors in [87] also obtained the following result.

Theorem 6.4.10. *If $a, b \in \mathcal{H}$, then*

$$\omega(a \otimes b) \leq \frac{\|a\|\|b\| + |\langle a, b\rangle|}{2}. \tag{6.25}$$

In this section, we will derive the corresponding formula for $\omega_q(a \otimes b)$.
The main result of this section is the following theorem.

Theorem 6.4.11. *Let $a, b \in \mathcal{H}$ and $q \in [0, 1]$. Then, we have*

$$\omega_q(a \otimes b) = \frac{\|a\|\|b\| + q|\langle a, b\rangle|}{2} + \frac{\sqrt{1-q^2}}{2} \cdot \sqrt{\|a\|^2\|b\|^2 - |\langle a, b\rangle|^2}. \tag{6.26}$$

Proof. If $a = 0$ or $b = 0$, the theorem trivially holds. Thus, let $a, b \in \mathcal{H} \setminus \{0\}$, and assume that $||a|| = ||b|| = 1$. From the equality $\langle (a \otimes b)x, y\rangle = \langle x, a\rangle\langle b, y\rangle$ for any $x, y \in \mathcal{H}$, and from (6.12), we have

$$\begin{aligned}
\omega_q(a \otimes b) &= \sup\left\{q|\,\langle (a \otimes b)y, y\rangle\,| + \sqrt{1-q^2}|\,\langle (a \otimes b)t, y\rangle\,| \,:\, \langle t, y\rangle = 0,\ ||t|| = ||y|| = 1\right\} \\
&= \sup\left\{q|\,\langle a, y\rangle\,\langle y, b\rangle\,| + \sqrt{1-q^2}|\,\langle t, a\rangle\,\langle y, b\rangle\,| \,:\, \langle t, y\rangle = 0,\ ||t|| = ||y|| = 1\right\}. \qquad (6.27)
\end{aligned}$$

Since $\mathcal{H} = \text{span}\,\{a\} \oplus \{a\}^{\perp}$, each of the vectors $t, y, b \in \mathcal{H}$ can be represented as

$$t = \langle t, a\rangle a + t_1, \tag{6.28}$$
$$y = \langle y, a\rangle a + y_1, \tag{6.29}$$
$$b = \langle b, a\rangle a + b_1, \tag{6.30}$$

where $t_1, y_1, b_1 \in \{a\}^{\perp}$ are uniquely determined. Now, from (6.28), (6.29), and (6.30) and $||a|| = 1$, it follows that

$$||t_1|| = \sqrt{1 - |\langle a, t\rangle|^2}, \tag{6.31}$$
$$||y_1|| = \sqrt{1 - |\langle a, y\rangle|^2}, \tag{6.32}$$
$$||b_1|| = \sqrt{1 - |\langle a, b\rangle|^2}. \tag{6.33}$$

By replacing (6.28) and (6.29) in $0 = \langle t, y\rangle$ and from $\langle a, y_1\rangle = 0$ and $\langle a, t_1\rangle = 0$, we obtain the equality

$$0 = \langle a, t\rangle\langle a, y\rangle + \langle t_1, y_1\rangle.$$

From here and from the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |\langle a, t\rangle\langle y, a\rangle| &= |\langle t_1, y_1\rangle| \\ &\leq \|t_1\|\|y_1\| \\ &= \sqrt{1 - |\langle a, t\rangle|^2}\sqrt{1 - |\langle a, y\rangle|^2} \\ &= \sqrt{1 - |\langle a, t\rangle|^2 - |\langle a, y\rangle|^2 + |\langle a, t\rangle|^2|\langle a, y\rangle|^2}. \end{aligned}$$

By squaring the last inequality, we get

$$|\langle a, t\rangle| \leq \sqrt{1 - |\langle a, y\rangle|^2}.$$

From here and from (6.27), we have that

$$\omega_q(a \otimes b) \leq \sup\left\{q \cdot |\langle a, y\rangle\langle y, b\rangle| + \sqrt{1 - q^2}\sqrt{1 - |\langle a, y\rangle|^2} \cdot |\langle y, b\rangle| \, : \, \|y\| = 1\right\}.$$

Similarly, we get the equality

$$\langle y, b\rangle = \langle y, a\rangle \cdot \langle a, b\rangle + \langle y_1, b_1\rangle,$$

and from the Cauchy-Schwarz inequality and (6.32) and (6.33), it follows that

$$\begin{aligned} |\langle y, b\rangle| &= |\langle y, a\rangle \cdot \langle a, b\rangle + \langle y_1, b_1\rangle| \\ &\leq |\langle y, a\rangle \cdot \langle a, b\rangle| + |\langle y_1, b_1\rangle| \\ &\leq |\langle y, a\rangle \cdot \langle a, b\rangle| + ||y_1|| \cdot ||b_1|| \\ &= |\langle y, a\rangle \cdot \langle a, b\rangle| + \sqrt{(1 - |\langle a, y\rangle|^2)(1 - |\langle a, b\rangle|^2)}. \end{aligned} \tag{6.34}$$

Therefore,

$$\omega_q(a \otimes b) \leq \sup_{\|y\|=1} I(y), \tag{6.35}$$

where we introduced the notation for $y \in \mathcal{H}$:

$$I(y) = \left(q|\langle a, y\rangle| + \sqrt{1 - q^2}\sqrt{1 - |\langle a, y\rangle|^2}\right)\left(|\langle y, a\rangle\langle a, b\rangle| + \sqrt{(1 - |\langle a, y\rangle|^2)(1 - |\langle a, b\rangle|^2)}\right).$$

Let $y \in \mathcal{H}$ be an arbitrary vector such that $||y|| = 1$, and let $\alpha, \beta, \gamma \in [0, \frac{\pi}{2}]$ such that

$$q = \cos\alpha, \ \ |\langle y, a\rangle| = \cos\beta \quad \text{and} \quad |\langle a, b\rangle| = \cos\gamma.$$

Using the introduced notation, it follows that

$$\begin{aligned} I(y) &= (\cos\alpha\cos\beta + \sin\alpha\sin\beta)(\cos\beta\cos\gamma + \sin\beta\sin\gamma) \\ &= \cos(\alpha-\beta)\cos(\beta-\gamma) \\ &= \frac{1}{2}\cos(\alpha-\gamma) + \frac{1}{2}\cos(\alpha+\gamma-2\beta) \\ &\le \frac{1}{2} + \frac{1}{2}\cos(\alpha-\gamma) \\ &= \frac{1}{2} + \frac{1}{2}(\cos\alpha\cos\gamma + \sin\alpha\sin\gamma) \\ &= \frac{1}{2} + \frac{1}{2}\left(q|\langle a,b\rangle| + \sqrt{(1-q^2)\left(1-|\langle a,b\rangle|^2\right)}\right) \end{aligned} , \tag{6.36}$$

for each unit vector $y \in \mathcal{H}$. From (6.35) and (6.36), we have the estimate

$$\omega_q(a\otimes b) \le \frac{1}{2} + \frac{1}{2}\left(q\cdot|\langle a,b\rangle| + \sqrt{(1-q^2)\left(1-|\langle a,b\rangle|^2\right)}\right). \tag{6.37}$$

We will prove that equality actually holds in (6.35). Let us take

$$y = \cos\frac{\alpha+\gamma}{2}a + \sin\frac{\alpha+\gamma}{2}e^{i\arg\langle a,b\rangle}\frac{b_1}{\|b_1\|},$$

where we take $\arg\langle a,b\rangle$ to be zero whenever a and b are orthogonal. Then,

$$y_1 = \sin\frac{\alpha+\gamma}{2}\cdot e^{i\arg\langle a,b\rangle}\frac{b_1}{\|b_1\|}$$

and the following are true:

$$\arg\langle y_1, b_1\rangle = \arg\langle a,b\rangle, \tag{6.38}$$

$$y_1 \in \operatorname{span}\{b_1\}, \tag{6.39}$$

$$\beta = \frac{\alpha+\gamma}{2}. \tag{6.40}$$

To achieve the equality in (6.35), it is sufficient that equalities in (6.34) and (6.36) hold. Due to (6.40), the equality in (6.36) holds, and due to (6.38) and (6.39), the equality in (6.34) holds. Now the equality holds in inequality (6.37). Therefore, we have

$$\omega_q(a\otimes b) = \frac{1}{2} + \frac{1}{2}\left(q\cdot|\langle a,b\rangle| + \sqrt{(1-q^2)\left(1-|\langle a,b\rangle|^2\right)}\right). \tag{6.41}$$

Since $a, b \in \mathcal{H} \setminus \{0\}$, from (6.41) and the homogeneity of ω_q, it follows that

$$\begin{aligned}
\omega_q(a \otimes b) &= \|a\|\|b\| \cdot \omega_q\left(\left(\frac{a}{\|a\|}\right) \otimes \left(\frac{b}{\|b\|}\right)\right) \\
&= \|a\|\|b\| \cdot \left[\frac{1}{2} + \frac{1}{2}\left(q\left|\langle\frac{a}{\|a\|}, \frac{b}{\|b\|}\rangle\right| + \sqrt{(1-q^2)\left(1 - \left|\langle\frac{a}{\|a\|}, \frac{b}{\|b\|}\rangle\right|^2\right)}\right)\right] \\
&= \frac{\|a\|\|b\| + q|\langle a, b\rangle|}{2} + \frac{\sqrt{1-q^2}}{2} \cdot \sqrt{\|a\|^2\|b\|^2 - |\langle a, b\rangle|^2}.
\end{aligned}$$

This completes the proof. ■

Remark 6.4.12. *When we choose $q = 1$ in Theorem 6.4.11, we obtain the formula (6.25). Moreover, Theorem 6.4.11 generalizes Theorem 6.4.10. Indeed, for given vectors $a, b \in \mathcal{H}$ and every $q \in [0, 1]$, we have that*

$$\begin{aligned}
|\langle a, x\rangle\langle b, y\rangle| &= |\langle (a \otimes b)x, y\rangle| \\
&\leq \frac{\|a\|\|b\| + q|\langle a, b\rangle|}{2} + \frac{\sqrt{1-q^2}}{2} \cdot \sqrt{\|a\|^2\|b\|^2 - |\langle a, b\rangle|^2}
\end{aligned} \tag{6.42}$$

whenever the vectors $x, y \in \mathcal{H}$ are such that $\|x\| = \|y\| = 1$ and $\langle x, y\rangle = q$. When we plug $q = 1$ in the (6.42), we get the inequality (6.24), since $y = \alpha x$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$. In other words, the inequality (6.42) represents a generalization of the classical Buzano inequality.

Corollary 6.4.13. *Let $a, b \in \mathcal{H}$ and $q \in [0, 1]$. If the operator $a \otimes b$ is nilpotent, then we have*

$$\omega_q(a \otimes b) = \frac{1 + \sqrt{1-q^2}}{2}\|a\|\|b\|.$$

Proof. It is easy to see that the operator $a \otimes b$ is nilpotent if and only if $\langle a, b\rangle = 0$. Applying directly Theorem 6.4.11, we obtain the wanted formula. ■

The following two corollaries are closely related to Theorem 6.4.6 and Theorem 6.4.9.

Corollary 6.4.14. *Let $a, b \in \mathcal{H}$ and $q \in [0, 1]$. Then, we have the equality*

$$\omega_q(a \otimes b) = \omega_q\left(\begin{bmatrix} a \otimes b & 0 \\ 0 & 0 \end{bmatrix}\right).$$

Proof. For each $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{H} \oplus \mathcal{H}$, we have

$$\begin{bmatrix} a \otimes b & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (a \otimes b)x \\ 0 \end{bmatrix} = \begin{bmatrix} \langle x, a\rangle b \\ 0 \end{bmatrix} = \langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} a \\ 0 \end{bmatrix}\rangle \begin{bmatrix} b \\ 0 \end{bmatrix} = \left(\begin{bmatrix} a \\ 0 \end{bmatrix} \otimes \begin{bmatrix} b \\ 0 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix}.$$

From the previous equality, we have that $\begin{bmatrix} a\otimes b & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} \otimes \begin{bmatrix} b \\ 0 \end{bmatrix}$. Theorem 6.4.11 now implies

$$\begin{aligned}
\omega_q\left(\begin{bmatrix} a\otimes b & 0 \\ 0 & 0 \end{bmatrix}\right) &= \omega_q\left(\begin{bmatrix} a \\ 0 \end{bmatrix} \otimes \begin{bmatrix} b \\ 0 \end{bmatrix}\right) \\
&= \frac{1}{2}\left(\left\|\begin{bmatrix} a \\ 0 \end{bmatrix}\right\| \cdot \left\|\begin{bmatrix} b \\ 0 \end{bmatrix}\right\| + q\left|\langle \begin{bmatrix} a \\ 0 \end{bmatrix}, \begin{bmatrix} b \\ 0 \end{bmatrix}\rangle\right|\right) + \\
&+ \frac{\sqrt{1-q^2}}{2} \cdot \sqrt{\left\|\begin{bmatrix} a \\ 0 \end{bmatrix}\right\|^2 \left\|\begin{bmatrix} b \\ 0 \end{bmatrix}\right\|^2 - \left|\left\langle \begin{bmatrix} a \\ 0 \end{bmatrix}, \begin{bmatrix} b \\ 0 \end{bmatrix}\right\rangle\right|^2} \\
&= \frac{\|a\|\|b\| + q|\langle a,b\rangle|}{2} + \frac{\sqrt{1-q^2}}{2} \cdot \sqrt{\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2} \\
&= \omega_q(a\otimes b),
\end{aligned}$$

which yields the wanted formula. ■

Corollary 6.4.15. *Let $a, b \in \mathcal{H}$ and $q \in [0,1]$. Then, we have the equality*

$$\omega_q\left(\begin{bmatrix} 0 & a\otimes b \\ 0 & 0 \end{bmatrix}\right) = \frac{1+\sqrt{1-q^2}}{2}\|a\|\|b\|.$$

Proof. For each $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{H} \oplus \mathcal{H}$, we have

$$\begin{bmatrix} 0 & a\otimes b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (a\otimes b)y \\ 0 \end{bmatrix} = \begin{bmatrix} \langle y,a\rangle b \\ 0 \end{bmatrix} = \langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} 0 \\ a \end{bmatrix}\rangle \begin{bmatrix} b \\ 0 \end{bmatrix} = \left(\begin{bmatrix} 0 \\ a \end{bmatrix} \otimes \begin{bmatrix} b \\ 0 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix},$$

from where it follows that $\begin{bmatrix} 0 & a\otimes b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ a \end{bmatrix} \otimes \begin{bmatrix} b \\ 0 \end{bmatrix}$. Using Theorem 6.4.11, we obtain

$$\begin{aligned}
\omega_q\left(\begin{bmatrix} 0 & a\otimes b \\ 0 & 0 \end{bmatrix}\right) &= \omega_q\left(\begin{bmatrix} 0 \\ a \end{bmatrix} \otimes \begin{bmatrix} b \\ 0 \end{bmatrix}\right) \\
&= \frac{1}{2}\left(\left\|\begin{bmatrix} 0 \\ a \end{bmatrix}\right\| \cdot \left\|\begin{bmatrix} b \\ 0 \end{bmatrix}\right\| + q\left|\langle \begin{bmatrix} 0 \\ a \end{bmatrix}, \begin{bmatrix} b \\ 0 \end{bmatrix}\rangle\right|\right) + \\
&+ \frac{\sqrt{1-q^2}}{2} \cdot \sqrt{\left\|\begin{bmatrix} 0 \\ a \end{bmatrix}\right\|^2 \left\|\begin{bmatrix} b \\ 0 \end{bmatrix}\right\|^2 - \left|\left\langle \begin{bmatrix} 0 \\ a \end{bmatrix}, \begin{bmatrix} b \\ 0 \end{bmatrix}\right\rangle\right|^2} \\
&= \frac{\|a\|\|b\| + q\cdot 0}{2} + \frac{\sqrt{1-q^2}}{2} \cdot \sqrt{\|a\|^2\|b\|^2 - 0^2} \\
&= \frac{1+\sqrt{1-q^2}}{2}\|a\|\|b\|.
\end{aligned}$$

This completes the proof. ■

Remark 6.4.16. *Note that we could have obtained the same formula by applying Corollary 6.4.13 since the operator* $\begin{bmatrix} 0 & a \otimes b \\ 0 & 0 \end{bmatrix} \in \mathfrak{B}(\mathcal{H} \oplus \mathcal{H})$ *is nilpotent.*

CHAPTER 7

CONCLUSION

In this concluding chapter, we recapitulate the significant insights and contributions that have emerged from our exploration of subnormal operators and related topics. More precisely, we shall go through all the chapters (except Chapter 1), emphasizing one more time our main results, and overall contribution of the dissertation.

Chapter 2. In this chapter, we affirmatively answered the question posed by R. E. Curto, S. H. Lee, and J. Yoon in [51], providing an alternative approach to the one applied in [146]. Our approach proved to be highly effective in the context of multivariable operator theory as well. Initially, we established characterizations of matricially and spherically quasinormal tuples in terms of their minimal normal extensions. This foundational insight paved the way for a multivariable adaptation of Theorem 2.1.2, which directly addressed the aforementioned question.

Later in the chapter, we shifted our focus to a broader approach to Problem 2.1.1, where we considered the square as a product. This led to a series of intriguing questions closely related to the renowned Fuglede-Putnam Theorem. Among other significant findings, we demonstrated that when the coordinate operators of a (jointly) quasinormal pair share both the same (closed) range and the same null space, the normality of their product implies the normality of each individual operator. We then extended our exploration by relaxing the quasinormality conditions to subnormality. Thus, it may be really interesting to go even further. More precisely, we may ask the following question:

Problem. *Let $\mathbf{T} = (T_1, T_2)$ be a jointly (hyponormal) pair such that $T_1 T_2$ is normal (quasinormal, subnormal). Find sufficient conditions for T_1 and T_2 to be normal (quasinormal, subnormal).*

Chapter 3. In this chapter, we introduced the concept of the spherical mean transform for operator tuples and elucidated several crucial properties of this transform. Among these properties, we demonstrated the preservation of the null space under the transform and investigated its behavior under unitary equivalence. Furthermore, we explored various spectral properties of the spherical mean transform. Notably, we established that while the transform does not generally preserve the Taylor spectrum, it does so in the case of spherical partial isometries, for instance. Additionally, we delved into the challenge of characterizing the mean transform of 2-variable weighted shifts, one of the most significant classes of operator pairs. Under specific conditions, we demonstrated that the p-hyponormality of such pairs remains intact through the transformation.

Overall, this chapter represents the start of the theory of spherical mean transform of operator tuples. From a practical standpoint, a similar observation can be made as in the case of the ordinary mean transform. Specifically, obtaining the spherical mean transform of an operator tuple is generally more straightforward than acquiring the spherical Aluthge transform of the same tuple. This distinction arises from the fact that the latter necessitates the determination of a square root of a positive operator, which is not a trivial task in general.

Chapter 4. The main focus of this chapter was the completion of upper triangular 2×2 operator matrices to normality, as similar completion problems exist for other operator properties such as invertibility, Fredholmness, regularity, and more. In pursuit of this goal, we introduced the concept of normal complements, a category of operator pairs that can be seen as an extension of the concept of subnormal duals. Within this chapter, we provided a comprehensive exploration of these newly introduced pairs. Our analysis covered various characterizations, with respect to the various Hilbert space decompositions, as well as in terms of the polar decompositions of operators.

After that, we explored different spectral properties shared among the coordinate operators of normal complements. We showed that there are significant spectral restrictions on operator C for which the operator matrix $\begin{bmatrix} A & C \\ 0 & B^* \end{bmatrix}$ can be normal. For instance, it was shown that $0 \in \sigma(C)$ for any such C. Additionally, we outlined the conditions under which $0 = \sigma(C)$, particularly in scenarios involving self-complemented subnormal operators.

Concluding this chapter, we turned our attention to the examination of situations in which the Aluthge and Duggal transforms of pure hyponormal operators are indeed self-dual subnormal operators. Furthermore, we provided some applications that stem from these results.

Considering that these topics introduce relatively novel concepts within the realm of completion problems in operator theory, numerous uncharted directions await further exploration. Although we present only a few in this context, the potential for future research and development in this field is substantial.

Problem. *An operator A is self-complemented if there exists C such that the operator matrix*

$$\begin{bmatrix} A & C \\ 0 & A^* \end{bmatrix} \tag{7.1}$$

is normal. In which way the properties of A are changed if we replace A^* with $B \in \{A^\dagger, (A^*)^\dagger, \widehat{A}, \widetilde{A}, \ldots\}$ in (7.1)?

Problem. *Consider an operator matrix n–tuple*

$$\mathbf{M_C} = \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{0} & \mathbf{B} \end{bmatrix} = \left(\begin{bmatrix} A_1 & C_1 \\ 0 & B_1 \end{bmatrix}, \begin{bmatrix} A_2 & C_2 \\ 0 & B_2 \end{bmatrix}, \ldots, \begin{bmatrix} A_n & C_n \\ 0 & B_n \end{bmatrix} \right).$$

Find the necessary and sufficient conditions (or provide the characterization of $\mathbf{C}$) under which the operator matrix $\mathbf{M}$ possesses specific properties like normality, Taylor invertibility, and more, when operator tuples $\mathbf{A}$ and $\mathbf{B}$ are known. This is closely related to the lifting problem for subnormal operators (see [120, 126, 141]).

Chapter 5. This chapter was split into three main sections, each of which deals with specific problems.

In Section 5.1, we delved into an examination of various properties associated with generalized powers of operators and introduced the notion of generalized logarithms for operators. In doing so, we extended certain findings from [12]. Our investigation revealed that numerous properties akin to those of the ordinary logarithmic function can be extended to the bounded operators setting. However, it's important to note that the applicability of several results hinges on specific commutativity conditions.

Further research can be extended in several ways. As mentioned in [12], one obvious way is to try to define a similar notion for the class of unbounded operators (see [132, 159]). Also, the current definitions hold only for positive operators. We may try to extend it for arbitrary self-adjoint, or even normal operators. Furthermore, commutativity assumptions may also be dropped in some cases (see [73, 131])

In Definition 5.1.1, the commutativity of A and B is not required, and therefore, in general $B \log A \neq \log(A)B$. This means that for a positive and invertible $A \in \mathfrak{B}(\mathcal{H})$ and an arbitrary $B \in \mathfrak{B}(\mathcal{H})$, it makes sense to actually define the left and right generalized powers:

$${}^{B}A = e^{B \log A} \quad \text{and} \quad A^B = e^{\log(A)B}.$$

Note that in that case, it means that in Section 5.1, we actually proved the results for ${}^{B}A$, although most of the result would also hold for A^B due to symmetry or some commutativity conditions.

These newly introduced concepts exhibit some interesting properties. For instance, using the fact that for any two invertible operators S and T we have that

$\sigma(ST) = \sigma(TS)$, we can establish that for any positive $A \in \mathfrak{B}(\mathcal{H})$ such that $0, 1 \notin \sigma(A)$ and any invertible $B \in \mathfrak{B}(\mathcal{H})$, the spectra of ${}^{B}A$ and A^B coincide, i.e.

$$\sigma({}^{B}A) = \sigma(A^B).$$

The conclusion immediately follows from the equality of spectra $\sigma(B \log A)$ and $\sigma(\log(A)B)$, combined with the Spectral Mapping Theorem.

Another noteworthy property is the intertwining relation

$$AB = BC \quad \Rightarrow \quad A^B = {}^{B}C,$$

which follows from the Spectral Theorem, provided $A, C \in \mathfrak{B}(\mathcal{H})$ are positive and invertible.

Similar concepts can also be defined for the generalized logarithms. Furthermore, the concepts regarding generalized powers have potential applications, such as defining the tetration operation on $\mathfrak{B}(\mathcal{H})$ and introducing an operator version of the Lambert $\mathcal{W}$ function.

In Section 5.2, we contributed to the ongoing research in operator theory, which explores various classes of operators closely tied to the fundamental concepts of normal and accretive operators ([13], [77]).

Within this context, we introduced a novel class known as polynomially accretive operators, expanding the classes of accretive and n-real power positive operators. Our central focus in this section revolved around giving some representation results, building upon the foundational Radjavi-Rosental Theorem [153].

One of the significant outcomes of our research lay in establishing a profound connection between normal and accretive operators. Specifically, we drew inspiration from recent findings, such as those presented in [74], which demonstrate that if the powers T^p and T^q of an invertible operator T are both normal, where p and q are coprime integers, then the operator T itself must be normal. In a similar vein, we derived results that align with the spirit of this theorem.

To be more specific, our research revealed that when T^2 is normal and T^n is accretive, where n is an odd integer greater than one, an intriguing consequence emerges: T^n must be normal for all values $n \geq 2$. Furthermore, under the additional assumption that $\mathcal{R}([T^*, T]) \subseteq \mathcal{N}(T^l)^{\perp}$ for some $l \geq 2$, we established that the operator T must be normal. Thus, a natural question arises:

Problem. *Let $T \in \mathfrak{B}(\mathcal{H})$ be an invertible operator. If T^p is normal and T^q is accretive for some comprime numbers p and q, does it follow that T is normal?*

We believe these results will not only contribute to the ongoing research in operator theory but will also pave the road for further exploration of the topic.

Lastly, Section 5.3 was dedicated to addressing the solvability of a general system of operator equations

$$A_1XB_1 = C_1,$$
$$A_2XB_2 = C_2.$$

Solving this system without additional assumptions poses formidable challenges at present. Nevertheless, we managed to make significant headway by providing the necessary and sufficient conditions for the solvability of such a system, shedding light on the specific scenarios in which solutions can be found. Special focus was put on finding the general forms of Hermitian and positive solutions.

Furthermore, we delved into the exploration of the solvability of the $*$-order operator inequality $C \overset{*}{\leq} AXB$. This inequality essentially corresponds to the following system of operator equations:

$$\begin{aligned} AXBC^* &= CC^*, \\ C^*AXB &= C^*C. \end{aligned}$$

We unveiled that the solvability of $C \overset{*}{\leq} AXB$ bears particular significance only when the ranges of the involved operators are not closed. In other instances, the solvability of the inequality is equivalent to the solvability of the operator equation $C = AXB$. Additionally, we provided the general forms of the solutions for $C \overset{*}{\leq} AX$ and $C \overset{*}{\leq} XB$.

It's worth noting that this section, while insightful, presents opportunities for refinement, given that some of the assumptions in our results currently limit their applicability to highly specific cases.

Chapter 6. In this chapter, our primary objective was to delve into the concept of the q-numerical radius, denoted as $\omega_q(\cdot)$, for operators on Hilbert spaces. Our exploration centered around investigating a range of inequalities associated with these values. As we progressed, we extended the well-established results related to the numerical radius, which is the special case when q equals 1.

For instance, when considering the equality for the ordinary numerical range of a direct sum of operators, it is expressed as

$$\omega\left(\bigoplus_{n=1}^{+\infty} A_n\right) = \sup_{n\in\mathbb{N}} \omega(A_n)$$

In this chapter, we demonstrated that for $q \in \overline{\mathbb{D}} \setminus \{0\}$, this equality translates to inequality

$$\sup_{n\in\mathbb{N}} \omega_q(A_n) \leq \omega_q\left(\bigoplus_{n=1}^{+\infty} A_n\right) \leq \frac{|q| + 2\sqrt{1-|q|^2}}{|q|} \sup_{n\in\mathbb{N}} \omega_q(A_n).$$

Problem. *Is it possible to establish tighter bounds for the inequality mentioned above?*

Furthermore, we provided the explicit formulas for computing $\omega_q(\cdot)$ in specific scenarios, such as operator matrices and rank one operators. Specifically, for a rank one operator $a \otimes b$, the formula is given by

$$\omega_q(a \otimes b) = \frac{\|a\|\|b\| + q|\langle a, b\rangle|}{2} + \frac{\sqrt{1-q^2}}{2} \cdot \sqrt{\|a\|^2\|b\|^2 - |\langle a, b\rangle|^2}.$$

Lastly, we explored various analytical properties of $\omega_q(\cdot)$ when treating it as a function of the parameter q.

This voyage has been one of unwavering determination, persistence, and fruitful collaborations. Once again, I extend my heartfelt gratitude to the supervisor, colleagues, and fellow researchers who have contributed to this academic odyssey. As this chapter draws to a close, I express our hope that the insights and revelations contained within these pages will serve as guiding lights for those who follow in our mathematical footsteps. The pursuit of knowledge is a continuum, and we are but one chapter in its enduring narrative.

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BIOGRAPHY

Hranislav Stanković was born on August 10, 1994, in Prokuplje, Serbia. He completed "Toplički heroji" Elementary School in Žitoradja as valedictorian, earning the "Vuk Karadžić" diploma. He also completed his high school education at "Gimnazija Prokuplje" Grammar School, again receiving the "Vuk Karadžić" diploma.

He actively participated in mathematics, Serbian language, and biology competitions, consistently achieving first place in regional competitions. Furthermore, he received numerous commendations, recognitions, and awards in various literary and music contests. As a sixth-grade pupil, he published a book of poetry in 2007.

Hranislav commenced his undergraduate studies in mathematics in 2013 at the Faculty of Science and Mathematics, University of Niš, graduating in 2016 with a perfect GPA of 10.00. In the same year, he enrolled in the master's program in mathematics, specializing in theoretical mathematics, which he completed in 2018. He successfully defended his master's thesis, titled "Characterization of Rank One Operators", earning a grade of 10.

During the 2018/2019 academic year, he began his doctoral studies in mathematics at the Faculty of Science and Mathematics, University of Niš. Throughout his doctoral studies, he excelled in all required exams, earning a perfect GPA of 10.

During his academic journey, he was twice awarded the "Dositeja" scholarship.

In 2019, he was appointed as a research associate trainee at the Faculty of Science and Mathematics, University of Niš. Since then, he has been actively involved in the scientific research project "Problems of Nonlinear Analysis, Operator Theory, Topology, and Applications" (OI 174025) funded by the Ministry of Education, Science, and Technological Development of the Republic of Serbia.

At the Faculty of Science and Mathematics, he was engaged in teaching various courses, including Mathematical Logic and Set Theory, Linear Algebra, Elementary Mathematics 1, Introduction to Topology, and Mathematical Analysis 3 at the undergraduate level. Additionally, he taught courses such as Operator Theory, Set Theory, Fixed Point Theory and Applications, and Algebraic Topology at the master's level.

Starting from 2021, he has been working as a teaching assistant at the Faculty of Electronic Engineering, University of Niš, where he teaches courses such as Mathematics 1, Mathematics 2, Mathematics 3, Mathematical Methods, Probability and Statistics, and Numerical Algorithms.

To date, he has published five research papers, and three more are currently under review in international journals with impact factor.

Publications

- D. S. Cvetković-Ilić, H. Stanković. *On normal complements*, Journal of Mathematical Analysis and Applications, (in review).
- D. Denčić, H. Stanković, I. Damnjanović, M. Krstić. *q-numerical radius of rank one operators*, (preprint).
- H. Stanković. *Converse of Fuglede Theorem*, Operators and Matrices, **17**(3) (2023), 705–714.
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List of Symbols

$[\mathbf{T}^*, \mathbf{T}]$	The self-commutator of an operator tuple $\mathbf{T}$
$[S, T]$	The commutator of operators S and T
$[T^*, T]$	The self-commutator of an operator T
$\mathcal{H}$	A complex Hilbert space
$\log T$	The logarithm of an operator T
$\log_A B$	The logarithm of an operator B to base A
$\mathbb{C}$	The set of complex numbers
$\mathbb{C}[z]$	The set of all non-trivial complex polynomials in one variable
$\mathbb{D}$	The open unit disk
$\mathbb{N}$	The set of natural numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{Z}$	The set of integers
$\mathbb{Z}_+$	The set of nonnegative integers
$\mathbf{T}^*$	An operator tuple $(T_1^*, \ldots, T_n^*)$
$\mathbf{T}$	An operator tuple $(T_1, \ldots, T_n)$
$\mathbf{T}^{(m,n)}$	An operator pair (T_1^m, T_2^n)
$\mathbf{W}_{(\alpha,\beta)}$	A 2-variable weighted shift
$\mathcal{D}(T)$	The domain of an (possibly unbounded) operator T
$\mathcal{K}(\mathbf{T}, \mathcal{H})$	The Koszul complex associated to $\mathbf{T}$ on $\mathcal{H}$
$\mathcal{M}(\mathbf{T})$	The spherical mean transform of an operator tuple $\mathbf{T}$

$\mathcal{M}(T)$	The mean transform of an operator T
$\mathcal{M} \oplus \mathcal{N}$	The orthogonal sum of $\mathcal{M}$ and $\mathcal{N}$
$\mathcal{M}^{\perp}$	The orthogonal complement of a subspace $\mathcal{M}$
$\mathcal{N}(\mathcal{H})$	The set of normal operators on $\mathcal{H}$
$\mathcal{N}(T)$	The null space of an operator T
$\mathcal{R}(T)$	The range of an operator T
$\mathcal{W}(\mathbf{T})$	Joint numerical range of an operator tuple $\mathbf{T}$
$\mathcal{W}(T)$	The numerical range of an operator T
$\mathcal{W}_q(T)$	The q-numerical range of an operator T
$\mathfrak{B}(\mathcal{H})$	The space of all bounded operators on $\mathcal{H}$
$\mathfrak{B}(\mathcal{H}, \mathcal{K})$	The space of all bounded operators from $\mathcal{H}$ to $\mathcal{K}$
$\mathfrak{B}(\mathcal{H})^n$	$\mathfrak{B}(\mathcal{H}) \times \cdots \mathfrak{B}(\mathcal{H})$ (n times)
$\mathfrak{C}(A, B)$	The set of all operators of the form $A^{1/2}UB^{1/2}$, where $\|U\| \leq 1$
$\mathfrak{N}(A)$	The set of all $C \in \mathfrak{B}(\mathcal{H})$ such that the matrix $\left[\begin{smallmatrix} A & C \\ 0 & A^* \end{smallmatrix}\right]$ is normal
$\mathfrak{N}(A, B)$	The set of all $C \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$ such that the matrix $\left[\begin{smallmatrix} A & C \\ 0 & B^* \end{smallmatrix}\right]$ is normal
$\text{Im}\,(T)$	The imaginary part of an operator T
$\text{Re}\,(T)$	The real part of an operator T
$\text{span}\{S\}$	The linear span of vectors in a finite set S
$\|\cdot\|$	The norm on $\mathcal{H}$
$\|\mathbf{T}\|$	Joint operator norm of an operator tuple $\mathbf{T}$
$\|\mathbf{T}\|^e$	Euclidean operator norm of an operator tuple $\mathbf{T}$
$\omega(\mathbf{T})$	Joint numerical radius of an operator tuple $\mathbf{T}$
$\omega(T)$	The numerical radius of an operator T
$\omega_q(T)$	The q-numerical radius of an operator T
$\overline{\mathcal{M}}$	The closure of a subspace $\mathcal{M}$

$\overline{p}(z)$	The conjugate of a polynomial p given by $\overline{p(\overline{z})},\ z \in \mathbb{C}$
$\overline{T}$	The continuous linear extension of an operator T
$\sigma(T)$	The spectrum of an operator T
$\sigma_l(T)$	The left spectrum of an operator T
$\sigma_p(T)$	The point spectrum of an operator T
$\sigma_r(T)$	The right spectrum of an operator T
$\sigma_T(\mathbf{T})$	The Taylor spectrum of an operator tuple $\mathbf{T}$
$\langle \cdot, \cdot \rangle$	The inner product on $\mathcal{H}$
$\overset{*}{\leq}$	The star order
$\text{Area}(A)$	The area of a set $A \subset \mathbb{C}$
$\text{Comm}(T)$	The commutant of an operator T
$\dim(\mathcal{H})$	The dimension of a Hilbert space $\mathcal{H}$
$\widehat{\mathbf{T}}$	The spherical Duggal transform of an operator tuple $\mathbf{T}$
$\widehat{T}$	The Duggal transform of an operator T
$\widetilde{\mathbf{T}}$	The spherical Aluthge transform of an operator tuple $\mathbf{T}$
$\widetilde{T}$	The Aluthge transform of an operator T
$a \otimes b$	The one-dimensional operator given by $(a \otimes b)x = \langle x, a \rangle b,\ x \in \mathcal{H}$
A^B	The generalized power of an operator A
e^T	The exponential of an operator T
$I_{\mathcal{H}}$	The identity operator on a Hilbert space $\mathcal{H}$
$l^2(\mathbb{Z}^2_+)$	The Hilbert space of square-summable sequences indexed by $\mathbb{Z}^2_+$
$l^\infty(\mathbb{Z}^2_+)$	The space of double-indexed non-negative bounded sequences
$P_{\mathcal{M},\mathcal{N}}$	The projection with the range $\mathcal{M}$ and the null space $\mathcal{N}$
$P_{\mathcal{M}}$	The orthogonal projection with the range $\mathcal{M}$
$r(T)$	The spectral radius of an operator T

$S \oplus T$	The direct sum of operators S and T
$T \restriction_{\mathcal{M}}$	The restriction of an operator T to a subspace $\mathcal{M}$
T^*	The adjoint operator of an operator T
$T^\dagger$	The Moore-Penrose (generalized) inverse of an operator T
T^{-1}	The inverse of an operator T
$T^{1/2}$	The square root of a positive operator T
$T^{1/n}$	The n-th root of a positive operator T
T^{cr}	The corestriction of an operator T

INDEX

КРАТАК ПРИКАЗ ДОКТОРСКЕ ДИСЕРТАЦИЈЕ НА СРПСКОМ ЈЕЗИКУ

Апстракт

Ова докторска дисертација примарно истражује комплексан домен субнормалних оператора, расветљавајући њихове различите аспекте и откривајући нова сазнања у оквиру вишедимензионалне теорије оператора.

Најпре, дисертација истражује однос између субнормалности и квазинормалности ограничених линеарних оператора. Испитују се услови под којима субнормалност оператора и квазинормалност његовог квадрата имплицирају квазинормалност самог оператора. Додатно, доказује се да квазинормални n-ти корени субнормалног оператора такође морају бити квазинормални. Истраживање даје довољне услове при којима је матрична и сферична квазинормалност парова оператора еквивалентна матричној и сферичној квазинормалности њихових степена. Такође се разматра обрат Фулеове теореме, утврђујући када субнормални оператори морају бити нормални, уколико је њихов производ нормалан.

Дисертација такође уводи концепт сферичне средње трансформације за парове оператора, проширујући појам средње трансформације са једнодимензионалног случаја на вишедимензионални. Истражују се различита спектрална својства ове трансформације, укључујући очување Тејлоровог спектра, као и нека аналитичка својства. Истраживање такође утврђује услове под којима трансформација очувава p-хипонормалност дводимензионалних тежинских оператора помераја.

Такође, у контексту субнормалних оператора и субнормалних дуала, дисертација се бави допуном горњетроугаоне операторске матрице до нормалности. Уводи се концепт нормалних комплемената и дају се карактеризације и репрезентационе теореме за ове парове. Истражују се заједничка спектрална својства нормалних комплемената, истичући заједничка својства између координатних оператора у пару нормалних комплемената. Такође се утврђује веза између теорије субнормалних дуала и Алутгеове и Дугалове трансформације.

Поред тога, дисертација истражује разне класе оператора повезаних са нормалним и субнормалним операторима, уводећи нове концепте и разматрајући решавање специфичних операторских једначина и система једначина. Такође се испитују неке неједнакости везане за q-нумерички радијус ограничених оператора и операторских матрица, проширујући добро познате једнакости које се односе на нумерички радијус.

Захвалност

Пре него што наставимо даље, желим изразити најдубљу захвалност свом ментору, професорки Драгани Цветковић-Илић, за њено непоколебљиво вођство и безгранично стрпљење током мојих докторских студија. Њено менторство није само обогатило моје математичко разумевање, већ ме је такође обликовало у бољег истраживача и мислиоца. Њена посвећеност развоју мог интелектуалног раста била је непроцењив дар.

Такође желим изразити искрену захвалност својој породици на њиховој непрестаној подршци, охрабрењу и вери у моје способности, које су биле главна покретачка снага мојих достигнућа.

Веома сам захвалан својим драгим пријатељима, чије пријатељство је било извор утехе, радости и мотивације током изазовних фаза овог путовања. Њихова присутност у мом животу га је учинила богатијим и дубљим.

Захвалан сам свим својим професорима, колегама и студентима који су делили своје знање и увиде са мном. Њихови заједнички доприноси значајно су обогатили моје академске искуство.

Такође желим изразити захвалност професору Каису Фекију на његовој великодушности у дељењу своје докторске дисертације о вишедимензионалној теорији оператора. Његов драгоцени допринос значајно је убрзао процес писања ове дисертације.

На крају, желим се захвалити за подршку и ресурсе које је пружио Универзитет у Нишу током мог докторског програма. Академска атмосфера и ресурси су играли кључну улогу у обликовању мојих академских и истраживачких тежњи.

Ова дисертација не би била могућа без подршке и охрабрења ових изузетних особа и институција. Дубоко сам захвалан на њиховом доприносу мом академском и личном расту.

Хвала свима што сте били неизоставни део овог изузетног путовања.

Са искреном захвалношћу,

Хранислав Станковић,

Јануар, 2024.

Садржај

Увод

Испитивање линеарних оператора у функционалној анализи и теорији оператора има богату историју која траје дуже од једног века. Ови оператори, често служећи као математички модели за многе физичке појаве, налазе примене у различитим областима као што су квантна механика, обрада сигнала и теорија управљања. Међу разним класама линеарних оператора, класа нормалних оператора се издваја као једна од најфундаменталнијих због своје суштинске везе са Спектралном теоремом. У том контексту, субнормални оператори, који природно генерализују класу нормалних оператора, заузимају значајно место. Они нуде комплексне изазове и уједно обећавајуће увиде у основне математичке структуре.

Ова докторска дисертација, под насловом "Субнормални оператори: Приступ из вишедимензионалне теорије оператора", истражује теорију субнормалних оператора са вишедимензионалног становишта. Субнормални оператори, као генерализација класе нормалних оператора, већ годинама фасцинирају математичаре. Међутим, њихово понашање, посебно када се проучава у оквиру вишедимензионалне теорије оператора, остаје подручје непрекидног истраживања са значајним потенцијалом.

Ова дисертација приступа проблему на свеж начин, уводећи нове методе и погледе како би се "одмрсили" комплексни аспекти субнормалних оператора. Циљ нам је да осветлимо њихове особине и откријемо везе и увиде који су можда претходно измакли. Истраживање је структурирано у седам глава, од којих је свака посвећена одређеним аспектима ове теме.

Наше истраживање почињемо увођењем појмова у **Глави 1**, основним прегледом теорије субнормалних оператора, као и теорије оператора уопште. Главни циљ је да опремимо наше читаоце неопходним знањем потребним за разумевање резултата изложених у овој дисертацији. Током овог процеса, тежимо да осигурамо да наша презентација остане што независнија, смањујући потребу за спољним референцама.

Глава 2 посвећена је истраживању везе између субнормалности и квазинормалности оператора. Конкретно, истражујемо да ли субнормалност оператора и квазинормалност његовог квадрата имплицирају квазинормалност самог оператора. Штавише, у Одељку 2.2, доказујемо да субнормални n-ти корени квазинормалног оператора такође морају бити квазинормални. Додатно, утврђујемо довољне услове под којима су матрична и сферична квазинормалност парова оператора еквивалентне са одговарајућим својствима њихових n-тих степена. У Одељку 2.3, разматрамо услове под којима субнормални оператори постају квазинормални уколико је њихов производ квазинормалан. Додатно, пружамо довољне услове под којима квазинормални (или субнормални) оператори морају бити нормални ако је њихов производ нормалан. У суштини, дајемо критеријуме за обрат Фулеове теореме, успут отривајући везу са вишедимензионалном теоријом субнормалних оператора. Резултати ове главе објављени су у [158, 163].

Глава 3 уводи концепт сферичне средње трансформације, који је дефинисан за парове оператора. Појам се заснива на концепту средње трансформације оператора, проширујући га на домен вишедимзионалне теорије оператора. У Одељку 3.2, анализирамо спектрална својства која су повезана са овом трансформацијом. Један од кључних аспеката које истражујемо је његова способност да очува Тејлоров спектар, један од кључних концепата у вишедимензионалној теорији оператора. Резултати обезбеђују да у неким посебним случајевима одређене спектралне карактеристике остају непромењене и након применење сферичне средње трансформације. Поред тога, разматрамо различита аналитичка својства везана за ову трансформацију. Одељак 3.3 бави се практичном применом сферичне средње трансформације. Утврђујемо неке довољне услове који гарантују очување p-хипонормалности, концепта који је посебно важан када се ради са дводимензионалним тежинским операторима помераја. Напомена да су добијени резултати у овој глави такође објављени у [162].

Глава 4 је посвећена проблему комплетирања горње-троугаоних операторских матрица до нормалности. Овај проблем има фундаменталну значајност у теорији субнормалних оператора, посебно у оквиру субнормалних дуала. С циљем давања одговора на поменути проблем комплетирања, уводимо појам нормалних комплемената. У Одељку 4.1, проучавамо карактеристике нормалних комплемената, откривајући њихове битне особине. Одељак 4.2 је усмерен на проучавање заједничних спектралних својстава нормалних комплемената и, као што ћемо видети, координатни оператори у пару нормалних комплемената деле многа заједничка својства. На крају, у Одељку 4.3, повезујемо теорију субнормалних дуала са теоријом Алутгеове и Дугалове трансформације. Ове трансформације су скренуле значајну пажњу на себе у последњих неколико деценија, чинећи ову везу значајним доприносом области. Резултати представљени у овој глави засновани су на заједничком раду [62].

Глава 5 истражује различите класе оператора повезаних са нормалним и субнормалним операторима. У недавном раду, А. Башир, М. Х. Мортад и Н. А. Сајаф [12] увели су опште степене линеарних оператора. Другим речима, оператори се не подижу на бројеве, већ на друге операторе. Они су навели неколико основних својстава везаних за овај концепт. У Одељку 5.1, проширујемо њихове резултате, дубље залазећи у својства овог новог операторског степена. Такође, уводимо појам општег логаритма у Одељку 5.1.3. Конкретно, за два позитивна и инвертибилна оператора, A и B, где $1 \notin \sigma(A)$, дефинишемо логаритам B у односу на базу A, у ознаци $\log_A B$. Наше истраживање обухвата обимну анализу његових својстава, додатно обогаћујући разумевање ових математичких конструкција. У Одељку 5.2, уводимо нову класу оператора на комплексном Хилбертовом простору $\mathcal{H}$ под именом полиномноакретивни оператори. Ова концепција проширује постојеће концепте акретивних и n-реално-позитивних оператора. Наше истраживање ове нове класе оператора открива нека фундаментална својства и генерализује познате резултате везане за акретивне операторе. Занимљиво откриће појављује се утврђивањем да сваки 2-нормални и $(2k+1)$-реално-позитивни оператор, за неки $k \in \mathbb{N}$, мора бити n-нормалан за свако $n \geq 2$. Додатно, пружамо довољне услове за нормалност T у контексту ове инклузије. Завршни одељак ове главе, Одељак 5.3, посвећен је испитивању решивости општег система операторских једначина: $A_iXB_i = C_i$ за $i = 1, 2$. У оквиру овог проблема, представљамо потребне и довољне услове за постојање решења, обухватајући хермитска решења и позитивна решења. Додатно, изводимо опште облике ових решења, омогућавајући истраживање операторских неједнакости $*$-поретка. Конкретно, испитујемо решивост $C \overset{*}{\leq} AXB$ и представљамо општи облик решења за $C \overset{*}{\leq} AX$ и $C \overset{*}{\leq} XB$. Већина резултата на којима је базирана ова глава већ је представљена у [159, 160, 161].

Напослетку, у **Глави 6**, разматрамо q-нумерички радијус $\omega_q(\cdot)$ операторских матрица дефинисаних на директном збиру Хилбертових простора и истражујемо различите неједнакости у вези са овим вредностима. Такође проширујемо неке добро познате једнакости везане за нумерички радијус које се јављају када узмемо да је $q = 1$. Након тога, дајемо експлицитне формуле за израчунавање $\omega_q(\cdot)$ за неке специјалне случајеве операторских матрица и такође утврђујемо нека аналитичка својства $\omega_q(\cdot)$ као функције у зависности од q. У Одељку 6.4.2, разматрамо једнодимензионалне операторе на Хилбертовом простору $\mathcal{H}$ и представљамо општу формулу за нумерички радијус оператора ранга 1. Такође доказујемо уопштење Бузанине неједнакости као последицу. Већина резултата ове главе може се наћи у [70] и [164].

Завршавамо презентацију сумирањем наших резултата и давањем неких завршних коментара у **Глави 7**.

Укратко, у овој дисертацији представљамо различите нове резултате, теореме и илустративне примере како бисмо дубље разумели ову тему. Иако је овај рад далеко од свеобухватног, због обимне теорије нормалних и субнормалних оператора, ипак тежи да пружи значајан допринос разумевању субнормалних оператора и субнормалних n-торки. Надамо се да ће увиди стечени из овог истраживања надахнути даље истраживање ове занимљиве области математике и довести до нових открића.

Важна напомена. *Главе 5, 6 и 7, као и сви докази који су изостављени у овој верзији дисертације, могу се пронаћи у верзији дисертације писаној на енглеском језику.*

Глава 1

Основни појмови

Ово поглавље почињемо представљањем основног оквира теорије оператора. Овај почетни корак служиће као основа за наше будуће истраживање субнормалних оператора. Такође, циљ је да опремимо читаоце основним знањем потребним за разумевање резултата представљених у овој дисертацији.

1.1 Оператори на Хилбертовим просторима

У овом одељку представљамо класичне резултате из теорије оператора, као и опис симбола који ће се користити у овој дисертацији.

Користимо стандардну нотацију $\mathbb{C}$ за означавање комплексне равни, док ће $\mathbb{R}$ означавати реалну осу. $\mathbb{N}$ представља скуп свих природних бројева, $\mathbb{Z}$ означава целе бројеве, док ће $\mathbb{Z}_+$ представљати $\mathbb{N} \cup \{0\}$. Такође ћемо користити $\mathbb{D}$ да бисмо означили отворени јединични диск око нуле, односно, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Са $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$, ... означавамо Хилбертове просторе, који се увек сматрају комплексним. Скаларни производ и норму на Хилбертовом простору означаваћемо са $\langle \cdot, \cdot \rangle$ и $\|\cdot\|$, респективно. Подпростор $\mathcal{M}$ од $\mathcal{H}$ увек подразумевамо да је линеарни подпростор, који не мора бити затворен у односу на топологију коју генерише скаларни производ на $\mathcal{H}$. Затворење подпростора $\mathcal{M}$ означавамо са $\overline{\mathcal{M}}$, док се подпростор $\mathcal{M}$ сматра густим ако је $\overline{\mathcal{M}} = \mathcal{H}$. Ортогонални комплемент подпростора $\mathcal{M}$ означаваћемо са $\mathcal{M}^\perp$, док ће $\mathcal{M} \oplus \mathcal{N}$ представљати ортогоналну суму два подпростора $\mathcal{M}$ и $\mathcal{N}$.

Ако су $\mathcal{H}$ и $\mathcal{K}$ Хилбертови простори, тада означавамо са $\mathfrak{B}(\mathcal{H}, \mathcal{K})$ Банахов простор свих ограничених линеарних оператора са $\mathcal{H}$ на $\mathcal{K}$. Ако је $\mathcal{H} = \mathcal{K}$, онда једноставно користимо $\mathfrak{B}(\mathcal{H})$ уместо $\mathfrak{B}(\mathcal{H}, \mathcal{H})$. Норму оператора T означавамо са $\|T\|$. За дати оператор $T \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$, означићемо са $\mathcal{R}(T)$ слику оператора T. $\mathcal{N}(T)$ ће означавати језгро (или кернел) оператора T. Кажемо да је оператор $T \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$ оператор са затвореном сликом (или има затворени опсег) ако је $\mathcal{R}(T)$ затворен подпростор од $\mathcal{K}$.

Адјунговани оператор оператора $T \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$ означавамо са $T^* \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$. Са

$\sigma(T)$ и $r(T)$ означавамо спектар и спектрални радијус оператора T, респективно. Нумерични ранг оператора T дефинишемо као скуп

$$\mathcal{W}(T) = \{\langle Tx, x\rangle\colon\ x \in \mathcal{H},\ \|x\| = 1\},$$

док се нумерични радијус дефинише као

$$\omega(T) = \sup_{w \in \mathcal{W}(T)} |w|.$$

Кажемо да је оператор $T \in \mathfrak{B}(\mathcal{H})$ нормалан ако је $TT^* = T^*T$, односно ако оператор T комутира са својим адјунгованим оператором T^*. Класа нормалних оператора је веома важна (ако не и најважнија!) у теорији оператора због значаја Спектралне теореме која важи за операторе из ове класе. Неки од основних примера нормалних оператора укључују унитарне операторе, Хермитске операторе (самоконјуговане) и позитивне операторе. Наведени термини имају уобичајено значење: T је унитаран ако је нормалан и инвертибилан; хермитски ако је $T = T^*$ и позитиван ако је $\langle Tx, x\rangle \geq 0$ за све $x \in \mathcal{H}$. Класа нормалних оператора на одређеном Хилбертовом простору $\mathcal{H}$ биће означена са $\mathcal{N}(\mathcal{H})$.

Скуп позитивних оператора представља конвексни конус у $\mathfrak{B}(\mathcal{H})$, а парцијални поредак на скупу хермитских оператора индукован овим конусом се зове Лефнерово уређење, у ознаци $\leq$. Сваки позитивни оператор T има јединствени позитивни квадратни корен, односно постоји јединствени позитивни оператор S такав да је $T = S^2$. Означавамо S са $T^{1/2}$. Користећи непрекидни функционални рачун, такође можемо дефинисати било који позитивни степен за T, односно за сваки $\alpha > 0$, оператор T^α има смисла дефинисати. Пошто је T^*T позитиван за сваки $T \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$, оператор $(T^*T)^{1/2}$ је добро дефинисан и назива се модул (или апсолутна вредност) оператора T, и означава се са $|T|$.

За сваки оператор $T \in \mathfrak{B}(\mathcal{H})$, са $\mathrm{Comm}(T)$ означавамо комутант оператора T, тј.

$$\mathrm{Comm}(T) = \{S \in \mathfrak{B}(\mathcal{H}) :\ TS = ST\}.$$

Добро је познато да за позитивни оператор $T \in \mathfrak{B}(\mathcal{H})$ важи

$$\mathrm{Comm}(T) = \mathrm{Comm}(T^{1/2}).$$

Штавише, имамо следеће трђење:

Теорема 1.1.1. *Ако је $n \in \mathbb{N}$, онда се комутанти позитивног оператора и његовог n-тог корена поклапају.*

Оператор $T \in \mathfrak{B}(\mathcal{H})$ се сматра пројекцијом ако је $T^2 = T$, тј. ако је T идемпотент. T је ортогонална пројекција ако је $T^2 = T = T^*$, тј. ако је T хермитски идемпотент. Пројекција са сликом $\mathcal{M}$ и језгром $\mathcal{N}$ биће означена са $P_{\mathcal{M},\mathcal{N}}$, док ће $P_{\mathcal{M}}$ означавати ортогоналну пројекцију са сликом $\mathcal{M}$.

Ако је $T : \mathcal{H} \to \mathcal{K}$ и $\mathcal{M} \subseteq \mathcal{H}$, рестрикција оператора T на $\mathcal{M}$ биће означена са $T \upharpoonright_{\mathcal{M}}$. Корестрикција T^{cr} оператора T је дефинисана као пресликавање са доменом $\mathcal{H}$, кодоменом $\mathcal{R}(A)$ и такво да важи

$$A^{cr}x = Ax, \quad x \in \mathcal{H}.$$

Затворени подпростор $\mathcal{M}$ у $\mathcal{H}$ се сматра инваријантним подпростором за $T \in \mathfrak{B}(\mathcal{H})$ ако $Tx \in \mathcal{M}$ за све $x \in \mathcal{M}$. Затворени подпростор $\mathcal{M}$ је је редукујући подпростор за $T \in \mathcal{B}(\mathcal{H})$ (или редукује $T \in \mathcal{B}(\mathcal{H})$) ако је инваријантан за T и T^*, тј. уколико је $T(\mathcal{M}) \subseteq \mathcal{M}$ и $T^*(\mathcal{M}) \subseteq \mathcal{M}$. Следећа једноставно, али корисно запажање, користиће се у неколико доказа.

Лема 1.1.2. *Нека су A, $P \in \mathfrak{B}(\mathcal{H})$ такви да је A самоконјугован а P је ортогонална пројекција. Тада је $\mathcal{R}(P)$ инваријантан за A ако и само ако A и P комутирају.*

Доказ. Ако је $\mathcal{R}(P)$ инваријантан за A, онда је очигледно $PAP = AP$. Конјуговањем последње једнакосте, добијамо $PAP = PA$, па је $AP = PA$.

Супротно, ако је $AP = PA$, онда је $PAP = AP$, што подразумева да је $\mathcal{R}(P)$ инваријантан за A. ■

Треба напоменути да претходна лема има општији облик у виду следеће теореме.

Теорема 1.1.3. *[85, стр. 62] Нека је T оператор на Хилбертовом простору $\mathcal{H}$, а $\mathcal{M}$ затворени подпростор од $\mathcal{H}$. Тада су следећи услови еквивалентни:*

(i) *$\mathcal{M}$ редукује T;*

(ii) *$\mathcal{M}^{\perp}$ редукује T;*

(iii) *$TP_{\mathcal{M}} = P_{\mathcal{M}}T$.*

За дати затворени подпростор $\mathcal{S}$ и за сваки $T \in \mathfrak{B}(\mathcal{H})$, декомпозиција операторске матрице T индукована подпростором $\mathcal{S}$ дата је са

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \tag{1.1}$$

где је $T_{11} = P_{\mathcal{S}}TP_{\mathcal{S}} \upharpoonright_{\mathcal{S}}$, $T_{12} = P_{\mathcal{S}}T(I - P_{\mathcal{S}}) \upharpoonright_{\mathcal{S}^{\perp}}$, $T_{21} = (I - P_{\mathcal{S}})TP_{\mathcal{S}} \upharpoonright_{\mathcal{S}}$ и $T_{22} = (I-P_{\mathcal{S}})T(I-P_{\mathcal{S}}) \upharpoonright_{\mathcal{S}^{\perp}}$. Ако је $T_{12} = 0$ и $T_{21} = 0$, једноставно ћемо писати $T = T_{11} \oplus T_{22}$.

За $T \in \mathfrak{B}(\mathcal{H})$ постоји линеарни оператор $T' : \mathcal{D}(T') \subseteq \mathcal{H} \mapsto \mathcal{H}$ такав да $\mathcal{R}(T) \subseteq \mathcal{D}(T')$ и

$$TT'T = T.$$

Оператор T' се зове унутрашњи инверз оператора T. У општем случају, напомињемо да T' може бити неограничен, односно $T' \notin \mathfrak{B}(\mathcal{H})$. Додатно, за $T \in \mathfrak{B}(\mathcal{H})$ постоји унутрашњи инверз T' такав да $T' \in \mathfrak{B}(\mathcal{H})$ ако и само ако T има затворену слику [135]. У том случају, оператор T се назива регуларним. Додатно, ако T' такође задовољава

$$T'TT' = T',$$

тада се T' зове рефлексивни инверз оператора T. Такође, постоји јединствени рефлексивни инверз X оператора T који задовољава систем једначина

$$XT = P_{\overline{R(T^*)}} \quad \text{и} \quad TX = P_{\overline{R(T)}} \restriction_{\mathcal{R}(T)\oplus\mathcal{R}(T)^{\perp}},$$

Такав оператор се зове Мур-Пенроузов (генералисани) инверз оператора T и означава се са $T^{\dagger}$. Еквивалентно, оператор $T^{\dagger}$ задовољава следећи систем једначина:

$$TT^{\dagger}T = T, \quad T^{\dagger}TT^{\dagger} = T^{\dagger}, \quad (TT^{\dagger})^* = TT^{\dagger}, \quad (T^{\dagger}T)^* = T^{\dagger}T,$$

које се зову Пенроузове једначине. Мур-Пенроузов инверз представља главни алат у решавању многих матричних и операторских једначина. Савремена теорија генералисаних инверза може се повезати са радом Бјерхамара у [22] и [23] када је истакао да је Муров "реципрочни елемент" [125] заправо најмање средње-квадратно решење једначине $AXB = C$. После тога, Пенроуз је у [139] и [140] проширио Бјерхамаров резултат и доказао следећу теорему:

Теорема 1.1.4. *[139] Нека су $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$ и $C \in \mathbb{C}^{m\times q}$. Тада је матрична једначина*

$$AXB = C$$

конзистентна ако и само ако постоје унутрашњи инверзи A', B' такви да важи

$$AA'CB'B = C,$$

и у том случају је општо решење

$$X = A'CB' + Y - A'AYBB',$$

за произвољно $Y \in \mathbb{C}^{n\times p}$.

Напомена 1.1.5. *Случај када су A и B оператори са затвореним сликама, у суштини је исти као случај матричне једначине $AXB = C$, и стога, Пенроузов алгебарски доказ се може применити и на операторски случај. Такоте, са неким модификацијама, може се доказати следећи резултат, за који су заслужни Ариас и Гонзалес [9]:*

Теорема 1.1.6. *[9] Нека су $A \in \mathfrak{B}(\mathcal{H},\mathcal{K})$, $B \in \mathfrak{B}(\mathcal{F},\mathcal{G})$ и $C \in \mathfrak{B}(\mathcal{F},\mathcal{K})$. Ако је $\mathcal{R}(A)$, $\mathcal{R}(B)$ или $\mathcal{R}(C)$ затворен, онда су следећи услови еквивалентни:*

(i) Једначина $AXB = C$ има решење;

(ii) $AA'CB'B = C$ за све унутрашње инверзе, A', B', од A и B, респективно;

(iii) $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ и $\mathcal{R}(C^*) \subseteq \mathcal{R}(B^*)$.

Такође износимо Дагласову лему, коју смемо слободно назвати незамењивим алатом кад год се ради о решавању једначина и инклузије слике оператора.

Теорема 1.1.7 (Дагласова лема [74]). *Нека су A и B ограничени оператори на Хилбертовом простору $\mathcal{H}$. Следећи услови су еквивалентни:*

(i) $\mathcal{R}(A) \subseteq \mathcal{R}(B)$;

(ii) $AA^* \leq \lambda^2 BB^*$ *за неки* $\lambda \geq 0$;

(iii) постоји ограничени оператор C *на* $\mathcal{H}$ *такав да је* $A = BC$.

Штавише, ако важи било који од претходних услова, онда постоји јединствени оператор C такав да је

1. $\|C\|^2 = \inf\{\mu : AA^* \leq \mu BB^*\}$;
2. $\mathcal{N}(A) = \mathcal{N}(C)$;
3. $\mathcal{R}(C) \subseteq \overline{\mathcal{R}(B^*)}$.

Оператор U на Хилбертовом простору $\mathcal{H}$ назива се парцијална изометрија ако постоји затворени подпростор $\mathcal{M}$ такав да важи

$$\|Ux\| = \|x\|$$

за свако $x \in \mathcal{M}$, и $Ux = 0$ за свако $x \in \mathcal{M}^\perp$, где се $\mathcal{M}$ назива иницијални простор оператора U, а $\mathcal{N} = \mathcal{R}(U)$ се назива финални простор оператора U. Пројекције на иницијални и финални простор се називају, респективно, иницијална и финална пројекција оператора U.

Теорема 1.1.8. *[85, стр. 53] Нека је U парцијална изометрија на Хилбертовом простору $\mathcal{H}$ са почетним простором $\mathcal{M}$ и коначним простором $\mathcal{N}$. Тада важи:*

(i) $UP_{\mathcal{M}} = U$ *и* $U^*U = P_{\mathcal{M}}$;

(ii) $\mathcal{N}$ *је затворени подпростор од* $\mathcal{H}$;

(iii) U^* *је парцијална изометрија са почетним простором* $\mathcal{N}$ *и коначним простором* $\mathcal{M}$, *тј.* $U^*P_{\mathcal{N}} = U^*$ *и* $UU^* = P_{\mathcal{N}}$.

За $T \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$, кажемо да је $T = UP$ поларна декомпозиција оператора T ако је P позитиван, U је парцијална изометрија, и $\mathcal{N}(T) = \mathcal{N}(U) = \mathcal{N}(P)$. У том случају, $P = |T|$.

Теорема 1.1.9. *[85, стр. 59] Ако је $T = UP$ поларна декомпозиција оператора T на Хилбертовом простору $\mathcal{H}$, онда је $T^* = U^*|T^*|$ поларна декомпозиција оператора T^*.*

Теорема 1.1.10. *[85, стр. 63] Ако је $T = UP$ поларна декомпозиција оператора T, онда U и P комутирају са A и A^*, где A означава било који оператор који комутира са T и T^*.*

Сваки $T \in \mathfrak{B}(\mathcal{H})$ можемо представити као

$$T = \mathrm{Re}\,(T) + i\,\mathrm{Im}\,(T),$$

где су $\mathrm{Re}\,(T)$ и $\mathrm{Im}\,(T)$ хермитски оператор. Таква декомпозиција је јединствена, и важи

$$\mathrm{Re}\,(T) = \frac{T + T^*}{2}, \quad \mathrm{Im}\,(T) = \frac{T - T^*}{2i}.$$

Оператори $\mathrm{Re}\,(T)$ и $\mathrm{Im}\,(T)$ се зову реални и имагинарни део оператора T, респективно. Ова декомпозиција се зове Декартова декомпозиција оператора T.

На крају, наводимо неке стандардне резултате у вези са нормалним и позитивним операторима.

Теорема 1.1.11. *[5] Нека је $\mathcal{S}$ затворен подпростор од $\mathcal{H}$ и нека $T \in \mathfrak{B}(\mathcal{H})$ има декомпозицију у облику операторске матрице индуковану са $\mathcal{S}$ и дату са* (1.1). *Тада је T позитиван ако и само ако важи:*

(i) $T_{11} \geq 0$;

(ii) $T_{21} = T_{12}^*$;

(iii) $\mathcal{R}(T_{12}) \subseteq \mathcal{R}(T_{11}^{1/2})$;

(iv) $T_{22} = \left((T_{11}^{1/2})^\dagger T_{12}\right)^* (T_{11}^{1/2})^\dagger T_{12} + F$, *где је* $F \geq 0$.

Теорема 1.1.12 (Фулеова теорема [83]). *Нека су T и N ограничени оператори на комплексном Хилбертовом простору такви да је N нормалан. Ако $TN = NT$, онда је $TN^* = N^*T$.*

Теорема 1.1.13 (Фуле-Патнамова теорема [144]). *Нека је $T \in \mathfrak{B}(\mathcal{H})$, и нека су M и N два комутирајућа нормална оператора. Тада важи*

$$TN = MT \iff TN^* = B^*T.$$

Последица 1.1.14. *[83] Ако су M и N комутирајући нормални оператори, онда је и оператор MN нормалан.*

Доказ. Нека су M, N из $\mathfrak{B}(\mathcal{H})$ нормални оператори такви да је $MN = NM$. Користећи Теорему 1.1.12, добијамо да је

$$\begin{aligned}(MN)(MN)^* &= MN(NM)^* = MNM^*N^* \\ &= MM^*NN^* = M^*MN^*N \\ &= M^*N^*MN = (NM)^*MN \\ &= (MN)^*(MN).\end{aligned}$$

Дакле, MN је нормалан оператор. ∎

За више информација о Фуле-Патнамовој теорији, упућујемо читаоца на [130]

Теорема 1.1.15. *[128, Последица 5.1.36] Ако су $A, B \in \mathfrak{B}(\mathcal{H})$ два комутирајућа позитивна оператора, онда је*

$$\sqrt[n]{AB} = \sqrt[n]{A}\sqrt[n]{B},$$

за свако $n \in \mathbb{N}$.

Теорема 1.1.16 (Лефнер-Хајнцова неједнакост [99, 119]). . *Ако су $A, B \in \mathfrak{B}(\mathcal{H})$ позитивни оператори такви да је $B \leq A$ и $p \in [0, 1]$, онда је $B^p \leq A^p$.*

Напомена 1.1.17. *Уопштено, претходна теорема не важи за $p > 1$ (видети, на пример, [129, стр. 55]). Метутим, ако A и B комутују, и $p \in \mathbb{N}$, онда $B \leq A$ повлачи да је $B^p \leq A^p$. Наиме, пошто A и B комутују, можемо написати*

$$A^p - B^p = (A - B)(A^{p-1} + A^{p-2}B + \ldots + B^{p-1}).$$

Обзиром на то да A и B комутују и да је $B \leq A$, имамо да су $A - B$ и $A^{p-1} + A^{p-2}B + \ldots + B^{p-1}$ два комутирајућа позитивна оператора, и стога је $A^p - B^p$ позитиван. Дакле, мора бити $B^p \leq A^p$.

1.2 Уопштења нормалних оператора

У теорији оператора постоји много општијих класа од класе нормалних оператора. Једна од најважнијих је класа субнормалних оператора. Субнормални оператори су ограничени линеарни оператори на Хилбертовом простору који се дефинишу слабљењем услова за нормалне операторе. Појам субнормалних оператора увео је Пол Р. Халмос [92], истовремено са дефинисањем хипонормалних оператора, још шире класе оператора. На то га је навело истраживање својстава оператора померaја, вероватно најбоље схваћеног оператора који није нормалан. У овом одељку представићемо нека основна својства субнормалних оператора и видети како су различите класе уопштења нормалних оператора повезане међусобно.

1.2.1 Субнормални оператори

Почињемо са дефиницијом субнормалности оператора.

Дефиниција 1.2.1. Оператор T на Хилбертовом простору $\mathcal{H}$ је *субнормалан* ако постоји Хилбертов простор $\mathcal{K}$ који садржи $\mathcal{H}$ и нормалан оператор $N : \mathcal{K} \mapsto \mathcal{K}$ такав да важи $N(\mathcal{H}) \subseteq \mathcal{H}$ и $Nx = Tx$ за сваки $x \in \mathcal{H}$.

Другим речима, оператор је субнормалан ако има нормалну екстензију (продужење), или еквивалентно, ако постоји Хилбертов простор $\mathcal{L}$ и нормалан оператор $N \in \mathcal{B}(\mathcal{H} \oplus \mathcal{L})$ такав да је

$$N = \begin{bmatrix} T & * \\ 0 & * \end{bmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{L} \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ \mathcal{L} \end{pmatrix}.$$

У раду [92], оператори који задовољавају Дефиницију 1.2.1 називају се *потпуно субнормални*. Термин *субнормалан*, како се користи овде, први пут је уведен у раду [93].

Као што је раније напоменуто, изучавање нормалних оператора има изузетан успех. Један од главних разлога је Спектрална теорема који важи за ову класу оператора. Такође, природно је покушати разумети структуру што више оператори који не припадају овој класи. Будући да се концепт субнормалности може посматрати као довољно близак нормалности, разумно је очекивати да теорија субнормалних оператора има потенцијал да прати сличан пут. Наравно, многа питања и концепти које се односе на субнормалне операторе, инспирисане су питањима која се односе на нормалне операторе и већ су добила одговоре. На пример, у раду [27] је показано да сваки субнормални оператор има нетривијалан инваријантан подпростор. Међутим, постоје неке битне разлике између ове две наведене класе, што је довело до тога да теорија субнормалних оператора прати сопствени пут. Наиме, у суштини, стубови теорије нормалних оператора леже на теорији мера и Спектралној теореми, док се теорија субнормалних оператора заснива на теорији аналитичких функција.

У литератури постоји много карактеризација субнормалних оператора. Видети, на пример, [92, 25, 78, 28].

Још једна елегантна карактеризација субнормалности која наглашава њену блискост са концептом нормалности у тополошком смислу долази од Бишопа [21].

Теорема 1.2.1. *[21] Ако је $T \in \mathfrak{B}(\mathcal{H})$, онда су следећи услови еквивалентни:*

(i) *T је субнормалан;*

(ii) *T је* SOT*-лимит низа нормалних оператора;*

(iii) *T припада* SOT*-затворењу скупа нормалних оператора.*

Ако је $T \in \mathfrak{B}(\mathcal{H})$ субнормалан оператор и $N \in \mathfrak{B}(\mathcal{K})$ је нормалан, онда је, очигледно, $S = T \oplus N$ такође субнормалан. Понекад је од интереса истражити само "не-нормални део" оператора S. Тачније, имамо следеће:

Теорема 1.2.2. *[45, Став 2.1] Ако је $T \in \mathfrak{B}(\mathcal{H})$, онда постоји подпростор $\mathcal{H}_0$ који редукује T такав да је*

$$T = \begin{bmatrix} T_n & 0 \\ 0 & T_p \end{bmatrix} : \begin{pmatrix} \mathcal{H}_0 \\ \mathcal{H}_0^\perp \end{pmatrix} \to \begin{pmatrix} \mathcal{H}_0 \\ \mathcal{H}_0^\perp \end{pmatrix},$$

где је T_n нормалан а T_p чист оператор.

Дефиниција 1.2.2. Оператор $T \in \mathfrak{B}(\mathcal{H})$ је *чист* ако нема нетривијалан редукујући подпростор $\mathcal{M}$ такав да је $T|_{\mathcal{M}}$ нормалан.

Напоменимо да је T чист ако је подпростор $\mathcal{H}_0$ из Теореме 1.2.2 уствари $\{0\}$. У наставку, редукујући простор $\mathcal{H}_0$ и његов ортогонални комплемент ћемо означити са $\mathcal{H}_n(T)$ и $\mathcal{H}_p(T)$, респективно, док T_p и T_n ћемо назвати, респективно, *чист* и *нормалан део* од T. Декомпозиција $T = T_n \oplus T_p$ ће се једноставно звати *чисто-нормална декомпозиција* од T.

Нормална екстензија субнормалног оператора никада није јединствена. Заправо, ако је N нормална екстензија од T, и M је било који нормалан оператор, онда је $M \oplus N$ такође нормална екстензија од T. Стога има смисла увести следећу дефиницију.

Дефиниција 1.2.3. Ако је T субнормалан оператор који делује на $\mathcal{H}$, и N је нормална екстензија од T која делује на $\mathcal{K} \supseteq \mathcal{H}$, кажемо да је N *минимална нормална екстензија* од ако $\mathcal{K}$ нема ни један прави подпростор који редукује N и садржи $\mathcal{H}$.

Следећа теорема показује да је минималне нормалне екстензије "јединствене". Стога можемо легитимно говорити о *минималној нормалној екстензији* субнормалног оператора T.

Теорема 1.2.3. *[45, Последица 2.7] Ако је $T \in \mathfrak{B}(\mathcal{H})$ субнормалан оператор и N_1 и N_2 су минималне нормалне екстензије од T, онда су N_1 и N_2 унитарно еквивалентне.*

Кажемо да су оператори $A \in \mathfrak{B}(\mathcal{H})$ и $B \in \mathfrak{B}(\mathcal{K})$ унитарно еквивалентни ако постоји унитарна трансформација $U \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ ($U^*U = I_{\mathcal{H}}, UU^* = I_{\mathcal{K}}$) таква да је $A = U^*BU$. Још један веома користан резултат повезује концепт минималне нормалне екстензије и чистоћу оператора.

Теорема 1.2.4. *[45, Став 2.10] Нека је $T \in \mathfrak{B}(\mathcal{H})$ субнормалан оператор и нека је*

$$N = \begin{bmatrix} T & A \\ 0 & B^* \end{bmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^{\perp} \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^{\perp} \end{pmatrix} \tag{1.2}$$

нормална екстензија од T. Следећи услови су еквивалентни:

(i) *T је чист;*

(ii) *N^* је минимална нормална екстензија од B;*

(iii) *Најмањи подпростор $\mathcal{H}$ који редукује T и садржи $\mathcal{R}(A)$ је $\mathcal{H}$;*

(iv) *Не постоји ненула пројекција P на $\mathcal{H}$ таква да је $PT = TP$ и $PA = 0$.*

Претпоставимо да је $T \in \mathcal{B}(\mathcal{H})$ субнормалан оператор и нека је $N \in \mathcal{B}(\mathcal{K})$ његова минимална нормална екстензија дата са (1.2). Ако је T чист субнормалан оператор, онда је S јединствен до на унитарну еквиваленцију и назива се дуални оператор оператору T (видети [44]). Кажемо да је T самодуалан ако је унитарно еквивалентан свом дуалу S. Дуал субнормалног оператора такође је истражен у радовима [131] и [180]. Имамо следеће карактеризације самодуалних оператора, које ће бити коришћене у Глави 4.

Теорема 1.2.5. *[131] Нека је T чист оператор на Хилбертовом простору $\mathcal{H}$. Тада је T самодуални субнормални оператор ако и само ако постоји нормални оператор A на $\mathcal{H}$ такав да важи*

$$[T^*, T] = AA^* \quad \text{и} \quad AT = T^*A.$$

Теорема 1.2.6. *[152] Нека је $T \in \mathfrak{B}(\mathcal{H})$ чист оператор. Тада је T самодуални субнормални оператор ако и само ако постоји оператор $A \in \mathfrak{B}(\mathcal{H})$ такав да је операторска матрица $\begin{bmatrix} T & A \\ 0 & T^* \end{bmatrix}$ на $\mathcal{H} \oplus \mathcal{H}$ нормална.*

1.2.2 Квазинормални оператори

Класа квазинормалних оператора је представљена у раду [26].

Дефиниција 1.2.4. Оператор T на Хилбертовом простору $\mathcal{H}$ је *квазинормалан* ако комутира са T^*T, тј. $TT^*T = T^*T^2$.

Теорема 1.2.7. *[45, Став 1.6] Ако је $T = U|T|$ поларна декомпозиција оператора T, онда је T квазинормалан ако и само ако U и $|T|$ комутирају.*

Очигледно, сваки нормалан оператор је квазинормалан, и класа квазинормалних оператора је тачно подскуп скупа $\mathfrak{B}(\mathcal{H})$ чији елементи имају комутирајуће поларне декомпозиције. Стога је класа квазинормалних оператора интересантна сама по себи. Исто тако, она има многе примене у теорији субнормалних оператора, јер представља "везу" између нормалности и субнормалности. Другим речима, имамо следеће:

Теорема 1.2.8. *Сваки квазинормални оператор је субнормалан.*

Теорема 1.2.9. *[131] Сваки чист квазинормалан оператор је самодуалан субнормалан оператор.*

Следећа лема, која се може наћи у [51], даје нам потребне и довољне услове за квазинормалност (и нормалност) субнормалног оператора.

Лема 1.2.10. *[51] Нека је $T \in \mathfrak{B}(\mathcal{H})$ субнормални оператор са нормалном екстензијом*

$$N = \begin{bmatrix} T & A \\ 0 & B^* \end{bmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{L} \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{H} \\ \mathcal{L} \end{pmatrix}.$$

Тада је T квазинормалан ако и само ако је $A^*T = 0$, и нормалан ако и само ако је $A = 0$.

Следећа лема, као што ћемо видети, показала се као много кориснија за доказивање разних резултата у овој дисертацији који се односе на квазинормалне операторе. Лема се први пут појавила у [30] (погледати [47]). Овде је представљамо у мало друкчијем облику користећи технику доказа базирану на Леми 1.1.2.

Лема 1.2.11. *[45, Лема 3.1] Нека је $T \in \mathfrak{B}(\mathcal{H})$ субнормалан оператор. Ако је N нормална екстензија за T, онда је T квазинормалан ако и само ако је $\mathcal{H}$ инваријантан за N^*N.*

Доказ. Нека је N нормална екстензија за T на $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$ дата са

$$N = \begin{bmatrix} T & A \\ 0 & B^* \end{bmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{pmatrix},$$

и нека је $P \in \mathfrak{B}(\mathcal{K})$ ортогонална пројекција на $\mathcal{H}$. уочимо да је $\mathcal{H}$ инваријантан за N^*N ако и само ако је $PN^*N = N^*NP$ (Лема 1.1.2). Директна рачунска провера показује да је

$$N^*NP = \begin{bmatrix} T^*T & 0 \\ A^*T & 0 \end{bmatrix} \quad \text{и} \quad PN^*N = \begin{bmatrix} T^*T & TA^* \\ 0 & 0 \end{bmatrix}.$$

Дакле, $PN^*N = N^*NP$ ако и само ако је $A^*T = 0$. Директно из Леме 1.2.10 сада следи закључак. ■

1.2.3 Хипонормални оператори

Како је већ споменуто, концепт хипонормалности је уведен у раду [92], док се термин "хипонормалан" први пут јавља у раду [19].

Дефиниција 1.2.5. Оператор T на Хилбертовом простору $\mathcal{H}$ је *хипонормалан* ако је $TT^* \leq T^*T$.

Класа хипонормалних оператора је шира од класе субнормалних оператора, као што показује следећа теорема.

Теорема 1.2.12. *Сваки субнормални оператор је хипонормалан.*

Директно из дефиниције, оператор на Хилбертовом простору $\mathcal{H}$ је хипонормалан ако и само ако је $\|x\| \geq \|^*x\|$ за свако $x \in \mathcal{H}$. Такође, користећи Теорему 1.1.7, следи да је $\mathcal{R}(T) \subseteq \mathcal{R}(T^*)$ за сваки хипонормални оператор $T \in \mathfrak{B}(\mathcal{H})$. Ако је T^* хипонормалан, онда кажемо да је *кохипонормалан.* Оператори који су хипонормални или кохипонормални се зову *семинормални.* Теорија семинормалних оператора је обимна и врло развијена област. Више информација о овој теми се може наћи у радовима [123], [43] и [176].

Многа својстава која важе за нормалне операторе важе и у аналогној форми за хипонормалне операторе, као што показују следећи резултати.

Теорема 1.2.13. *[45, Став 4.4] Нека је $T \in \mathfrak{B}(\mathcal{H})$ хипонормалан оператор.*

(a) *Ако је T инвертибилан, онда је T^{-1} хипонормалан.*

(b) *Ако је $\lambda \in \mathbb{C}$, онда је $T - \lambda$ хипонормалан.*

(c) *Ако су* $\lambda \in \sigma_p(T)$ *и* $x \in \mathcal{H}$ *такви да је* $Tx = \lambda x$*, онда је* $T^*x = \overline{\lambda}x$.

(d) *Ако су* x *и* y *сопствени вектори који одговарају различитим сопственим вредностима оператора* T*, онда је* $x \perp y$.

Теорема 1.2.14. *[155] Ако је* A *хипонормалан, онда је* $\|A^n\| = \|A\|^n$*, и последично,* $\|A\| = r(A)$.

Важно је напоменути да је теорија хипонормалних оператора (и стога субнормалних оператора) строго теорија са бесконачно димензионалним просторима. Тачније, класа хипонормалних оператора се поклапа са класом нормалних оператора, ако је Хилбертов простор коначне димензије. Наиме, ако је $\mathcal{H}$ коначно-димензионалан простор и $T \in \mathfrak{B}(\mathcal{H})$ је хипонормалан, онда је $T^*T - TT^* \geq 0$, док је траг матрице $T^*T - TT^*$ је 0. Стога, $T^*T = TT^*$, односно T је нормалан. Следећа теорема Патнама [147] показује да имамо још јачи резултат.

Теорема 1.2.15. *[147] Ако је* $T \in \mathfrak{B}(\mathcal{H})$ *хипонормалан, онда важи*

$$\|[T^*, T]\| \leq \frac{1}{\pi}\,\mathrm{Area}(\sigma(T)).$$

1.2.4 p-хипонормални оператори и Алутгеова трансформација

У овом одељку кратко помињемо друге генерализације нормалних оператора и везе између њих.

Дефиниција 1.2.6. Оператор T на Хилбертовом простору $\mathcal{H}$ се назива *p-хипонормалан оператор* ако важи $(TT^*)^p \leq (T^*T)^p$ за неко $p \in (0, 1]$.

p-хипонормалан оператор T се назива *полухипонормални* ако је $p = \frac{1}{2}$, а јасно је да је T хипонормалан ако је $p = 1$. Користећи Теорему 1.1.16, напомињемо да сваки хипонормалан оператор мора бити p-хипонормалан за све $p \in (0, 1]$. Општије, ако је $0 < q \leq p \leq 1$ и $T \in \mathfrak{B}(\mathcal{H})$ је p-хипонормалан, онда је такође q-хипонормалан. Стога је класа p-хипонормалних оператора дефинисана као проширење хипонормалних оператора у [176], и она је истраживана од стране многих аутора од тада. Видети, на пример, [1, 2, 175].

Обједињавајући претходна разматрања, добијамо следећи низ импликација:

$$\begin{aligned} \text{нормалан } &\Rightarrow \text{ квазинормалан } \Rightarrow \text{ субнормалан} \\ &\Rightarrow \text{ хипонормалан } \Rightarrow p\text{-хипонормалан.} \end{aligned}$$

У тесној вези са p-хипонормалним операторима су *Алутгеова трансформација* и *Дугалова трансформација.* Алутгеова трансформација $\widetilde{T}$ оператора $T \in \mathfrak{B}(\mathcal{H})$ са поларном декомпозицијом $T = U|T|$ дефинисана је као $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$, док је

Дугалова трансформација $\widehat{T}$ од T дата са $\widehat{T} = |T|U$. За више детаља о Алутгеовој и Дугаловој трансформацији, видети, на пример, [1, 7, 39, 81, 105].

Алутгеова трансформација $\widetilde{T}$ оператора $T \in \mathfrak{B}(\mathcal{H})$ се показала као веома интересантна и корисна идеја у изучавању линеарних оператора. На пример, имамо да је $\sigma(\widetilde{T}) = \sigma(T)$. Ово произилази из чињенице да је $\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}$ за било које $A, B \in \mathfrak{B}(\mathcal{H})$. Можда још занимљивија и изненађујућа чињеница је следећа:

Теорема 1.2.16. *[1] Нека је $T = U|T|$ p-хипонормалан за неки $0 < p \leq 1$ и нека је U унитаран. Тада*

(i) *$\widetilde{T}$ је $(p + \frac{1}{2})$-хипонормалан ако је $0 < p < \frac{1}{2}$.*

(ii) *$\widetilde{T}$ је хипонормалан ако је $\frac{1}{2} \leq p < 1$.*

Дакле, Алутгеова трансформација "шаље" p-хипонормалан оператор у класу мању од почетне класе p-хипонормалних оператора. Ово је један од главних разлога примењивости Алутгеове трансформације у различитим областима теорије оператора.

1.3 Субнормалне и квазинормалне n-торке

Нека је $n \in \mathbb{N}$. Ако су $T_i \in \mathfrak{B}(\mathcal{H})$, $i = \overline{1, n}$, онда $\mathbf{T} = (T_1, \ldots, T_n) \in \mathfrak{B}(\mathcal{H})^n$ означава n-торку оператора који делују на $\mathcal{H}$. Под $\mathbf{T}^*$ подразумевамо n-торку оператора $\mathbf{T}^* = (T_1^*, \ldots, T_n^*) \in \mathfrak{B}(\mathcal{H})^n$. n-торка оператора $\mathbf{T} = (T_1, \ldots, T_n) \in \mathfrak{B}(\mathcal{H})^n$ се сматра *комутирајућом* ако је $T_iT_j = T_jT_i$, за све $i, j \in \{1, \ldots, n\}$.

Многи концепти и идеје из теорије оператора једне променљиве пренети су у поставку теорије оператора са више променљивих. На пример, класична операторска норма има свој вишедимензионални аналог у облику *заједничке операторске норме* и *еуклидске операторске норме.* Заједничка операторска норма за n-торку $\mathbf{T} = (T_1, \ldots, T_n)$ уведена је у [42] и дефинисана је као

$$\|\mathbf{T}\| = \sup \left\{ \left(\sum_{k=1}^{n} \|T_k x\|^2 \right)^{1/2} : x \in \mathcal{H}, \|x\| = 1 \right\},$$

док се еуклидска операторска норма први пут појављује у [138] и дефинисана је као

$$\|\mathbf{T}\|^e = \left(\sum_{k=1}^{n} \|T_k\|^2 \right)^{1/2}.$$

Појмови нумеричког ранга и нумеричког радијуса следе сличан пут. Наиме, *заједнички нумерички ранг* за n-торку $\mathbf{T} = (T_1, \ldots, T_n)$ дефинисан је као

$$\mathcal{W}(\mathbf{T}) = \{(\langle T_1x, x\rangle, \ldots, \langle T_nx, x\rangle) : x \in \mathcal{H}, \|x\| = 1\},$$

а *заједнички нумерички радијус* (такође познат и као *еуклидски операторски радијус*) за $\mathbf{T}$ дефинисан је као

$$\omega(\mathbf{T}) = \sup\left\{\left(\sum_{k=1}^{n}|\langle T_k x, x\rangle|^2\right)^{1/2} : x \in \mathcal{H},\ \|x\| = 1\right\}.$$

Појам заједничког нумеричког ранга првобитно је истраживао Халмос [94, Проблем 166], док је $\omega(\mathbf{T})$ изучаван у [138]. За више информација о овим концептима, саветујемо читаоцу да се консултује са [42, 65, 75, 107, 133, 134, 142, 157].

За разлику од теорије оператора са једном промењљивом, спектар n-торке оператора има више дефиниција. Видети, на пример, [8], [66] и [165]. У овој дисертацији, ограничићемо се на Тејлорову инвертибилност само за пар оператора. Она се дефинише на следећи начин: нека је $\mathbf{T} = (T_1, T_2)$ комутирајући пар. Посматрајмо Кошулов комплекс $\mathcal{K}(\mathbf{T}, \mathcal{H})$ придружен оператору $\mathbf{T}$ и простору $\mathcal{H}$:

$$\mathcal{K}(\mathbf{T}, \mathcal{H}) : \quad 0 \longrightarrow \mathcal{H} \xrightarrow{\mathbf{T}} \mathcal{H} \oplus \mathcal{H} \overset{(-T_2\ T_1)}{\longrightarrow} \mathcal{H} \longrightarrow 0,$$

где је $\mathbf{T} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$. Тада се $\mathbf{T}$ сматра *инвертибилним у Тејлоровом смислу* ако је његов придружени Кошулов комплекс $\mathcal{K}(\mathbf{T}, \mathcal{H})$ егзактан. Дефинишемо *Тејлоров спектар* $\sigma_T(\mathbf{T})$ на следећи начин:

$$\sigma_T(\mathbf{T}) = \left\{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : \mathcal{K}((T_1 - \lambda_1, T_2 - \lambda_2), \mathcal{H}) \text{ није тачан}\right\}.$$

За више информација о Тејлоровој инвертибилности и Кошуловим комплексима, саветујемо читаоцу да се консултује са [97, 98, 110, 132, 165, 166].

За $S, T \in \mathfrak{B}(\mathcal{H})$, нека је $[S, T] = ST - TS$. Оператор $[S, T]$ се назива *комутатор* оператора S и T. Ако је $S = T^*$, онда се $[T^*, T]$ назива *самокомутатор* оператора T. Аналогно, ако је $\mathbf{T} = (T_1, \ldots, T_n) \in \mathfrak{B}(\mathcal{H})^n$ n-торка оператора, са $[\mathbf{T}^*, \mathbf{T}]$ означавамо самокомутатор n-торке $\mathbf{T}$ и дефинишемо га као

$$[\mathbf{T}^*, \mathbf{T}]_{i,j} = [T_j^*, T_i] = T_j^* T_i - T_i T_j^*,$$

за све $i, j \in \{1, \ldots, n\}$. Кажемо да је n-торка $\mathbf{T} = (T_1, \ldots, T_n)$ оператора на $\mathcal{H}$ (заједнички) *хипонормална* ако је операторска матрица

$$[\mathbf{T}^*, \mathbf{T}] := \begin{bmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{bmatrix}$$

позитивна на директној суми n копија $\mathcal{H}$ (упореди са [10, 48, 52]). n-торка $\mathbf{T}$ се назива *нормалном* ако је $\mathbf{T}$ комутирајућа и сваки T_i је нормалан, и *субнормалном* ако постоји

Хилбертов простор $\mathcal{K}$ који садржи $\mathcal{H}$ и нормална n-торка $\mathbf{N} = (N_1, \dots, N_n) \in \mathfrak{B}(\mathcal{K})^n$ таква да је $N_i(\mathcal{H}) \subseteq \mathcal{H}$ и $N_i x = T_i x$ за свако $x \in \mathcal{H}$ и свако $i \in \{1, \dots, n\}$. За $i, j, k \in \{1, 2, \dots, n\}$, $\mathbf{T}$ се назива *матрично-квазинормалном* ако сваки T_i комутира са сваким $T_j^* T_k$, $\mathbf{T}$ је (заједнички) *квазинормална* ако сваки T_i комутира са сваким $T_j^* T_j$, и *сферично-квазинормална* ако сваки T_i комутира са $\sum_{j=1}^n T_j^* T_j$. Као што је показано у [11] и [87], имамо да важи

нормална $\Rightarrow$ матрично квазинормална $\Rightarrow$ (заједнички) квазинормална
$\Rightarrow$ сферично квазинормална $\Rightarrow$ субнормална
$\Rightarrow$ (заједнички) хипонормална

С друге стране, резултати из [55] и [87] показују да супротне импликације не важе у општем случају.

За $T_1, T_2 \in \mathfrak{B}(\mathcal{H})$, посматрајмо пар $\mathbf{T} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$ као оператор из $\mathcal{H}$ у $\mathcal{H} \oplus \mathcal{H}$, односно,

$$\mathbf{T} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} : \mathcal{H} \to \begin{array}{c} \mathcal{H} \\ \oplus \\ \mathcal{H} \end{array}.$$

Дефинишемо *(канонску) сферичну поларну декомпозицију* за $\mathbf{T}$ (погледати [54], [55], [109]) као

$$\mathbf{T} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} P = \begin{pmatrix} V_1 P \\ V_2 P \end{pmatrix} = \mathbf{V} P,$$

где је $P = (\mathbf{T}^* \mathbf{T})^{1/2} = \sqrt{T_1^* T_1 + T_2^* T_2}$ позитиван оператор на $\mathcal{H}$, и

$$\mathbf{V} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} : \mathcal{H} \to \begin{array}{c} \mathcal{H} \\ \oplus \\ \mathcal{H} \end{array},$$

је сферична парцијална изометрија из $\mathcal{H}$ у $\mathcal{H} \oplus \mathcal{H}$. Тада, $\mathbf{V}^* \mathbf{V} = V_1^* V_1 + V_2^* V_2$ представља (ортогоналну) пројекцију на почетни простор парцијалне изометрије $\mathbf{V}$ који је дат са

$$\mathcal{N}(\mathbf{T})^\perp = (\mathcal{N}(T_1) \cap \mathcal{N}(T_2))^\perp = \mathcal{N}(P)^\perp = (\mathcal{N}(V_1) \cap \mathcal{N}(V_2))^\perp.$$

У односу на поларну декомпозицију, сферично-квазинормални парови се могу окарактерисати на следећи начин:

Теорема 1.3.1. *[56] Нека је $\mathbf{T} = (V_1 P, V_2 P)$ поларна декомпозиција n-торке $\mathbf{T}$. Тада је $\mathbf{T}$ сферично-квазинормалан ако и само ако је $V_i P = P V_i$, $i = 1, 2$.*

На крају, подсетимо се класе дводимензионалних тежинских оператора померајa. Посматрајмо двоструко индексиране ненегативне ограничене низове $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in l^{\infty}(\mathbb{Z}_{+}^{2})$, где је $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}_{+}^{2}$, и нека је $l^2(\mathbb{Z}_{+}^{2})$ Хилбертов простор квадратно-сумабилних комплексних низова индексираних са $\mathbb{Z}_{+}^{2}$. Дефинишемо *дводимензионалних тежински оператор померајa* $\mathbf{W}_{(\alpha,\beta)} = (T_1, T_2)$ на следећи начин:

$$T_1 e_{(k_1,k_2)} = \alpha_{(k_1,k_2)} e_{(k_1+1,k_2)}$$

и

$$T_2 e_{(k_1,k_2)} = \beta_{(k_1,k_2)} e_{(k_1,k_2+1)},$$

где је $\{e_{(k,l)}\}_{k,l=0}^{\infty}$ канонска ортонормирана база у $l^2(\mathbb{Z}_{+}^{2})$. За све $(k_1, k_2) \in \mathbb{Z}_{+}^{2}$, лако је видети да

$$T_1T_2 = T_2T_1 \Longleftrightarrow \beta_{(k_1+1,k_2)}\alpha_{(k_1,k_2)} = \alpha_{(k_1,k_2+1)}\beta_{(k_1,k_2)}.$$

За основна својства дводимензионалних тежинских оператора померајa $\mathbf{W}_{(\alpha,\beta)}$, видети [49] и [53].

Глава 2

Субнормални фактори квазинормалних оператора

У овом поглављу разматрамо питање да ли је субнормалност оператора и квазинормалност његовог квадрата довољна за квазинормалност самог оператора. Штавише, у Одељку 2.2 биће показано да субнормални n-ти корени квазинормалног оператора такође морају бити квазинормални. Додатно, дајемо неке довољне услове под којима су матрично и сферично квазинормални парови оператора еквивалентни матричној и сферичној квазинормалности њихових n-тих степена. У Одељку 2.3 бавимо се питањем налажења услова под којима субнормални оператори морају бити квазинормални уколико је њихов производ квазинормалан. Додатно, дајемо довољне услове под којима квазинормални (субнормални) оператори морају бити нормални уколико је њихов производ нормалан. Другим речима, налазимо довољне услове за обрат Фулеове теореме, правећи при томе везу са вишедимензионалном теоријом субнормалних оператора.

2.1 Проблем корена квазинормалних оператора

У недавном раду [51], Р. Е. Курто, С. Х. Ли и Џ. Јун, делимично подстакнути резултатима својих претходних радова [49] и [50], поставили су следеће питање:

Проблем 2.1.1. *Нека је T субнормалан оператор такав да је T^2 квазинормалан. Да ли из тога следи да је T квазинормалан?*

С додатном претпоставком леве инвертибилности, показали су да лево инвертибилан субнормалан оператор T чији је квадрат квазинормалан мора бити квазинормалан (видети Теорему 2.1.5 испод). Остало је отворено питање да ли је ово важи без претпоставке о левој инвертибилности све до објављивања рада [141]. Штавише, аутори су доказали још јачи резултат:

Теорема 2.1.2. *[141] Нека је $T \in \mathfrak{B}(\mathcal{H})$ субнормалан оператор такав да је T^n квазинормалан за неко $n \in \mathbb{N}$. Тада је T квазинормалан.*

Доказ се заснива на теорији операторско-монотоних функција и Хансеновој неједнакости. Конкретно, Теорема 1.1.1, Теорема 1.1.16, и следеће теореме биле су кључне за доказ.

Теорема 2.1.3. *[78] Нека је T ограничени оператор на $\mathcal{H}$. Тада су следећи услови еквивалентни:*

(i) T је квазинормалан;

(ii) $(T^*)^nT^n = (T^*T)^n$, $n \in \mathbb{N}$;

(iii) постоји (јединствена) спектрална Борелова мера E на $\mathbb{R}_+$ таква да

$$(T^*)^nT^n = \int_{\mathbb{R}_+} x^n \, E(dx), \quad n \in \mathbb{Z}_+.$$

Теорема 2.1.4 (Хансенова неједнакост [96, 168]). *Нека је $A \in \mathfrak{B}(\mathcal{H})$ позитиван оператор, $T \in \mathfrak{B}(\mathcal{H})$ контракција и $f : [0,\infty) \to \mathbb{R}$ непрекидна операторско-монотона функција таква да је $f(0) \geq 0$. Тада важи*

$$T^* f(A)T \leq f(T^*AT).$$

Штавише, ако f није афина функција и T је ортогонална пројекција таква да $T \neq I_{\mathcal{H}}$, онда једнакост важи ако и само ако $TA = AT$ и $f(0) = 0$.

У литератури су познате сличне карактеристике које се јављају у Проблему 2.1.1 за друге класе оператора. На пример, хипонормални корени нормалних оператора су нормални (видети [155, Теорема 5]). Аутор је користио технику засновану на Спектралној теореми. Један једноставнији доказ може се наћи у [3]. Конкретно, аутори су показали да за било који p-хипонормалан оператор $T \in \mathfrak{B}(\mathcal{H})$ и било које $n \in \mathbb{N}$ важи:

$$((T^n)^*T^n)^{p/n} \geq (T^*T)^p \geq (TT^*)^p \geq (T^n(T^n)^*)^{p/n}.$$

Ако се додатно претпостави да је T^n нормалан за неко $n \in \mathbb{N}$, онда имамо:

$$((T^n)^*T^n)^{p/n} = (T^*T)^p = (TT^*)^p = (T^n(T^n)^*)^{p/n},$$

и стога T мора бити нормалан. Још једно проширење [155, Теорема 5] може се наћи у [6]. Међутим, ако заменимо нормалност оператора неком слабијом претпоставком, аналогни закључци неће важити увек. Постоји оператор T који је хипонормалан и T^n је субнормалан, али T није субнормалан (видети [156]).

Мотивисани оваквим врстама проблема, у овом одељку такође разматрамо проблеме када матрицијална (сферична) квазинормалност $\mathbf{T}^{(n,n)} := (T_1^n, T_2^n)$ подразумева матрицијалну (сферичну) квазинормалност $\mathbf{T} = (T_1, T_2)$.

Вратимо се накратко на Проблем 2.1.1. Да би одговорили на постављено питање, аутори у раду [51] прво су доказали Лему 1.2.10.

Иако можда изгледа као практичан алат за одређивање да ли је неки оператор квазинормалан или не, овај приступ не даје одговор на Проблем 2.1.1 без додатних претпоставки о T и постаје још непрактичнији ако заменимо квадрат оператора његовим произвољним степеном.

Ипак, користећи поменуту лему, поменути аутори доказали су следећи резултат:

Теорема 2.1.5. *[51] Нека је $T \in \mathfrak{B}(\mathcal{H})$ субнормалан оператор и претпоставимо да је T^2 квазинормалан. Ако је T ограничен одоздо, онда је T квазинормалан.*

Теорема 2.1.5 и Лема 1.2.10 представљају основу за доказивање вишедимензионалних аналога ових резултата. Наиме, за субнормалан пар $\mathbf{T} = (T_1, T_2)$ са нормалном екстензијом $\mathbf{N} = (N_1, N_2)$, где је

$$N_i = \begin{bmatrix} T_i & A_i \\ 0 & B_i^* \end{bmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{pmatrix},$$

аутори су доказали следеће резултате:

Последица 2.1.6. *[51] Нека је $\mathbf{T}$ субнормалан и претпоставимо да је T_i ограничен одозго и T_i^2 квазинормалан, $i = 1, 2$. Тада је $\mathbf{T}$ сферично-квазинормалан.*

Теорема 2.1.7. *[51] Нека је $\mathbf{T}$ субнормалан, са нормалном екстензијом $\mathbf{N}$. Тада је $\mathbf{T}$ сферично-квазинормалан ако и само ако је $A_1^*T_1 + A_2^*T_2 = 0$.*

Теорема 2.1.8. *[51] Нека је $\mathbf{T}$ субнормалан, са нормалном екстензијом $\mathbf{N}$. Тада је $\mathbf{T}$ (заједнички) квазинормалан ако и само ако је $A_i^*T_j = 0, i, j = 1, 2$.*

Последица 2.1.9. *[51] Нека је $\mathbf{T}$ субнормалан пар са нормалном екстензијом $\mathbf{N}$. Тада је $\mathbf{T}$ матрично-квазинормалан ако и само ако је $A_iA_j^*T_k = 0, i, j, k = 1, 2$.*

Овде користимо прилику да нагласимо да је Теорема 2.1.8 уствари нетачна. Наиме, ако је $A_j^*T_k = 0, j, k = 1, 2$, онда очигледно важи $A_iA_j^*T_k = 0, i, j, k = 1, 2$. Ако би Теорема 2.1.8 била тачна, онда ово имплицира да сваки (заједнички) квазинормална n-торка мора бити матрично-квазинормална. Међутим, као што је поменуто раније, резултати у [55] и [87] показују да то није случај. Исправка и друге "неочекиване" импликације ове грешке биће представљене касније у одељку 2.2.2.

2.2 Субнормални n-ти корени квазинормалних оператора

У овом одељку даћемо одговор на Проблем 2.1.1 користећи елементарну технику. Између осталог, такође показујемо да можемо ослабити услов у дефиницији матрично-квазинормалних n-торки и дајемо исправку за Теорему 2.1.8.

2.2.1 Једнодимензионални случај

Доказ Теореме 2.1.2. Нека је $N \in \mathfrak{B}(\mathcal{K})$ нормална екстензија за T, где је $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^{\perp}$, и нека је $P \in \mathfrak{B}(\mathcal{K})$ ортогонална пројекција на $\mathcal{H}$. Тада, N^n је нормална екстензија за T^n, и како је $\mathcal{H}$ инваријантан за $(N^n)^* N^n = (N^* N)^n$ (Лема 1.2.11), произилази да P комутира са $(N^* N)^n$ (Лема 1.1.2). Дакле, P такође комутира са $N^* N$, према Теореми 1.1.1. Стога, $\mathcal{H}$ је инваријантан за $N^* N$, и примењујући опет Лему 1.2.11, закључујемо да је T квазинормалан. ■

Следећа лема је општија верзија леме [51, Тврђење 2.4].

Последица 2.2.1. *Нека је $T \in \mathfrak{B}(\mathcal{H})$ субнормалан оператор такав да је T^n чист квазинормалан за неко $n \in \mathbb{N}$. Тада је T чист квазинормалан.*

Доказ. Квазинормалност следи из Теореме 2.1.2. Ако T није чист, онда постоји ненула редукујући подпростор $\mathcal{M}$ од $\mathcal{H}$ такав да је $P^{cr}_{\mathcal{M}} T \restriction_{\mathcal{M}}$ нормалан. Обзиром на то да је $P^{cr}_{\mathcal{M}} T^n \restriction_{\mathcal{M}} = (P^{cr}_{\mathcal{M}} T \restriction_{\mathcal{M}})^n$ такође нормалан, T^n не може бити чист, што је у супротности са претпоставкама. Дакле, T мора бити чист. ■

2.2.2 Вишедимензионални случај

Сада можемо пребацити фокус на вишедимензионални случај. Иако представљамо наше резултате за комутирајуће парове оператора, читалац ће лако уочити да иста (или слична) тврђења важе и за комутирајуће n-торке оператора, када је $n > 2$.

Теорема 2.1.2 нам омогућава да уклонимо претпоставку леве инвертибилности из Последице 2.1.6. Осим тога, можемо доказати још јачи резултат:

Последица 2.2.2. *Нека је $\mathbf{T} = (T_1, T_2)$ субнормалан пар и нека је T_1^k и T_2^l квазинормалан за неке $k, l \in \mathbb{N}$. Тада је $\mathbf{T}$ сферично квазинормалан.*

Доказ. Пошто су T_i, $i = 1, 2$ субнормални, а T_1^k и T_2^l квазинормални, Теорема 2.1.2 имплицира да су T_i, $i = 1, 2$ квазинормални. Стога је $\mathbf{T}$ сферично-квазинормалан (видети [51, Напомена 2.6]). ■

Следећа лема се може посматрати као вишедимензионални аналог Леме 1.2.11 (видети Напомену 2.2.4 испод).

Лема 2.2.3. *Нека је $\mathbf{T} = (T_1, T_2)$ субнормалан пар, са нормалним проширењем $\mathbf{N} = (N_1, N_2)$. Пар $\mathbf{T}$ сферично-квазинормалан ако и само ако је инваријантан за $N_1^* N_1 + N_2^* N_2$.*

Напомена 2.2.4. *Ако посматрамо $\mathbf{N} = (N_1, N_2)$ као вектор колоне, можемо користити нотацију $\mathbf{N}^* \mathbf{N} = N_1^* N_1 + N_2^* N_2$, што нам даје следећи аналог Леме 1.2.11:*

Лема 2.2.5. *Нека је $\mathbf{T}$ субнормалан пар, са нормалним проширењем $\mathbf{N}$. Тада је $\mathbf{T}$ сферно-квазинормалан ако и само ако је $\mathcal{H}$ инваријантан за $\mathbf{N}^* \mathbf{N}$.*

Како је показано у [51, Пример 3.6], постоји сферично-квазинормалан дводимензионални тежински оператор помераја $W_{(\alpha,\beta)}$ такав да $W_{(\alpha,\beta)}^{(2,1)}$ није сферично-квазинормалан. Другим речима, ако је $\mathbf{T} = (T_1, T_2)$ сферично-квазинормалан пар, онда $\mathbf{T}^{(m,n)} = (T_1^m, T_2^n)$ не мора бити сферично-квазинормалан.

Следећа теорема даје довољан услов за еквивалентност сферичне квазинормалности $\mathbf{T}^{(n,n)} = (T_1^n, T_2^n)$ и сферичне квазинормалности $\mathbf{T} = (T_1, T_2)$.

Теорема 2.2.6. *Нека је $\mathbf{T} = (T_1, T_2)$ субнормалан пар са нормалним продужењем $\mathbf{N} = (N_1, N_2)$ такав да је $N_1 N_2 = 0$. Тада је $\mathbf{T}^{(n,n)} = (T_1^n, T_2^n)$ сферично-квазинормалан за неко $n \in \mathbb{N}$ ако и само ако је $\mathbf{T}$ сферично-квазинормалан.*

Сада дајемо још једну карактеризацију матрично-квазинормалних n-торки и исправљамо грешку из [51]:

Лема 2.2.7. *Нека је $\mathbf{T} = (T_1, T_2)$ субнормалан пар, са нормалним продужењем $\mathbf{N} = (N_1, N_2)$. Тада је $\mathbf{T}$ матрично-квазинормалан ако и само ако је $A_i^* T_j = 0$, $i, j = 1, 2$.*

Као последица претходног резултата, уочавамо да можемо ослабити услов у дефиницији матрично-квазинормалних парова:

Последица 2.2.8. *$\mathbf{T} = (T_1, T_2)$ је матрично-квазинормалан ако и само ако T_i комутира са $T_i^* T_j$, $i, j = 1, 2$.*

Ево исправке за Теорему 2.1.8:

Последица 2.2.9. *Нека је $\mathbf{T} = (T_1, T_2)$ субнормалан пар, са нормалним продужењем $\mathbf{N} = (N_1, N_2)$. Тада је $\mathbf{T}$ (заједнички) квазинормалан ако и само ако је $A_i A_j^* T_j = 0$, $i, j = 1, 2$.*

Подстакнути Лемом 1.2.11, дајемо још један аналоган резултат у случају више промењљивих.

Лема 2.2.10. *Нека је $\mathbf{T} = (T_1, T_2)$ субнормалан пар, са нормалним продужењем $\mathbf{N} = (N_1, N_2)$. Тада је $\mathbf{T}$ матрично-квазинормалан ако и само ако је $\mathcal{H}$ инваријантан за $N_i^* N_j$, $i, j = 1, 2$.*

Следећа теорема даје довољне услове за еквивалентност матричне квазинормалности $\mathbf{T}^{(n,n)} = (T_1^n, T_2^n)$ и матричне квазинормалности $\mathbf{T} = (T_1, T_2)$:

Теорема 2.2.11. *Нека је $\mathbf{T} = (T_1, T_2)$ субнормалан пар са нормалним продужењем $\mathbf{N} = (N_1, N_2)$, такав да је $N_1^* N_2 \geq 0$. Тада је $\mathbf{T}^{(n,n)} = (T_1^n, T_2^n)$ матрично-квазинормалан за неко $n \in \mathbb{N}$ ако и само ако је $\mathbf{T}$ матрично-квазинормалан.*

Напомена 2.2.12. *Нагласимо да је Теорема 2.2.11 општија верзија Теореме 2.1.2. Конкретно, добијамо Теорему 2.1.2 као последицу, узимајући да је $T_1 = T_2 = T$ и $N_1 = N_2$ у Теореми 2.2.11.*

2.3 Субнормални фактори нормалних оператора

У овом одељку фокусирамо се на општији приступ Проблему 2.1.1. Прецизније, посматрамо квадрат као производ и премештамо проблем из једнодимензионалног слушаја у вишедимензионални контекст. Третирамо нови (општи) проблем као обрат Фулеове теореме, посебно Последице 1.1.14. Како ћемо видети, добијене верзије, у специјалним случајевима, доводе до раније познатих резултата. Кључни корак је следеће опажање:

Можемо реформулисати Проблем 2.1.1 на следећи начин: *Нека је* $\mathbf{T} = (T, T)$ *субнормалан пар и претпоставимо да је* $T \cdot T$ *квазинормалан. Да ли следи да је* T *квазинормалан?*

Ово нас такође мотивише да поставимо следећа питања:

Проблем 2.3.1. *Нека је* $\mathbf{T} = (T_1, T_2)$ *субнормалан пар такав да је* T_1T_2 *квазинормалан. Које услове треба да испуњавају оператори* T_1 *и* T_2 *да би они били квазинормални?*

Проблем 2.3.2. *Нека је* $\mathbf{T} = (T_1, T_2)$ *(заједнички) квазинормалан пар такав да је* T_1T_2 *нормалан. Које услове треба да испуњавају оператори* T_1 *и* T_2 *да би они били нормални?*

Проблем 2.3.3. *Нека је* $\mathbf{T} = (T_1, T_2)$ *субнормалан пар такав да је* T_1T_2 *нормалан. Које услове треба да испуњавају оператори* T_1 *и* T_2 *да би они били нормални?*

Као што видимо, Проблем 2.3.3 може се третирати као обрат Последице 1.1.14.

2.3.1 Квазинормални фактори нормалних оператора

Почетна тачка у нашем разматрању биће следећа лема:

Лема 2.3.4. *Нека је* $\mathbf{T} = (T_1, T_2)$ *субнормалан пар са нормалним продужењем* $\mathbf{N} = (N_1, N_2)$ *такав да је* T_2 *квазинормалан и* T_1T_2 *нормалан. Ако је* T_1 *лево инвертибилан, онда је* T_2 *нормалан.*

Лема 2.3.5. *Нека је* $\mathbf{T} = (T_1, T_2)$ *субнормалан пар са нормалним продужењем* $\mathbf{N} = (N_1, N_2)$ *такав да је* T_2 *квазинормалан и* T_1T_2 *нормалан. Ако је* $\mathcal{R}(T_1) = \overline{\mathcal{R}(T_1)} \subseteq \overline{\mathcal{R}(T_2^*)}$ *и* $\mathcal{N}(T_1) = \mathcal{N}(T_2)$*, онда је* T_2 *нормалан.*

Последица 2.3.6. *Нека је* $\mathbf{T} = (T_1, T_2)$ *(заједнички) квазинормалан пар такав да је* T_1T_2 *нормалан. Ако је* $\mathcal{R}(T_1) = \mathcal{R}(T_2) = \overline{\mathcal{R}(T_2)}$ *и* $\mathcal{N}(T_1) = \mathcal{N}(T_2)$*, онда је* $\mathbf{T}$ *нормалан.*

Комбинујући претходне резултате, добијамо следећу теорему:

Теорема 2.3.7. *Нека је* $\mathbf{T} = (T_1, T_2)$ *(заједнички) квазинормалан пар такав да је* T_1T_2 *нормалан. Тада је* $\mathbf{T}$ *нормалан ако важи бар један од следећих услова:*

(i) T_1 *или* T_2 *је десно инвертибилан;*

(ii) T_1 *и* T_2 *су лево инвертибилни;*

(iii) $\mathcal{R}(T_i) = \overline{\mathcal{R}(T_i)} \subseteq \overline{\mathcal{R}(T_j^*)}$ *за* $i \neq j$, *и* $\mathcal{N}(T_1) = \mathcal{N}(T_2)$;

(iv) $\mathcal{R}(T_1) = \mathcal{R}(T_2) = \overline{\mathcal{R}(T_2)}$ *и* $\mathcal{N}(T_1) = \mathcal{N}(T_2)$.

Напомена 2.3.8. *У Последици 2.3.6 и Теореми 2.3.7 довољно је претпоставити да су* T_1 *и* T_2 *квазинормални, уместо (заједничке) квазинормалности пара* $\mathbf{T} = (T_1, T_2)$. *У наставку ћемо показати да можемо уклонити услов квазинормалности са једног (или оба) од координатних оператора.*

Напомена 2.3.9. *Иако услов* (iv) *Теореме 2.3.7 имплицира услов* (iii) *исте теореме (како је показано у доказу Последице 2.3.6), наведен је због своје елегантне форме.*

2.3.2 Субнормални фактори квазинормалних оператора и обрат Фулеове теореме

Претходни одељак завршава наше разматрање Проблема 2.3.3. Сада прелазимо на "проблем имплициране квазинормалности" и обрат Фулеове теореме, односно бавимо се Проблемом 2.3.1 и Проблемом 2.3.3.

Лема 2.3.10. *Нека је* $\mathbf{T} = (T_1, T_2)$ *субнормалан пар са нормалним продужењем* $\mathbf{N} = (N_1, N_2)$ *такав да је* T_1T_2 *квазинормалан. Тада је* T_2 *квазинормалан ако важи бар један од следећих услова:*

(i) $\mathrm{Comm}(|N_1N_2|) \subseteq \mathrm{Comm}(|N_2|)$;

(ii) T_1 *је квазинормалан и десно инвертибилан;*

(iii) T_1 *је квазинормалан и* N_1 *је ињективан.*

Теорема 2.1.2 сада следи као једноставна последица:

Доказ теореме 2.1.2. Нека је N нормално продужење за T, и нека је $T_1 = T^{n-1}$ и $T_2 = T$. Онда је $\mathbf{T} = (T_1, T_2)$ субнормалан пар са нормалним продужењем $\mathbf{N} = (N_1, N_2) = (N^{n-1}, N)$. Приметимо да је $(N_1N_2)^*(N_1N_2) = (N^*N)^n$ и стога је прва ставка из Леме 2.3.10 испуњена, према Теореми 1.1.1. Дакле, $T_2 = T$ је квазинормалан. ∎

Коришћењем Леме 2.3.10 и исте технике као у доказу Леме 2.3.5, можемо доказати наредну лему:

Лема 2.3.11. *Нека је* $\mathbf{T} = (T_1, T_2)$ *субнормалан пар са нормалним продужењем* $\mathbf{N} = (N_1, N_2)$ *такав да су* T_1 *и* T_1T_2 *квазинормални. Ако је* $\mathcal{R}(T_1) = \overline{\mathcal{R}(T_2^*)}$ *и* $\mathcal{N}(T_2) \subseteq \mathcal{N}(T_1)$, *онда је* T_2 *квазинормалан.*

Како бисмо доказали наш следећи резултат, сличан Леми 2.3.10, али такође и од независног интереса, потребна нам је следеће теорема:

Теорема 2.3.12. *[77] Нека су A и B оператори са $\sigma(A)\cap\sigma(B) = \emptyset$. Тада сваки оператор који комутира са $A + B$ и са AB такође комутира са A и B.*

Теорема 2.3.13. *Нека је $\mathbf{T} = (T_1, T_2)$ сферично-квазинормалан пар са нормалним продужењем $\mathbf{N} = (N_1, N_2)$ такав да је $\sigma(|N_1|)\cap\sigma(|N_2|) = \emptyset$. Ако је T_1T_2 квазинормалан, онда је $\mathbf{T}$ (заједнички) квазинормалан.*

Коначно, стижемо до главног резултата овог одељка:

Теорема 2.3.14 (Обрат Фулеове теореме). *Нека је $\mathbf{T} = (T_1, T_2)$ субнормалан пар са нормалним продужењем $\mathbf{N} = (N_1, N_2)$ такав да је T_1T_2 нормалан. Тада је $\mathbf{T}$ нормалан ако важи један од следећих услова:*

(i) *T_1 или T_2 је десно инвертибилан квазинормалан оператор;*

(ii) *T_1 је квазинормалан и N_1 и T_2 су лево инвертибилни, или T_2 је квазинормалан и T_1 и N_2 су лево инвертибилни;*

(iii) *T_1 или T_2 је квазинормалан, $\mathcal{R}(T_i) = \overline{\mathcal{R}(T_j^*)}$ за $i \neq j$, и $\mathcal{N}(T_1) = \mathcal{N}(T_2)$.*

(iv) $\mathrm{Comm}(|N_1N_2|) \subseteq \mathrm{Comm}(|N_1|)\cap\mathrm{Comm}(|N_2|)$ *и важи било који од услова $(i) - (iv)$ из Теореме 2.3.7;*

(v) *$\mathbf{T}$ је сферично-квазинормалан, $\sigma(|N_1|) \cap \sigma(|N_2|) = \emptyset$ и важи било који од услова $(i) - (iv)$ из Теореме 2.3.7.*

Глава 3

Сферична средња трансформација парова оператора

У овом поглављу, уводимо концепт сферичне средње трансформације за комутирајуће парове оператора. Ово нам омогућава да проширимо дефиницију средње трансформације, која је првобитно дефинисана у једнодимензионалним оквирима, на домен вишедимензионалне теорије оператора. Наш главни циљ је да истражимо различите спектралне особине ове трансформације, укључујући њену способност да очува Тејлоров спектар, као и неке аналитичке карактеристике. Осим тога, дајемо конкретне услове под којима трансформација задржава својство p-хипонормалности за дводимензионалне тежинске операторе померања.

3.1 Мотивација и претходни резултати

Нека је $T = U|T|$ поларна декомпозиција оператора $T \in \mathfrak{B}(\mathcal{H})$. У одељку 1.2.4, дали смо дефиниције Алутгеове и Дугалове трансформације оператора T. Недавно су аутори у раду [114] представили још једну трансформацију оператора. *Средња трансформација* оператора T, у ознаци $\mathcal{M}(T)$, дефинише се као:

$$\mathcal{M}(T) = \frac{1}{2}(U|T| + |T|U) = \frac{1}{2}(T + \widehat{T}).$$

У последњих неколико година, осим Алутгеове и Дугалове трансформације, средња трансформација је такође привукла значајну пажњу (видети, на пример, [14, 31, 32, 33, 104, 137, 179, 181]). Са становишта практичне употребе, једна од главних предности средње трансформације је следећа: понекад може бити веома захтевно пронаћи Алутгеову трансформацију датог оператора, јер онаукључује израчунавање корена позитивног оператора, док средња трансформација се заснива на сабирању два оператора, па је самим тим лакше добити средњу трансформацију ако је позната поларна декомпозиција оператора.

Нека је $\mathbf{T} = (V_1P, V_2P)$ (канонска) сферична поларна декомпозиција операторског пара $\mathbf{T}$ (видети Одељак 1.3). На аналоган начин као у једнодимензионом случају, добијамо *сферичну Алутгеову трансформацију* $\widetilde{\mathbf{T}}$ као $\widetilde{\mathbf{T}} = (\sqrt{P}V_1\sqrt{P}, \sqrt{P}V_2\sqrt{P})$ и *сферичну Дугалову трансформацију* $\widehat{\mathbf{T}}$ као $\widehat{\mathbf{T}} = (PV_1, PV_2)$ (погледати [16, 54, 55, 79, 109]). Природно, проширујемо појам средње трансформације на вишедимензионални случај.

Дефиниција 3.1.1. Нека је $\mathbf{T} = (T_1, T_2) = (V_1P, V_2P)$ канонска сферична поларна декомпозиција n-торке $\mathbf{T}$. *Сферичну средњу трансформацију* од $\mathbf{T}$ дефинишемо као

$$\mathcal{M}(\mathbf{T}) = (\mathcal{M}_1(\mathbf{T}), \mathcal{M}_2(\mathbf{T})) = \frac{1}{2}(V_1P + PV_1, V_2P + PV_2).$$

Појам се може лако општити на било коју n-торку оператора.

Напомена 3.1.1. *Обратимо пажњу да $\mathcal{M}_1(\mathbf{T})$ и $\mathcal{M}_2(\mathbf{T})$ у претходној дефиницији нису средње трансформације за T_1 и T_2, респективно, јер $T_i = V_iP$, $i = 1, 2$, нису стандардне поларне декомпозиције оператора на $\mathcal{H}$.*

У недавном раду (види [108]), аутори су увели појам *сферичне p-хипонормалности* на следећи начин: кажемо да је комутирајући пар $\mathbf{T} = (T_1, T_2)$ оператора на $\mathcal{H}$ *сферично p-хипонормалан* $(0 < p \leq 1)$, ако важи

$$(T_1^*T_1 + T_2^*T_2)^p \geq (T_1T_1^* + T_2T_2^*)^p.$$

Такође су показали следећу теорему:

Теорема 3.1.2. *[108] Нека је $\mathbf{W}_{(\alpha,\beta)} = (T_1, T_2)$ дводимензионални тежински оператор помераја. Тада, за $0 < p \leq 1$, важи да је $\mathbf{W}_{(\alpha,\beta)}$ сферично p-хипонормалан уколико је*

$$\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)} \geq \alpha^2_{(k_1-1,k_2)} + \beta^2_{(k_1,k_2-1)}, \quad \text{за све } k_1, k_2 \geq 0, \tag{3.1}$$

где је $\alpha_{(-1,0)} = \beta_{(0,-1)} = 0$.

У овом поглављу, ради краћег записа, користићемо следећу нотацију: за пар оператора $\mathbf{T} = (T_1, T_2)$, и $A, B \in \mathfrak{B}(\mathcal{H})$, са $A\mathbf{T}B$ означавамо

$$A\mathbf{T}B = (AT_1B, AT_2B).$$

Исто тако, за два пара оператора $\mathbf{A} = (A_1, A_2)$ и $\mathbf{B} = (B_1, B_2)$, пишемо

$$\mathbf{AB} = (A_1B_1, A_2B_2).$$

Коначно, $\mathbf{0}$ и $\mathbf{I}$ означавају парове оператора $(0, 0)$ и (I, I), респективно.

Глава је организована на следећи начин. У одељку 3.2, дајемо неке особине сферичне средње трансформације, које представљају основу за даље испитивање ове теме. У одељку 3.3, описујемо како се дводимензионални тежински оператори помераја понашају под овом трансформацијом. Конкретно, фокусирамо се на p-хипонормалност дводимензионалних тежинских оператора помераја и њихових сферичних средњих трансформација.

3.2 Општа својства

Почињемо овај одељак следећим једноставним запажањем.

Теорема 3.2.1. *Нека је* $\mathbf{T} = (T_1, T_2)$ *комутирајући пар оператора на* $\mathcal{H}$. *Следећи услови су еквивалентни:*

(i) $\mathbf{T}$ *је сферично-квазинормалан;*

(ii) $\mathcal{M}(\mathbf{T}) = \mathbf{T}$.

Следећа теорема тврди да је језгро пара оператора $\mathbf{T}$ сачувано под сферичном средњом трансформацијом.

Теорема 3.2.2. *Нека је* $\mathbf{T} = (T_1, T_2)$ *пар оператора на* $\mathcal{H}$. *Тада важи:*

$$\ker(\mathcal{M}(\mathbf{T})) = \ker(\mathbf{T}).$$

Последица 3.2.3. *Нека је* $\mathbf{T} = (T_1, T_2)$ *пар оператора на* $\mathcal{H}$. *Следећи услови су еквивалентни:*

(i) $\mathbf{T} = \mathbf{0}$*;*

(ii) $\mathcal{M}(\mathbf{T}) = \mathbf{0}$.

У духу претходне последице, следећа теорема се бави аналогним проблемом за операторски пар $\mathbf{I}$.

Теорема 3.2.4. *Нека је* $\mathbf{T} = (T_1, T_2)$ *пар оператора на* $\mathcal{H}$ *са сферничном поларним декомпозицијом* $\mathbf{T} = (V_1 P, V_2 P)$. *Следећи услови су еквивалентни:*

(i) $\mathbf{T} = \mathbf{I}$*;*

(ii) $\mathcal{M}(\mathbf{T}) = \mathbf{I}$ *и* $\operatorname{Re}(V_1^* V_2) = \frac{1}{2} I$.

Теорема 3.2.5. *Нека је* $\mathbf{T} = (T_1, T_2)$ *пар оператора на* $\mathcal{H}$ *и нека је* $U \in \mathfrak{B}(\mathcal{H})$ *унитарни оператор. Тада важи*

$$\mathcal{M}(U\mathbf{T}U^*) = U\mathcal{M}(\mathbf{T})U^*.$$

На основу претходне теореме видимо да се сферична средња трансформација "лепо понаша" у односу на унитарну еквиваленцију.

У општем случају, не мора бити $\sigma_T(\mathbf{T}) \neq \sigma_T(\mathcal{M}(\mathbf{T}))$ (видети Пример 3.3.8 испод). Међутим, ако је $\mathbf{T}$ сферична парцијална изометрија, добијамо потврдан одговор. Да бисмо доказали нашу тврдњу, користићемо следећу дефиницију и теорему.

Дефиниција 3.2.1. [15] Нека су $\mathbf{A} = (A_1, \dots, A_n)$ и $\mathbf{B} = (B_1, \dots, B_n)$ две n-торке оператора на $\mathcal{H}$. Кажемо да $\mathbf{A}$ и $\mathbf{B}$ *укрштено комутирају* (или да $\mathbf{A}$ *укрштено комутира* са $\mathbf{B}$) ако $A_i B_j A_k = A_k B_j A_i$ и $B_i A_j B_k = B_k A_j B_i$, за све $i, j, k = 1, \dots, n$.

Теорема 3.2.6. *(cf. [17, 18]) Нека* $\mathbf{A}$ *укрштено комутира са* $\mathbf{B}$ *на* $\mathcal{H}$*, и претпоставимо да је пар* $\mathbf{AB}$ *комутирајући. Тада важи*

$$\sigma_T(\mathbf{BA}) \setminus \{\mathbf{0}\} = \sigma_T(\mathbf{AB}) \setminus \{\mathbf{0}\}.$$

Теорема 3.2.7. *Нека је* $\mathbf{V} = (V_1, V_2)$ *сферична парцијална изометрија. Тада важи*

$$\mathcal{M}(\mathbf{V}) = \frac{1}{2}\left((I+P)V_1, (I+P)V_2\right),$$

где је $P = \sqrt{V_1^*V_1 + V_2^*V_2}$*.*
При томе,

$$\sigma_T(\mathbf{V}) = \sigma_T(\mathcal{M}(\mathbf{V})).$$

Усмеравамо сада нашу пажњу на тополошке особине сферичне средње трансформације. Како бисмо доказали наш наредни резултат, искористићемо следећу теорему.

Теорема 3.2.8. *[55] Сферична Алутгеова трансформација* $(T_1, T_2) \mapsto \widetilde{(T_1, T_2)}$ *је* $(\|\cdot\|, \|\cdot\|)$*-непрекидно пресликавање на* $\mathfrak{B}(\mathcal{H})$*.*

Теорема 3.2.9. *Нека је* $\mathbf{T} = (T_1, T_2)$ *комутирајући пар оператора са* $\ker(\mathbf{T}) = \{0\}$*. Сферична средња трансформација* $(T_1, T_2) \mapsto (\mathcal{M}_1(\mathbf{T}), \mathcal{M}_2(\mathbf{T}))$ *је* $(\|\cdot\|, \mathrm{SOT})$*-непрекидно пресликавање.*

3.3 Сферична средња трансформација дводимезионалних тежинских оператора помераја

Почињемо ову секцију извођењем опште формуле за сферичну средњу трансформацију за произвољан дводимензионални тежински оператор помераја.

Теорема 3.3.1. *Нека је* $\mathbf{W}_{(\alpha,\beta)} = (T_1, T_2)$ *дводимензионални тежински оператор помераја. Тада је* $\mathcal{M}(\mathbf{W}_{(\alpha,\beta)}) = (\mathcal{M}_1(\mathbf{W}_{(\alpha,\beta)}), \mathcal{M}_2(\mathbf{W}_{(\alpha,\beta)}))$ *дата са*

$$\mathcal{M}_1(\mathbf{W}_{(\alpha,\beta)})e_{\mathbf{k}} = \gamma_{\mathbf{k}} \frac{\sqrt{\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2} + \sqrt{\alpha_{\mathbf{k}+\varepsilon_1}^2 + \beta_{\mathbf{k}+\varepsilon_1}^2}}{2} e_{\mathbf{k}+\varepsilon_1},$$
$$\mathcal{M}_2(\mathbf{W}_{(\alpha,\beta)})e_{\mathbf{k}} = \delta_{\mathbf{k}} \frac{\sqrt{\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2} + \sqrt{\alpha_{\mathbf{k}+\varepsilon_2}^2 + \beta_{\mathbf{k}+\varepsilon_2}^2}}{2} e_{\mathbf{k}+\varepsilon_2},$$

за све $\mathbf{k} \in \mathbb{Z}_+^2$*, где је* $\varepsilon_1 = (1,0)$*,* $\varepsilon_2 = (0,1)$*,*

$$\gamma_{\mathbf{k}} = \begin{cases} \dfrac{\alpha_{\mathbf{k}}}{\sqrt{\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2}}, & \text{ако } \alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2 \neq 0, \\ 0, & \text{ако } \alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2 = 0, \end{cases} \tag{3.2}$$

и

$$\delta_{\mathbf{k}}=\begin{cases}\dfrac{\beta_{\mathbf{k}}}{\sqrt{\alpha_{\mathbf{k}}^2+\beta_{\mathbf{k}}^2}}, & \text{ако } \alpha_{\mathbf{k}}^2+\beta_{\mathbf{k}}^2\neq 0,\\ 0, & \text{ако } \alpha_{\mathbf{k}}^2+\beta_{\mathbf{k}}^2=0.\end{cases} \tag{3.3}$$

Напомена 3.3.2. *Нека је $\mathbf{W}_{(\alpha,\beta)}=(T_1,T_2)$ дводимензионални тежински оператор помераја. Из претходне теореме видимо да је и $\mathcal{M}(\mathbf{W}_{(\alpha,\beta)})=(\mathcal{M}_1(\mathbf{W}_{(\alpha,\beta)}),\mathcal{M}_2(\mathbf{W}_{(\alpha,\beta)}))$ дводимензионални тежински оператор помераја.*

Последица 3.3.3. *Нека је $\mathbf{W}_{(\alpha,\beta)}=(T_1,T_2)$ дводимензионални тежински оператор помераја такав да је $\alpha_{\mathbf{k}}^2+\beta_{\mathbf{k}}^2\neq 0$ за сваки $\mathbf{k}\in\mathbb{Z}_+^2$.*
Тада је $\mathcal{M}(\mathbf{W}_{(\alpha,\beta)})=(\mathcal{M}_1(\mathbf{W}_{(\alpha,\beta)}),\mathcal{M}_2(\mathbf{W}_{(\alpha,\beta)}))$ дата са

$$\mathcal{M}_1(\mathbf{W}_{(\alpha,\beta)})e_{\mathbf{k}}=\frac{\alpha_{\mathbf{k}}}{2}\left(1+\sqrt{\frac{\alpha_{\mathbf{k}+\varepsilon_1}^2+\beta_{\mathbf{k}+\varepsilon_1}^2}{\alpha_{\mathbf{k}}^2+\beta_{\mathbf{k}}^2}}\right)e_{\mathbf{k}+\varepsilon_1},$$
$$\mathcal{M}_2(\mathbf{W}_{(\alpha,\beta)})e_{\mathbf{k}}=\frac{\beta_{\mathbf{k}}}{2}\left(1+\sqrt{\frac{\alpha_{\mathbf{k}+\varepsilon_2}^2+\beta_{\mathbf{k}+\varepsilon_2}^2}{\alpha_{\mathbf{k}}^2+\beta_{\mathbf{k}}^2}}\right)e_{\mathbf{k}+\varepsilon_2},$$

за све $\mathbf{k}\in\mathbb{Z}_+^2$, где је $\varepsilon_1=(1,0)$, $\varepsilon_2=(0,1)$.

У наставку ћемо, због једноставности, увек претпостављати да је $\alpha_{\mathbf{k}}^2+\beta_{\mathbf{k}}^2\neq 0$ за све $\mathbf{k}\in\mathbb{Z}_+^2$.

Најпре, напомињемо да из Теореме 3.1.2 директно имамо следећи резултат.

Последица 3.3.4. *Нека је $\mathbf{W}_{(\alpha,\beta)}=(T_1,T_2)$ дводимензионални тежински оператор помераја. Тада је $\mathcal{M}(\mathbf{W}_{(\alpha,\beta)})=(\mathcal{M}_1(\mathbf{W}_{(\alpha,\beta)}),\mathcal{M}_2(\mathbf{W}_{(\alpha,\beta)}))$ сферично p-хипонормалан (за $0<p\leq 1$) ако и само ако важи*

$$m(\alpha)_{\mathbf{k}}^2+m(\beta)_{\mathbf{k}}^2\geq m(\alpha)_{\mathbf{k}-\varepsilon_1}^2+m(\beta)_{\mathbf{k}-\varepsilon_2}^2, \tag{3.4}$$

за све $\mathbf{k}\in\mathbb{Z}_+^2$, где је $\varepsilon_1=(1,0)$, $\varepsilon_2=(0,1)$,

$$m(\alpha)_{\mathbf{k}}=\alpha_{\mathbf{k}}\left(1+\sqrt{\frac{\alpha_{\mathbf{k}+\varepsilon_1}^2+\beta_{\mathbf{k}+\varepsilon_1}^2}{\alpha_{\mathbf{k}}^2+\beta_{\mathbf{k}}^2}}\right),$$
$$m(\beta)_{\mathbf{k}}=\beta_{\mathbf{k}}\left(1+\sqrt{\frac{\alpha_{\mathbf{k}+\varepsilon_2}^2+\beta_{\mathbf{k}+\varepsilon_2}^2}{\alpha_{\mathbf{k}}^2+\beta_{\mathbf{k}}^2}}\right),$$

и $m(\alpha)_{(-1,0)}=m(\beta)_{(0,-1)}=0$.

У општем случају, сферична (Алутгеова, Дугалова, средња) трансформација не очувавају хипонормалност операторских парова. У следећој теореми дајемо неке довољне услове под којима трансформација $\mathbf{W}_{(\alpha,\beta)} \mapsto \mathcal{M}(\mathbf{W}_{(\alpha,\beta)})$ очувава хипонормалност.

Теорема 3.3.5. *Нека је* $\mathbf{W}_{(\alpha,\beta)} = (T_1, T_2)$ *дводимензионални тежински оператор померaја. Претпоставимо да важе следећи услови:*

(i) $\mathbf{W}_{(\alpha,\beta)}$ *је хипонормалан;*

(ii) $\alpha_{(k_1+1,k_2)} = \alpha_{(k_1,k_2+1)}$ *и* $\beta_{(k_1+1,k_2)} = \beta_{(k_1,k_2+1)}$ *за све* $k_1, k_2 \geq 0$*;*

(iii) за све $k_1, k_2 \geq 0$*,*

$$\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)} \leq \sqrt{\left(\alpha^2_{(k_1-1,k_2)} + \beta^2_{(k_1-1,k_2)}\right)\left(\alpha^2_{(k_1+1,k_2)} + \beta^2_{(k_1+1,k_2)}\right)}.$$

Тада је $\mathcal{M}(\mathbf{W}_{(\alpha,\beta)})$ *хипонормалан.*

Напомена 3.3.6. *Аналог претходне теореме важи и када се хипонормалност замени са* p*-хипонормалношћу, за ма које* $0 < p \leq 1$.

Сетимо се следеће дефиниције:

Дефиниција 3.3.1. Низ $\{\sigma_k\}_{k\in\mathbb{Z}_+}$ реалних бројева се зове *Стилтјесов низ момената* ако постоји позитивна Борелова мера μ на затвореној полуправој $[0, +\infty)$ таква да важи

$$\sigma_k = \int_0^{+\infty} t^k \, d\mu(t), \quad k \in \mathbb{Z}_+.$$

Мера μ се зове *репрезентациона мера* за $\{\sigma_k\}_{k\in\mathbb{Z}_+}$.

Последица 3.3.7. *Нека је* $\mathbf{W}_{(\alpha,\beta)} = (T_1, T_2)$ *дводимензионални тежински оператор померaја. Претпоставимо да важе следећи услови:*

(i) $\mathbf{W}_{(\alpha,\beta)}$ *је хипонормалан;*

(ii) $\alpha_{(k_1+1,k_2)} = \alpha_{(k_1,k_2+1)}$ *и* $\beta_{(k_1+1,k_2)} = \beta_{(k_1,k_2+1)}$ *за све* $k_1, k_2 \geq 0$*;*

(iii) За све $k_2 \geq 0$*, низ* $\{\sigma^{(k_2)}_k\}_{k\in\mathbb{Z}_+}$ *дат са*

$$\sigma^{(k_2)}_k = \alpha^2_{(k,k_2)} + \beta^2_{(k,k_2)}, \quad k \in \mathbb{Z}_+,$$

је Стилтјесов низ момената.

Тада је $\mathcal{M}(\mathbf{W}_{(\alpha,\beta)})$ *хипонормалан.*

Завршавамо ово поглавље давањем примера пара оператора **T** таквог да $\sigma_T(\mathbf{T}) \neq \sigma_T(\mathcal{M}(\mathbf{T}))$. Штавише, показаћемо да је $\mathcal{M}(\mathbf{T})$ инвертибилан у Тејлоровом смислу, иако **T** то није.

Пример 3.3.8. *Нека је $\{e_{(k_1,k_2)}\}_{(k_1,k_2)\in\mathbb{Z}^2}$ канонска база простора $l^2(\mathbb{Z}^2)$ и за $n \in \mathbb{Z}$ нека је*

$$\theta_n = \begin{cases} 1, & \text{ако је } n \text{ паран}, \\ \frac{1}{n^2}, & \text{ако је } n \text{ непаран}. \end{cases}$$

Нека је $\mathbf{W}_{(\alpha,\beta)} = (T_1, T_2)$ дводимензионални тежински оператор померајa дефинисан на следећи начин:

$$T_1 e_{(k_1,k_2)} = \alpha_{(k_1,k_2)} e_{(k_1+1,k_2)},$$

и

$$T_2 e_{(k_1,k_2)} = \alpha_{(k_1,k_2)} e_{(k_1,k_2+1)},$$

где је $\alpha_{(k_1,k_2)} = \theta_{k_1+k_2}$ за $(k_1,k_2) \in \mathbb{Z}^2$. Тада $\mathbf{W}_{(\alpha,\beta)} = (T_1, T_2)$ није инвертибилан у Тејлоровом смислу, док $\mathcal{M}(\mathbf{W}_{(\alpha,\beta)})$ јесте инвертибилан (у Тејлоровом смислу).

Глава 4

Субнормални дуали и комплетирање до нормалности

Мотивисани дефиницијама субнормалних дуала (видети одељак 1.2.1) и са основним циљем да разматрамо комплетирање горње-троугаоне 2×2 операторске матрице (са познатим дијагоналним блоковима) до нормалног оператора, уводимо следећу дефиницију:

Дефиниција 4.0.1. Нека је $A \in \mathfrak{B}(\mathcal{H})$ и $B \in \mathfrak{B}(\mathcal{K})$. Кажемо да су оператори A и B *нормални комплементи* ако постоји $C \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$ тако да је операторска матрица

$$M_C = \begin{bmatrix} A & C \\ 0 & B^* \end{bmatrix} \qquad (\star)$$

нормална.

Такође, нека је

$$\mathfrak{N}(A, B) = \{C \in \mathfrak{B}(\mathcal{K}, \mathcal{H}) : \ M_C \text{ дата са } (\star) \text{ је нормална}\}.$$

Јасно је да су оператори A и B нормални комплементи ако и само ако постоји $C \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$ тако да важе следећи једнакости:

$$A^*A - AA^* = CC^* \qquad (4.1)$$

$$B^*B - BB^* = C^*C \qquad (4.2)$$

$$A^*C = CB. \qquad (4.3)$$

Важно је приметити разлику између дуала и нормалних комплемената, јер ако су A и B нормални комплементи према Дефиницији 4.0.1, онда не следи да је B дуал од A, према дефиницији уведеној у [44], јер A не мора да буде ни чист. Као што је већ поменуто, дефиниција 4.0.1 уведена је са циљем да се одговори на питање о комплетирању операторских матрица до нормалности и зато не намећемо никаква додатна ограничења операторима A и B.

4.1 Различите карактеризације нормалних комплемената

У следеће две теореме, представићемо неке карактеризације оператора A и B као нормалних комплемената у односу на декомпозиције ових оператора. Најпре, подсетимо да је оператор $T \in \mathfrak{B}(\mathcal{H})$ назива *позинормалним* ако постоји позитиван оператор $Q \in \mathfrak{B}(\mathcal{H})$ такав да важи $TT^* = T^*QT$. Сваки хипонормалан оператор је позинормалан, што директно следи из Тврђења 1.1.7 и чињенице да је T позинормалан ако и само ако је $\mathcal{R}(T) \subseteq \mathcal{R}(T^*)$ (видети [150, Тврђење 2.1]).

Теорема 4.1.1. *Нека су $A \in \mathfrak{B}(\mathcal{H})$, $B \in \mathfrak{B}(\mathcal{K})$. Нека су још $A_1 = P^{cr}_{\mathcal{N}(A)^\perp} A|_{\mathcal{N}(A)^\perp}$ и $B_1 = P^{cr}_{\mathcal{N}(B)^\perp} B|_{\mathcal{N}(B)^\perp}$. Следећи услови су еквивалентни:*

(i) *Оператори A и B су нормални комплементи;*

(ii) *Оператори A и B су позинормални, и A_1 и B_1 су нормални комплементи.*

Штавише,

$$\mathfrak{N}(A,B) = \{C_1 \oplus 0 : \, C_1 \in \mathfrak{N}(A_1, B_1)\}.$$

Теорема 4.1.2. *Нека су $A \in \mathfrak{B}(\mathcal{H})$ и $B \in \mathfrak{B}(\mathcal{K})$ оператори са чистим деловима A_p и B_p, респективно. Следећи усллови су еквивалентни:*

(i) *Оператори A и B су нормални комплементи;*

(ii) *Оператори A_p и B_p су нормални комплементи.*

Штавише,

$$\mathfrak{N}(A,B) = \{C_1 \oplus 0 : \, C_1 \in \mathfrak{N}(A_p, B_p)\}.$$

Следеће тврђење излаже потребне и довољне услове за комплетирање операторске матрице M_C, дате изразом ($\star$), до нормалног оператора, у односу на постојање парцијалне изометрије с прописаним иницијалним и финалним просторима. Напомињемо да је претпоставка да су $A \in \mathfrak{B}(\mathcal{H})$ и $B \in \mathfrak{B}(\mathcal{K})$ хипонормални оператори природна, јер је то неопходан услов да би A и B били нормални комплементи.

Теорема 4.1.3. *Нека су $A \in \mathfrak{B}(\mathcal{H})$ и $B \in \mathfrak{B}(\mathcal{K})$ хипонормални оператори. Следећи услови су еквивалентни:*

(i) *Оператори A и B су нормални комплементи;*

(ii) *Постоји парцијална изометрија $U \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$ са иницијалним простором $\mathcal{M} \supseteq \overline{\mathcal{R}([B^*, B])}$ и финалним простором $\mathcal{N} \supseteq \overline{\mathcal{R}([A^*, A])}$ таква да важи*

$$[A^*, A]U = U[B^*, B], \tag{4.4}$$

$$A^*[A^*, A]^{1/2}U = U[B^*, B]^{1/2}B. \tag{4.5}$$

Штавише,

$$\mathfrak{N}(A,B) = \{U[B^*,B]^{1/2} : \ U \text{ је парцијална изометрија из дела } (ii)\}.$$

Размотримо сада следећу ознаку уведену у [101]: За два позитивна оператора $A \in \mathfrak{B}(\mathcal{H})$ и $B \in \mathfrak{B}(\mathcal{K})$, дефинишемо

$$\mathfrak{C}(A,B) = \{A^{1/2}UB^{1/2} : U \in \mathfrak{B}(\mathcal{K},\mathcal{H}), \ \|U\| \leq 1\}.$$

У наредној теореми, показујемо да за нормалне комплементе A и B из $\mathfrak{B}(\mathcal{H})$, оператор $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ за који важи да је M_C нормалан може бити представљен за свако $\lambda \in [0,1]$ у облику $C = [A^*,A]^{\frac{\lambda}{2}} U_\lambda [B^*,B]^{\frac{1-\lambda}{2}}$, где је $U_\lambda \in \mathfrak{B}(\mathcal{K},\mathcal{H})$ контракција.

Теорема 4.1.4. *Нека су $A, B \in \mathfrak{B}(\mathcal{H})$ нормални комплементи. Тада важи:*

$$\mathfrak{N}(A,B) \subseteq \bigcap_{\lambda\in[0,1]} \mathfrak{C}([A^*,A]^\lambda, [B^*,B]^{1-\lambda}). \tag{4.6}$$

У случају када су $A \in \mathfrak{B}(\mathcal{H})$ и $B \in \mathfrak{B}(\mathcal{K})$ нормални комплементи и један од њих је квазинормалан, онда су оба квазинормална. Такође, у том случају, чисти делови од A и B су унитарно еквивалентни. Истичемо да и обрнута тврдња важи, што је показано у наредној теореми.

Теорема 4.1.5. *Нека су $A \in \mathfrak{B}(\mathcal{H})$ и $B \in \mathfrak{B}(\mathcal{K})$ такви да је један од њих квазинормалан. Тада важи:*

(i) *A и B су нормални комплементи;*

(ii) *Чисти делови од A и B су унитарно еквивалентни.*

Претходна теорема може се тумачити као уопштење следећег једноставног резултата.

Последица 4.1.6. *Нека је $A \in \mathfrak{B}(\mathcal{H})$ нормалан оператор и $B \in \mathfrak{B}(\mathcal{K})$ субнормалан оператор. Тада важи:*

(i) *A и B су нормални комплементи;*

(ii) *B је нормалан.*

4.2 Заједничка спектрална својства нормалних комплемената

У случају када су $A \in \mathfrak{B}(\mathcal{H})$ и $B \in \mathfrak{B}(\mathcal{K})$ дати оператори, проблем комплетирања операторске матрице M_C до Фредхолмовог оператора има једну веома интересантну особину. Наиме, постојање оператора $C \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$ таквог да је M_C Фредхолмов оператор еквивалентно је постојању инвертибилног таквог $C \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$ (видети [61]). Ова особина важи и за нека друга комплетирања (видети [58, 59, 184]). Као што ћемо видети у следећем резултату, ово није случај код комплетирања M_C до нормалног оператора. Другим речима, ако су $A \in \mathfrak{B}(\mathcal{H})$ и $B \in \mathfrak{B}(\mathcal{K})$ нормални комплементи, онда $C \in \mathfrak{N}(A, B)$ не може бити инвертибилан.

Теорема 4.2.1. *Нека су $A, B \in \mathfrak{B}(\mathcal{H})$ нормални комплементи. Тада $0 \in \sigma(C)$ за свако $C \in \mathfrak{N}(A, B)$.*

Додатно, у следећој теореми, показаћемо да за било који субнормални оператор $A \in \mathcal{B}(\mathcal{H})$ спектар његовог самокомутатора садржи нулу, односно, самокомутатор није инвертибилан.

Теорема 4.2.2. *Нека је $A \in \mathfrak{B}(\mathcal{H})$ субнормалан (или хипонормалан) оператор. Тада $0 \in \sigma([A^*, A])$.*

Из горе наведене теореме можемо закључити да за $C \in \mathfrak{N}(A, B)$ имамо да $0 \in \sigma_l(C) \cap \sigma_r(C)$, тј. важи следећа последица:

Последица 4.2.3. *Нека су $A, B \in \mathfrak{B}(\mathcal{H})$ нормални комплементи. Тада $0 \in \sigma([A^*, A]) \cap \sigma([B^*, B])$ и $\sigma([A^*, A]) = \sigma([B^*, B])$.*

Уопштено, када разматрамо различите особине M_C, можемо уочити сличност између A и B у погледу неких својстава. Имајући на уму одређене резултате о допунама горње-троугаоне операторске матрице до инвертибилности и Фредхолмности, можемо доћи до следећих закључака:

Теорема 4.2.4. *Нека су $A \in \mathfrak{B}(\mathcal{H})$ и $B \in \mathfrak{B}(\mathcal{K})$ дати оператори. Тада важе следеће тврдње:*

(i) *Ако је операторска матрица M_C дата са $(\star)$ инвертибилна за неко $C \in \mathfrak{N}(A, B)$, онда су и A и B лево инвертибилни. Штавише, ако је M_C инвертибилна, онда инвертибилност једног од оператора A и B подразумева инвертибилност другог.*

(ii) *Ако је операторска матрица M_C дата са $(\star)$ Фредхолмова за неко $C \in \mathfrak{N}(A, B)$, онда су A и B лево полу-Фредхолмови. Штавише, ако је M_C Фредхолмова, онда Фредхолност једног од оператора A и B подразумева Фредхолмност другог.*

У следећим теоремама, разматраћемо специфичне случајеве када су $A \in \mathfrak{B}(\mathcal{H})$ и $B \in \mathfrak{B}(\mathcal{K})$ нормални комплементи, или када су $A \in \mathfrak{B}(\mathcal{H})$ и $B \in \mathfrak{B}(\mathcal{K})$ нормални комплементи такви да је један од њих квазинормалан. Најпре, видећемо да ако су $A \in \mathfrak{B}(\mathcal{H})$ и $B \in \mathfrak{B}(\mathcal{K})$ нормални комплементи и ако за неки $C \in \mathfrak{N}(A, B)$ имамо да је M_C инјективан, онда су и A и B инјективни (што у општем случају не важи). Такође, показаћемо да ћемо уместо импликација које имамо у ставкама (i) – (ii) Теореме 4.2.4, добити еквиваленције када су $A \in \mathfrak{B}(\mathcal{H})$ и $B \in \mathfrak{B}(\mathcal{K})$ нормални комплементи и када је један од њих квазинормалан.

Теорема 4.2.5. *Нека су $A \in \mathfrak{B}(\mathcal{H})$ и $B \in \mathfrak{B}(\mathcal{K})$ нормални комплементи. Ако је операторска матрица M_C ињективна за неко $C \in \mathfrak{N}(A, B)$, онда су и A и B ињективни.*

Случај када је један од оператора $A \in \mathfrak{B}(\mathcal{H})$ и $B \in \mathfrak{B}(\mathcal{K})$, који су нормални комплементи, квазинормалан, разматрамо у наредној теореми:

Теорема 4.2.6. *Нека су $A \in \mathfrak{B}(\mathcal{H})$ и $B \in \mathfrak{B}(\mathcal{K})$ нормални комплементи такви да је један од њих квазинормалан, и нека је $C \in \mathfrak{N}(A, B)$. Тада важи:*

(i) *M_C је инвертибилан ако и само ако су A и B лево инвертибилни оператори;*

(ii) *M_C је Фредхолмов ако и само ако су A и B лево полу-Фредхолмови оператори;*

(iii) *M_C је регуларан ако и само ако су A и B регуларни оператори.*

На основу Теореме 4.2.6, можемо закључити да, ако су $A \in \mathfrak{B}(\mathcal{H})$ и $B \in \mathfrak{B}(\mathcal{K})$ нормални комплементи и ако је један од њих квазинормалан, онда је операторска матрица M_C инвертибилна (Фредхолмова или регуларна) за неко $C \in \mathfrak{N}(A, B)$ ако и само ако је инвертибилна (Фредхолмова или регуларна) за свако $C \in \mathfrak{N}(A, B)$.

Што се тиче регуларности матрице M_C у случају када $C \in \mathfrak{N}(A, B)$, имамо следећи резултат:

Теорема 4.2.7. *Нека су $A \in \mathfrak{B}(\mathcal{H})$ и $B \in \mathfrak{B}(\mathcal{K})$ нормални комплементи такви да је један од њих квазинормалан и нека је $C \in \mathfrak{N}(A, B)$. Ако су A, B и C регуларни, онда су $(A^*)^\dagger$ и $(B^*)^\dagger$ нормални комплементи и $(C^*)^\dagger \in \mathfrak{N}((A^*)^\dagger, (B^*)^\dagger)$.*

Дефиниција 4.2.1. Оператор $A \in \mathfrak{B}(\mathcal{H})$ се зове *самокомплементаран* субнормалан оператор ако је $\mathfrak{N}(A, A)$ непразан скуп.

У наставку ћемо писати $\mathfrak{N}(A, A) = \mathfrak{N}(A)$. Из $(4.1)-(4.3)$, имамо да је A самокомплементаран субнормалан оператор ако и само ако постоји нормалан оператор $C \in \mathfrak{B}(\mathcal{H})$ такав да важи

$$[A^*, A] = CC^* \quad \text{и} \quad A^*C = CA. \tag{4.7}$$

У том случају, $\mathfrak{N}(A)$ се састоји само од нормалних оператора.

Очигледно је да је сваки квазинормалан оператор A самокомплементаран субнормалан (можемо узети $C = (A^*A - AA^*)^{1/2}$) и A је самокомплементаран ако и само ако је чисти део оператора A самодуалан.

Следећа теорема каже да ниједна линеарна комбинација оператора $C \in \mathfrak{N}(A)$ и његовог адјунгованог оператора није инвертибилан оператор.

Теорема 4.2.8. *Нека је $A \in \mathfrak{B}(\mathcal{H})$ самокомплементаран субнормалан оператор. Тада*

$$0 \in \sigma(\lambda C + \mu C^*)$$

за свако $C \in \mathfrak{N}(A)$ и свако $\lambda, \mu \in \mathbb{C}$.

Као последицу, имамо да реални и имагинарни делови оператора $C \in \mathfrak{N}(A)$ нису инвертибилни.

Последица 4.2.9. *Нека је $A \in \mathfrak{B}(\mathcal{H})$ самокомплементаран субнормалан оператор и нека је $C \in \mathfrak{N}(A)$. Тада $\mathrm{Re}\,(C)$ и $\mathrm{Im}\,(C)$ нису инвертибилни.*

Ако је A самокомплементаран субнормалан оператор и $C \in \mathfrak{N}(A)$, Теорема 4.2.1 нам каже да $0 \in \sigma(C)$. Следећи резултати дају неке довољне услове да је $\sigma(C) = \{0\}$, што је уствари еквивалентно нормалности оператора A.

Теорема 4.2.10. *Нека је $A \in \mathfrak{B}(\mathcal{H})$ самокомплементаран субнормалан оператор. Ако постоји $C \in \mathfrak{N}(A)$ такав да A и C^n комутирају за неко $n \in \mathbb{N}$, онда је A нормалан.*

Теорема 4.2.11. *Нека је $A \in \mathfrak{B}(\mathcal{H})$ самокомплементаран субнормалан оператор. Ако $C \in \mathfrak{N}(A)$ задовољава бар један од следећих услова:*

(i) $AC = CA$,

(ii) $A^*C = AC$,

(iii) $\mathcal{R}(C) \perp \mathcal{R}(\mathrm{Im}\,(A))$,

(iv) $\mathrm{Re}\,(C)$ *и* $\mathrm{Re}\,(C^2)$ *комутирају са* A,

онда је A нормалан.

Последица 4.2.12. *Нека је $A \in \mathfrak{B}(\mathcal{H})$ самодуалан субнормалан оператор. Тада A не комутира ни са једним $C \in \mathfrak{N}(A)$.*

4.3 Самодуалност Алутгеове и Дугалове трансформације

На почетку ове секције, дајемо два посебна резултата која обезбеђују довољне услове да Алутгеова и Дугалова трансформација чистих хипонормалних (полухипонормалних) оператора $A \in \mathfrak{B}(\mathcal{H})$ буду самодуални субнормални оператори.

Најпре, ако је $A \in \mathfrak{B}(\mathcal{H})$ чист хипонормалан оператор са густом сликом, онда је A инјективан и у поларној декомпозицији $A = U|A|$, имамо да је U унитаран оператор. Наиме, из $A^*A \geq AA^*$ следи да је $\mathcal{N}(A) \subseteq \mathcal{N}(A^*) = \mathcal{R}(A)^\perp = \{0\}$. Такође, користећи $U^*|A|^2U \geq |A|^2$ и поларну декомпозицију $A = U|A|$, закључујемо да је хипонормалност оператора A еквивалентна следећем услову:

$$U^*|A|^2U \geq |A|^2.$$

Спремни смо да дамо први резултат:

Теорема 4.3.1. *Нека је $A \in \mathfrak{B}(\mathcal{H})$ чист хипонормалан оператор са густом сликом и поларном декомпозицијом $A = U|A|$, и нека је $P = (U^*|A|^2U - |A|^2)^{1/2}$. Ако је $P\widehat{A}$ самоконјугован оператор, онда је $\widehat{A}$ самодуалан субнормалан оператор.*

Да бисмо доказали следећи резултат који разматра случај када је Алутгеова трансформација чистог полухипонормалног оператора самодуалан субнормалан оператор, потребан нам је следећи помоћни резултат:

Теорема 4.3.2. *[38, Lemma 4] Нека је $T = U|T|$ чист p-хипонормалан оператор са густом сликом. Тада је Алутгеова трансформација $\widetilde{T}$ чист $(p + \frac{1}{2})$-хипонормалан оператор.*

Теорема 4.3.3. *Нека је $A \in \mathfrak{B}(\mathcal{H})$ чист полухипонормалан оператор са густом сликом и поларном декомпозицијом $A = U|A|$, и нека је $P := (U^*|A|U - U|A|U^*)^{1/2}$. Ако су $[A^*, \widehat{A}]$ и $P\widehat{A}|A|^{1/2}$ самоконјуговани оператори, онда је $\widetilde{A}$ самодуалан субнормалан оператор.*

Дефиниција 4.3.1 ([114]). Нека је $T = U|T|$ поларна декомпозиција оператора $T \in \mathfrak{B}(\mathcal{H})$. Оператор T је оператор δ-класе ако је $U^2|T| = |T|U^2$.

Мотивисани дефиницијом и Теоремом 4.3.3, имамо следећи закључак:

Последица 4.3.4. *Нека је $A \in \mathfrak{B}(\mathcal{H})$ чист полухипонормалан оператор δ-класе са густом сликом, и са поларном декомпозицијом $A = U|A|$. Тада је $\widetilde{A}$ самодуалан субнормалан оператор.*

Теорема 4.3.5. *Нека је $A \in \mathfrak{B}(\mathcal{H})$. Ако је поларна декомпозиција оператора A дата као $A = U|A|$, где је U унитаран оператор, онда су следећи услови еквивалентни:*

(i) *A је нормалан;*

(ii) *A је оператор δ-класе и $\mathcal{N}(\mathcal{H}) \cap \mathfrak{N}(A, \widehat{A}) \neq \emptyset$.*

Лема 4.3.6. *Нека је $A \in \mathfrak{B}(\mathcal{H})$. Тада је сваки редукујући подпростор за A редукујући подпростор и за $\widehat{A}$ и $\widetilde{A}$.*

Нека је $A \in \mathcal{B}(\mathcal{H})$ самодуалан оператор са поларном декомпозицијом $A = U|A|$, и нека је $C \in \mathfrak{N}(A)$. Ако претпоставимо да A има густу слику, онда је U унитаран оператор. Оператор $C \in \mathfrak{N}(A)$ је нормалан па се може се представити као $C = V'|C|$, за неки унитаран оператор V' (видети [85, p. 66]). Ипак, следећа теорема показује да у неким случајевима, за сваки нетривијалан заједнички редукујући подпростор за A и C, рестрикције U и V морају бити различите.

Теорема 4.3.7. *Нека је $A \in \mathfrak{B}(\mathcal{H})$ самодуалан оператор δ-класе са густом сликом и поларном декомпозицијом $A = U|A|$, и нека је $C \in \mathfrak{N}(A)$. Ако је $C = V|C|$ декомпозиција за C где је V унитаран оператор, и $\mathcal{M}$ је нетривијалан заједнички редукујући подпростор за A и C, онда је $U|_{\mathcal{M}} \neq V|_{\mathcal{M}}$.*

Напомена 4.3.8. *Напоменимо да претходна теорема важи и у случају када је $C = V|C|$ "обична" поларна декомпозиција оператора C.*

Биографија

Хранислав Станковић је рођен 10. августа 1994. године у Прокупљу. Основну школу ”Топлички хероји“ у Житорађи је завршио као носилац дипломе ”Вук Караџић“ и титуле ђака генерације. Гимназију у Прокупљу је такође завршио као носилац дипломе ”Вук Караџић“.

Активно је учествовао на такмичењима из математике, српског језика и биологије, редовно освајајући прва места на регионалним такмичењима. Такође је добио бројне похвале, признања и награде на различитим литерарним и музичким такмичењима. Као ученик шестог разреда, објавио је збирку поезије 2007. године.

Хранислав је започео студије математике 2013. године на Природно-математичком факултету Универзитета у Нишу, и завршио их је 2016. године са просеком 10,00. Исте године уписао се на мастер студије математике, специјализујући се у области теоријске математике, које је успешно завршио 2018. године. Успешно је одбранио мастер тезу, под насловом ”Карактеризација оператора ранга 1“, добивши оцену 10.

Током академске године 2018/2019, започео је докторске студије математике на Природно-математичком факултету Универзитета у Нишу. Положио је све предвиђене испите, остваривши просечну оцену 10.

Током студија, био је корисник стипендије ”Доситеја“ у два наврата.

У 2019. години је постао истраживач-приправник на Природно-математичком факултету Универзитета у Нишу. Од тада је активно укључен у научно-истраживачки пројекат ”Проблеми нелинеарне анализе, теорије оператора, топологије и примена“ (OI 174025), који финансира Министарство просвете, науке и технолошког развоја Републике Србије.

На Природно-математичком факултету, предавао је различите курсеве, укључујући Математичку логику и теорију скупова, Линеарну алгебру, Елементарну математику 1, Увод у топологију и Математичку анализу 3 на основним студијама. Додатно, предавао је курсеве као што су Теорија оператора, Теорија скупова, Теорија фиксне тачке и примене и Алгебарска топологија на мастер студијама.

Почевши од 2021. године, ради као асистент на Електронском факултету Универзитета у Нишу, где предаје курсеве као што су Математика 1, Математика 2, Математика 3, Математички методи, Теорија вероватноће и статистика и Нумерички алгоритми.

До сада је објавио пет научних радова, а још три су тренутно на рецензији у међународним часописима са импакт фактором.

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ИЗЈАВА О АУТОРСТВУ

Изјављујем да је докторска дисертација, под насловом

SUBNORMAL OPERATORS: A MULTIVARIABLE OPERATOR THEORY PERSPECTIVE

која је одбрањена на Природно-математичком факултету Универзитета у Нишу:

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Дозвољавам да се објаве моји лични подаци, који су у вези са ауторством и добијањем академског звања доктора наука, као што су име и презиме, година и место рођења и датум одбране рада, и то у каталогу Библиотеке, Дигиталном репозиторијуму Универзитета у Нишу, као и у публикацијама Универзитета у Нишу.

У Нишу, 09.01.2024.

Потпис аутора дисертације:

Хранислав М. Станковић

ИЗЈАВА О ИСТОВЕТНОСТИ ШТАМПАНОГ И ЕЛЕКТРОНСКОГ ОБЛИКА ДОКТОРСКЕ ДИСЕРТАЦИЈЕ

Наслов дисертације:

SUBNORMAL OPERATORS: A MULTIVARIABLE OPERATOR THEORY PERSPECTIVE

Изјављујем да је електронски облик моје докторске дисертације, коју сам предао/ла за уношење у **Дигитални репозиторијум Универзитета у Нишу,** истоветан штампаном облику.

У Нишу, 09.01.2024.

Потпис аутора дисертације:

Хранислав М. Станковић

ИЗЈАВА О КОРИШЋЕЊУ

Овлашћујем Универзитетску библиотеку „Никола Тесла“ да у Дигитални репозиторијум Универзитета у Нишу унесе моју докторску дисертацију, под насловом:

SUBNORMAL OPERATORS: A MULTIVARIABLE OPERATOR THEORY PERSPECTIVE

Дисертацију са свим прилозима предао/ла сам у електронском облику, погодном за трајно архивирање.

Моју докторску дисертацију, унету у Дигитални репозиторијум Универзитета у Нишу, могу користити сви који поштују одредбе садржане у одабраном типу лиценце Креативне заједнице (Creative Commons), за коју сам се одлучио/ла.

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